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ON CALABI'S INHOMOGENEOUS EINSTEIN-KAEHLER MANIFOLDS[1]

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ABSTRACT. We use some information on Lie groups to replace a long computation of Calabi, proving that certain complete Einstein-Kaehler manifolds are not locally homogeneous, and finding their isometry groups.

E. Calabi [1] constructed a complete Einstein-Kaehler metric on the tube domain

$$M = \{z = x + iy \in \mathbf{R}^n + i\mathbf{R}^n: \|y\| < r\} \subset \mathbf{C}^n$$

which is invariant under the natural action $(v, g): x + iy \longmapsto v + gx + igy$ of the proper euclidean group $\mathbf{E}(n) = \mathbf{R}^n \cdot SO(n)$. He used a rather complicated calculation to show that M is not homogeneous with that metric. We are going to replace his calculation by a simple group-theoretic argument and obtain a slightly stronger result:

THEOREM. *If $n \geqslant 2$ and ds^2 is an $\mathbf{E}(n)$-invariant Kaehler metric on M, then (M, ds^2) cannot be both complete and locally homogeneous, in particular, cannot be homogeneous.*

Here note that Calabi's metric [1] is complete and the flat metric is locally homogeneous.

Finally, we will show that the theorem implies

COROLLARY. *If $n \geqslant 2$ and ds^2 is an $\mathbf{E}(n)$-invariant Kaehler metric on M, then $\mathbf{E}(n)$ is the largest connected group of holomorphic isometries of (M, ds^2).*

PROOF OF THEOREM. Let ds^2 be an $\mathbf{E}(n)$-invariant Kaehler metric on M and $T^{1,0}(0)$ the holomorphic tangent space at 0. The curvature transformation of (M, ds^2) at 0 is a linear tranformation of $\Lambda^2 T^{1,0}(0)$ that commutes with the irreducible action of $SO(n)$ on $\Lambda^2 T^{1,0}(0)$, hence is scalar. So (M, ds^2) has constant holomorphic sectional curvature at 0.

Suppose that (M, ds^2) is complete and locally homogeneous. Then (M, ds^2) is complete and simply connected with some constant holomorphic sectional curvature c, hence holomorphically isometric to a complex projective space ($c > 0$), a complex euclidean space ($c = 0$), or a complex hyperbolic space

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($c < 0$). The first two possibilities cannot occur because M is noncompact and admits nonconstant bounded holomorphic functions such as $f(z) = 1/(z_1 - 2ir)$. Thus (M, ds^2) has holomorphic isometry group $SU(1, n)/(\text{scalars})$, and so $\mathbf{E}(n)$ is contained in that group locally isomorphic to $SU(1, n)$. The next two lemmas show that this is impossible.

LEMMA. *Let G be a reductive Lie group. If E is an analytic subgroup of G then the solvable radical of $[E, E]$ is a unipotent subgroup of G. In particular, if $n \geqslant 2$ and $\mathbf{E}(n)$ is a subgroup of G, then the translation subgroup $\mathbf{R}^n \subset \mathbf{E}(n)$ is unipotent in G.*

PROOF. We may cut G down to its identity component and then divide out its center, so we may assume that G is a semisimple linear group. One knows [2, Theorem 3.2, p. 128] that a finite dimensional linear representation of a Lie algebra carries the radical of the derived algebra to an algebra of nilpotent transformations. So the radical of $[E, E]$ is unipotent in G. Q.E.D.

LEMMA. *If $n \geqslant 2$ and G is locally isomorphic to the special unitary group $SU(1, n)$ of Lorentz signature, then G has no subgroup isomorphic to $\mathbf{E}(n)$.*

PROOF. It is known [3, §3] that the maximal unipotent subgroups of G are isomorphic to the $(2n - 1)$-dimensional Heisenberg group $H_{2n-1} = \mathbf{R} + \mathbf{C}^{n-1}$ with product $(z, w)(z', w') = (z + z' + \operatorname{Im} w \cdot w', w + w')$ where $w \cdot w'$ is the usual $U(n - 1)$-invariant hermitian scalar product on $\mathbf{C}^{n-1}$. An easy calculation with the real symplectic structure underlying $\mathbf{C}^{n-1}$ shows that every abelian subgroup of H_{2n-1} is $U(n - 1)$-conjugate, hence [3, §3] G-conjugate, to $\mathbf{R} + \mathbf{R}^{n-1}$. Again by [3, §3], the latter has G-normalizer locally isomorphic to $H_{2n-1} \cdot (SO(n - 1) \times \mathbf{R})$, and the latter has no subgroup locally isomorphic to $SO(n)$. Q.E.D.

PROOF OF COROLLARY. Let G be the largest connected group of holomorphic isometries of (M, ds^2) and $K = \{g \in G: g(0) = 0\}$. Then $z = x + iy \in M$ has $\mathbf{E}(n)$-orbit $\{z' = x' + iy' \in \mathbf{R}^n + i\mathbf{R}^n: \|y'\| = \|y\|\}$, which has real codimension 1 whenever $y \neq 0$. As $G(z)$ and $\mathbf{E}(n)(z)$ are complete riemannian submanifolds, and the Theorem ensures that $G(z)$ has positive real codimension in M, now $G(z) = \mathbf{E}(n)(z)$ for $y \neq 0$, and, hence, also for the other orbit $y = 0$. Now that other orbit $\mathbf{R}^n = G/K$, so K acts on the tangent space to M at 0 as a subgroup of $U(n)$ that stabilizes $\mathbf{R}^n$. Thus K coincides with its subgroup $SO(n)$, and so by dimension G coincides with its subgroup $\mathbf{E}(n)$. Q.E.D.

REFERENCES

1. E. Calabi, *A construction of nonhomogeneous Einstein metrics*, Proc. Sympos. Pure Math., vol. 27, Part II, Amer. Math. Soc., Providence, R. I., 1975, pp. 17–24. MR **52** #816.

2. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, Calif., 1965. MR **34** #7696.

3. J. A. Wolf, *Representations of certain semidirect product groups*, J. Functional Analysis **19** (1975), 339–372.

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