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# Poincaré series and automorphic cohomology on flag domains

By R. O. WELLS, JR.[1] and JOSEPH A. WOLF[2]

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## 1. Introduction

Let $D$ be a bounded symmetric domain and $\mathbf{K} \to D$ its canonical line bundle. If $\Gamma$ is a discontinuous group of analytic automorphisms of $D$, then the $\Gamma$-invariant holomorphic sections of $\mathbf{K}^m \to D$ are naturally identified (by a holomorphic trivialization of the bundle) with the automorphic forms of weight $m$ on $D$; here one imposes a growth condition at $\infty$ if $\dim_{\mathbf{C}} D = 1$. Denote the space of holomorphic sections by $H^0(D; \mathcal{O}(\mathbf{K}^m))$ and the subspace of $\Gamma$-invariant sections by $H^0_\Gamma(D; \mathcal{O}(\mathbf{K}^m))$. If $\varphi \in H^0(D; \mathcal{O}(\mathbf{K}^m))$ is absolutely integrable relative to the natural Hermitian metric on $\mathbf{K}^m$, and if $m \geqq m_0 > 0$, then the Poincaré series

$$\theta(\varphi) = \sum_{\gamma \in \Gamma} \gamma^* \varphi$$

converges absolutely and uniformly on compact subsets of $D$ to an automorphic form of weight $m$. See Borel ([6], [8]) for a discussion of these

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matters. This classical construction, initiated by Poincaré [27] for the unit disc in C, is the primary source of automorphic forms on $D$ and automorphic functions on the quotient space $\Gamma\backslash D$, leading to quasi-projective embeddings and analytic space compactifications of $\Gamma\backslash D$ in a natural way (cf. [5]).

In Griffiths' study ([15], [16]) of periods of integrals on algebraic manifolds, he looked at a class of open homogeneous complex manifolds (period matrix domains) that are not bounded symmetric domains and in fact [45] carry no nonconstant holomorphic functions. Results of Schmid ([28]) show that if $D$ is one of these period matrix domains and $\mathbf{E} \to D$ is a "nondegenerate" homogeneous holomorphic vector bundle, then the cohomology $H^q(D; \mathcal{O}(\mathbf{E}))$ vanishes for $q \neq s = \dim_{\mathbf{C}} Y > 0$, where $Y$ is a maximal compact subvariety of $D$, and that $H^s(D; \mathcal{O}(\mathbf{E}))$ is an infinite dimensional Fréchet space. In particular there are no automorphic forms in the classical sense on $D$. In view of this, Griffiths investigated the space $H^s_\Gamma(D; \mathcal{O}(\mathbf{E}))$ of $\Gamma$-invariant classes in $H^s(D; \mathcal{O}(\mathbf{E}))$, calling that the space of *automorphic cohomology* classes ([15], [16], [14]). While the geometric role of automorphic cohomology is still unclear, its natural presence on the domains arising in algebraic geometry makes it a primary object of interest.

Let $\mathbf{E} \to D$ be nondegenerate over a period matrix domain. Griffiths [15] conjectured a geometric representation for $H^s_\Gamma(D; \mathcal{O}(\mathbf{E}))$ as the space of holomorphic $\Gamma$-invariant sections of an associated vector bundle $\widetilde{\mathbf{E}} \to M$ on the space of all compact (linearly deformed) subvarieties of complex dimension $s$ in $D$. More precisely, he conjectured that $M$ with its natural complex structure is a Stein manifold, that the disjoint union $\mathcal{Y}_D$ of the varieties parameterized by $M$ sits in a diagram

$$M \xleftarrow{\pi} \mathcal{Y}_D \xrightarrow{\tau} D \tag{1.1}$$

with $\pi$ and $\tau$ holomorphic, surjective, and of maximal rank, and that $\pi^s_* \tau^* \mathcal{O}(\mathbf{E})$ is locally free ($\pi^s_* = s^{\text{th}}$ direct image) and thus corresponds to a holomorphic vector bundle $\widetilde{\mathbf{E}} \to M$. These conjectures are the period matrix domain cases of our results in Section 2 below, especially Theorem 2.5.6. Griffiths further conjectured, and we prove as Theorem 3.4.7 below, that the induced map on cohomology

$$\sigma\colon H^s(D; \mathcal{O}(\mathbf{E})) \longrightarrow H^0(M; \mathcal{O}(\widetilde{\mathbf{E}})) \tag{1.2}$$

is a $\Gamma$-equivariant topological injection of Fréchet spaces, so that it induces a topological injection

$$\sigma\colon H^s_\Gamma(D; \mathcal{O}(\mathbf{E})) \longrightarrow H^0_\Gamma(M; \mathcal{O}(\widetilde{\mathbf{E}}))\,. \tag{1.3}$$

The map $\sigma$ transfers a Poincaré series on $D$,

$$\theta(c) = \sum_{\gamma \in \Gamma} \gamma^*(c)\,, \ c \in H^s\big(D; \mathcal{O}(\mathbf{E})\big)\,, \tag{1.4}$$

to a Poincaré series on $M$ that only involves sections,

$$\theta\big(\sigma(c)\big) = \sum_{\gamma \in \Gamma} \gamma^*\big(\sigma(c)\big)\,. \tag{1.5}$$

As Griffiths illustrated [15], (1.5) converges when $c$ is absolutely integrable (i.e., $||c||$ is $L_1$) with respect to the natural hermitian metrics on $D$ and E. In Theorem 4.1.7 below, we prove this convergence of (1.5), and it ensures convergence of (1.4) by the Equivariant Representation Theorem (1.2).

In order to yield automorphic cohomology, our convergence result (Theorem 4.1.6) for Poincaré series $\theta(c) = \sum_{\gamma \in \Gamma} \gamma^*(c)$ requires nonzero $L_1$ classes $c \in H^s\big(D; \mathcal{O}(\mathbf{E})\big)$. Griffiths [15] asserted that such $L_1$ classes could be constructed by the methods of Schmid's thesis [28] when $\mathbf{E} \to D$ is a high power of the canonical line bundle. But the situation is somewhat more delicate; the correct setting for $L_1$ cohomology is discussed below. Originally we had combined some results and techniques of Harish-Chandra, of Schmid [28], and of Trombi and Varadarajan [36], carrying out a program outlined by Schmid in a letter to us, constructing nontrivial $L_1$ cohomology classes for a specific large family of nondegenerate vector bundles $\mathbf{E} \to D$. The final result is Theorem 4.3.12; here we follow a shorter route, suggested by the referee, using more recent results of Schmid.

Some of the results in this paper were announced in [40] and [43].

We now turn to a section by section description.

Chapter 2 establishes our basic geometric setting. In Sections 2.1 and 2.2 we recall the definitions and basic facts concerning the complex flag manifolds $X$ and the flag domains $D \subset X$. Here $X$ is a compact complex homogeneous space $G_{\mathbf{C}}/P$ where $G_{\mathbf{C}}$ is a complex Lie group and $P$ is a complex parabolic subgroup, and $D$ is an open orbit of a noncompact real form $G$ of $G_{\mathbf{C}}$. The class of flag domains includes the bounded symmetric domains (such as the Poincaré half plane and Siegel's generalized half planes) and also the period matrix domains of Griffiths [15] that arise in algebraic geometry. In Section 2.3 we discuss the maximal compact complex submanifolds of $D$ and the fibration $D \to G/K$ over the symmetric space associated to $G$; there are nonconstant holomorphic functions on $D$ only when $G/K$ is Hermitian and $D \to G/K$ is holomorphic. Then in Section 2.4 we introduce a deformation space $\mathcal{Y}_D \xrightarrow{\pi} M$ whose fibres are the maximal compact linear subvarieties of $D$ and put a complex structure on $\mathcal{Y}_D$ so that $\pi$ is holomorphic and of maximal rank. Section 2.5 then is devoted to showing that $M$ is a Stein manifold, thus verifying Griffiths' conjecture cited above. Here our principal tools are Schmid's exhaustion function for $D$ $\big([28], [14]\big)$, the Andreotti-Norguet solution to the generalized Levi problem for strictly $q$-convex manifolds ([2], [3]), and the theorem of Docquier-Grauert [11].

