THE PLANCHEREL FORMULA FOR PARABOLIC SUBGROUPS OF THE CLASSICAL GROUPS

By

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0. Introduction

In [6] we worked out explicit Plancherel formulae for the parabolic subgroups of real rank one simple Lie groups. Here we continue that work by considering a class of non-unimodular groups that includes most of the maximal parabolic subgroups of the classical groups. In those maximal parabolics $P = MAN$, M need not be compact. This has two important consequences. First, the compact extension procedure of [6, §4], based on [7, §4], must be replaced by a procedure based on [8, §2]. Second, and more important, the global operator in our Plancherel formula, viewed as an operator on $Z = \text{center}\ (N)$, becomes non-elliptic, e.g., the wave operator. This causes L_1 problems that are not yet completely resolved. Despite that, we obtain explicit Plancherel formulae (Theorem 4.9) for virtually all the maximal parabolic subgroups of the classical groups. In addition, we describe the nature of the global operator (Theorem 5.11) that occurs in the Plancherel formula of any parabolic subgroup of a semisimple Lie group whose nilradical is nonabelian and has square integrable representations.

§1 contains a discussion of non-unimodular Plancherel formulae in general and the domain problem for the global operator that compensates lack of unimodularity. In §2 we describe the maximal parabolic subgroups in a large family of classical groups and specify their generic representations. Then in §3 we define the global operators for those parabolic groups and examine their analytic and algebraic properties. §4 consists of the Plancherel formula (using the global operator) for that family of parabolics and a discussion of the extent to which analogous formulae hold for the other maximal parabolic subgroups of classical groups. In §5 we introduce another type of global operator — it exists whenever the nilradical is

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noncommutative but has square integrable representations. We compare the new operators with the old ones and verify that they agree when both are defined. We then apply the new operators to obtain Plancherel formulae for several non-maximal parabolics. Finally, in §6, we return to the domain question for certain of the global operators.

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1. The non-unimodular Plancherel formula

As in [6] we emphasize the global (on the group) operator that appears in the Plancherel formula, rather than the infinitesimal (on each representation space) operators. In effect, the global operator carries more information and yields the infinitesimal operators directly. In any case, the infinitesimal operators are fairly transparent — for example, they are multiplication by the modular function when the representation is induced from the kernel of the modular function ([14], [3]), or multiplication by canonical semi-invariants when the representation has a Kirillov model [4]. In our situation the global operator turns out to be an extremely interesting sort of generalized pseudo-differential operator, and this sheds some light (but many questions) on problems of harmonic analysis on the group.

Here is a global formulation of the non-unimodular Plancherel theorem (see also [7, theorem 6.4]).

1.1 Theorem. *Let G be a locally compact group of type I with right Haar measure dg and modular function δ_G. Then there exist* (i) *a positive selfadjoint invertible operator D on $L_2(G)$, affiliated with the left ring of G and semi-invariant of weight δ_G, and* (ii) *a positive standard Borel measure μ on $\hat{G}$, such that*

$$(1.1\mathrm{a})\quad \int_G |f(g)|^2 dg = \int_{\hat{G}} \|\pi(D^{\frac{1}{2}}f)\|_2^2 d\mu(\pi) \qquad \textit{for all } f \in \mathrm{Dom}(D^{\frac{1}{2}}) \cap D^{-\frac{1}{2}}L_1(G).$$

Remarks. (1) Implicit in (1.1a) is that, for $f \in \mathrm{Dom}(D^{\frac{1}{2}}) \cap D^{-\frac{1}{2}}L_1(G)$, $\pi(D^{\frac{1}{2}}f)$ is Hilbert–Schmidt for μ-almost all $[\pi] \in \hat{G}$ and $[\pi] \mapsto \|\pi(D^{\frac{1}{2}}f)\|_2$ is in $L_2(\hat{G}, \mu)$. But Theorem 1.1 makes no assertion about the size of $\mathrm{Dom}(D^{\frac{1}{2}}) \cap D^{-\frac{1}{2}}L_1(G)$. In the Lie group cases known so far ([13], [6]), $C_c^\infty(G) \subset \mathrm{Dom}(D^{\frac{1}{2}}) \cap D^{-\frac{1}{2}}L_1(G)$; but we will see in §6 that this is not always the case.

(2) If (D_1, μ_1) is another pair as in Theorem 1.1, then it is equivalent to (D, μ) in the following sense. There is a positive selfadjoint invertible operator C on $L_2(G)$, affiliated with both the left and right rings of G, such that $D_1 = CD$. From the

affiliation, $\pi(C)$ is defined and is a scalar $c(\pi)I$ for μ-almost-all $[\pi] \in \hat{G}$. Also, μ_1 is equivalent to μ with $d\mu/d\mu_1 = c(\pi)$.

(3) Since D is affiliated with the left ring, $D_\pi = \pi(D)$ is defined for μ-almost-all $[\pi] \in \hat{G}$. Those D_π are the infinitesimal operators of the non-unimodular Plancherel theorem ([3], [14]; or see [6, theorem 1.1]). For all $f \in C_c^\infty(G)$, $D_\pi^{1/2} \cdot \pi(f)$ is Hilbert–Schmidt for μ-almost-all $[\pi]$ and

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} \|D_\pi^{1/2} \cdot \pi(f)\|_2^2 \, d\mu(\pi). \tag{1.2}$$

(4) One needs the global operator D to understand the canonical trace on the left ring of G, and in fact it is the global operator that passes more naturally to the semi-finite non-type-I situation. But the price is the domain problem. (See [7, p. 486], [8, pp. 129–130], [3, p. 228], and §6 below.) However, the following seems to be the case, and we verify it for the groups that appear in this paper:

1.3 Conjecture. *If G is type I and D is as in Theorem* 1.1, *then* $\mathrm{Dom}(D^{\frac{1}{2}}) \cap D^{-\frac{1}{2}}L_1(G)$ *is dense in* $L_2(G)$.

(5) By [7, theorem 6.4] there exists a unitary map $\Upsilon: L_2(G) \to \int_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \bar{\mathcal{H}}_\pi d\mu(\pi)$ that simultaneously decomposes the left and right regular representations into irreducible constituents (with multiplicity equal dimension). That is half the point of the Plancherel Theorem. The other half amounts to specifying the intertwining operator Υ. By (1.1a)

$$(\Upsilon f)_\pi = \pi(D^{\frac{1}{2}}f) = \int_G (D^{\frac{1}{2}}f)(g)\pi(g)dg, \qquad f \in \mathrm{Dom}\, D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1(G).$$

This is one reason why we are interested in Conjecture 1.3. On the other hand, by (1.2) one does have

$$(\Upsilon f)_\pi = D_\pi^{\frac{1}{2}}\pi(f), \qquad f \in C_c^\infty(G).$$

(6) We now recast the Plancherel formula as an expansion of the Dirac trace. This is a much more subtle procedure than in the unimodular case. The point is that to evaluate the Dirac trace $\delta_G(\varphi)$ for $\varphi \in C^*(G)^+$, one must factor φ into a convolution product of left bounded elements of $L_2(G)$, and then use the corresponding bitrace (see [7, lemma 4.3]). For example in the unimodular situation, elements $f \in A(G) \cap L_1(G)$ are factored in [9, corol. 4.3] into convolu-

tions of L_2 functions; and in the infinitesimal non-unimodular situation, elements $f \in C_c^\infty(G)$ are factored in [3] into convolutions of $C_c^{(m)}(G)$ functions. In the global non-unimodular situation, this matter of factorization touches on domain questions, and so it is more delicate.

Let $f^*(g) = \bar{f}(g^{-1})\delta(g)^{-1}$ and write $\Omega f = f^*$, $\Lambda f = f\delta^{-\frac{1}{2}}$. Define $D' = \Omega D \Omega^{-1}$. (In the notation of [7], $D^{\frac{1}{2}} = M'^{-1}$, $D'^{\frac{1}{2}} = M^{-1}$.) D' satisfies $D'^{\frac{1}{2}} = \Lambda^{-1} D^{\frac{1}{2}}$ and $D'\Lambda = \Lambda D'$. Then, setting $h = f * f^*$, we calculate

$$h(e) = \int |f|^2 = \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f)\pi(D^{\frac{1}{2}}f)^* d\mu(\pi)$$

$$= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f)\pi(\Omega D^{\frac{1}{2}}f) d\mu(\pi)$$

$$= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f * D'^{\frac{1}{2}}f^*) d\mu(\pi).$$

We use the fact that D (resp. D') is affiliated with the left (resp. right) ring to write

$$h(e) = \int \operatorname{Tr} \pi(D^{\frac{1}{2}}D'^{\frac{1}{2}}h) d\mu(\pi). \tag{1.4}$$

We can replace h by $\Lambda^{-1}h$ to get

$$h(e) = \int \operatorname{Tr} \pi(Dh) d\mu(\pi). \tag{1.5}$$

But these computations are purely formal. We now make precise for which kinds of functions formulas (1.4) and (1.5) actually hold.

Here are some notational conventions (see [7, 8]):

$$P(G) = \text{continuous positive-definite functions on } G;$$

$$L_2(G)^{\mathcal{L}} = \text{left bounded elements in } L_2(G)$$

$$= \{f \in L_2(G) \colon \|f * h\|_2 \leq c\|h\|_2,\ \forall h \in L_2(G)\};$$

$$\mathfrak{A}_f(G) = \{h \in L_2(G) \colon \Lambda^n h \in L_2(G)^{\mathcal{L}},\ \forall n \in \mathbf{Z}\}.$$

1.6 Proposition. *Assume* $\operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1(G) \cap L_2(G)^{\mathcal{L}}$ *is dense in* $L_2(G)$. *Then*

$$h(e) = \int \operatorname{Tr} \pi(D^{\frac{1}{2}}D'^{\frac{1}{2}}h)d\mu(\pi), \tag{1.6a}$$

$$\forall h \in P(G) \cap \operatorname{Dom} D^{\frac{1}{2}}D'^{\frac{1}{2}} \cap D^{-\frac{1}{2}}D'^{-\frac{1}{2}}L_1(G) \cap \mathfrak{A}_f(G);$$

$$h(e) = \int \operatorname{Tr} \pi(Dh)d\mu(\pi), \tag{1.6b}$$

$$\forall h \in P(G) \cap \operatorname{Dom} D \cap D^{-1}L_1(G) \cap \mathfrak{A}_f(G).$$

Proof. Since $D'^{\frac{1}{2}} = \Lambda^{-1}D^{\frac{1}{2}}\Lambda^{-1}$ and Λ^{-1} preserves $P(G)$ and $\mathfrak{A}_f(G)$, we see that (1.6a) and (1.6b) are equivalent. We shall prove the former.

Let $h \in P(G) \cap \operatorname{Dom} D^{\frac{1}{2}}D'^{\frac{1}{2}} \cap D^{-\frac{1}{2}}D'^{-\frac{1}{2}}L_1 \cap \mathfrak{A}_f$. The proof of [2, theorem 13.8.6] shows that we may factor $h = f * f$, where $f \in P(G) \cap L_2(G)^{\mathscr{L}}$. Furthermore f is constructed as follows. There exist non-negative polynomial functions p_i which vanish at 0 such that $f_i = p_i(h)$ (multiplication is group convolution) and $f_i \to f$ in $L_2(G)$. But $p_i(\Lambda^n h) = \Lambda^n p_i(h)$. Thus [2, theorem 13.8.6] also applies to $\Lambda^n h$ to yield

$$\Lambda^n h = f_n * f_n, \qquad f_n = \lim_i p_i(\Lambda^n h).$$

Then $f_n = \Lambda^n f$; so $f \in P(G) \cap \mathfrak{A}_f(G)$ and $f^* = \Lambda f \in L_2(G)$.

Put $g = f^*$, so that $h = f * g^*$ with $f, g, g^* \in P(G) \cap \mathfrak{A}_f(G)$. Now $h \in \operatorname{Dom} D'^{\frac{1}{2}}$, and $D'^{\frac{1}{2}}$ is affiliated with the right ring. Thus $g^* \in \operatorname{Dom} D'^{\frac{1}{2}}$ and $D'^{\frac{1}{2}}h = f * D'^{\frac{1}{2}}g^* = f * (D^{\frac{1}{2}}g)^*$. Moreover $D'^{\frac{1}{2}}h \in \operatorname{Dom} D^{\frac{1}{2}}$ and, since $D^{\frac{1}{2}}$ is affiliated with the left ring, $f \in \operatorname{Dom} D^{\frac{1}{2}}$ and

$$D^{\frac{1}{2}}D'^{\frac{1}{2}}h = D^{\frac{1}{2}}f * (D^{\frac{1}{2}}g)^*.$$

By the Plancherel formula (1.1a),

$$\begin{aligned} h(e) = (f * g^*)(e) = (f, g) & \\ &= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f)\pi(D^{\frac{1}{2}}g)^* d\mu(\pi) \\ &= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f * (D^{\frac{1}{2}}g)^*)d\mu(\pi) \\ &= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}D'^{\frac{1}{2}}h)d\mu(\pi). \end{aligned}$$

Thus the proof is done once we establish

1.7 Lemma. *Let* $h = f * g^*$ *where* $h \in \operatorname{Dom} D^{\frac{1}{2}}D'^{\frac{1}{2}} \cap D^{-\frac{1}{2}}D'^{-\frac{1}{2}}L_1$ *and* $f, g, g^* \in L_2$. *Then*

$$\pi(D^{\frac{1}{2}}D'^{\frac{1}{2}}h) = \pi(D^{\frac{1}{2}}f)\pi(D^{\frac{1}{2}}g)^* \qquad a.a.\ \pi \in \hat{G}.$$

To prove this we require another

1.8 Sublemma. *Let* $k \in \operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1 \cap L_2(G)^{\mathscr{L}}$. *Then for every* $u \in \operatorname{Dom} D^{\frac{1}{2}}$

$$\pi(D^{\frac{1}{2}}k * D^{\frac{1}{2}}u) = \pi(D^{\frac{1}{2}}k)\pi(D^{\frac{1}{2}}u) \qquad a.a.\ \pi \in \hat{G}.$$

Proof. Indeed, how is $\pi(D^{\frac{1}{2}}u)$ defined? By the density assumption, we may choose $u_j \in \operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1$ such that

$$u_j \to u, \qquad D^{\frac{1}{2}}u_j \to D^{\frac{1}{2}}u, \qquad \text{both in } L_2.$$

Then $\pi(D^{\frac{1}{2}}u_j) \to \pi(D^{\frac{1}{2}}u)$ a.a. π (actually as Hilbert–Schmidt operators). Now since k is left bounded and $D^{\frac{1}{2}}u_j \in L_1$ we have

$$k * D^{\frac{1}{2}}u_j \to k * D^{\frac{1}{2}}u \qquad \text{in } L_2,$$
$$D^{\frac{1}{2}}(k * D^{\frac{1}{2}}u_j) = D^{\frac{1}{2}}k * D^{\frac{1}{2}}u_j \in L_1.$$

Therefore

$$\begin{aligned}\pi(D^{\frac{1}{2}}k * D^{\frac{1}{2}}u) = \pi(D^{\frac{1}{2}}(k * D^{\frac{1}{2}}u)) &= \lim \pi(D^{\frac{1}{2}}(k * D^{\frac{1}{2}}u_j))\\ &= \lim \pi(D^{\frac{1}{2}}k * D^{\frac{1}{2}}u_j)\\ &= \lim \pi(D^{\frac{1}{2}}k)\pi(D^{\frac{1}{2}}u_j)\\ &= \pi(D^{\frac{1}{2}}k)\pi(D^{\frac{1}{2}}u).\end{aligned}$$

q.e.d.

