ON CALABI'S INHOMOGENEOUS EINSTEIN-KÄHLER MANIFOLDS

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ABSTRACT. We use some information on Lie groups to replace a long computation of Calabi, proving that certain complete Einstein-Kähler manifolds are not locally homogeneous, and finding their isometry groups.

E. Calabi [1] constructed a complete Einstein-Kähler metric on the tube domain

\[ M = \{ z = x + iy \in \mathbb{R}^n + i\mathbb{R}^n : \| y \| < r \} \subset \mathbb{C}^n \]

which is invariant under the natural action \( (v, g) : x + iy \mapsto v + gx + igy \) of the proper euclidean group \( E(n) = \mathbb{R}^n \cdot SO(n) \). He used a rather complicated calculation to show that \( M \) is not homogeneous with that metric. We are going to replace his calculation by a simple group-theoretic argument and obtain a slightly stronger result:

**Theorem.** If \( n > 2 \) and \( ds^2 \) is an \( E(n) \)-invariant Kaehler metric on \( M \), then \( (M, ds^2) \) cannot be both complete and locally homogeneous, in particular, cannot be homogeneous.

Here note that Calabi’s metric [1] is complete and the flat metric is locally homogeneous.

Finally, we will show that the theorem implies

**Corollary.** If \( n > 2 \) and \( ds^2 \) is an \( E(n) \)-invariant Kaehler metric on \( M \), then \( E(n) \) is the largest connected group of holomorphic isometries of \( (M, ds^2) \).

**Proof of Theorem.** Let \( ds^2 \) be an \( E(n) \)-invariant Kaehler metric on \( M \) and \( T^{1,0}(0) \) the holomorphic tangent space at 0. The curvature transformation of \( (M, ds^2) \) at 0 is a linear transformation of \( \Lambda^2 T^{1,0}(0) \) that commutes with the irreducible action of \( SO(n) \) on \( \Lambda^2 T^{1,0}(0) \), hence is scalar. So \( (M, ds^2) \) has constant holomorphic sectional curvature at 0.

Suppose that \( (M, ds^2) \) is complete and locally homogeneous. Then \( (M, ds^2) \) is complete and simply connected with some constant holomorphic sectional curvature \( c \), hence holomorphically isometric to a complex projective space \( (c > 0) \), a complex euclidean space \( (c = 0) \), or a complex hyperbolic space...
(c < 0). The first two possibilities cannot occur because $M$ is noncompact and admits nonconstant bounded holomorphic functions such as $f(z) = 1/(z - 2i)$. Thus $(M, ds^2)$ has holomorphic isometry group $SU(1, n)/(\text{scalars})$, and so $E(n)$ is contained in that group locally isomorphic to $SU(1, n)$. The next two lemmas show that this is impossible.

**Lemma.** Let $G$ be a reductive Lie group. If $E$ is an analytic subgroup of $G$ then the solvable radical of $[E, E]$ is a unipotent subgroup of $G$. In particular, if $n > 2$ and $E(n)$ is a subgroup of $G$, then the translation subgroup $\mathbb{R}^n \subset E(n)$ is unipotent in $G$.

**Proof.** We may cut $G$ down to its identity component and then divide out its center, so we may assume that $G$ is a semisimple linear group. One knows [2, Theorem 3.2, p. 128] that a finite dimensional linear representation of a Lie algebra carries the radical of the derived algebra to an algebra of nilpotent transformations. So the radical of $[E, E]$ is unipotent in $G$. Q.E.D.

**Lemma.** If $n > 2$ and $G$ is locally isomorphic to the special unitary group $SU(1, n)$ of Lorentz signature, then $G$ has no subgroup isomorphic to $E(n)$.

**Proof.** It is known [3, §3] that the maximal unipotent subgroups of $G$ are isomorphic to the $(2n - 1)$-dimensional Heisenberg group $H_{2n-1} = \mathbb{R} + \mathbb{C}^{n-1}$ with product $(z, w)(z', w') = (z + z' + \text{Im } w \cdot w', w + w')$ where $w \cdot w'$ is the usual $U(n - 1)$-invariant hermitian scalar product on $\mathbb{C}^{n-1}$. An easy calculation with the real symplectic structure underlying $\mathbb{C}^{n-1}$ shows that every abelian subgroup of $H_{2n-1}$ is $U(n - 1)$-conjugate, hence [3, §3] $G$-conjugate, to $\mathbb{R} + \mathbb{R}^{n-1}$. Again by [3, §3], the latter has $G$-normalizer locally isomorphic to $H_{2n-1} \cdot (SO(n-1) \times \mathbb{R})$, and the latter has no subgroup locally isomorphic to $SO(n)$. Q.E.D.

**Proof of Corollary.** Let $G$ be the largest connected group of holomorphic isometries of $(M, ds^2)$ and $K = \{ g \in G: g(0) = 0 \}$. Then $z = x + iy \in M$ has $E(n)$-orbit $\{ z' = x' + iy' \in \mathbb{R}^n + i\mathbb{R}^n: \| y' \| = \| y \| \}$, which has real codimension 1 whenever $y \neq 0$. As $G(z)$ and $E(n)(z)$ are complete riemannian submanifolds, and the Theorem ensures that $G(z)$ has positive real codimension in $M$, now $G(z) = E(n)(z)$ for $y \neq 0$, and, hence, also for the other orbit $y = 0$. Now that other orbit $\mathbb{R}^n = G/K$, so $K$ acts on the tangent space to $M$ at 0 as a subgroup of $U(n)$ that stabilizes $\mathbb{R}^n$. Thus $K$ coincides with its subgroup $SO(n)$, and so by dimension $G$ coincides with its subgroup $E(n)$. Q.E.D.

**References**


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