Chapter 3 sets up our vector bundles and establishes the information we need on cohomology without bounds. In Section 3.1 we collect the basic facts on homogeneous holomorphic vector bundles, including the Bott-Borel-Weil Theorem [10]. The notion of a nondegenerate bundle $\mathbf{E} \to D$ is developed in Section 3.2. Its defining condition (3.2.1) is just what one needs to conclude from the Bott-Borel-Weil Theorem that $H^q(Y; \mathcal{O}(\mathbf{E} \otimes \Lambda^l \mathbf{N})) = 0$, for $0 \leqq q < s$ and all $l$, where $Y$ is our maximal compact subvariety of $D$, $s = \dim_{\mathbf{C}} Y$, and $\mathbf{N} \to Y$ is the holomorphic normal bundle of $Y$ in $D$. Our modification of Schmid's Identity Theorem [28] appears as Theorem 3.2.2, and Theorem 3.2.3 is an associated vanishing theorem. Their proofs, which are in Section 3.3, are minor variations on Schmid's proofs. These theorems state that if $\mathbf{E} \to D$ is nondegenerate

(i) then $H^q(D, \mathcal{O}(\mathbf{E})) = 0$ for $q \neq s$ and

(ii) if $c \in H^s(D, \mathcal{O}(\mathbf{E}))$ vanishes on every fibre of $D \to G/K$ then $c = 0$.

The proof is a finite recursion, considering representatives of the class that vanish to successively higher order along the fibres. This technique is used repeatedly in Chapter 4 where growth of cohomology classes (square integrable, absolutely integrable, etc.) is taken into account. In Section 3.4 we use the Identity Theorem, the Leray spectral sequence, and the fact that the parameter space $M$ is Stein, to obtain the Equivariant Representation Theorem (Theorem 3.4.7). It says that $H^s(D; \mathcal{O}(\mathbf{E})) \to H^0(M; \mathcal{O}(\widetilde{\mathbf{E}}))$ is a $G$-equivariant topological injection of Fréchet spaces whenever $\mathbf{E}$ is a nondegenerate homogeneous vector bundle over a flag domain, thus establishing Griffiths' conjecture on this matter.

Chapter 4 gives the convergence and nontriviality of the Poincaré series (1.4) when $\mathbf{E} \to D$ is nondegenerate and satisfies certain simple nonsingularity conditions.

First, in Section 4.1 we establish convergence for the Poincaré series $\theta(c) = \sum_{\gamma \in \Gamma} \gamma^*(c)$ whenever $\mathbf{E} \to D$ is nondegenerate and $c \in H^s(D; \mathcal{O}(\mathbf{E}))$ is $L_1$. This is an application of the Equivariant Representation Theorem, and is a modification of Griffiths' considerations in [15].

Second, in Sections 4.2 and 4.3 we establish the conditions for existence of nonzero $L_1$ classes in $H^s(D; \mathcal{O}(\mathbf{E}))$. Square integrable cohomology, the recent solution of the Langlands Conjecture by Schmid [32], and the work of Trombi-Varadarajan [36] and Hecht-Schmid [24] on integrable discrete series representations, are collected in Section 4.2. This allows us to specify the nondegenerate bundles $\mathbf{E} \to D$ on whose square integrable cohomology spaces $\mathcal{H}^{0,s}(D; \mathbf{E})$ the action on $G$ is an integrable discrete series representation. Then we consider the natural map $\mathcal{H}^{0,q}(D; \mathbf{E}) \to H^q(D; \mathcal{O}(\mathbf{E}))$ from square integrable cohomology through Dolbeault cohomology to ordinary sheaf cohomology. A close look at the mechanism of Schmid's proof of the Langlands Conjecture gives us (Theorem 4.3.8): when $G$ acts on $\mathcal{H}^{0,q}(D; \mathbf{E})$ by an integrable discrete series representation, the $K$-finite elements of $\mathcal{H}^{0,q}(D; \mathbf{E})$ map to $L_1$ classes in $H^q(D; \mathcal{O}(\mathbf{E}))$. Of course this is useful only when $\mathcal{H}^{0,q}(D; \mathbf{E}) \to H^q(D; \mathcal{O}(\mathbf{E}))$ is nonzero. We then use some methods of Harish-Chandra ([19], [20]) to work out an $L_2$ version of the recursion procedure (§3.3) in the proof of the Identity Theorem, and use it to show (Theorem 4.3.9): if $\mathbf{E} \to D$ is nondegenerate and $q \leqq s$ then $\mathcal{H}^{0,q}(D; \mathbf{E}) \to H^q(D; \mathcal{O}(\mathbf{E}))$ is injective. Combining these we have (Theorem 4.3.12): if $\mathbf{E} \to D$ is nondegenerate and such that $G$ acts on $\mathcal{H}^{0,s}(D; \mathbf{E})$ by an integrable discrete series representation $\pi$, then $H^s(D; \mathcal{O}(\mathbf{E}))$ has an infinite dimensional subspace $H^s_2(D; \mathcal{O}(\mathbf{E}))$ on which $G$ acts by $\pi$, and the $K$-finite classes in $H^s_2(D; \mathcal{O}(\mathbf{E}))$ (which form a dense subspace) are absolutely integrable. This provides the $L_1$ cohomology classes that can be summed in the Poincaré series (1.4).

Finally we mention some open problems.

1. Is $H^s_\Gamma(D; \mathcal{O}(\mathbf{E}))$ finite dimensional, say, when $\Gamma$ is arithmetic?

2. Which Poincaré series $\theta(c)$ are nonzero? What is the dimension of the space of Poincaré series arising from a given bundle $\mathbf{E} \to D$? How does that space compare with the full automorphic cohomology space $H^s_\Gamma(D; \mathcal{O}(\mathbf{E}))$?

3. How does one obtain quasi-projective embeddings from automorphic cohomology? Are the holomorphic arc components of the boundary orbits [44] the correct counterparts to the boundary components [46] as used by Bailey and Borel [5]?

It is a pleasure to express our thanks to Wilfried Schmid, Gregory Eskin and the referee. W. Schmid gave us permission to use the proof of the Identity Theorem (3.3.2) from his unpublished thesis [28], and he suggested our original program of using a certain direct image map to prove Theorem 4.3.8 for $\mathbf{E} \to D$ nondegenerate and $q = s$. That program required some $L_1$ a priori estimates, concerning which one of us had several helpful conversations with G. Eskin. In the present version of this paper we follow a different program for Theorem 4.3.8, suggested by the referee and based on results of Schmid that were not available at first writing.

Finally we thank Wanna King of Rice University for her careful job of typing the manuscript.