Proof of Lemma 1.7. Let $k \in \operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1 \cap L_2(G)^{\mathscr{L}}$. Applying the Plancherel formula, left boundedness of k, and Sublemma 1.8, we compute

$$\begin{aligned}(D'^{\frac{1}{2}}h, k) = (f * D'^{\frac{1}{2}}g^*, k) &= (f * (D^{\frac{1}{2}}g)^*, k)\\ &= (f, k * D^{\frac{1}{2}}g)\\ &= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f)\pi(D^{\frac{1}{2}}(k * D^{\frac{1}{2}}g))^* d\mu(\pi)\\ &= \int \operatorname{Tr} \pi(D^{\frac{1}{2}}f)\pi(D^{\frac{1}{2}}g)^*\pi(D^{\frac{1}{2}}k)^* d\mu(\pi).\end{aligned}$$

The Lemma follows by the density assumption and the unitarity of Y. q.e.d.

We may consider (1.6a) and (1.6b) to be alternate forms of the Plancherel formula (1.1a). (1.6b) is neater to work with; but it is the distributions $h \to \operatorname{Tr} \pi(D^{\frac{1}{2}}D'^{\frac{1}{2}}h)$ which are $\operatorname{Ad}G$-invariant, $\mathfrak{Z}(\mathfrak{g})$-eigendistributions on G. (Note also that the expression $D^{\frac{1}{2}}D'^{\frac{1}{2}}h$ equals $D^{\frac{1}{2}} * h * D^{\frac{1}{2}}$ in the notation of [6, (3.3a)].) As with (1.1a), there are non-trivial domain questions to be dealt with in both (1.6a) and (1.6b).

We wish to give a name to the operators of Theorem 1.1. It seems appropriate, since such operators were first considered in [1] and [11], to make the

1.9 Definition. By a *Dixmier–Pukanszky operator* on a locally compact group G we mean a positive, self-adjoint, invertible operator on $L_2(G)$, affiliated with the left ring and semi-invariant of weight δ_G.

Then, by combining [13, lemmas 7.1, 7.2] and [7, theorem 6.4], we have the following result.

1.10 Theorem. *Let G be type I and let D be a Dixmier–Pukanszky operator on G. Then in fact D does occur in the Plancherel formula; i.e., there exists a positive standard measure μ on $\hat{G}$ such that*

$$\int |f(g)|^2 dg = \int \| \pi(D^{\frac{1}{2}}f)\|_2^2 d\mu(\pi), \qquad \forall f \in \operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}}L_1(G).$$

Now in deriving the Plancherel formula for a specific group, there are two levels at which one can operate. The first — and more detailed — procedure is the one used in [6]. Specifically, from an explicit knowledge of the irreducibles, one finds the equivalence class μ of Plancherel measure by the group extension technique [7]. One then computes $\operatorname{Tr} \pi(f)$ *formally* (usually via [7, theorem 3.2]) and uses the accumulated data to guess what operator D will work in formula (1.6b) say. Then having guessed D, one goes back, adjusts μ appropriately and proves rigorously one of (1.1a), (1.6a) or (1.6b). We carry out that process for the maximal parabolic subgroups of the classical groups in §4, and for minimal parabolic subgroups of certain split rank 2 groups in §5c. In the second procedure, one ignores the irreducibles and simply produces a Dixmier–Pukanszky operator. Then by Theorem 1.10 one knows there exists a measure $\tilde{\mu}$, equivalent to μ, such that the Plancherel formula holds with the pair $(D, \tilde{\mu})$. This is less precise than the first method since one doesn't identify the Radon–Nikodym derivative $d\mu/d\tilde{\mu}$. We shall implement this second procedure in §5 for an *arbitrary* parabolic whose nilradical is non-abelian and has square integrable representations mod its center.

Finally we remark that the matter of uniqueness is up in the air. The question of whether there is a "best" pair (D, μ) in any sense is — although it has been studied by several people — completely unresolved. We shall say a little more about this at the end of §4.

2. The indefinite unitary groups

Our technique for obtaining the Plancherel formula can be applied to almost all of the maximal parabolic subgroups of classical groups. But in order to avoid repetition, we present the details only for the (indefinite) unitary groups. See §4e for a summary of the results on the other classical groups.

Let $\mathbf{F}$ denote one of the fields $\mathbf{R}$ (reals), $\mathbf{C}$ (complexes), or $\mathbf{Q}$ (quaternions). For $n \geqq 1$, we view $\mathbf{F}^n$ as a right vector space. Then for $u \geqq 1$, $v \geqq 0$, $u + v = n$, we set $\mathbf{F}^{u,v} = \mathbf{F}^n$ with hermitian form

$$\langle x, y \rangle = \sum_{1}^{u} x_i \bar{y}_i - \sum_{u+1}^{u+v} x_i \bar{y}_i. \tag{2.1}$$

The indefinite unitary groups are

$U(u, v; \mathbf{F})$ = the $\mathbf{F}$ linear transformations of $\mathbf{F}^{u,v}$ that preserve $\langle \cdot, \cdot \rangle$. Note that

$$U(u, v; \mathbf{F}) = \begin{cases} \mathrm{O}(u, v) & \mathbf{F} = \mathbf{R} \\ \mathrm{U}(u, v) & \mathbf{F} = \mathbf{C} \\ \mathrm{Sp}(u, v) & \mathbf{F} = \mathbf{Q} \end{cases}$$

is always a reductive real Lie group of $\mathbf{R}$-rank $\min(u, v)$.

Next let $\mathbf{F}^{s \times n}$ = the space of $s \times n$ matrices over $\mathbf{F}$. For $A \in \mathbf{F}^{s \times n}$, we denote $A^* = {}^t\bar{A} \in \mathbf{F}^{n \times s}$. Then

$$U(u, v; \mathbf{F}) = \left\{ g \in \mathbf{F}^{(u+v)\times(u+v)} \colon g \begin{pmatrix} I_u & 0 \\ 0 & -I_v \end{pmatrix} g^* = \begin{pmatrix} I_u & 0 \\ 0 & -I_v \end{pmatrix} \right\}.$$

If $n = u + v$ we have a hermitian map

$$\mathscr{H} \colon \mathbf{F}^{s \times n} \times \mathbf{F}^{s \times n} \to \mathbf{F}^{s \times s}$$

given by

$$\mathscr{H}((A_1, B_1), (A_2, B_2)) = A_1 A_2^* - B_1 B_2^*, \tag{2.2}$$

where $A_i \in \mathbf{F}^{s\times u}$, $B_i \in \mathbf{F}^{s\times v}$. We write

$$\mathbf{F}^{s\times(u,v)} = \mathbf{F}^{s\times n} \qquad \text{with hermitian map } \mathscr{H}.$$

Next for $A \in \mathbf{F}^{s\times s}$, put

$$\operatorname{Re} A = \frac{1}{2}(A + A^*), \qquad \operatorname{Im} A = \frac{1}{2}(A - A^*).$$

Then for $s \geqq 1$, $u + v = n$, $u \geqq v \geqq 0$, define the simply connected nilpotent Lie group

$$N_{s;u,v}(\mathbf{F}) = \operatorname{Im} \mathbf{F}^{s\times s} + \mathbf{F}^{s\times(u,v)}$$

with group composition

$$(z_1, x_1)(z_2, x_2) = (z_1 + z_2 + \tfrac{1}{2}\operatorname{Im} \mathscr{H}(x_1, x_2), x_1 + x_2), \qquad z_i \in \operatorname{Im} \mathbf{F}^{s\times s}, x_i \in \mathbf{F}^{s\times(u,v)}.$$

The group $\mathrm{GL}(s, \mathbf{F}) \times \mathrm{U}(u, v; \mathbf{F})$ acts by automorphisms on $N_{s;u,v}(\mathbf{F})$ via

$$(2.3) \qquad (\gamma, g)\cdot(z, x) = (\gamma z \gamma^*, \gamma x g^*), \qquad \gamma \in \mathrm{GL}(s, \mathbf{F}), \quad g \in \mathrm{U}(u, v; \mathbf{F});$$

and so we have a semidirect product

$$P_{s;u,v}(\mathbf{F}) = N_{s;u,v}(\mathbf{F}) \cdot (\mathrm{GL}(s, \mathbf{F}) \times \mathrm{U}(u, v; \mathbf{F})).$$

Note that the choice $s = 1$, $v = 0$ gives as a special case the parabolic groups considered in [6].

2.4 Proposition [15, 17]. *Let $p \geqq q \geqq 1$. Then the groups $P_{s;p-s,q-s}(\mathbf{F})$, $s = 1, \cdots, q$, constitute a complete set of representatives for the conjugacy classes of maximal parabolic subgroups of* $\mathrm{U}(p, q; \mathbf{F})$, *except that $P_{n-1;1,1}(\mathbf{R})$ is not maximal in* $\mathrm{O}(n, n)$.

Denote $\varepsilon = \dim_{\mathbf{R}} \mathbf{F}$. For $\gamma \in \mathrm{GL}(s, \mathbf{F})$, let $\det_{\mathbf{R}} \gamma$ denote the module (with respect to Lebesgue measure) for the action of γ on $\mathbf{F}^s$. Set $\mathrm{GL}'(s, \mathbf{F}) = \{\gamma \in \mathrm{GL}(s, \mathbf{F}) \colon \det_{\mathbf{R}} = 1\}$. We have $\det_{\mathbf{R}} \gamma = |\psi(\gamma)|$ where

$$(2.5) \qquad \psi(\gamma) = \begin{cases} \text{usual real determinant} & \mathbf{F} = \mathbf{R} \\ |\text{usual complex determinant}|^2 & \mathbf{F} = \mathbf{C} \\ |\text{usual complex determinant viewing } \mathbf{Q} = \mathbf{C}^2|^2 & \mathbf{F} = \mathbf{Q}. \end{cases}$$

We now exclude from consideration the situation: $\mathbf{F} = \mathbf{R}$ *and* s *odd*. Then the function $z \to \psi(z)$, $z \in \operatorname{Im} \mathbf{F}^s$, is a non-trivial real polynomial function on $\operatorname{Im} \mathbf{F}^{s\times s}$ of degree εs. In case $\mathbf{F} \neq \mathbf{R}$, ψ is actually non-negative.

Next denote the multiplicative group of positive real numbers by $\mathbf{R}_+^*$ and view it as the group of positive real scalar matrices in $\mathrm{GL}(s, \mathbf{F})$. Then $\mathrm{GL}(s, \mathbf{F}) \cong \mathbf{R}_+^* \times \mathrm{GL}'(s, \mathbf{F})$. The Langlands decomposition of $P_{s;u,v}(\mathbf{F})$ is as follows: set $N = N_{s;u,v}(\mathbf{F})$, $A = \mathbf{R}_+^*$, $M = \mathrm{GL}'(s, \mathbf{F}) \times \mathrm{U}(u, v; \mathbf{F})$. Then $P = P_{s;u,v}(\mathbf{F}) = NAM$. (We continue to write NAM for the same reasons as in [6].) The group multiplication is

$$(2.6) \qquad (z_1, x_1, a_{r_1}, \gamma_1, g_1)(z_2, x_2, a_{r_2}, \gamma_2, g_2) =$$

$$(z_1 + r_1^2\gamma_1 z_2 \gamma_1^* + \tfrac{1}{2}\operatorname{Im} \mathcal{H}(x_1, r_1\gamma_1 x_2 g_1^*), x_1 + r_1\gamma_1 x_2 g_1^*, a_{r_1 r_2}, \gamma_1\gamma_2, g_1 g_2).$$

The modular function is given by

$$(2.7) \qquad \delta_{NAM}(z, x, a_r, \gamma, g) = \delta_{NA}(z, x, a_r) = r^{2q}, \quad q = \dim_{\mathbf{R}} \operatorname{Im} \mathbf{F}^{s\times s} + \frac{1}{2} \dim_{\mathbf{R}} \mathbf{F}^{s\times(u,v)}.$$

Also, the polynomial $\psi(z)$ is M-invariant and A-homogeneous. Indeed

$$(2.8) \qquad \psi(ma_r \cdot z) = r^{2\varepsilon s}\psi(z) = r^{2\deg\psi}\psi(z).$$

The representation theory of the groups $P_{s;u,v}(\mathbf{F})$ has been completely described in [15]. Since we are interested in the Plancherel formula, we only need consider *generic* representations. Put $Z = \operatorname{Cent} N$, $\mathfrak{z}$ = Lie algebra of Z, $\mathfrak{z}^* = \operatorname{Hom}_{\mathbf{R}}(\mathfrak{z}, \mathbf{R})$ and $k = \dim Z$, $l = \dim N/Z$. We identify $\hat{Z}$ with $\mathfrak{z}^*$ by the abuse of notation $\lambda(z) = e^{i\lambda(\log z)}$, $\lambda \in \mathfrak{z}^*$. We also identify $\mathfrak{z}^*$ with $\mathfrak{z}$ (or Z) via the non-degenerate bilinear form

$$(2.9) \qquad (z, \lambda) = \operatorname{Re} \operatorname{trace} z\lambda^*, \qquad z, \lambda \in \operatorname{Im} \mathbf{F}^{s\times s}$$

(the notation is as in [15, p. 41]). Put $\mathfrak{z}_0^* = \mathfrak{z}^* \cap \mathrm{GL}(s, \mathbf{F}) = \{\lambda \in \mathfrak{z}^* : \psi(\lambda) \neq 0\}$, the Zariski open subset of $\operatorname{Im} \mathbf{F}^{s\times s}$ consisting of maximal rank matrices. To each $\lambda \in \mathfrak{z}_0^*$, there exists an irreducible unitary representation class $[\gamma_\lambda]$ of N, uniquely determined by the equation

$$\gamma_\lambda(zn) = \lambda(z)\gamma_\lambda(n), \qquad z \in Z, \quad n \in N.$$

Moreover $[\gamma_\lambda] \neq [\gamma_{\lambda'}]$ if $\lambda \neq \lambda'$. These are the generic representations of N. The generic representations of NA are obtained by induction

$$\eta_\lambda = \operatorname{Ind}_N^{NA} \gamma_\lambda, \qquad \lambda \in \mathfrak{z}_0^*.$$

Since

$$a_r \cdot [\gamma_\lambda] = [\gamma_{r^{-2}\lambda}]$$

we have certain equivalences. Put

$$S = \{\lambda \in \mathfrak{z}_0^* : |\psi(\lambda)| = 1\}. \tag{2.10}$$

Then S parametrizes a generic set of inequivalent irreducible unitary representations of NA. The action of M commutes with that of A and is essentially transitive on S. In fact, if $\mathbf{F} = \mathbf{R}$ or $\mathbf{Q}$, M is transitive on S. If $\mathbf{F} = \mathbf{C}$, there are $s + 1$ orbits; and a cross-section for these orbits is the set

$$\tilde{S} = \left\{\sqrt{-1}\begin{pmatrix} I_i & 0 \\ 0 & -I_{s-i}\end{pmatrix} : i = 0, 1, \cdots, s\right\}.$$