## 2. Homogeneous complex manifolds

### 2.1. Complex flag manifolds

In this section we define the compact homogeneous complex manifolds

$$X = G_{\mathbf{C}}/P = G_u/V$$

which are the natural ambient spaces for our open homogeneous complex manifolds

$$D = G(x_0) \subset X, \quad D \cong G/V$$

of primary interest. Briefly, $G_{\mathbf{C}}$ is a connected complex semisimple group and $P$ is a parabolic subgroup, so $X = G_{\mathbf{C}}/P$ is a complex flag manifold. $G_u$ is a compact real form of $G_{\mathbf{C}}$ and $V = G_u \cap P$ is the centralizer of a torus subgroup. In the applications, $G$ will be a noncompact real form of $G_{\mathbf{C}}$ with $G \cap P = V$, and $D$ will be an open $G$-orbit on $X$. An example: $X$ is complex projective space

$$\mathbf{P}_n(\mathbf{C}) = \mathrm{SL}(n+1, \mathbf{C})/P = \mathrm{SU}(n+1)/U(n)$$

where $P = \left\{g \in \mathrm{SL}(n+1, \mathbf{C}) \colon g \text{ has form } \begin{pmatrix} a & b \\ 0 & A \end{pmatrix}, \det A = a^{-1}\right\}$, and $D$ is the unit ball in $\mathbf{C}^n$ given by

$$B_n(\mathbf{C}) = \mathrm{SU}(1, n)/U(n) .$$

But in our applications $D$ will have nontrivial compact subvarieties.

Let $\mathfrak{g}_{\mathbf{C}}$ be a complex semisimple Lie algebra. Its maximal solvable subalgebras are the *Borel subalgebras*

$$\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} + \textstyle\sum_{\alpha \in \Delta^+} \mathfrak{g}_{\mathbf{C}}^{-\alpha} \tag{2.1.1}$$

where $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra, $\Delta^+$ is a positive root system, and $\mathfrak{g}_{\mathbf{C}}^{\alpha}$ denotes the complex root space for a root $\alpha$. Evidently any two Borel subalgebras are conjugate. The subalgebras of $\mathfrak{g}_{\mathbf{C}}$ that contain Borel subalgebras are called *parabolic subalgebras*. The ones containing $\mathfrak{b}$ (2.1.1) are the

$$\mathfrak{p}_{\Phi} = \mathfrak{b} + \textstyle\sum_{\beta \in \langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{\beta} \tag{2.1.2}$$

where $\Phi$ is any set of simple roots and $\langle\Phi\rangle$ consists of all positive roots $\sum n_i \varphi_i$ with $\varphi_i \in \Phi$. Note $\mathfrak{b} = \mathfrak{p}_{\varnothing}$ where $\varnothing$ is the empty set, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{p}_{\Psi}$ where $\Psi$ is the entire simple root system. Further $\mathfrak{p}_{\Phi} = \mathfrak{p}_{\Phi}^{r} + \mathfrak{p}_{\Phi}^{n}$, the sum of a nilpotent ideal and a reductive complement given by

$$\mathfrak{p}_{\Phi}^{r} = \mathfrak{h}_{\mathbf{C}} + \textstyle\sum_{\beta \in \langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{\beta} + \mathfrak{g}_{\mathbf{C}}^{-\beta} \quad\text{and}\quad \mathfrak{p}_{\Phi}^{n} = \sum_{\substack{\alpha \in \Delta^+ \\ \alpha \notin \langle\Phi\rangle}} \mathfrak{g}_{\mathbf{C}}^{-\alpha} . \tag{2.1.3}$$

For example $\mathfrak{b}^r = \mathfrak{h}_{\mathbf{C}}$ and $\mathfrak{b}^n = \sum_{\alpha \in \Delta} \mathfrak{g}_{\mathbf{C}}^{-\alpha}$. For a general reference on this topic, see Humphreys [25].

If $r = \operatorname{rank} \mathfrak{g}_{\mathbf{C}}$, so that $|\Psi| = r$, then in (2.1.2) there are $2^r$ choices of $\Phi$. Every parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ is conjugate to just one of the $2^r$ algebras $\mathfrak{p}_\Phi$ of (2.1.2).

Let $G_{\mathbf{C}}$ be a connected complex semisimple Lie group and $\mathfrak{g}_{\mathbf{C}}$ its Lie algebra. A Lie subgroup of $G_{\mathbf{C}}$ is called *Borel* (resp. *parabolic*) if its Lie algebra is a Borel (resp. parabolic) subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Using the Levi-Whitehead decomposition and simple transitivity of the Weyl group on the set of positive root systems, one finds that

$$(2.1.4) \qquad P_\Phi = \{g \in G_{\mathbf{C}},\ \mathrm{Ad}(g)\mathfrak{p}_\Phi = \mathfrak{p}_\Phi\} \text{ is the analytic subgroup for } \mathfrak{p}_\Phi\ .$$

It follows that parabolic subgroups are closed in $G_{\mathbf{C}}$, are connected, are self-normalizing, and are conjugate each to just one of the $2^r$ groups $P_\Phi$ of (2.1.4).

The basic facts about parabolic subgroups, essentially proved by Tits [35], are summarized as

**2.1.5.** PROPOSITION. *Let $P$ be a complex Lie subgroup of $G_{\mathbf{C}}$ with only finitely many topological components. Then the following are equivalent*:

(i) *$P$ is a parabolic subgroup of $G_{\mathbf{C}}$*;

(ii) *$X = G_{\mathbf{C}}/P$ is compact*;

(iii) *$X$ is a compact simply-connected homogeneous Kähler manifold*;

(iv) *$X$ is a homogeneous complex projective algebraic variety*;

(v) *$X$ is a closed $G_{\mathbf{C}}$-orbit in a projective representation.*

We will refer to the spaces $X = G_{\mathbf{C}}/P$ as *complex flag manifolds*. The complex flag manifolds $X = G_{\mathbf{C}}/P$ here are the Kähler $C$-spaces of Wang [38]. In effect, if $G_u$ is the compact real form of $G_{\mathbf{C}}$ whose Lie algebra $\mathfrak{g}_u$ has $\mathfrak{g}_u \cap \mathfrak{h}_{\mathbf{C}}$ as a Cartan subalgebra, then $G_u$ is transitive on $X = G_{\mathbf{C}}/P_\Phi$ by dimension and compactness, so

$$X = G_u/V \text{ where } V = G_u \cap P_\Phi \text{ is the centralizer of a torus } \big(\text{Borel [7], Tits [35]}\big)\ .$$

These complex flag manifolds include the Hermitian symmetric spaces, such as the complex projective spaces, complex Grassmanians and complex quadrics. They include the classical flag manifolds

$$G_{\mathbf{C}}/(\text{Borel}) = G_u/(\text{maximal torus})\ .$$

But the important fact for us is that they include the Zariski closures of the universal coverings of certain period matrix domains.

**2.1.6.** *Example.* Here is the example that comes up in the study of variation of Hodge structure (Griffiths [15], [16]; see Griffiths-Schmid [14], Wells [41, Example V.5.8]). Let $r, s \geqq 0$ integers and consider the Grassmanian

$$G(r, 2r+s; \mathbf{C}) = \{r\text{-planes through } 0 \text{ in } \mathbf{C}^{2r+s}\} .$$

Let $b(\cdot,\cdot)$ be the bilinear form on $\mathbf{C}^{2r+s}$ with matrix

$$Q = \begin{pmatrix} I_{2r} & 0 \\ 0 & -I_s \end{pmatrix} . \tag{2.1.7}$$

It defines a quadric in $G(r, 2r+s; \mathbf{C})$ consisting of the $b$-isotropic $r$-planes,

$$X = \{S \in G(r, 2r+s; \mathbf{C}) \colon b(S, S) = 0\} .$$

This is a complex flag manifold of the special orthogonal group of $b$,

$$X = G_{\mathbf{C}}/P \text{ where } G_{\mathbf{C}} = \mathrm{SO}(2r, s; \mathbf{C}) = \{g \in \mathrm{SL}(2r+s; \mathbf{C}) \colon {}^t gQg = Q\} .$$

It has compact presentation

$$X = G_u/V = \mathrm{SO}(2r+s)/U(r) \times \mathrm{SO}(s) .$$

## 2.2. Flag domains

Fix a complex flag manifold $X = G_{\mathbf{C}}/P$ as in Section 2.1. In this section we discuss open orbits

$$D = G(x_0) \subset X,\ D \cong G/V$$

where $G$ is a real form of $G_{\mathbf{C}}$ whose stability subgroup $V$ at $x_0$ is compact. An open orbit of this form will be called a *flag domain.*

Let $\mathfrak{g}$ be a real form of $\mathfrak{g}_{\mathbf{C}}$ and define

$$G\colon \text{analytic subgroup of } G_{\mathbf{C}} \text{ for } \mathfrak{g} . \tag{2.2.1}$$

Then $G$ is a closed subgroup of $G_{\mathbf{C}}$; in fact it is the topological identity component of the real algebraic group

$$G_{\mathbf{R}} = \{g \in G_{\mathbf{C}} \colon \mathrm{Ad}(g)\mathfrak{g} = \mathfrak{g}\} .$$

General considerations (Wolf [44, § 2]) tell us that $G$ has open orbits on $X$ and that every stability subgroup

$$G \cap P_x \text{ where } P_x = \{g \in G_{\mathbf{C}} \colon g(x) = x\}$$

contains a Cartan subgroup of $G$. We assume from now on that

$$G \text{ has a compact Cartan subgroup} . \tag{2.2.2}$$

The maximal compact subgroups of $G$ all are conjugate. If $K$ is one of them, then (2.2.2) is equivalent to the condition

$$\mathrm{rank}\, K = \mathrm{rank}\, G, \text{ i.e., } K \text{ contains a Cartan subgroup of } G . \tag{2.2.3}$$

Fix a maximal compact subgroup $K \subset G$ and

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g} \text{ where } \mathfrak{h} \text{ is a Cartan subalgebra of } \mathfrak{g} . \tag{2.2.4}$$

Then the $\mathfrak{h}_{\mathbf{C}}$-root system of $\mathfrak{g}_{\mathbf{C}}$ decomposes as a disjoint union

$$\Delta = \Delta_K \cup \Delta_S \quad \text{where} \quad \mathfrak{k}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} + \sum_{\alpha \in \Delta_K} \mathfrak{g}_{\mathbf{C}}^{\alpha} .$$

This corresponds to the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s} \quad \text{where} \quad \mathfrak{s}_{\mathbf{C}} = \sum_{\alpha \in \Delta_S} \mathfrak{g}_{\mathbf{C}}^{\alpha}$$

and specifies a particular compact real form $\mathfrak{g}_u = \mathfrak{k} + i\mathfrak{s}$ of $\mathfrak{g}_{\mathbf{C}}$. We denote

(2.2.5) $$G_u\text{: analytic subgroup of } G_{\mathbf{C}} \text{ for } \mathfrak{g}_u .$$

It is a compact real form of $G_{\mathbf{C}}$ and $K = G \cap G_u$.

Fix a positive root system $\Delta^+$ for $\mathfrak{g}_{\mathbf{C}}$. Then $P$ has a unique conjugate $g_0 P g_0^{-1}$, $g_0 \in G_{\mathbf{C}}$, with Lie algebra $\mathfrak{p}_\Phi$ given as in Section 2.1. We replace $P$ by that conjugate so $\mathfrak{p} = \mathfrak{p}_\Phi$, and we denote

$$x_0 = 1 \cdot P \in G_{\mathbf{C}}/P, \text{ a “base point” of } X .$$

Then $X$ has the compact presentation

$$X = G_u/V ,$$

where $V = G_u \cap P$ is the centralizer of a torus in $G_u$. Note that $V$ has Lie algebra

$$\mathfrak{v} = \mathfrak{g}_u \cap \mathfrak{p} = \mathfrak{h} + \sum_{\beta \in \langle\Phi\rangle} \mathfrak{g}_u \cap (\mathfrak{g}_{\mathbf{C}}^{\beta} + \mathfrak{g}_{\mathbf{C}}^{-\beta})$$

which is a real form of $\mathfrak{p}^r = \mathfrak{p}_\Phi^r$. Similarly

$$\mathfrak{g} \cap \mathfrak{p} = \mathfrak{h} + \sum_{\beta \in \langle\Phi\rangle} \mathfrak{g} \cap (\mathfrak{g}_{\mathbf{C}}^{\beta} + \mathfrak{g}_{\mathbf{C}}^{-\beta}) ,$$

another real form of $\mathfrak{p}^r$, because complex conjugation over $\mathfrak{g}$ sends every root to its negative. Now, by dimension,

(2.2.6) $$D = G(x_0) \cong G/G \cap P \text{ is open in } X = G_u/V$$

and

(2.2.7) $$G \cap P \text{ is a real form of } V_{\mathbf{C}} .$$

Here of course $G \cap P$ is compact if and only if $G \cap P = V$, which is the case if and only if $\Phi \subset \Delta_K$.

In Example 2.1.6 above we have a specific compact homogeneous complex manifold $X$ defined by the quadratic form $Q$ in (2.1.7). Let

(2.2.8) $$D = \{S \in G(r, 2r + s; \mathbf{C}) \colon b(S, S) = 0,\, b(S, \bar{S}) > 0\} ,$$

which is an open subset of $X$. Then $D$ is the open orbit of $S_0$ where $S_0 \in G(r, 2r + s; \mathbf{C})$ is the span of the columns of the $(2r + s)$ matrix $\begin{pmatrix} I_r \\ iI_r \\ 0 \end{pmatrix}$, under the action of $G \subset G_{\mathbf{C}}$, where $G = \mathrm{SO}(2r, s)$. In fact, $D$ has two components corresponding to the components of $G$. Moreover,

$$D = \mathrm{SO}(2r, s)/U(r) \times \mathrm{SO}(s)$$

and there is a natural fibration

$$\text{(2.2.9)} \qquad D = \mathrm{SO}(2r, s)/U(r) \times \mathrm{SO}(s) \longrightarrow \mathrm{SO}(2r, s)/\mathrm{SO}(2r) \times \mathrm{SO}(s)\ ,$$

which is nontrivial for $r>1$. This fibering has compact fibers isomorphic to $\mathrm{SO}(2r)/U(r)$ which turn out to be complex submanifolds of $D$ with respect to the complex structure on $D$ when $D$ is considered as an open subset of $X = G_{\mathbf{C}}/P$. This is discussed for the general case in the next section. This particular domain $D$ arises in algebraic geometry as a period matrix domain, the two defining relations in (2.2.8) being generalizations due to Hodge of Riemann's classical period relation (see the survey by Griffiths [16] and Wells [41, Chap. 5]).

The domain $D$ itself is a generalization of the classical upper half plane, which classifies periods of holomorphic 1-forms on an elliptic curve and the Siegel upper half space of rank $r$, which classifies periods of holomorphic 1 forms on an algebraic curve (Riemann surface) of genus $r$. In particular, $D$ is the classifying space for periods of holomorphic 2-forms on a compact Kähler surface of complex dimension 2.