Fix $\lambda_1 \in S$ ($\mathbf{F} = \mathbf{R}$ or $\mathbf{Q}$), and $\lambda_i \in \tilde{S}$ ($\mathbf{F} = \mathbf{C}$), $i = 0, 1, \cdots, s$. Then the M-stabilizer of $[\eta_{\lambda_i}]$ is the same as the M-stabilizer of either $[\gamma_{\lambda_i}]$ or λ_i itself; it equals

$$M_1 = \begin{cases} \mathrm{Sp}\left(\frac{s}{2}, \mathbf{R}\right) \times \mathrm{O}(u, v) & \mathbf{F} = \mathbf{R}, \quad s \text{ even} \\ \mathrm{SO}^*(2s) \times \mathrm{Sp}(u, v) & \mathbf{F} = \mathbf{Q} \end{cases} \tag{2.11}$$

$$M_i = \mathrm{U}(i, s-i) \times \mathrm{U}(u, v) \qquad \mathbf{F} = \mathbf{C}, \quad i = 0, 1, \cdots, s.$$

Note that in all cases the stability group is reductive. Wolf [15, p. 52 ff] has proven that η_{λ_i} extends to an ordinary representation $\tilde{\eta}_i$ of M_i. Then the generic representations of NAM are:

$$\begin{aligned} \pi_\tau &= \mathrm{Ind}_{NAM_1}^P \tilde{\eta}_1 \otimes \tau, & \tau \in \hat{M}_1 \quad \mathbf{F} = \mathbf{R}, \mathbf{Q}, \\ \pi_\tau^i &= \mathrm{Ind}_{NAM_i}^P \tilde{\eta}_i \otimes \tau, & \tau \in \hat{M}_i \quad \mathbf{F} = \mathbf{C}. \end{aligned} \tag{2.12}$$

The Plancherel formula we will deduce for P looks as follows. Let $q = k + \frac{1}{2}l$, $k = \dim_{\mathbf{R}} Z$, $l = \dim_{\mathbf{R}} N/Z$. Consider the operator E defined on P by letting it act on the direct factor Z according to

$$E = (2\pi)^{-q} |\Psi|^{q/\deg\psi},$$

where Ψ is the constant coefficient differential operator on Z which corresponds under Fourier transform to multiplication by ψ — see §3 for the precise definition of E. Then

$$(2.13)\qquad f(1_p)=\begin{cases}\displaystyle\int_{\hat{M}_1}\operatorname{Tr}\pi_\tau(E^{\frac12}f)\,d\mu_{M_1}(\tau) & \mathbf{F}=\mathbf{R},\mathbf{Q}\\[2ex] \displaystyle\sum_{i=0}^{s}\int_{\hat{M}_i}\operatorname{Tr}\pi_\tau^i(E^{\frac12}f)\,d\mu_{M_i}(\tau) & \mathbf{F}=\mathbf{C}.\end{cases}$$

3. Definition and properties of certain unbounded operators

Now we define the global unbounded operators that will appear in our Plancherel formulae for $P_{s;u,v}(\mathbf{F})$. When $s=1$, $v=0$ they are the fractional powers of the Laplace operator that occurred already in [6]. In general they will be fractional powers of the absolute value of a non-elliptic constant coefficient differential operator on Z. We develop the algebraic and analytic properties of such operators in this section.

3a. *Definition of the Operators.* Consider a differentiable manifold $V=Z\times W$ where Z has a fixed identification with an euclidean vector space $\mathbf{R}^k$. In our applications, we will have $V=NA$ or $V=NAM$ and $Z=\operatorname{Cent}N$. The euclidean structure on Z defines an operation of partial Fourier transfer on V via

$$(3.1)\qquad \mathcal{F}(f)(\xi,w)=\int_{\mathbf{R}^k} f(z,w)e^{i(z,\xi)}dz,\qquad \xi\in Z,\quad w\in W.$$

Suppose next that $\theta(\xi)$ is a polynomial function of $\xi\in\mathbf{R}^k$. Then there is a unique "constant coefficient" differential operator Θ on V which is related to θ by the equation

$$(3.2)\qquad \mathcal{F}(\Theta f)(\xi,w)=\theta(\xi)\mathcal{F}(f)(\xi,w).$$

We may utilize the Fourier transform then to define positive powers $|\Theta|^t$, namely

$$(3.3)\qquad |\Theta|^t f(z,w)=\mathcal{F}^{-1}\{|\theta(\xi)|^t\mathcal{F}(f)\}(z,w),\qquad t\geqq 0.$$

These operators do not increase the W-projection of the support of f, but they may increase the Z-projection.

Now fix a positive Radon measure dw on W. That determines a positive Radon measure on V by $dv=dzdw$, where dz is Lebesgue measure on Z. If V is NA or NAM, Haar measure is of this form. Then, exactly as in [6, prop. 2.6], we have

3.4 Proposition. *View $|\Theta|^t$, $t \geqq 0$, as an operator on $L_2(V, dv)$ with domain $C_c^\infty(V)$. Then $|\Theta|^t$ is symmetric, and its closure is a positive self-adjoint operator.*

The unbounded operators in the Plancherel formula of NA and NAM are special cases of the above construction. With $k = \dim_{\mathbf{R}} Z$, $l = \dim_{\mathbf{R}} N/Z$ and $q = k + \frac{1}{2} l$, we have diffeomorphic splittings

(3.5) $Z = \operatorname{Im} \mathbf{F}^{s \times s} \cong \mathbf{R}^k$, $\quad NA \cong Z \times (\mathbf{R}^l \times \mathbf{R}_+^*)$, $\quad NAM \cong Z \times (\mathbf{R}^l \times \mathbf{R}_+^* \times M)$.

Recall the polynomial function ψ on Z defined in (2.5). The corresponding differential operator Ψ is specified by (3.2); and the powers $|\Psi|^t$ by (3.3). Then our operators are defined, relative to (3.5), by

$$D = (2\pi)^{-q} |\Psi|^{q/\deg \psi} \quad \text{on } NA, \tag{3.6}$$

$$E = (2\pi)^{-q} |\Psi|^{q/\deg \psi} \quad \text{on } NAM. \tag{3.7}$$

3b. *Density Properties of the Operators.* We recall the result of [6, §2b].

3.8 Theorem. *If $t \geqq 0$ and $f \in C_c^{(m)}(V)$ with $m > 2t + k$, then $\Delta^t f \in L_1(V, dv)$.*

We need an analogous L_1 property for the one parameter family $|\Psi|^t$ — first in order to know that $\pi(D^{\frac{1}{2}} f)$ means $\int_G (D^{\frac{1}{2}} f)(g) \pi(g) dg$, and second to verify Conjecture 1.3 in the cases under consideration. Well, the analog of Theorem 3.8 for $|\Psi|^t$ is simply not true in general. Experience indicates the likelihood of a critical value α such that

$$|\Psi|^t C_c^\infty(V) \subseteq L_1(V), \qquad t > \alpha,$$

$$|\Psi|^t C_c^\infty(V) \not\subseteq L_1(V), \qquad t \leqq \alpha.$$

(We state a specific result in the Appendix §6.) Nevertheless, we can verify Conjecture 1.3. Let $\mathscr{C}_\psi = \{\lambda \in Z : \psi(\lambda) = 0\}$, a Zariski-closed subvariety of Z.

3.9 Definition. Put $\mathscr{S}_\psi(V) = \{f(z, w) : f$ has properties (i)–(iv)$\}$, where

(i) $f \in C^\infty(V)$;

(ii) $\exists L \subseteq W$ compact subset such that $f(z, w) = 0$ if $w \notin L$;

(iii) $\forall$ polynomial $p(z)$ and $\forall$ constant coefficient differential operator T on Z

$$\sup_{z, w} |p(z)(Tf)(z, w)| < \infty;$$

(iv) $\exists\mathcal{N}$ neighborhood of $\mathscr{C}_\psi$ such that $\mathscr{F}(f)(\xi, w) = 0$ if $\xi \in \mathcal{N}$.
Clearly $\mathscr{S}_\psi(V) \subseteqq \operatorname{Dom}|\Psi|^t$, $\forall t \geqq 0$. Moreover we have

3.10 Lemma. *$\mathscr{S}_\psi(V)$ is dense in $L_2(V)$, and $|\Psi|^t f \in L_1(V)$, $\forall t \geqq 0$, $\forall f \in \mathscr{S}_\psi(V)$.*

Proof. Let $\mathscr{S}(V)$ denote the set of functions on V having properties (i)–(iii) of Definition 3.9. The partial Fourier transform is a linear isomorphism of $\mathscr{S}(V)$ onto itself. (It's actually a topological isomorphism if we put seminorms on $\mathscr{S}(V)$ appropriately.) Now the fact that $|\Psi|^t \mathscr{S}_\psi(V) \subseteq L_1(V)$ is easy to see from the definition of $\mathscr{S}_\psi(V)$ and equation (3.3). Indeed for $f \in \mathscr{S}_\psi(V)$, we have $|\Psi|^t f \in \mathscr{S}(V) \subseteq L_1(V)$.

Next let $f(z, w) \in L_2(V)$. We may approximate f arbitrarily closely in L_2 norm by a finite linear combination of functions of the form $g(z)h(w)$, $g \in L_2(Z)$, $h \in L_2(W)$. Furthermore the functions $g_1(z)h_1(w)$, $g_1 \in \mathscr{S}_\psi(Z)$, $h_1 \in C_c^\infty(W)$ belong to $\mathscr{S}_\psi(V)$. So it is enough to prove density in case W is trivial, i.e. $V = \mathbf{R}^k$. But this follows because the Fourier transform is a unitary map of L_2, and $\{f \in \mathscr{S}(\mathbf{R}^k): f = 0$ near $\mathscr{C}_\psi\}$ is dense in $L_2(\mathbf{R}^k)$. q.e.d.

3.11 Corollary. (1) $\operatorname{Dom} D^t \cap D^{-t}L_1(NA) \cap L_2(NA)^{\mathscr{L}}$ *is dense in* $L_2(NA)$, $\forall t \geqq 0$;

(2) $\operatorname{Dom} E^t \cap E^{-t}L_1(NAM) \cap L_2(NAM)^{\mathscr{L}}$ *is dense in* $L_2(NAM)$, $\forall t \geqq 0$;

(3) *Let $V = ZW$ be a Lie group with Z a closed normal subgroup and $dv = dzdw$ right Haar measure. Then if $f \in \mathscr{S}_\psi(V)$, we have $f *_V f^* \in \mathscr{S}_\psi(V)$.*

Proof. Parts (1) and (2) are immediate consequences of equations (3.6) and (3.7) and Lemma 3.10. Let us now prove (3). We observe

$$(f * f^*)(v) = \int f(v\nu^{-1}) f^*(\nu)\, d\nu$$

$$= \int f(v\nu^{-1}) \bar{f}(\nu^{-1}) \delta_V(\nu^{-1})\, d\nu$$

$$= \int f(v\nu) \bar{f}(\nu)\, d\nu.$$

Now we expand the integral in Z–W coordinates. We are slightly hampered by the fact that W may not be a group. For any $v \in V$, we write $v = z_v w_v$, $z_v \in Z$, $w_v \in W$. Then

$$
\begin{aligned}
(f*f^*)(zw) &= \int_{Z\times W} f(zw\zeta\omega)\bar{f}(\zeta\omega)d\zeta d\omega \\
&= \int f(zw\zeta w^{-1}z_{w\omega}w_{w\omega})\bar{f}(\zeta\omega)d\zeta d\omega.
\end{aligned} \tag{3.11a}
$$

Consider the four defining properties of $\mathscr{S}_\psi(V)$.

(i) $f*f^*$ is clearly C^∞.

(ii) The integrand in (3.11a) vanishes unless $\omega\in L$ and $w_{w\omega}\in L$. But then the equation $z_{w\omega}^{-1}w = w_{w\omega}\omega^{-1}$ guarantees that the values of w for which the integrand is non-zero are also restricted to a compact set. That is, $f*f^*$ is compactly supported mod Z.

(iii) Let n be a non-negative integer and T a constant coefficient differential operator on Z. We show

$$\sup_{z,w}|(1+\|z\|)^nT(f*f^*)(zw)|<\infty.$$

Indeed the supremum can be estimated:

$$
\begin{aligned}
&\sup_{z,w}\left|(1+\|z\|)^n\int Tf(zw\zeta w^{-1}z_{w\omega}w_{w\omega})\bar{f}(\zeta\omega)d\zeta d\omega\right| \\
&\leqq \sup_{z;\,w,\omega\in\text{cpt}}\left|(1+\|z\|)^n\int Tf(zw\zeta w^{-1}z_{w\omega}w_{w\omega})\bar{f}(\zeta\omega)d\zeta\right|\operatorname{meas}(L) \\
&\leqq \sup_{z;w,\omega\in\text{cpt}}\left|\int (1+\|zw\zeta^{-1}wz_{w\omega}^{-1}\|)^nTf(zw\zeta w^{-1}z_{w\omega}w_{w\omega})\cdot\right. \\
&\qquad\qquad \left.\cdot(1+\|w\zeta w^{-1}z_{w\omega}\|)^n\bar{f}(\zeta\omega)d\zeta\right|\operatorname{meas}(L) \\
&\leqq \sup_{u;\,w,\omega\in\text{cpt}}\left|(1+\|u\|)^nTf(uw_{w\omega})\int(1+\|w\zeta w^{-1}z_{w\omega}\|)^n\bar{f}(\zeta\omega)d\zeta\right|\operatorname{meas}(L) \\
&\leqq \text{constant}\cdot\sup_{w,\omega\in\text{cpt}}\left|\int(1+\|w\zeta w^{-1}z_{w\omega}\|)^n\bar{f}(\zeta\omega)d\zeta\right| \\
&\leqq \text{constant}\cdot\sup_{w,\omega\in\text{cpt}}\left|\int(1+\|w\zeta w^{-1}\|)^n\bar{f}(\zeta\omega)d\zeta\right|<\infty.
\end{aligned}
$$

Note we have used (twice) that $f \in \mathcal{S}_\psi(V)$ and that, since $\zeta \to w\zeta w^{-1}$ is a linear transformation, $\| w\zeta w^{-1} \| \leqq C_w \| \zeta \|$ where $w \to C_w$ is continuous.

$$\text{(iv)} \quad \int (f * f^*)(zw)\lambda(z)dz = \int f(zw\zeta w^{-1} z_{w\omega} w_{w\omega})\bar{f}(\zeta\omega)\lambda(z)d\zeta d\omega dz$$

$$= \int f(zw_{w\omega})\bar{f}(\zeta\omega)\lambda(z)\lambda(w\zeta^{-1}w^{-1})\lambda(z_{w\omega}^{-1})dzd\zeta d\omega$$

$$= 0 \quad \text{if} \quad \lambda \quad \text{is near } \mathscr{C}_\psi.$$

q.e.d.