## 2.3 Compact linear subvarieties

Retain the setup of Section 2.2 and define

$$\text{(2.3.1)} \qquad Y = K(x) \subset G(x) = D \subset X$$

and

$$\text{(2.3.2)} \qquad L = \{g \in G_{\mathbf{C}} \colon gY = Y\}\ .$$

We check that $Y$ is a (maximal) compact subvariety of $D$ and $L$ is a complex Lie subgroup of $G_{\mathbf{C}}$, and then work out the structure of $L$. In the next section we will use this to study the $G_{\mathbf{C}}$-deformation space $\{g \in G_{\mathbf{C}} \colon gY \subset D\}/L$ of $Y$ in $D$.

**2.3.3.** PROPOSITION. *$Y$ is a complex submanifold of $X$, and is a complex flag manifold $K_{\mathbf{C}}/K_{\mathbf{C}} \cap P$ where $K_{\mathbf{C}}$ is the complex analytic subgroup of $G_{\mathbf{C}}$ for $\mathfrak{k}_{\mathbf{C}}$. $L$ is a complex Lie subgroup of $G_{\mathbf{C}}$ given by*

$$\text{(2.3.4)} \qquad L = K_{\mathbf{C}} \cdot E \quad \textit{where} \quad E = \textstyle\bigcap_{k \in K} kPk^{-1} = \bigcap_{k \in K_{\mathbf{C}}} kPk^{-1}\ ,$$

*and $E$ is a closed normal complex subgroup of $L$.*

*Proof.* The first assertion is clear: $\mathfrak{k}_{\mathbf{C}} \cap \mathfrak{p}$ is the parabolic subalgebra

$$\mathfrak{h}_{\mathbf{C}} + \textstyle\sum_{\langle\Phi\rangle \cap \Delta_K} (\mathfrak{g}_{\mathbf{C}}^{\beta} + \mathfrak{g}_{\mathbf{C}}^{-\beta}) + \sum_{\Delta_K^+ \setminus \langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{-\alpha}$$

of $\mathfrak{k}_{\mathbf{C}}$, so $K_{\mathbf{C}} \cap P$ is the corresponding parabolic subgroup, and evidently $K$ is transitive on the flag manifold $K_{\mathbf{C}}(x_0) = K_{\mathbf{C}}/K_{\mathbf{C}} \cap P$, which thus must be $Y$.

$L$ is closed in $G_{\mathbf{C}}$ because $Y$ is closed in $X$. Now it has a Lie algebra $\mathfrak{l}$.

Since $Y$ is complex, so is $\mathfrak{l}$: if $y \in Y$ and $\xi \in \mathfrak{l}$ then $\xi_y$ is in the holomorphic tangent space, and thus also is $(i\xi)_y$. Thus $L$ is a complex Lie subgroup of $G_{\mathbf{C}}$.

$K \subset L$ by construction of $Y$ and $L$. As $L$ is complex now, $K_{\mathbf{C}} \subset L$. $E$ is the kernel of the action of $L$ on $Y$, hence is a closed complex normal subgroup. Now compute

$$\begin{aligned} L &= \{g \in G_{\mathbf{C}}: gY = Y\} \subset \{g \in G_{\mathbf{C}}: gY \subset Y\} \\ &= \{g \in G_{\mathbf{C}}: gK \subset K_{\mathbf{C}}P\} = \textstyle\bigcap_{k \in K} K_{\mathbf{C}}Pk^{-1} \\ &= \textstyle\bigcap_{k \in K} K_{\mathbf{C}}kPk^{-1} = K_{\mathbf{C}}E \subset L\,. \end{aligned}$$ Q.E.D.

We now assume that the Lie group $G$ is simple, i.e., that $\mathfrak{g}$ has no proper ideal. For a local direct product splitting of $G$ would give the same for $G_{\mathbf{C}}$ and $P$ and would give a global direct product splitting of $X$ and $D$. Also without loss of generality we suppose that $G$ acts on $X$ as a noncompact group, which in view of simplicity just says

$$G \neq K,\ \text{i.e.},\ L \neq G_{\mathbf{C}},\ \text{i.e.},\ Y \neq X,\ \text{i.e.},\ D \neq X\,.$$

These assumptions made, there are two cases, distinguished by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$:

(I) *non-Hermitian*: $\mathrm{Ad}_G(K)$ is absolutely irreducible on $\mathfrak{s}$;

(II) *Hermitian*: $\mathfrak{s}_{\mathbf{C}} = \mathfrak{s}_+ + \mathfrak{s}_-$ with $\mathrm{Ad}_G(K)$ irreducible on $\mathfrak{s}_\pm$.

In the Hermitian case, $G/K$ has just two invariant complex structures, induced from its Borel embedding $gK \mapsto gK_{\mathbf{C}}S_\pm$ as an open $G$-orbit on the complex flag manifold $G_{\mathbf{C}}/K_{\mathbf{C}}S_\pm$ where $S_\pm = \exp(\mathfrak{s}_\pm)$.

**2.3.5.** PROPOSITION. *In the non-Hermitian case, $L$ is a finite extension of $K_{\mathbf{C}}$, so in particular $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$.*

*In the Hermitian case, each of the following conditions implies the next, and if $G \cap P$ is compact then all four are equivalent.*

1. *$D \to G/K$ by $g(x_0) \mapsto gK$ is well defined* (i.e., $G \cap P \subset K$), *and is holomorphic for one of the two invariant complex structures on $G/K$.*
2. *$L = K_{\mathbf{C}}S_\pm$ for a choice of sign $\pm$.*
3. *$\mathfrak{l} \neq \mathfrak{k}_{\mathbf{C}}$, i.e., $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}} + \mathfrak{s}_\pm$ for a choice of sign $\pm$.*
4. *One of the $\mathfrak{p} \cap (\mathfrak{k}_{\mathbf{C}} + \mathfrak{s}_\pm)$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$.*

*If $\mathfrak{l} \neq \mathfrak{k}_{\mathbf{C}}$ then $\mathfrak{l}$ is the $\mathfrak{l}_{\mathbf{C}} + \mathfrak{s}_\pm$ whose intersection with $\mathfrak{p}$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$.*

*Proof.* In the non-Hermitian case, $\mathfrak{k}_{\mathbf{C}}$ is a maximal subalgebra of $\mathfrak{g}_{\mathbf{C}}$. As $\mathfrak{l} \neq \mathfrak{g}_{\mathbf{C}}$ then $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$. Now the identity component $L^0 = K_{\mathbf{C}}$. As $L$ is a stabilizer, it is algebraic over $\mathbf{R}$ inside $G_{\mathbf{C}}$, so $L/L^0$ is finite, and $L$ is a finite extension of $K_{\mathbf{C}}$.

We go to the Hermitian case. Given (1), the holomorphic map $D \to G/K$ is proper; in fact $Y$ is the inverse image of a point; so $K_{\mathbf{C}}S_{\pm} \subset L \subsetneqq G_{\mathbf{C}}$, which forces (2). Evidently (2) implies (3). In general $Y = L/L \cap P$ shows that $L \cap P$ contains the solvable radical of $L$ and maps to a parabolic subgroup of the quotient. Given (3), $L$ is parabolic in $G_{\mathbf{C}}$, and now $L \cap P$ is parabolic in $G_{\mathbf{C}}$, i.e., $\mathfrak{p} \cap (\mathfrak{l} = \mathfrak{k}_{\mathbf{C}} + \mathfrak{s}_{\pm})$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$, which is (4). That also proves the remark following (4). Finally, given (4) we have the obvious $G_{\mathbf{C}}$-equivariant holomorphic maps

$$G_{\mathbf{C}}/P \cap K_{\mathbf{C}}S_{\pm} \begin{array}{l} \nearrow G_{\mathbf{C}}/P \\ \searrow G_{\mathbf{C}}/K_{\mathbf{C}}S_{\pm}\ , \end{array}$$

and we can fill in $G_{\mathbf{C}}/P \to G_{\mathbf{C}}/K_{\mathbf{C}}S_{\pm}$ just when $P \subset K_{\mathbf{C}}S_{\pm}$, which is when $G \cap P$ is compact. So (4) implies (1) when $G \cap P$ is compact. Q.E.D.

**2.3.6.** COROLLARY. *If $G \cap P$ is compact,* i.e., *if $G \cap P = V$, then $D \to G/K$ by $g(x_0) \mapsto gK$ is well defined, and either $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$ and $D \to G/K$ is not (anti)-holomorphic or $\mathfrak{l} \neq \mathfrak{k}_{\mathbf{C}}$ and $D \to G/K$ is (anti)-holomorphic.*

$L$ is given globally as follows. If $\mathfrak{l} \neq \mathfrak{k}_{\mathbf{C}}$ then $L$ is the $K_{\mathbf{C}}S_{\pm}$ such that $\mathfrak{p} \cap (\mathfrak{k}_{\mathbf{C}} + \mathfrak{s}_{\pm})$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$. If $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$ then the identity component $L^0 = K_{\mathbf{C}}$, so $K_{\mathbf{C}}$ is normal in $L$, and $L \subset N_{G_{\mathbf{C}}}(K_{\mathbf{C}})$ the $G_{\mathbf{C}}$-normalizer of $K_{\mathbf{C}}$. By Proposition 2.3.3 every component of $L$ is represented by an element of $E = \bigcap kPk^{-1} \subset P$; and if $p \in P$ normalizes $K_{\mathbf{C}}$ then

$$p(Y) = pK_{\mathbf{C}}(x_0) = K_{\mathbf{C}}p(x_0) = K_{\mathbf{C}}(x_0) = Y$$

so $p \in L$; now $L = K_{\mathbf{C}} \cdot N_P(K_{\mathbf{C}})$. Here we can replace $P$ by $P^r$ because $L^0$ contains every unipotent element of $N_P(K_{\mathbf{C}})$, and we can replace $K_{\mathbf{C}}$ by $K$ in the normalizer because compact real forms are conjugate, so finally $L = K_{\mathbf{C}} \cdot N_{P^r}(K)$.

The fibering given in (2.2.9) is an example of the case where $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$ and hence an example of a nonholomorphic fibering. The fact that $\mathfrak{l} = \mathfrak{k}_{\mathbf{C}}$ for this special case is verified by simple matrix algebra in Wells [39, Theorem 3.3].

## 2.4. Linear deformation spaces

Retain the setup of Sections 2.2 and 2.3. We are going to define and examine the $G_{\mathbf{C}}$-deformation space of $Y$ inside $D$. In the next section we will assume $G \cap P$ compact and prove that the base of the deformation space is a Stein manifold.

If $Z$ is a subset of $X$ we set $G_{\mathbf{C}}\{Z\} = \{g \in G_{\mathbf{C}}: gY \subset Z\}$. Evidently $G_{\mathbf{C}}\{Z\} \cdot L = G_{\mathbf{C}}\{Z\}$. Also, if $Z$ is open in $X$ then $G_{\mathbf{C}}\{Z\}$ is open in $G_{\mathbf{C}}$.

Fix a subset $Z \subset X$ with $G_{\mathbf{C}}\{Z\} = \{g \in G_{\mathbf{C}}: gY \subset Z\}$ not empty, and fix a closed complex subgroup $L' \subset G_{\mathbf{C}}$ with $K_{\mathbf{C}} \subset L' \subset L$. These data specify the $G_{\mathbf{C}}$*-deformation space of* $Y$ *inside* $Z$ *with respect to* $L'$, denoted

$$\pi\colon \mathcal{Y}_{Z,L'} \longrightarrow M_Z\ ,$$

as follows. $M_Z$ is the set of all $G_{\mathbf{C}}$-translates of $Y$ inside $Z$; that is,

$$M_Z = G_{\mathbf{C}}\{Z\}/L,\ \text{ subset of } G_{\mathbf{C}}/L\ . \tag{2.4.1}$$

$\mathcal{Y}_{Z,L'}$ is the disjoint union of those $G_{\mathbf{C}}$-translates $gY \in M_Z$ where we only identify $gY$ to $g'Y$ when $g' \in gL'$; that is,

$$\mathcal{Y}_{Z,L'} = G_{\mathbf{C}}\{Z\}/L' \cap P\ . \tag{2.4.2}$$

Now of course the projection is

$$\pi\colon \mathcal{Y}_{Z,L'} \longrightarrow M_Z \text{ by } \pi\big(g(L' \cap P)\big) = gL\ . \tag{2.4.3}$$

Observe that the inclusion $Z \subset X$ induces inclusions $\mathcal{Y}_{Z,L'} \subset \mathcal{Y}_{X,L'}$ and $M_Z \subset M_X$. If we denote

$$\tau\colon \mathcal{Y}_{Z,L'} \longrightarrow Z \text{ by } \tau\big(g(L' \cap P)\big) = gP = g(x_0) \tag{2.4.4}$$

then these inclusions induce a commutative diagram

$$\begin{tikzcd}
& \mathcal{Y}_{Z,L'} = G_{\mathbf{C}}\{Z\}/L' \cap P \arrow[rr] \arrow[dl, "\tau"'] \arrow[ddr, "\pi"] & & \mathcal{Y}_{X,L'} = G_{\mathbf{C}}/L' \cap P \arrow[dl, "\tau"'] \arrow[ddr, "\pi"] & \\
Z \arrow[rr] & & X = G_{\mathbf{C}}/P & & \\
& & M_Z = G_{\mathbf{C}}\{Z\}/L \arrow[rr] & & M_X = G_{\mathbf{C}}/L
\end{tikzcd} \tag{2.4.5}$$

whose horizontal maps are inclusions. As $L$, $P$ and $L'$ all are closed complex subgroups of $G_{\mathbf{C}}$, the coset realizations of (2.4.5) specify complex structures on $\mathcal{Y}_{X,L'}$ and $M_X$ such that $\tau\colon \mathcal{Y}_{X,L'} \to X$ and $\pi\colon \mathcal{Y}_{X,L'} \to M_X$ are holomorphic fibre bundles. When $Z$ is open in $X$, and $G_{\mathbf{C}}\{Z\}$ is open in $G_{\mathbf{C}}$, the horizontal maps of (2.4.5) are inclusions of open subsets. Thus we have the lemma.