3.12 Remark. The proofs of Lemma 3.10 and Corollary 3.11 (3) work just as well with the pair (ψ, Ψ) replaced by any polynomial θ and associated constant coefficient differential operator Θ.

3c. *Algebraic Properties of the Operators.* We now extend [6, §2c], with Ψ in place of Δ. The technique is similar, so we do not supply full details. Consult [6, §2c] for undefined notation and terminology — except that for a Lie group V, we write $\mathfrak{U}(\mathfrak{v})$ for its (complexified) universal enveloping algebra. Observe that if V is NA or NAM, then the splittings (3.5) determine a cannonical embedding $\mathfrak{U}(\mathfrak{z}) \to \mathfrak{U}(\mathfrak{v})$ that respects right invariance.

3.13 Lemma. *Let V be NA or NAM and view $\Psi \in \mathfrak{U}(\mathfrak{v})$. Define $\alpha\colon V \to \mathbf{R}_+^*$ by $\alpha(z, x, a_r, \cdots) = r$. Then if $f \in C^\infty(V)$, we have*

$$\Psi * f = \Psi f, \tag{3.13a}$$

$$f * \Psi = \alpha^{2\deg\psi}\Psi f. \tag{3.13b}$$

Proof. We argue as in [6, lemma 2.11, $\mathbf{F} \neq \mathbf{R}$]. If ξ is any element of $\mathfrak{z}$ and $\zeta = \exp \xi$, then

$$(\xi * f)(z, x, a_r, m) = \frac{d}{dt} f(z - t\zeta, x, a_r, m)\big|_{t=0}$$

and

$$(f * \xi)(z, x, a_r, m) = \frac{d}{dt} f(z - r^2 tm \cdot \zeta, x, a_r, m)\big|_{t=0}$$

$$= r^2 \frac{d}{dt} f(z - tm \cdot \zeta, x, a_r, m)\big|_{t=0}$$

$$= r^2((m \cdot \xi) * f)(z, x, a_r, m).$$

The result follows immediately from the definition of Ψ and the fact that ψ is M-invariant. q.e.d.

Now as in [6, §2c], we use Lemma 3.13 to define the right and left actions of positive powers of Ψ:

$$|\Psi|^t * f = |\Psi|^t f, \qquad f * |\Psi|^t = \alpha^{2t\deg\psi}|\Psi|^t f, \qquad t \geqq 0. \tag{3.14}$$

The next result is

3.15 Lemma. *We have*

$$(|\Psi|^t * f)^* = f^* * |\Psi|^t, \qquad t \geqq 0. \tag{3.15a}$$

Proof. It suffices to prove

$$|\Psi|^t * f^* = \alpha^{-2t\deg\psi}(|\Psi|^t f)^*, \tag{3.16}$$

for (3.14) and (3.16) combine to give

$$\begin{aligned} f^* * |\Psi|^t &= \alpha^{2t\deg\psi}|\Psi|^t * f^* \\ &= \alpha^{2t\deg\psi}\alpha^{-2t\deg\psi}(|\Psi|^t f)^* \\ &= (|\Psi|^t * f)^*. \end{aligned}$$

The proof of (3.16) is analogous to that of [6, prop. 2.14]. Let $\xi \in \mathfrak{z}$, $\zeta = \exp\xi \in Z$. Then

$$\begin{aligned} (\xi * f^*)(z, x, a_r, m) &= \frac{d}{dt} f^*(z - t\zeta, x, a_r, m)\big|_{t=0} \\ &= \frac{d}{dt}\bar{f}(-r^{-2}m^{-1}\cdot(z - t\zeta), -r^{-1}m^{-1}\cdot x, a_{r^{-1}}, m^{-1})r^{-2q}\big|_{t=0} \\ &= r^{-2}\frac{d}{dt}\bar{f}(-r^{-2}m^{-1}\cdot z + m^{-1}\cdot t\zeta, -r^{-1}m^{-1}\cdot x, a_{r^{-1}}, m^{-1})r^{-2q}\big|_{t=0} \\ &= r^{-2}(m^{-1}\cdot\zeta * f)^*(z, x, a_r, m). \end{aligned}$$

Once again we invoke M-invariance to obtain

$$\Psi * f^* = \alpha^{-2\deg\psi}(\Psi * f)^* = \alpha^{-2\deg\psi}(\Psi f)^*.$$

Rewrite this as

$$\alpha^{2\deg\psi}\Psi f = (\Psi f^*)^*.$$

Now Ψ and the operation of "multiplication by α" commute. Hence

$$\alpha^{4\deg\psi}\Psi^2 f = (\Psi^2 f^*)^*.$$

Moreover Ψ is self adjoint and $|\Psi| = \sqrt{\Psi^2}$. Hence

$$\alpha^{2\deg\psi}|\Psi| f = (|\Psi| f^*)^*.$$

Equation (3.16) follows easily now. q.e.d.

Finally, since the operators D on NA and E on NAM are positive multiples of $|\Psi|^{q/\deg\psi}$, we obtain from (3.14) and (2.7)

3.17 Proposition. (1) *D is a semi-invariant of weight δ_{NA} on NA.*
(2) *E is a semi-invariant of weight δ_{NAM} on NAM.*
(3) $(E^{\frac{1}{2}} * f * E^{\frac{1}{2}})|_{NA} = D^{\frac{1}{2}} * f|_{NA} * D^{\frac{1}{2}}$.

4. Plancherel formula for $P_{s;u,v}(\mathbf{F})$

As in [6] we first derive the Plancherel formula for NA, and then via an extension technique we pass to NAM. Here the compact extension technique of [6] must be replaced by a non-compact extension technique. We continue to exclude the case $\mathbf{F} = \mathbf{R}$, s odd, but at the end of this section, we comment briefly on the case $\mathbf{F} = \mathbf{R}$, $s = 1$ — which we can handle — and the maximal parabolics of the other classical groups.

4a. *Some Results on Disintegration of Measures.* Recall the basic facts on the groups $P_{s;u,v}(\mathbf{F})$ and their representations from §2. Now we normalize Haar measures. Put dz = Lebesgue measure on $Z = \operatorname{Im}\mathbf{F}^{s\times s}$, dx = Lebesgue measure on $\mathbf{F}^{s\times(u,v)}$, $da_r = dr/r$ where dr is Lebesgue measure on $A = \mathbf{R}_+^*$, and dm a fixed choice of Haar Measure on $M = \mathrm{GL}'(s,\mathbf{F}) \times \mathrm{U}(u,v;\mathbf{F})$. Then $dn = dzdx$ is Haar measure on N, $dnda_r$ is right invariant on NA and $dnda_rdm$ is right invariant on NAM. The identification (2.9) gives us Lebesgue measure $d\lambda$ on $\hat{Z}$ so that

$$\int_{\hat{Z}}\int_{Z} f(z)\lambda(z)dzd\lambda = (2\pi)^k f(1_Z), \qquad f \in C_c^\infty(Z). \tag{4.0}$$

We disintegrate $d\lambda$ under the action of A. Note that $d\lambda$ is quasi-invariant with modulus r^{2k}. The principal stability group for the action of A on $\hat{Z}$ is trivial. Fix the measure $r^{2k}da_r = r^{2k-1}dr$. This is a quasi-invariant measure on A, also with modulus r^{2k}. Thus by [7, theorem 2.1], there exists a *unique* quasi-invariant measure σ on $\hat{Z}/A$ such that

$$(4.1) \qquad \int_{\hat{Z}} f(\lambda)d\lambda = \int_{\hat{Z}/A} \int_{A} f(a_r \cdot \lambda) r^{2k} da_r d\sigma(\bar{\lambda}).$$

The set S defined by (2.10) is a Borel cross-section for a co-null set in $\hat{Z}/A$. If we restrict the canonical projection $\hat{Z} \to \hat{Z}/A$ to S we get a Borel isomorphism onto a co-null set. We transfer σ via this isomorphism, and continue to write it σ. σ is the unique measure on S satisfying

$$(4.2) \qquad \int_{\hat{Z}} f(\lambda)d\lambda = \int_0^\infty \int_S f(r^2\lambda) r^{2k-1} d\sigma(\lambda)dr.$$

4.3 Lemma. *σ is M-invariant.*

Proof. This follows instantly from the facts: M and A commute, $d\lambda$ is M-invariant, and the uniqueness of σ in equation 4.2.

Assume momentarily $\mathbf{F} = \mathbf{R}$ or $\mathbf{Q}$. Then we have a Borel isomorphism

$$M_1\backslash M \to S, \qquad mM_1 \to m \cdot \lambda_1.$$

We transfer σ to $M_1\backslash M$ via this isomorphism. Since Haar measure on M is already fixed, there is a uniquely determined Haar measure on M_1 such that

$$\int_M f(m)dm = \int_{M_1\backslash M} \int_{M_1} f(m_1 m)dm_1 d\sigma(\bar{m}).$$

Finally (since the group M_1 is unimodular and type I), there is a unique Plancherel measure μ_{M_1} on $\hat{M}_1$ such that

$$\int_{M_1} |f(m_1)|^2 dm_1 = \int_{\hat{M}_1} \|\tau(f)\|_2^2 d\mu_{M_1}(\tau).$$

If $\mathbf{F} = \mathbf{C}$, put $S_i = M \cdot \lambda_i$, $i = 0, 1, \cdots, s$; transfer $\sigma|_{S_i}$ to $M_i\backslash M$ via the isomorphism $mM_i \to m \cdot \lambda_i$; and then choose Haar and Plancherel measures on M_i and $\hat{M}_i$ accordingly.

4b. *The Characters of N.* Here we derive an expression for the character $\operatorname{Tr}\gamma_\lambda(f)$ as a Fourier transform over Z. For fixed $\lambda \in \hat{Z}$, the functional

$$f \to \int_Z f(z)\lambda(z)dz, \qquad C_c^\infty(N) \to \mathbf{C}$$

is easily checked to be an Ad N-invariant, $\mathfrak{Z}(\mathfrak{n})$-eigendistribution. (Note the center $\mathfrak{Z}(\mathfrak{n})$ of $\mathfrak{U}(\mathfrak{n})$ equals $\mathfrak{U}(\mathfrak{z})$ in this case.) Moreover for $T \in \mathfrak{U}(\mathfrak{z})$ the eigenvalue is $\hat{T}(\lambda)$. For those λ which are in general position (the set $\mathfrak{z}_0^*$), the infinitesimal character uniquely determines the global character. Thus there is a number $c(\lambda)$ such that

$$\operatorname{Tr}\gamma_\lambda(f) = c(\lambda)\int_Z f(z)\lambda(z)dz.$$

Let $Pf(\lambda)$ be the Pfaffian polynomial on $\mathfrak{z}^*$ in the sense of [10]. According to [10, p. 455], the Plancherel formula for N may be written

$$f(e) = c\int \operatorname{Tr}\gamma_\lambda(f)|Pf(\lambda)|d\lambda.$$

It follows from the inversion formula on Z that

$$c(\lambda) = c_1|Pf(\lambda)|^{-1}, \qquad \lambda \in \mathfrak{z}_0^*.$$

We wish to compute c_1 and $|Pf(\lambda)|$ explicitly. That can be done by evaluating $Pf(\lambda)$, then computing the Kostant measure and using [12]. It can also be done directly.

4.4 Proposition. *For $\lambda \in \mathfrak{z}_0^*$ and $f \in C_c^\infty(N)$,*

$$\operatorname{Tr}\gamma_\lambda(f) = (2\pi)^{l/2}|\psi(\lambda)|^{-l/2\deg\psi}\int_Z f(z)\lambda(z)dz. \tag{4.4a}$$

Proof. Fix $\lambda \in \mathfrak{z}_0^*$. Let $\mathfrak{q}$ be any maximal totally isotropic subspace of $\mathbf{F}^{s\times(u,v)}$ with respect to the antisymmetric form $B_\lambda(\eta,\zeta) = \lambda([\eta,\zeta]) = (\lambda, \operatorname{Im}\mathscr{H}(\eta,\zeta))$ — refer to (2.9). Then $\mathfrak{z}+\mathfrak{q}$ is a real polarization for λ. Letting $Y = \exp\mathfrak{q}$ we know that $\gamma_\lambda = \operatorname{Ind}_{ZY}^N\lambda$, where $\lambda(z\exp\eta) = \lambda(z)$, $z \in Z$, $\eta \in \mathfrak{q}$. Let $\mathfrak{x}$ be any real complement for $\mathfrak{q}$ in $\mathbf{F}^{s\times(u,v)}$, $X = \exp\mathfrak{x}$. Then γ_λ can be realized on $L_2(X)$, and a straightforward computation reveals

$$\gamma_\lambda(zyx)h(u) = \lambda\left(z + \frac{1}{2}\operatorname{Im}\mathcal{H}(u, y + x) - \frac{1}{2}\operatorname{Im}\mathcal{H}(y, u + x)\right)h(u + x), \quad h \in L_2(X).$$

Lifting to functions, we get the kernel operator

$$\gamma_\lambda(f)h(u) = \int \lambda\left(z + \frac{1}{2}\operatorname{Im}\mathcal{H}(u, y + x - u) - \frac{1}{2}\operatorname{Im}\mathcal{H}(y, x)\right)h(x)f(z, y, x - u)dzdydx.$$

The trace is computed by integrating down the diagonal

$$\operatorname{Tr}\gamma_\lambda(f) = \int \lambda\left(z + \frac{1}{2}\operatorname{Im}\mathcal{H}(x, y) - \frac{1}{2}\operatorname{Im}\mathcal{H}(y, x)\right)f(z, y, 0)dzdydx$$

$$= \int \lambda(z)\lambda(\operatorname{Im}\mathcal{H}(x, y))f(z, y, 0)dzdydx.$$

Now the pairing $(x, y) = \operatorname{Re}\operatorname{trace}\mathcal{H}(x, y)$ is a non-degenerate bilinear form on X, and the inversion formula on X takes the form

$$\int_X \int_X h(y)e^{i(x,y)}dydx = (2\pi)^{l/2}h(0), \qquad h \in C_c^\infty(X).$$

Moreover a straightforward computation shows that

$$\operatorname{Re}\operatorname{trace}[\operatorname{Im}\mathcal{H}(x, y)\lambda^*] = -(\lambda x, y).$$

Therefore

$$\operatorname{Tr}\gamma_\lambda(f) = (2\pi)^{l/2}|\Psi(\lambda)|^{-\frac{1}{2}(u+v)}\int \lambda(z)f(z)dz.$$

The proof is completed by the observation that $u + v = l/\varepsilon s = l/\deg\psi$. q.e.d.

Remarks. (1) The comments made in the first paragraph of the proof of [6, lemma 3.1] apply here — namely, the formal computation of $\operatorname{Tr}\gamma_\lambda(f)$ is legitimate and equation (4.4a) actually holds for sufficiently differentiable, sufficiently rapidly decreasing functions on N.

(2) We do *not* exclude the possibility that $N = Z = \operatorname{Im}\mathbf{R}^{s\times s}$ is abelian, i.e., $u + v = 0$. Proposition 4.4 is still valid (trivially since $l = 0$).