**2.4.6.** LEMMA. *Let $Z$ be open in $X$. Let $\mathcal{Y}_{Z,L'}$, $Z$ and $M_Z$ carry the complex structures as open subsets of $\mathcal{Y}_{X,L'}$, $X$ and $M_X$. Then*

(i) *$\pi\colon \mathcal{Y}_{Z,L'} \to M_Z$ is a holomorphic fibre bundle, and*

(ii) *$\tau\colon \mathcal{Y}_{Z,L'} \to Z$ has image open in $Z$ and is a holomorphic fibre bundle over that image.*

One of our principal objectives is to show that the parameter space $M_D$ is a Stein manifold under the assumption that $G \cap P = V$ (that is, $G \cap P$ is compact). When $D \to G/K$ is holomorphic we do this by proving $M_D = G/K$, which is known to be Stein. When $D \to G/K$ is not holomorphic, we show that $M_X$ is an affine algebraic variety, thus Stein, and then in Section 2.5 we show that $M_D$ is an open Stein submanifold of $M_X$.

We now assume $G$ simple with $G \neq K$, so the dichotomy of Section 2.3 holds. Incidently, were $G = K$, $M_D$ would be reduced to a point and so would trivially be a Stein manifold.

Suppose that $\mathfrak{l} \neq \mathfrak{k}_C$, i.e., that $p\colon D \to G/K$ is holomorphic for one of the two choices of invariant complex structure on the bounded domain $G/K$. If $Z \subset D$ is a connected compact subvariety, then $p(Z)$ is reduced to a point by the maximum modulus principle, so $Z \subset p^{-1}(gK) = gY$ for some $g \in G$. If further $\dim_C Z = \dim_C Y$ then $Z = p^{-1}(gK) = gY$. In other words,

**2.4.7.** PROPOSITION. *If $\mathfrak{l} \neq \mathfrak{k}_C$ then $\mathcal{Y}_{D,L} \to M_D$ coincides with $D \to G/K$, in particular $M_D$ is a Stein manifold.*

Now suppose $\mathfrak{l} = \mathfrak{k}_C$, i.e., that $p\colon D \to G/K$ is not holomorphic. Then $M_X = G_C/L$ is the quotient of a connected semisimple group /C by a reductive algebraic subgroup. In effect, $L$ is algebraic by its definition as $G_C$-stabilizer of an algebraic subvariety of $X$, and now $L$ is reductive because its topological identity component $K_C$ is reductive. It follows (see [9, § 3]) that $M_X$ is an affine variety and a Stein manifold. We record

**2.4.8.** LEMMA. *If $\mathfrak{l} = \mathfrak{k}_C$ then $M_X$ is a homogeneous affine algebraic variety, in particular is a Stein manifold.*

We will prove that $M_D$, which is an open subset of $M_X$, is itself a Stein manifold, by proving holomorphic convexity for $M_D$ in $M_X$. That is the subject of Section 2.5 below. In Wells [39] a specific representation of $M_D$ (for the special case of $D$ given by (2.2.8)) is computed for the case $r = 2$, $s = 1$, and the fact that $M_D$ is Stein in this case is verified directly.

## 2.5. A Stein parameter space

In this section we will establish that $M_D$ is holomorphically convex. In preparation for this we need the following important lemma which shows how compact sets in $D$ and $M_D$ are related. From now on, $G \cap P$ is compact, i.e., $G \cap P = V$.

**2.5.1.** LEMMA. *Suppose $Z$ is a relatively compact subset of $D$, then $M_Z$ is a relatively compact subset of $M_D$.*

*Proof*. We may assume that (i) $G_C$ is simple and that $Y \subsetneqq X = G_C/P$ and (ii) $M_X$ is noncompact (cf. Lemma 2.4.6). Up to finite cover, we can assume that $M_X = G_C/K_C$ since $L$ is a finite extension of $K_C$. We now rephrase the statement of the lemma in group-theoretic terms using compactness of $G \cap P$:

$$(2.5.2)\qquad \begin{cases} \text{if } S \text{ is a relatively compact subset of } G, \\ \text{if } x_0 = 1\cdot P \in X = G_{\mathbf{C}}/P, \text{ then } M_{S(x_0)} \text{ is a} \\ \text{relatively compact subset of } G_{\mathbf{C}}/K_{\mathbf{C}}\ . \end{cases}$$

We may take $S$ to be in the form $SK$ (a larger set), so

$$M_{S(x_0)} = \{g \in G_{\mathbf{C}}\colon gK_{\mathbf{C}}P \subset SKP(=SK_{\mathbf{C}}P)\}/K_{\mathbf{C}}\ .$$

Thus the assertion of the lemma is equivalent to

$$(2.5.3)\qquad \begin{cases} \text{if } S = SK \text{ is a relatively compact subset of } G, \text{ then} \\ \qquad\qquad \{g \in G_{\mathbf{C}}\colon gK_{\mathbf{C}}P \subset SK_{\mathbf{C}}P\} \\ \text{is relatively compact modulo } K_{\mathbf{C}} \text{ (in } G_{\mathbf{C}}/K_{\mathbf{C}})\ . \end{cases}$$

Passing to closures, we see that (2.5.3) will follow from:

$$(2.5.4)\qquad \begin{cases} \text{if } S \text{ is a compact subset of } G, \text{ then} \\ \qquad \{g \in G_{\mathbf{C}}\colon gK_{\mathbf{C}}P \subset SK_{\mathbf{C}}P\} \\ \text{is compact modulo } K_{\mathbf{C}} \text{ (in } G_{\mathbf{C}}/K_{\mathbf{C}})\ . \end{cases}$$

We recall from Section 2.3, $E = \bigcap_{k \in K_{\mathbf{C}}} kPk^{-1}$, and we observe that $G_{\mathbf{C}}\{S\}$, which is by definition $= \{g \in G_{\mathbf{C}}\colon gK_{\mathbf{C}}P \subset SK_{\mathbf{C}}P\}$, has the property that:

$$(2.5.5)\qquad G_{\mathbf{C}}\{S\} = SK_{\mathbf{C}}E = SEK_{\mathbf{C}} \ \text{ and } \ kE = Ek \ \text{ for all } k \in K_{\mathbf{C}}\ .$$

That is,
$$\begin{aligned} g \in G_{\mathbf{C}}\{S\} &\Leftrightarrow \text{if } k \in K_{\mathbf{C}} \text{ and } p \in P, \text{ then } gkp \in SK_{\mathbf{C}}P\ , \\ &\Leftrightarrow \text{if } k \in K_{\mathbf{C}} \text{ and } p \in P, \text{ then } g \in SK_{\mathbf{C}}Pp^{-1}k^{-1}\ , \\ &\Leftrightarrow \text{if } k \in K_{\mathbf{C}}, \text{ then } g \in SK_{\mathbf{C}}kPk^{-1}\ , \\ &\Leftrightarrow g \in SK_{\mathbf{C}}E\ . \end{aligned}$$

This verifies (2.5.5). We know that $L = K_{\mathbf{C}}\cdot E$ is a finite extension of $K_{\mathbf{C}}$ (in fact we reduced to the case $L = K_{\mathbf{C}}$ in this proof). Using the representation of $G_{\mathbf{C}}\{S\}$ as given in (2.5.5) we see that

$$G_{\mathbf{C}}\{S\}/K_{\mathbf{C}} = SK_{\mathbf{C}}E/K_{\mathbf{C}} = SEK_{\mathbf{C}}/K_{\mathbf{C}}\ ,$$

and this last representation is the orbit of a finite set in $G_{\mathbf{C}}/K_{\mathbf{C}}$ under a compact set $S$, which is therefore compact. Q.E.D.

Our main result in this section is the following theorem.

**2.5.6.** THEOREM. *Assume that $G \cap P = V$, i.e., that $G \cap P$ is compact. Then $M_D$ is an open Stein submanifold of $G_{\mathbf{C}}/L$.*

The rest of this section will be devoted to proving this theorem.