4c. *The Plancherel Formula for NA.* We start by evaluating $\operatorname{Tr} \eta_\lambda(D^{\frac{1}{2}} * f * D^{\frac{1}{2}})$ for suitable f.

4.5 Lemma. *Let $\lambda \in \mathfrak{z}_0^*$ and $\eta_\lambda = \operatorname{Ind}_N^{NA} \gamma_\lambda$. Then for any $f \in \mathcal{S}_\psi(NA)$, $\eta_\lambda(D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}})$ is trace class and*

$$\text{(4.5a)} \qquad \operatorname{Tr} \eta_\lambda(D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}}) = (2\pi)^{-k} |\psi(\lambda)|^{+k/\deg\psi} \int_0^\infty \hat{h}_0(r^2\lambda) r^{2k-1} dr,$$

where $h = f * f^*$, $h_0 = h|_Z$ and $\hat{h}_0(\lambda) = \int_Z h(z)\lambda(z)dz$.

Proof. We first observe that according to Corollary 3.11, $f * f^* \in \mathcal{S}_\psi(NA)$ and the expression $\eta_\lambda(D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}})$ makes good sense. We shall employ (as we did many times in [6]) theorem 3.2 of [7]. This is legitimate since $D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}} \in L_1(NA) \cap P(NA)$. Putting $\varphi = D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}}$ for convenience, we compute

$$\begin{aligned}
\operatorname{Tr} \eta_\lambda(\varphi) &= \int_A \delta(a)^{-1} \operatorname{Tr}\left[\int_N \varphi(a^{-1}na)\gamma_\lambda(n)dn\right] da \\
&= \int_A \alpha(a)^{-2q} (2\pi)^{l/2} |\psi(\lambda)|^{-l/2\deg\psi} \int_Z \varphi(a^{-1}za)\lambda(z)dzda \\
&= \int_A \alpha(a)^{-l} (2\pi)^{l/2} |\psi(\lambda)|^{-l/2\deg\psi} \int_Z \varphi_0(z)\lambda(aza^{-1})dzda \\
&= \int_0^\infty r^{-l} (2\pi)^{l/2} |\psi(\lambda)|^{-l/2\deg\psi} \int_Z \varphi_0(z)\lambda(r^2 z)dz \frac{dr}{r} \\
&= \int_0^\infty r^{-l} (2\pi)^{l/2} |\psi(\lambda)|^{-l/2\deg\psi} \hat{\varphi}_0(r^2\lambda) \frac{dr}{r} \\
&= \int_0^\infty r^{-l} (2\pi)^{-k} |\psi(\lambda)|^{-l/2\deg\psi} |\psi(r^2\lambda)|^{q/\deg\psi} \hat{h}_0(r^2\lambda) \frac{dr}{r} \\
&= \int_0^\infty r^{-l+2q} (2\pi)^{-k} |\psi(\lambda)|^{(q-l/2)/\deg\psi} \hat{h}_0(r^2\lambda) \frac{dr}{r} \\
&= (2\pi)^{-k} |\psi(\lambda)|^{k/\deg\psi} \int_0^\infty r^{2k} \hat{h}_0(r^2\lambda) \frac{dr}{r}.
\end{aligned}$$

The last integral is absolutely convergent — thus the positivity of $\eta_\lambda(\varphi)$ guarantees that it is trace class. Since $a_r \cdot \eta_\lambda = \eta_{r^{-2}\lambda} \cong \eta_\lambda$, the left side of (4.5a) is invariant under the transformation $\lambda \to t\lambda$, $t > 0$. We leave to the reader the verification that the right side is also. q.e.d.

We are now ready for

4.6 Theorem (Plancherel formula for NA). *Let D be the operator on NA defined by* (3.6). *Then for any $f \in \mathcal{S}_\psi(NA)$ we have*

$$\text{(4.6a)} \qquad \int_{NA} |f|^2 = \int_S \|\eta_\lambda(D^{\frac{1}{2}} * f)\|_2^2 d\sigma(\lambda).$$

Proof. Using Lemma 4.5, Lemma 3.15, equations (4.0) and (4.2), we compute

$$\begin{aligned}
\int_S \|\eta_\lambda(D^{\frac{1}{2}} * f)\|_2^2 d\sigma(\lambda) &= \int_S \operatorname{Tr} \eta_\lambda(D^{\frac{1}{2}} * f * f^* * D^{\frac{1}{2}}) d\sigma(\lambda) \\
&= \int_S (2\pi)^{-k} \int_0^\infty (f * f^*)_0^\wedge(r^2\lambda) r^{2k-1} dr d\sigma(\lambda) \\
&= (2\pi)^{-k} \int_0^\infty \int_S (f * f^*)_0^\wedge(r^2\lambda) r^{2k-1} d\sigma(\lambda) dr \\
&= (2\pi)^{-k} \int_{\hat{Z}} (f * f^*)_0^\wedge(\lambda) d\lambda \\
&= (f * f^*)_0(1_Z) \\
&= (f * f^*)(1_{NA}) \\
&= \int_{NA} |f|^2. \qquad \text{q.e.d.}
\end{aligned}$$

Here is a case where Proposition 1.6 applies (because of Corollary 3.11). The set $\mathcal{S}_\psi(NA)$ is dense in $L_2(NA)$ and is contained in $\operatorname{Dom} D^{\frac{1}{2}} \cap D^{-\frac{1}{2}} L_1(NA) \cap \mathfrak{A}_f(NA)$. Hence we can write the Plancherel formula:

$$\text{(4.7)} \qquad h(1_{NA}) = \int_S \operatorname{Tr} \eta_\lambda(D^{\frac{1}{2}} * h * D^{\frac{1}{2}}) d\sigma(\lambda), \qquad h \in P(NA) \cap \mathcal{S}_\psi(NA),$$

$$h(1_{NA}) = \int_S \operatorname{Tr} \eta_\lambda(Dh)\,d\sigma(\lambda), \qquad h \in P(NA) \cap \mathcal{S}_\psi(NA). \tag{4.8}$$

4d. *The Plancherel Formula for P.* We are now ready for the proof of the Plancherel formula on the parabolic group itself. As we said in the introduction, we have to replace the compact extension procedure of [6, §4] by a non-compact procedure. Our model for this passage will be [6, §2]. Here is the main result.

4.9 Theorem (Plancherel formula for NAM). *Let E be the operator on NAM defined by* (3.7). *Then for any $f \in \mathcal{S}_\psi(NAM)$ we have*

$$\int_{NAM} |f|^2 = \int_{\tau \in \hat{M}_1} \|\pi_\tau(E^{\frac{1}{2}} * f)\|_2^2\,d\mu_{M_1}(\tau), \qquad \mathbf{F} = \mathbf{R}, \mathbf{Q}, \tag{4.9a}$$

$$\int_{NAM} |f|^2 = \sum_{i=0}^{s} \int_{\tau \in \hat{M}_i} \|\pi_\tau^i(E^{\frac{1}{2}} * f)\|_2^2\,d\mu_{M_i}(\tau), \ \mathbf{F} = \mathbf{C}. \tag{4.9b}$$

Proof. We give an outline of the computation, followed by a justification of each of the steps, for (4.9a); the proof of (4.9b) is basically the same.

Let $f \in \mathcal{S}_\psi(NAM)$, $h = f * f^*$, $\varphi = h|_{NA}$. Then we calculate

$$\int_{NAM} |f|^2 = h(1_{NAM}) = \varphi(1_{NA})$$

$$= \int_S \operatorname{Tr} \eta_\lambda(D^{\frac{1}{2}} * \varphi * D^{\frac{1}{2}})\,d\sigma(\lambda) \tag{4.10}$$

$$= \int_{M_1 \backslash M} \operatorname{Tr} \eta_{m \cdot \lambda_1}(D^{\frac{1}{2}} * \varphi * D^{\frac{1}{2}})\,d\sigma(\bar{m}) \tag{4.11}$$

$$= \int_{\hat{M}_1} \operatorname{Tr} \pi_\tau(E^{\frac{1}{2}} * h * E^{\frac{1}{2}})\,d\mu_{M_1}(\tau) \tag{4.12}$$

$$= \int_{\hat{M}_1} \|\pi_\tau(E^{\frac{1}{2}} * f)\|_2^2\,d\mu_{M_1}(\tau). \tag{4.13}$$

We must substantiate equations (4.10)–(4.13).

(4.13) holds because of Lemma (3.15). (4.10) is valid because of (4.7) — indeed the function $\varphi = (f * f^*)|_{NA}$ is in the set $P(NA) \cap \mathcal{S}_\psi(NA)$. (Note: Proposition 1.6 is critical here — without it, the proof would be blocked at this stage.) Formula (4.11) is a trivial consequence of the discussion at the end of §4a. Now we come to (4.12) — this is the heart of the proof. Here $NA \backslash P$ is not compact so we reason as in [8, theorem 2.3].

We begin by applying the trace formula of [7] to the representation π_τ. Since $\delta_P|_{NAM_1} = \delta_{NAM_1}$, and $\delta_{NAM}|_M \equiv 1$, [7, theorem 3.2] gives

$$\operatorname{Tr} \pi_\tau (E^{\frac{1}{2}} * h * E^{\frac{1}{2}})$$

$$= \int_{M_1\backslash M} \operatorname{Tr} \int_{NAM_1} (E^{\frac{1}{2}} * h * E^{\frac{1}{2}})(m^{-1}nam_1m)(\tilde{\eta}_1 \otimes \tau)(nam_1)d(nam_1)d\sigma(\bar{m}).$$

For convenience we write $h_1 = E^{\frac{1}{2}} * h * E^{\frac{1}{2}}$, $\varphi_1 = D^{\frac{1}{2}} * \varphi * D^{\frac{1}{2}}$. Set

$$\Lambda_\tau(m) = \operatorname{Tr} \int_{NAM_1} h_1(m^{-1}nam_1m)(\tilde{\eta}_1 \otimes \tau)(nam_1)d(nam_1).$$

This is a non-negative (possibly ∞-valued) Borel function of τ and m. Therefore by Tonelli's Theorem

$$\int_{\hat{M}_1} \operatorname{Tr} \pi_\tau(h_1)d\mu_{M_1}(\tau) = \int_{\hat{M}_1} \int_{M_1\backslash M} \Lambda_\tau(m)d\sigma(\bar{m})d\mu_{M_1}(\tau)$$

$$= \int_{M_1\backslash M} \int_{\hat{M}_1} \Lambda_\tau(m)d\mu_{M_1}(\tau)d\sigma(\bar{m}).$$

Thus we are reduced to proving

$$\operatorname{Tr} \eta_{m\cdot\lambda_1}(\varphi_1) = \int_{\hat{M}_1} \Lambda_\tau(m)d\mu_{M_1}(\tau). \tag{4.14}$$

Let $\{\xi_i\}$ denote an orthonormal basis for the space of η_1, and let $\{\zeta_j^\tau\}$ denote an orthonormal basis for the space of τ. Then

$$\Lambda_\tau(m) = \sum_{i,j} \int_{NAM_1} h_1(m^{-1}nam_1m)\langle(\tilde{\eta}_1 \otimes \tau)(nam_1)\xi_i \otimes \zeta_j^\tau, \xi_i \otimes \zeta_j^\tau\rangle d(nam_1)$$

$$= \sum_{i,j} \int_{NAM_1} h_1(m^{-1}nam_1m)\langle\tilde{\eta}_1(nam_1)\xi_i, \xi_i\rangle\langle\tau(m_1)\zeta_j^\tau, \zeta_j^\tau\rangle d(nam_1).$$

Set

$$\Omega_{i,m}(m_1) = \int_{NA} h_1(m^{-1}nam_1m)\langle\tilde{\eta}_1(nam_1)\xi_i, \xi_i\rangle d(na).$$

Then

$$\int_{\hat{M}_1} \Lambda_\tau(m)d\mu_{M_1}(\tau) = \int_{\hat{M}_1} \sum_{i,j} \int_{M_1} \Omega_{i,m}(m_1)\langle\tau(m_1)\zeta_j^\tau, \zeta_j^\tau\rangle dm_1.$$

But

$$\sum_j \int_{M_1} \Omega_{i,m}(m_1)\langle\tau(m_1)\zeta_j^\tau, \zeta_j^\tau\rangle dm_1 = \operatorname{Tr}\tau(\Omega_{i,m}).$$

So

$$\int_{\hat{M}_1} \Lambda_\tau(m)d\mu_{M_1}(\tau) = \int_{\hat{M}_1} \sum_i \operatorname{Tr}\tau(\Omega_{i,m})d\mu_{M_1}(\tau).$$

Hence we are further reduced to proving

$$\operatorname{Tr}\eta_{m\cdot\lambda_1}(\varphi) = \int_{\hat{M}_1} \sum_i \operatorname{Tr}\tau(\Omega_{i,m})d\mu_{M_1}(\tau). \tag{4.15}$$

The right side of equation (4.15) can be computed as follows:

$$\begin{aligned}
&\int_{\hat{M}_1} \sum_i \operatorname{Tr}\tau(\Omega_{i,m})d\mu_{M_1}(\tau) = \sum_i \int_{\hat{M}_1} \operatorname{Tr}\tau(\Omega_{i,m})d\mu_{M_1}(\tau) && (4.16)\\
&= \sum_i \Omega_{i,m}(1_{M_1}) && (4.17)\\
&= \sum_i \int_{NA} h_1(m^{-1}nam)\langle\eta_1(na)\xi_i, \xi_i\rangle d(na) \\
&= \sum_i \int_{NA} \varphi_1(na)\langle\eta_1(mnam^{-1})\xi_i, \xi_i\rangle d(na) && (4.18)\\
&= \operatorname{Tr}\eta_{m\cdot\lambda_1}(\varphi_1).
\end{aligned}$$

In (4.18) we used M-invariance of the measure $d(na)$ and Proposition 3.17(3). We can justify (4.16) and (4.17) simultaneously by appealing to [8, lemma 2.1] (in the case of no multiplier). For that we must prove that $\Omega_{i,m}$ is L_1, continuous and that for any unitary representation τ of M_1, $\tau(\Omega_{i,m})$ is a positive operator. Well, continuity of $\Omega_{i,m}$ follows from that of h_1. Integrability holds because h_1 is compactly supported mod N, i.e., $\Omega_{i,m}$ is actually compactly supported. And the last condition can be established by relating $\tau(\Omega_{i,m})$ to kernels of positive operators (exactly as in [8, pp. 110–112]; but there is a more direct way. Let τ be any unitary representation of M_1, ζ a vector in the space of τ. We need to show $\langle \tau(\Omega_{i,m})\zeta, \zeta\rangle \geqq 0$. But

$$\begin{aligned}\langle \tau(\Omega_{i,m})\zeta, \zeta\rangle &= \int_{M_1} \Omega_{i,m}(m_1)\langle \tau(m_1)\zeta, \zeta\rangle dm_1 \\ &= \int_{NAM_1} h_1(m^{-1}nam_1m)\langle \tilde{\eta}(nam_1)\xi_i, \xi_i\rangle\langle \tau(m_1)\zeta, \zeta\rangle d(nam_1) \\ &= \langle (\tilde{\eta}_1 \otimes \tau)(h_1^m)\xi_i \otimes \zeta, \xi_i \otimes \zeta\rangle,\end{aligned}$$

where $h_1^m(nam_1) = h_1(m^{-1}nam_1m)$. The conclusion follows because $\tilde{\eta}_1 \otimes \tau$ is a unitary representation of NAM_1 and h_1^m is positive-definite. This completes the proof. q.e.d.

4e. *Other Maximal Parabolics.* Theorem 4.9 applies to all the maximal parabolic subgroups $P_{s;p-s,q-s}(\mathbf{F}) \subseteqq U(p, q; \mathbf{F})$ except for: $\mathbf{F} = \mathbf{R}$ and s odd. In addition, we have the Plancherel formula in the case $\mathbf{F} = \mathbf{R}$, $s = 1$. The parabolics in that case are

$$P = P_{1;u,v}(\mathbf{R}) = \mathbf{R}^{u,v} \cdot (GL(1, \mathbf{R}) \times O(u, v)), \qquad u + v \geqq 1.$$

The analog of the polynomial ψ is $\psi_0(\lambda) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_u^2 - \lambda_{u+1}^2 - \cdots - \lambda_{u+v}^2$, $\lambda \in \mathbf{R}^{u,v}$; and the corresponding differential operator (analog of Ψ) is the wave operator

$$\Box = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_u^2} + \frac{\partial^2}{\partial x_{u+1}^2} + \cdots + \frac{\partial^2}{\partial x_{u+v}^2}.$$

The generic representations of NA are parametrized by $S^+ \cup S^-$, $S^\pm = \{\lambda \in \mathbf{R}^{u,v} : \psi_0(\lambda) = \pm 1\}$; the generic representations of P are parametrized (essentially) by $O(u-1, v)^\wedge \cup O(u, v-1)^\wedge$. The Plancherel formula has the form:

$$\int_P |f|^2 = \int_{O(u-1,v)^\wedge} \| \pi_\tau(E^{\frac{1}{2}} * f)\|_2^2 d\mu_{O(u-1,v)}(\tau) \tag{4.19}$$
$$+ \int_{O(u,v-1)^\wedge} \| \pi_\tau(E^{\frac{1}{2}} * f)\|_2^2 d\mu_{O(u,v-1)}(\tau)$$

where $E = c|\Box|^{\frac{1}{2}(u+v)}$ acts on $\mathbf{R}^{u,v}$ in the usual way. The method of proof of (4.19) is the same as that of Theorem 4.9 — the details are actually a little less complicated since N is abelian.

The remaining cases are: $\mathbf{F} = \mathbf{R}$, s odd and $s \geqq 3$. There the situation is still unsettled. Consider the simplest example

$$O(3,3) \supset P_{3;0,0}(\mathbf{R}) \cong \operatorname{Im} \mathbf{R}^{3\times 3} \cdot GL(3;\mathbf{R}) \cong \mathbf{R}^3 \cdot GL(3;\mathbf{R}). \tag{4.20}$$

This parabolic group has one generic irreducible representation, carries a unique Dixmier–Pukanszky operator (up to scalar), and that operator *cannot* live on the nilradical. We will return to these groups on another occasion.

The maximal parabolic subgroups of the other classical groups are similar in structure to those of the unitary groups. Most of them can be treated by the methods used here in §§2–4, and the ones not amenable to such methods resemble the example (4.20).

The "good" maximal parabolic subgroups of classical groups fall into two categories. In the first category, the nilradical is of the form $N = Z + X$ where $Z = \operatorname{Cent} N$ is a nonzero $\mathbf{R}$-linear subspace of an $\mathbf{F}^{s\times s}$, $s \geqq 1$, which contains invertible matrices, and X is a subspace of an $\mathbf{F}^{s\times n}$, $n \geqq 0$. There, the module for the action of Z on $\mathbf{F}^s$ plays the role of ψ, and E is a positive power of the absolute value of the corresponding constant coefficient differential operator Ψ on Z. Here is the list, in the notation of [15].

(4.21)

G	P	see [15], pages
$GL(2n;\mathbf{F})$	$L_{n,n}(\mathbf{F})$	14–15
$U(p,q;\mathbf{F})$	$P_{s;p-s,q-s}(\mathbf{F})$, except $\mathbf{F}=\mathbf{R}$, s odd	26–28
$Sp(n;\mathbf{F})$	$P_{s;2(n-s)}(\mathbf{F})$	83–85
$O(n;\mathbf{C})$	$P_{s;m-s,n-m-s}(\mathbf{R})_{\mathbf{C}}$ with s even	126–127
$SO^*(2n)$	$P^*_{2s,n-2s}$	147–149

Note that N is abelian in some of these cases.

The second category of "good" maximal parabolic subgroups of classical groups consists of those resembling $P_{1;u,v}(\mathbf{R})$ — the nilradical is abelian and has an obvious semi-invariant not like a determinant. Those are just

(4.22)

G	P	N	see [15], pages
$O(p,q)$	$P_{1,p-1,q-1}(\mathbf{R})$	$\mathbf{R}^{p-1,q-1}$	26–28
$O(n;\mathbf{C})$	$P_{1,m-1,n-m-1}(\mathbf{R})_{\mathbf{C}}$	$\mathbf{C}^{n-2}$	126–127

In each case, Ψ is the wave operator $\square$.

The Plancherel formula for both "good" classes (4.21) and (4.22) is derived, as above for the $P_{s;u,v}(\mathbf{F})$ (with s even in case $\mathbf{F}=\mathbf{R}$) and the $P_{1,u,v}(\mathbf{R})$, using the explicit structural information in [15].

To avoid mis-impression, we mention the parabolics not covered by the methods of this paper:

(4.23)

G	P	see [15], pages
$GL(a+b;\mathbf{F})$	$L_{a,b}(\mathbf{F}),\ a \neq b$	14–15
$O(p,q)$	$P_{s;p-s,q-s}(\mathbf{R}),\ s$ odd, $s \geqq 3$	26–28
$O(n;\mathbf{C})$	$P_{s;m-s,n-m-s}(\mathbf{R})_{\mathbf{C}},\ s$ odd, $s \geqq 3$	126–127

Here $L_{a,b}(\mathbf{F})$ has abelian nilradical $\mathbf{F}^{a\times b}$, and $P_{s;0,0}(\mathbf{R})$ and $P_{s;0,0}(\mathbf{R})_{\mathbf{C}}$ also have abelian nilradical.

Finally let us note that an examination of the subgroup $NA \cong \mathbf{R}^2 \cdot \mathbf{R}_+^*$, which occurs in both $P_{1;1,1}(\mathbf{R})$ and $P_{1;2,0}(\mathbf{R})$, seems to indicate that there is no "best" choice of the pair (D,μ) in Theorem 1.1. In effect, any Borel cross section $S \subset \mathbf{R}^2 - \{0\} = \mathfrak{n}^* - \{0\}$ to the action of $\mathbf{R}_+^* = A$ determines a measure $\mu = \mu_S$ on $(NA)^\wedge$ by

$$\int_{\mathbf{R}^2} f(\lambda)\,d\lambda = \int_0^\infty \left(\int_S f(r\lambda)\,d\mu_S(\lambda)\right) r\,dr,$$

and that in turn determines an operator D. The section $\{\lambda : \lambda_1^2 + \lambda_2^2 = 1\}$ leads to $D = \Delta$, which is suitable for $P_{1;2,0}(\mathbf{R})$; the section $\{\lambda : \lambda_1^2 - \lambda_2^2 = \pm 1\}$ leads to $D = |\square|$, which is suitable for $P_{1;1,1}(\mathbf{R})$.

5. Pfaffian polynomials and operators

The maximal parabolic subgroups of the classical groups, in which the nilradical is noncommutative but has representations square integrable modulo the center, are just the ones listed in Table (4.21) for which the nilradical is noncommutative. In those cases the Pfaffian polynomial of [10] gives a canonical element Ψ in the enveloping algebra of the center of the nilradical. In Section 5a, we prove that $|\Psi|$ is semi-invariant under the full parabolic, and that an appropriate power $|\Psi|^r$ is a Dixmier–Pukanszky operator. In fact, we do this for all parabolic subgroups $P \subset G$ where: (i) G is a reductive real Lie group such that every $\mathrm{Ad}(x)$, $x \in G$, is an inner automorphism of $\mathfrak{g}_{\mathbf{C}}$; and (ii) if G_i is a simple local factor of G then the nilradical of $P \cap G_i$ is noncommutative but has square integrable (mod center) representations. In §5b we return to the groups (4.21) with noncommutative nilradical and show that their Dixmier–Pukanszky operators of §§3 and 4 agree with the ones defined here from the Pfaffian polynomials. Then finally, in §5c, we illustrate the use of the Pfaffian operators, describing the explicit Plancherel formula for minimal parabolics in simple groups with restricted root system of type A_2.

5a. *The Pfaffian as a Dixmier–Pukanszky Operator.* We will use, without further remarks, the following straightforward facts about extensions of operators from a normal subgroup N to a semidirect product $P = N \cdot Y$. Every operator T on N can be viewed as an operator $\tilde{T}$ on P by

$$(\tilde{T}f)(xy) = T(f_y)(x), \quad f_y(x) = f(xy), \qquad x \in N, \quad y \in Y.$$

If T is right N-invariant then $\tilde{T}$ is right P-invariant. If T is left N-invariant then (i) $\tilde{T}$ is left N-invariant and (ii) $\tilde{T}$ is left P-invariant just when T is Y-invariant. If T is right N-invariant and Y-semi-invariant, $T(f^y) = \alpha(y)T(f)$ where $f^y(x) = f(yxy^{-1})$ and $\alpha: Y \to \mathbf{C}^*$, then $\tilde{T}$ is P-semi-invariant with module $\tilde{\alpha}(xy) = \alpha(y)$. Let P have right Haar measure $dxdy$ and let T be densely defined on $L_2(N, dx)$. If T is symmetric with positive self adjoint closure on $L_2(N, dx)$, the same holds for $\tilde{T}$ on $L_2(P, dxdy)$. Also invertibility of T guarantees that of $\tilde{T}$. Finally, by right invariance, $\tilde{T}$ is affiliated with the left ring of P.

Now let G be a reductive real Lie group such that every $\mathrm{Ad}(x)$, $x \in G$, is an inner automorphism on $\mathfrak{g}_{\mathbf{C}}$. Let P be a parabolic subgroup of G, say with Langlands decomposition $P = NAM$, and suppose that N has square integrable (modulo its center Z) representations [10]. Fix a volume element ω on $\mathfrak{n}/\mathfrak{z}$. It defines the Pfaffian polynomial on $\mathfrak{z}^*$ by (i) if $\mathfrak{z} = \mathfrak{n} \neq 0$ then $Pf \equiv 1$ and (ii) if $\mathfrak{n}/\mathfrak{z} \neq 0$ then:

$$(5.1) \qquad \begin{cases} \text{if } \lambda \in \mathfrak{z}^*, \ \varphi \in \mathfrak{n}^* \text{ with } \varphi|_{\mathfrak{z}} = \lambda, \text{ and } l = \dim \mathfrak{n}/\mathfrak{z} \\ \text{then } \psi(\lambda) = Pf(\lambda) \text{ is given by } b_\varphi^{l/2} = \psi(\lambda)\omega, \\ \text{where } b_\varphi \text{ is the 2-form } b_\varphi(x+\mathfrak{z}, y+\mathfrak{z}) = \varphi[x,y] \text{ on } \mathfrak{n}/\mathfrak{z}. \end{cases}$$

See [10, §3] for the fact that $Pf(\lambda)$ is well defined, and is nonconstant when N is noncommutative.

The Poincaré–Birkhoff–Witt map gives an element $\Psi \in \mathfrak{U}(\mathfrak{z}) = \mathcal{Z}(\mathfrak{n})$ corresponding to $\Psi = Pf$. Its action as a differential operator on N is determined by

$$\int_{\mathfrak{z}} \Psi f(\exp x) e^{i\varphi(x)} dx = Pf(\varphi|_{\mathfrak{z}}) \int_{\mathfrak{z}} f(\exp x) e^{i\varphi(x)} dx, \qquad \varphi \in \mathfrak{n}^*. \tag{5.2}$$

So Ψ is a densely defined, symmetric, conjugation-invariant, invertible operator on $L_2(N)$.

5.3 Lemma. *$|Pf|$ and $|\Psi|$ are M-invariant and A-semi-invariant.*

Proof. Let g be an automorphism of N. It preserves Z and multiplies the volume element ω of $\mathfrak{n}/\mathfrak{z}$ by a scalar which we denote $\alpha(g)$. If $\varphi \in \mathfrak{n}^*$, then

$$Pf(g * \varphi)|_{\mathfrak{z}})\omega = b_{g*\varphi}^{l/2} = (\Lambda^2(g)b_\varphi)^{l/2} = \Lambda^l(g) b_\varphi^{l/2} = Pf(\varphi|_{\mathfrak{z}})\Lambda^l(g)\omega;$$

so

$$Pf(g * \lambda) = \alpha(g) Pf(\lambda) \quad \text{and} \quad |Pf(g * \lambda)| = |\alpha(g)|\,|Pf(\lambda)| \quad \text{for } \lambda \in \mathfrak{z}^*. \tag{5.4}$$

That means

$$g \text{ sends } \Psi \text{ to } \alpha(g)\Psi \text{ and } |\Psi| \text{ to } |\alpha(g)|\,|\Psi|. \tag{5.5}$$

But $x \to |\alpha(\mathrm{Ad}(x)|_N)|$ is a homomorphism of MA to the positive reals. Our assertions follow because M has no nontrivial positive character that factors through $\mathrm{Ad}_G|_M$. q.e.d.

5.6 Remark. One can also derive Lemma 5.3 from the fact that $|Pf(\lambda)|d\lambda$ is Plancherel measure on $\hat{N}$.

Now let $\alpha: A \to \mathbf{R}_+^*$ (as in Lemma 5.5) and $\beta: A \to \mathbf{R}_+^*$ denote the respective moduli of $\mathrm{Ad}_G|_A$ on $\mathfrak{n}/\mathfrak{z}$ and on $\mathfrak{z}$, so

$$\delta_p(a) = \alpha(a)\beta(a) \qquad \text{for } a \in A. \tag{5.7}$$

As outlined at the beginning of the section, we extend $|\Psi|$ to a semi-invariant operator on $P = NAM$ with module $\alpha = \delta_p \beta^{-1}$. The next lemma will enable us to deduce that $|\Psi|^{2q/l}$ is a Dixmier–Pukanszky operator.

5.8 Lemma. *If G is simple then $\alpha(a) = \delta_P(a)^{l/2q}$ for all $a \in A$, where $k = \dim Z$, $l = \dim N/Z$ and $q = k + \frac{1}{2}l$.*

Proof. Let Q be the positive $\mathfrak{a}$-root system on $\mathfrak{g}$ such that $\mathfrak{n} = \Sigma_{\nu \in Q} \mathfrak{g}_\nu$, and let ν_0 be the highest $\mathfrak{a}$-root. We first show (this is valid for any real parabolic) that $\mathfrak{z} = \mathfrak{g}_{\nu_0}$. Since M acts irreducibly on each $\mathfrak{g}_\nu$ ([16, p. 296] and the assumption that every $\mathrm{Ad}(g)$ is inner on $\mathfrak{g}_C$), $\mathfrak{z} = \Sigma_{\nu \in S} \mathfrak{g}_\nu$ for some subset $S \subset Q$. Clearly $\mathfrak{g}_{\nu_0} \subset \mathfrak{z}$, i.e., $\nu_0 \in S$. If $\nu \in Q$ with $\nu < \nu_0$ we claim there exists a sequence $\nu_0, \nu_1, \cdots, \nu_t = \nu$ in Q such that each $\nu_i - \nu_{i+1} \in Q$. For that, let $\mathfrak{h}$ be a maximally split Cartan subalgebra of $\mathfrak{g}$ that contains $\mathfrak{a}$; let Σ be a positive $\mathfrak{h}_C$-root system on $\mathfrak{g}_C$ consistent with Q in the sense

$$Q = \{\gamma|_\mathfrak{a} : \gamma \in \Sigma \quad \text{and} \quad \gamma|_\mathfrak{a} \neq 0\};$$

let γ_0 be the maximal $\mathfrak{h}_C$-root; choose $\gamma \in \Sigma$ with $\gamma|_\mathfrak{a} = \nu$, and take a corresponding sequence in Σ. This is where we use simplicity of G. If $\nu < \nu_0$ in Q now, we have $\nu' \in Q$ with $[\mathfrak{g}_\nu, \mathfrak{g}_{\nu'}] \neq 0$, so $\nu \notin S$. Thus $\mathfrak{z} = \mathfrak{g}_{\nu_0}$.

Now we use the fact that N has square integrable representations. If $\mathfrak{n}/\mathfrak{z} = 0$ the Lemma just says $1 = 1$, so we may assume N nonabelian. Choose a basis $\{\lambda_1, \cdots, \lambda_k\}$ of $\mathfrak{z}^*$ with each $Pf(\lambda_i) \neq 0$ and let $\{z_1, \cdots, z_k\}$ be the dual basis of $\mathfrak{z}$. Fix $r \in \{1, \cdots, k\}$. Given $\nu \in Q$, $\nu \neq \nu_0$, then $b_{\lambda_r}(x, y) = \lambda_r[x, y]$ pairs $\mathfrak{g}_\nu$ nondegenerately with $\mathfrak{g}_{\nu_0 - \nu}$; so if $\nu_0 \neq 2\nu$ we have bases $\{x_{1,\nu,r}, \cdots, x_{m_\nu,\nu,r}\}$ of $\mathfrak{g}_\nu$ and $\{y_{1,\nu,r}, \cdots, y_{m_\nu,\nu,r}\}$ of $\mathfrak{g}_{\nu_0 - \nu}$ such that

$$\lambda_r[x_{i,\nu,r}, y_{j,\nu,r}] = \delta_{ij}, \quad \text{i.e.} \quad [x_{i,\nu,r}, y_{j,\nu,r}] = \delta_{ij} z_r + \sum_{s \neq r} b_{ijrs} z_s. \tag{5.9}$$

If $\nu_0 = 2\nu$ then $\mathfrak{g}_\nu = \mathfrak{g}_{\nu_0 - \nu}$ has a basis $\{x_{i,\nu,r}, y_{j,\nu,r}\}$ that satisfies (5.9). Selecting one root from each pair $\nu, \nu_0 - \nu$ where $\nu_0 \neq \nu \in Q$, we have a basis $\{x_{i,r}; y_{j,r}\}$ of $\mathfrak{n}$ modulo $\mathfrak{z}$ of size l. If $\zeta \in \mathfrak{a}$ then define a, b, c by

$$[\zeta, x_{i,r}] = a_{i,r} x_{i,r}, \quad [\zeta, y_{j,r}] = b_{j,r} y_{j,r}, \quad [\zeta, z_r] = c_r z_r.$$

Now

$$(a_{i,r} + b_{i,r})\left(z_r + \sum_{s \neq r} b_{iirs} z_s\right) = [\zeta, [x_{i,r}, y_{i,r}]] = c_r z_r + \sum_{s \neq r} b_{iirs} c_s z_s.$$

Equate coefficients of z_r: $a_{i,r} + b_{i,r} = c_r$. Sum over i:

$$\text{trace } (\mathrm{ad}(\zeta)|_{\mathfrak{n}/\mathfrak{z}}) = (l/2) c_r.$$

Now sum over r:

$$k \cdot \text{trace}(\mathrm{ad}(\zeta)|_{\mathfrak{n}/\mathfrak{z}}) = (l/2) \cdot \text{trace}(\mathrm{ad}(\zeta)|_\mathfrak{z}). \tag{5.10}$$

Because of (5.7) and the notation just before it, (5.10) exponentiates to

$$\alpha(a)^k = \{\delta_P(a)\cdot\alpha(a)^{-1}\}^{l/2} \qquad \text{for } a \in A.$$

So $\alpha(a)^{2k+l} = \delta_P(a)^l$, i.e., $\alpha(a) = \delta_P(a)^{l/2q}$ as asserted. q.e.d.

In the general case, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_p$ where $\mathfrak{g}_0$ is the center and the other $\mathfrak{g}_i$ are simple ideals. The parabolic $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_p$ where $\mathfrak{p}_i = \mathfrak{n}_i + \mathfrak{a}_i + \mathfrak{m}_i$ is parabolic in $\mathfrak{g}_i$ and $N_i = \exp_G(\mathfrak{n}_i)$ has square integrable representations modulo its center Z_i. Here $N = N_1 \times \cdots \times N_p$ and $Z = Z_1 \times \cdots \times Z_p$. Set $k_i = \dim Z_i$, $l_i = \dim N_i/Z_i$ and $q_i = k_i + \frac{1}{2}l_i$. Then Lemma 5.8 says that $A_i = \exp_G(\mathfrak{a}_i)$ acts on $\mathfrak{n}_i/\mathfrak{z}_i$ with modulus $\delta_{P_i}^{l_i/2q_i}$, where P_i is the parabolic subgroup with Lie algebra $\mathfrak{p}_i$ in a local direct factor G_i of G with Lie algebra $\mathfrak{g}_i$. So we have

5.11 Theorem. *If the derived group $[G,G]$ is simple, and if $\Psi \in \mathfrak{U}(\mathfrak{p})$ corresponds to the Pfaffian polynomial on $\mathfrak{z}^*$, then $|\Psi|^{2q/l}$ is a Dixmier–Pukanszky operator on P. More generally, if for each simple ideal $\mathfrak{g}_i$ in $\mathfrak{g}$, $\mathfrak{p} \cap \mathfrak{g}_i = \mathfrak{n}_i + \mathfrak{a}_i + \mathfrak{m}_i$ with $\mathfrak{n}_i/\mathfrak{z}_i \neq 0$, and if $\Psi_i \in \mathfrak{U}(\mathfrak{z}_i)$ is the operator on P determined by the Pfaffian polynomial on $\mathfrak{z}_i^*$, then $\Pi|\Psi_i|^{2q_i/l_i}$ is a Dixmier–Pukanszky operator on P.*

Remarks. (1) The reader can check Conjecture 1.3 for the operator $\Pi|\Psi_i|^{2q_i/l_i}$ on P that occurs in Theorem 5.11 (using the partial Schwartz space $\mathcal{S}_\psi$ of (3.9) corresponding to the Pfaffian $\psi(\lambda) = Pf(\lambda)$ on $\mathfrak{z}^*$).

(2) Theorem 5.11 applies to many parabolics not listed in (4.21). For example, in a split A_l or E_6, it applies to those of parabolic rank 2 in which the root system of M is obtained by removing a pair of simple roots symmetrical in the Dynkin diagram of G. A complete classification of the cases to which it applies will appear in [17]. See §5c below for some interesting special cases.

5b. *Comparison of Operators.* In this section $P = NAM$ is one of the groups $P_{s;u,v}(\mathbf{F})$ where $u + v > 0$ and where $s \geqq 1$ is even in case $\mathbf{F} = \mathbf{R}$. In other words, P is a maximal parabolic subgroup of a unitary group $\mathrm{U}(s+u, s+v; \mathbf{F})$ whose nilradical is noncommutative but has square integrable representations. We will prove that the operators D on NA and E on $P = NAM$, defined in (3.6) and (3.7) and used in the Plancherel formulae (4.6) and (4.9), are equal to the operators $|\Psi|^{2q/l}$ of Theorem 5.11 which are defined (5.2) by the Praffian polynomial (5.1).

It suffices to show that the "real determinant" function on $\mathfrak{z} = \mathrm{Im}\,\mathbf{F}^{s\times s}$, given by (2.5),

$$\text{(5.12)} \qquad \det_{\mathbf{R}}(z) = \text{module of } z \in \mathfrak{z} \text{ on } \mathbf{F}^s \text{ relative to Lebesgue measure,}$$

and the Pfaffian polynomial on $\mathfrak{z}^*$, $Pf(\lambda)$ defined in (5.1), are related under the pairing (2.9)

$$\mathfrak{z} \cong \mathfrak{z}^*, \quad \text{by } z \mapsto \lambda_z, \quad \text{where } \lambda_z(z_0) = (z, z_0) = \operatorname{Re}\operatorname{trace}(zz_0^*),$$

in the manner

$$|\det_{\mathbf{R}}(z)|^{q/\varepsilon s} = |Pf(\lambda_z)|^{2q/l} \tag{5.13}$$

where $\varepsilon = \dim_{\mathbf{R}}\mathbf{F}$, $k = \dim_{\mathbf{R}} Z$, $l = \dim_{\mathbf{R}} N/Z$ and $q = k + \frac{1}{2}l$.

Let $\{x_1, \cdots, x_l\}$ be an $\mathbf{R}$-basis of $\mathbf{F}^{s \times (u,v)} \cong \mathfrak{n}/\mathfrak{z}$. We normalize the volume element on $\mathfrak{n}/\mathfrak{z}$ used to define the Pfaffian of an antisymmetric $\mathbf{R}$-bilinear form b, so that the Pfaffian has value equal to the classical Pfaffian of the matrix $(b(x_i, x_j))$. It will also be convenient to use the nondegenerate inner product $(x, x_0) = \operatorname{Re}$ trace $(aa_0^* - bb_0^*)$ on $\mathbf{F}^{s \times (u,v)}$, where $x = (a, b)$ and $x_0 = (a_0, b_0)$ with $a, a_0 \in \mathbf{F}^{s \times u}$ and $b, b_0 \in \mathbf{F}^{s \times v}$.

Let $z \in \mathfrak{z}$ and $\lambda = \lambda_z \in \mathfrak{z}^*$. Now $Pf(\lambda)$ is the Pfaffian of the matrix.

$$(\lambda[x_i, x_j]) = (z, [x_i, x_j]).$$

We compute

$$\begin{aligned}
(z, [x_i, x_j]) &= \operatorname{Re}\operatorname{trace}\{z \cdot \operatorname{Im} \mathscr{H}(x_i, x_j)^*\} \\
&= \frac{1}{2}\operatorname{Re}\operatorname{trace}\{-za_ia_j^* + za_ja_i^* + zb_ib_j^* - zb_jb_i^*\} \\
&= \frac{1}{2}\operatorname{Re}\operatorname{trace}\{-(za_i)a_j^* - a_j(za_i)^* + (zb_i)b_j^* + b_j(zb_i)^*\} \\
&= -(zx_i, x_j).
\end{aligned}$$

Now we have

$$\begin{aligned}
Pf(\lambda)^2 &= \det(-(zx_i, x_j)) = (-1)^{-l}\det(x \to zx \quad \text{on } \mathfrak{n}/\mathfrak{z}) \\
&= (-1)^{-l}\{\det_{\mathbf{R}}(x \mapsto zx) \quad \text{on } \mathbf{F}^s\}^{(u+v)=(l/\varepsilon s)};
\end{aligned}$$

so

$$|Pf(\lambda)|^{2q/l} = \{|\det_{\mathbf{R}}(z)|^{(l/\varepsilon s)}\}^{(q/l)} = |\det_{\mathbf{R}}(z)|^{q/\varepsilon s} \qquad \text{q.e.d.}$$

Virtually identical considerations apply to the maximal parabolics $P_{s;2(n-s)}(\mathbf{F}) \subset \operatorname{Sp}(n; \mathbf{F})$, $1 \leqq s < n$, and $P^*_{2s;n-2s} \subset \mathrm{SO}^*(2n)$, $1 \leqq s < n$, listed in (4.21).

5c. *Application to Minimal Parabolic Subgroups.* Let G be a simple real Lie group and $P = NAM$ a minimal parabolic subgroup. One knows [5] that N has representations square integrable modulo its center Z if, and only if, the restricted root ($\mathfrak{a}$-root) system of $\mathfrak{g}$ is of type A_1 or A_2. In the A_1 case, the Plancherel formula for P is given in [6], and except when $\mathfrak{g}$ is of type F_4 it is a special case of Theorem 4.9 and (4.19) above. Here we are going to use Theorem 5.11 to write an explicit Plancherel formula in the A_2 case.

The real simple Lie algebras with restricted root system of type A_2 are the $\mathfrak{sl}(3;\mathbf{F})$, where $\mathbf{F}$ is real, complex, quaternion or Cayley, as follows.

$$(5.14)\quad \begin{cases} \mathfrak{sl}(3;\mathbf{R}): & 3\times 3 \text{ real matrices of trace zero} \\ \mathfrak{sl}(3;\mathbf{C}): & 3\times 3 \text{ complex matrices of trace zero} \\ \mathfrak{sl}(3;\mathbf{Q}): & 3\times 3 \text{ quaternion matrices of real trace zero} \\ \mathfrak{sl}(3;\mathrm{Cay}): & \text{this just means } \mathfrak{e}_{6(-26)}, \text{ the Lie algebra of} \\ & \text{type } E_6 \text{ with maximal compact of type } F_4. \end{cases}$$

A convenient choice of groups with these Lie algebras is

$$(5.15)\quad \begin{cases} SL(3;\mathbf{R}): & 3\times 3 \text{ real matrices of determinant 1} \\ SL(3;\mathbf{C}): & 3\times 3 \text{ complex matrices of determinant 1} \\ SL(3;\mathbf{Q}): & GL'(3;\mathbf{Q}), \text{ the real form of } SL(6;\mathbf{C}) \text{ with} \\ & \text{maximal compact subgroup } Sp(3) \\ SL(3;\mathrm{Cay}): & \text{this just means the connected simple Lie} \\ & \text{group of type } E_6 \text{ with maximal compact subgroup} \\ & \text{of type } F_4. \end{cases}$$

They have minimal parabolic subgroups $P = NAM$ as follows:

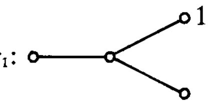
(5.16)
$$N = \left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{F}\right\} \quad\text{and}\quad A = \left\{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_i \in \mathbf{R}_+^*, a_1a_2a_3 = 1\right\}$$

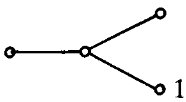
with ordinary matrix multiplication. Also

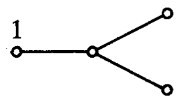
$$(5.17a)\quad \begin{cases} \mathbf{F}=\mathbf{R} \text{ or } \mathbf{C}: M = \left\{\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} : m_i \in \mathbf{F}, |m_i| = 1, m_1m_2m_3 = 1\right\} \\ \mathbf{F}=\mathbf{Q}: \qquad M = \left\{\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} : m_i \in \mathbf{F}, |m_i| = 1\right\} \end{cases}$$

again with ordinary matrix multiplication. And

$$\mathbf{F} = \mathrm{Cay}\colon M \cong \mathrm{Spin}(8) \text{ with } \mathrm{Ad}(m)\cdot\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \sigma_1(m)x & \sigma_3(m)z\\ 0 & 1 & \sigma_2(m)y\\ 0 & 0 & 1\end{pmatrix}$$

(5.17b) where σ_1: 

σ_2: 

σ_3: 

so σ_1, σ_2 are the half spin representations, σ_3 is the vector representation, and the Triality Principle says that $\mathrm{Ad}(m)$ is an automorphism on N.

Let M_1 denote the M-centralizer of

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in N.$$

From (5.17), M_1 and a section Σ to the action of M on M/M_1 are given by

$$(5.18)\begin{cases} \mathbf{F} \neq \mathrm{Cay}\colon & M_1 = \{m \in M\colon m_1 = m_3\} \text{ and } \Sigma = \{m \in M\colon m_1 m_2 = m_3 = 1\} \\ \mathbf{F} = \mathrm{Cay}\colon & M_1 \cong \mathrm{Spin}(7) \text{ and } \sigma_3(\Sigma) \text{ consists of the} \\ & \text{multiplications } z \to wz,\ |w| = 1. \end{cases}$$

Thus for $\mathbf{R}$, M_1 and Σ are cyclic groups of order 2; for $\mathbf{C}$, M_1 and Σ are circle groups: for $\mathbf{Q}$, $M_1 \cong \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ and $\Sigma \cong \mathrm{Sp}(1)$; and for Cay, $M_1 \cong \mathrm{Spin}(7)$ and Σ is the 7-sphere Moufang loop.

Let

$$U = \left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\colon x, z \in \mathbf{F}\right\},$$

a normal subgroup of P. We shall describe the representation theory and Plancherel data of P via the group extension $U \subseteqq P$. For $\xi, \zeta \in \mathbf{F}$, let $\gamma_{\xi,\zeta} \in \hat{U}$ be defined by

$$\gamma_{\xi,\zeta}\begin{pmatrix}1 & x & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = e^{i\,\mathrm{Re}(x\bar{\xi} + z\bar{\zeta})}.$$

The set $\{\gamma_{\xi,\zeta}\colon \zeta \neq 0\}$ is a single P-orbit. The stability group of $\gamma_{0,1}$ is $L = UA_1M_1$ where

$$A_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & a \end{pmatrix} : a > 0 \right\} \quad \text{and } M_1 \text{ is given by (5.18).}$$

The generic representations of P are given by $\pi_{t,\chi} = \operatorname{Ind}_{UA_1M_1}^{P} \gamma_{0,1} \times \rho_t \times \chi$, $t \in \mathbf{R}$, $\chi \in \hat{M}_1$, where

$$\rho_t \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & a \end{pmatrix} = a^{it}.$$

We now apply the results of §5a. Write $\varepsilon = \dim_{\mathbf{R}} \mathbf{F}$. Then

$$\operatorname{Cent} N = Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbf{F} \right\},$$

and $k = \dim Z = \varepsilon$, $l = \dim N/Z = 2\varepsilon$, $q = k + \frac{1}{2}l = 2\varepsilon$. Therefore $2q/l = 2$. Let $\Psi \in \mathfrak{Z}(\mathfrak{n})$ correspond to the Pfaffian polynomial on $\mathfrak{z}^*$. Then Theorem 5.11 says that

$$E = |\Psi|^2 = \Psi^2 \text{ is a Dixmier–Pukanszky operator on } P.$$

Since this is a differential operator, there are no domain problems in this case. Also it's easy to see that

$$(5.19) \qquad (Ef)^{\wedge}(\zeta) = |\zeta|^{2\varepsilon} \hat{f}(\zeta).$$

We have the following Plancherel formula.

5.20 Proposition. *There is a (computable) constant $c > 0$ so that*

$$(5.20a) \qquad f(1_P) = c \int_{-\infty}^{\infty} \sum_{\chi \in \hat{M}_1} \operatorname{Tr} \pi_{t,\chi}(Ef)\,dt, \qquad f \in C_c^{\infty}(P).$$

The proof is similar to the argument of §§3 and 4. We remark only that the group

$$V = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbf{F} \right\} \cdot \Sigma \qquad (\Sigma \text{ given by (5.18)})$$

is a cross-section for $UA_1M_1 \backslash P$. Therefore we can realize the representations $\pi_{t,\chi}$ on the space $L_2(V)$. It is then possible, using: [7, theorem 3.2], Duflo's factorization theorem for C_c^∞ functions, equation (5.19) and the inversion formula on $\mathbf{F}$ itself, to derive formula (5.20a). The computations are a bit tedious. In fact, formula (5.20a) represents a very special case of more general Plancherel formulae that one of us has worked out for parabolic subgroups of Chevalley groups.

6. Appendix

We present two results as evidence for the existence of the critical value α in §3b. The first was shown to us by R. Prosser; the second by C. Fefferman. As a corollary of the first, we deduce that E moves test functions to integrable functions for "most" parabolic subgroups of the unitary groups; we deduce a negative result about the maximal parabolics $P_{1;p,q}(\mathbf{R})$ in $\mathrm{O}(p+1, q+1)$ from the second.

6a. $|\Theta|^t C_c^\infty(V) \subseteqq L_1(V)$ *for t Large.* The following result is valid in the setup of §3, i.e., $V = Z \times V$, etc. But for simplicity we take $V = Z = \mathbf{R}^k$. Let $\theta(\xi)$ be a polynomial function of $\xi \in \mathbf{R}^k$ and define $|\Theta|^t$ as in (3.3), i.e.

$$|\Theta|^t f(z) = \mathcal{F}^{-1}\{|\theta(\xi)|^t \mathcal{F}(f)\}(z), \qquad t \geqq 0.$$

6.1 Proposition. $|\Theta|^t C_c^\infty(\mathbf{R}^k) \subseteq L_1(\mathbf{R}^k)$ *if* $t > 2[k/4] + 2$.

Proof. Let $\beta(\xi) = |\theta(\xi)|^t (1 + \|\xi\|^2)^{-n}$ where $n > 0$ is large enough to insure that $\beta(\xi) \in L_w(\mathbf{R}^k)$, $\forall w \geqq 1$. Now let v be a positive integer satisfying $2v \leqq t$. If we set

$$\gamma(\xi) = (1 + \Delta)^v \beta(\xi),$$

then (since $(1+\Delta)^v |\theta(\xi)|^t$ is bounded by a polynomial function) we have $\gamma(\xi) \in L_w(\mathbf{R}^k)$, $\forall w \geqq 1$. In particular $\hat{\gamma}(x) \in L_2(\mathbf{R}^k)$. But

$$\hat{\beta}(x) = (1 + \|x\|^2)^{-v} \hat{\gamma}(x)$$

is a product of L_2 functions — and hence is in $L_1(\mathbf{R}^k)$ — as long as $4v > k$. Therefore

$$\begin{aligned} |\Theta|^t f &= \mathcal{F}^{-1}\{|\theta|^t \hat{f}\} = \mathcal{F}^{-1}\{(1 + \|\xi\|^2)^n \beta \hat{f}\} \\ &= \mathcal{F}^{-1}(\beta) * \mathcal{F}^{-1}\{(1 + \|\xi\|^2)^n \hat{f}\} \\ &= \mathcal{F}^{-1}(\beta) * (1 + \Delta)^n f. \end{aligned}$$

The latter is a convolution of two L_1 functions; hence $|\Theta|^t f$ is integrable, so long as we can find v satisfying $2v \leqq t$ and $4v > k$. That is the case whenever $t > 2[k/4] + 2$.

We can apply this proposition to obtain an analog of [6, corollary 2.8] for "most" of the parabolic subgroups $P_{s;u,v}(\mathbf{F})$ of the unitary groups.

6.2 Corollary. *Let $t \geqq 0$ and let E be the Dixmier–Pukanszky operator on $P = P_{s;u,v}(\mathbf{F})$ defined in (3.7) — $\mathbf{F} = \mathbf{R}$, s odd is excluded as usual. Then for any fixed s, we have $E^t C_c^\infty(P) \subseteqq L_1(P)$ as long as $u + v$ is sufficiently large.*

6b. $|\Box|^t(\mathbf{R}^k) \not\subseteqq L_1(\mathbf{R}^k)$ *if* $t \leqq \frac{1}{2}k - 1$. We consider the indefinite quadratic polynomial

$$\psi_0(\lambda) = \lambda_1^2 + \cdots + \lambda_u^2 - \lambda_{u+1}^2 - \cdots - \lambda_{u+v}^2,$$

and the associated wave operator

$$\Box = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_u^2} + \frac{\partial^2}{\partial x_{u+1}^2} + \cdots + \frac{\partial^2}{\partial x_{u+v}^2}.$$

Put $\mathscr{C} = \mathscr{C}_{\psi_0} = \{\xi \in \mathbf{R}^k : \psi_0(\xi) = 0\}$ and consider the distribution

$$K(x) = \{|\psi_0(\xi)|^t\}^\wedge(x).$$

K is homogeneous of degree $-k-2t$ and is invariant under the action of $\mathrm{O}(u, v)$. Hence, away from the cone $\mathscr{C}$, it must be a smooth function of the form

$$A(x) = c\,|\psi_0(x)|^{-\frac{1}{2}k - t}, \qquad x \notin \mathscr{C}.$$

Note the constant c may vary from component to component in $\mathbf{R}^k - \mathscr{C}$; but that is irrelevant to the ensuing argument. Of course we have

$$|\Box|^t \varphi = K * \varphi \qquad \text{if} \quad \varphi \in C_c^\infty(\mathbf{R}^k).$$

We produce a function $\varphi \in C_c^\infty(\mathbf{R}^k)$ such that $\int |(K * \varphi)(x)|\,dx = \infty$.

Let $\varphi \in C_c^\infty(\mathbf{R}^k)$ have the following properties:

(i) $\operatorname{Supp}\varphi = \{x : \|x\| \leqq 1\}$;

(ii) $\varphi(x) = 1$ if $\|x\| \leqq \dfrac{1}{2}$;

(iii) $\varphi(x) \geqq 0$, $\forall x \in \mathbf{R}^k$.

Then

$$\int |(K*\varphi)(x)|dx \geqq \int_{d(x,\mathscr{C})\geqq 2} (K*\varphi)(x)dx$$

$$= \int_{d(x,\mathscr{C})\geqq 2} \int_{\|y\|\leqq 1} A(x-y)dydx$$

$$\geqq \int_{d(x,\mathscr{C})\geqq 2} \int_{\|y\|\leqq \frac{1}{2}} A(x-y)dydx$$

$$= \int_{\|y\|\leqq \frac{1}{2}} \int_{d(x+y,\mathscr{C})\geqq 2} A(x)dxdy$$

$$\geqq c_k \int_{d(x,\mathscr{C})\geqq 3} A(x)dx.$$

Now it's a simple max-min exercise to see that $d(x,\mathscr{C})^2 = \text{constant} \times (r-\rho)^2$, $r = (x_1^2 + \cdots + x_u^2)^{\frac{1}{2}}$, $\rho = (x_{u+1}^2 + \cdots + x_{u+v}^2)^{\frac{1}{2}}$. Thus we are done if we can prove that the integral

$$\int_{|r-\rho|\geqq\delta>0} |\psi_0(x)|^{-\frac{1}{2}k-t}dx \tag{6.3}$$

is divergent. In fact

$$\int_{|r-\rho|\geqq\delta} |\psi_0(x)|^{-\frac{1}{2}k-t}dx = c_{u,v}\int\int_{|r-\rho|\geqq\delta} \frac{r^{u-1}\rho^{v-1}}{|r^2-\rho^2|^{\frac{1}{2}k+t}}drd\rho$$

$$\geqq c_{u,v}\int_{\delta}^{\infty}\int_{0}^{\rho-\delta} \frac{r^{u-1}\rho^{v-1}}{|r^2-\rho^2|^{\frac{1}{2}k+t}}drd\rho$$

$$= c_{u,v}\int_{\delta}^{\infty}\int_{0}^{(\rho-\delta)/\rho} \frac{(\rho\sigma)^{u-1}\rho^{v-1}}{\rho^{k+2t}|\sigma^2-1|^{\frac{1}{2}k+t}}\rho d\sigma d\rho$$

$$= c_{u,v}\int_{\delta}^{\infty}\left(\int_{0}^{1-\delta/\rho} \frac{\sigma^{u-1}}{|\sigma^2-1|^{\frac{1}{2}k+t}}d\sigma\right)\rho^{-2t-1}d\rho.$$

Now as $\rho\to\infty$, the inner integral is asymptotic to

$$\int_0^{1-\delta/\rho} \frac{d\sigma}{|\sigma-1|^{\frac{1}{2}k+t}} = \int_{-1}^{-\delta/\rho} \frac{ds}{|s|^{\frac{1}{2}k+t}} \sim \rho^{\frac{1}{2}k+t-1}.$$

Thus the entire expression is asymptotic to

$$\int_\delta^\infty \rho^{\frac{1}{2}k+t-1}\rho^{-2t-1}d\rho = \int_\delta^\infty \rho^{\frac{1}{2}k-t-2}d\rho$$

$$= \begin{cases} \infty & t+2-\dfrac{1}{2}k \leqq 1 \\ \text{finite} & t+2-\dfrac{1}{2}k > 1. \end{cases}$$

Hence the integral (6.3) is divergent for $t+2-\frac{1}{2}k \leqq 1$, i.e., $t \leqq \frac{1}{2}k-1$.

Fefferman conjectured that the critical value for $|\Box|^t$ on $\mathbf{R}^k$ is exactly $t = \frac{1}{2}k - 1$. Proser's observation only handles $t > 2[k/4]+2$. At present we do not know how to close the gap.

Remark. If we consider the parabolic group $P_{1;u,v}(\mathbf{R})$, we know (see §4e) that $D = c|\Box|^{k/2}$, $k = u+v$. Therefore $D^{\frac{1}{2}} = c^{\frac{1}{2}}|\Box|^{k/4}$. According to Fefferman's example, $C_c^\infty \not\subseteq \operatorname{Dom} D^{1/2} \cap D^{-1/2}L_1$ if $k/4 \leqq k/2-1$, i.e., if $k \geqq 4$. On the other hand, the example says nothing about whether $C_c^\infty \subseteq \operatorname{Dom} D \cap D^{-1}L_1$. If his conjecture holds, the inclusion would follow. And so we would like to close by posing one more

6.4 Conjecture. *Let G be type I. Then there exists a Dixmier–Pukanszky operator D on G such that $C_c^\infty(G) \subseteq \operatorname{Dom} D \cap D^{-1}(L_1(G))$.*

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