Letting $M = M_D$, $\mathcal{Y}_X = \mathcal{Y}_{X,L}$, and $\mathcal{Y}_D = \mathcal{Y}_{D,L}$, we rewrite diagram (2.4.5).

$$
\begin{tikzcd}
 & \mathcal{Y}_X \arrow[r, "\tau"] & X = G_{\mathbf{C}}/P \\
\mathcal{Y}_X \arrow[d, "\pi"] & \mathcal{Y}_D \arrow[l] \arrow[u] \arrow[r, "\tau"] \arrow[d, "\pi"] & D = G/V \arrow[u] \\
G_{\mathbf{C}}/L = M_X & M \arrow[l] &
\end{tikzcd} \tag{2.5.7}
$$

Here the unmarked arrows in the diagram are open inclusions, and $\tau$ and $\pi$ are appropriately restricted from $\mathcal{Y}_X$ to the open subset $\mathcal{Y}_D$.

Let $\mathbf{K}_X$ be the canonical bundle of $X$, and let $\mathbf{K}_D = \mathbf{K}_X|_D$. Then there is a natural $G$-invariant metric $h_G$ on $\mathbf{K}_D$ and a natural $G_u$-invariant metric $h_{G_u}$ on $\mathbf{K}_X$. Letting

$$\varphi = -\log \frac{h_G}{h_{G_u}} , \tag{2.5.8}$$

we see that $\varphi$ is a $C^\infty$ function defined on $D$. In fact $\varphi$ is a strongly $q$-pseudo-convex exhaustion function for $D$ (Schmid [28], Griffiths-Schmid [14]), which we formulate in the following theorem.

**2.5.9.** THEOREM (Schmid). *Let* $q = \dim_{\mathbf{C}} Y = (1/2)\dim_{\mathbf{R}} K/V$, *then the function* $\varphi$ *given in* (2.5.8) *has the following properties*:

(a) $\varphi \in C^\infty(D)$ *and is an exhaustion function for* $D$, i.e., $\{x \in D\colon \varphi(x) < c\}$ *is relatively compact in* $D$ *for all* $c \in \mathbf{R}$.

(b) *At each point* $x \in D$, *the complex Hessian* (*Levi form*) $L(\varphi) = i\partial\bar{\partial}\varphi$ *has* $n - q$ *positive eigenvalues and* $q$ *negative eigenvalues.*

(c) *There is a G-invariant splitting of the tangent bundle* $\mathbf{T}(D) = \mathbf{T}_v(D) \oplus \mathbf{T}_h(D)$, *where* $\mathbf{T}_v(D)$ *are vertical fibres* (*tangential to the fibres of* $D \to G/K$) *and* $\mathbf{T}_{v,x}(D) \cong \mathbf{T}_x(Y_x)$, *where* $Y_{g(x_0)} = gY$, *and* $\mathbf{T}_h(D)$ *is a G-invariant subbundle of* $\mathbf{T}(D)$ *which is transversal to the fibres of* $D$.

(d) $L(\varphi)|_{\mathbf{T}_v(D)}$ *is negative definite and* $L(\varphi)|_{\mathbf{T}_h(D)}$ *is positive definite.*

We will use the $\varphi$ given by the above theorem to construct a continuous exhaustion function for $M$. First let $\varphi_{\mathcal{Y}} = \tau^*\varphi$, and define, for $g \in G_{\mathbf{C}}\{D\}$,

$$\varphi_M(gY) = \sup_{y \in Y} \varphi(gy) = \sup_{k \in K} \varphi_{\mathcal{Y}}\big(g \cdot k(L \cap P)\big) . \tag{2.5.10}$$

**2.5.11.** LEMMA. $\varphi_M$ *is a continuous function on* $M$.

*Proof.* Let $\psi\colon G_{\mathbf{C}}\{D\} \to \mathbf{R}$ be the continuous (in fact $C^\infty$) function given by $\psi(g) = \varphi(gP)$. Clearly we have

$$\varphi_{\mathcal{Y}}\big(g \cdot (L \cap P)\big) = \psi(g) ,$$
$$\varphi_M(gY) = \sup_{k \in K} \psi(g \cdot k) .$$

Thus it suffices to take $A = G_{\mathbf{C}}\{D\}$ equipped with any metric and verify the following proposition, the proof of which is a simple exercise in uniform continuity and is omitted.

**2.5.12.** LEMMA. *Let $A$ and $B$ be metric spaces, $A$ locally compact, and $B$ compact. Suppose $\eta: A \times B \rightarrow \mathbf{R}$ is continuous, then $\tilde{\eta}: A \rightarrow \mathbf{R}$ given by $\tilde{\eta}(a) = \sup_{b \in B} \eta(a, b)$ is continuous.*

We now proceed to use the exhaustion function $\varphi$ to show that $M = M_D$ is Stein. First we consider the relatively compact subdomains of $D$ defined by

$$D_c = \{x \in D: \varphi(x) < c\} .$$

Let $M_c = \pi(\tau^{-1}(D_c))$. By Lemma 2.5.1, we see that $M_c$ is relatively compact in $M$. Our first objective is to show that $M_c$ is Stein, and then later we will use this to conclude that $M$ is Stein.

We shall be constructing holomorphic functions on $M_c$ to show that $M_c$ is Stein. Our basic principle will be to integrate cohomology classes over the fibre to obtain holomorphic functions. We will discuss this principle now with regard to the domains $D$ and $M$, but it applies with no change to $D_c$ and $M_c$. Let $\xi$ be a Dolbeault cohomology class represented by the $\bar{\partial}$-closed $(q, q)$-form $\psi$ on $D$, then $\tau^*\psi$ is a $\bar{\partial}$-closed $(q, q)$-form on $\mathcal{Y}_D$, and we may consider $\tau^*\psi$ as a $(q, q)$-current on $\mathcal{Y}_D$. Since $\pi$ is a proper holomorphic mapping, the push forward $\pi_*\tau^*\psi = f$ is a $\bar{\partial}$-closed current of type $(0, 0)$ on $M$. Thus, by the regularity theorems for currents, we see that $f$ is a holomorphic function on $M$. Moreover, if $\lambda \in M$, then

$$f(\lambda) = \int_{Y_\lambda \subset \mathcal{Y}_D} \tau^*\psi = \int \psi$$

is simply integration over the fibre. Since $\int_{Y_\lambda} \psi$ does not depend on the representative differential form used, by Stokes' theorem, we write simply

$$f(\lambda) = \int_{Y_\lambda} \xi , \quad \xi \in H^q(D, \Omega^q_D), \lambda \in M .$$

For references to the recent literature on the interaction of currents and complex analysis see the survey paper by Harvey [22] (cf. also Wells [42], for more details with regard to the specific discussion here).

We shall need the following deep result due to Andreotti and Norguet [2, Prop. 7]. Recall that a function $\varphi$ is called strongly $q$-pseudoconvex on a complex manifold if $\varphi$ is real-valued, $C^2$, and $i\partial\bar{\partial}\varphi$ has at least $n - q$ eigenvalues $>0$ at each point.

**2.5.13.** THEOREM (Andreotti-Norguet). *Let $\varphi$ be a strongly $q$-pseudo-*

*convex function on a complex manifold $Z$, and let $B = \{x \in Z: \varphi(x) < 0\}$. Suppose that at $x \in \partial B$, $d\varphi_x \neq 0$, and $i\partial\bar{\partial}\varphi|_{T_x(\partial B)}$ has $n - q - 1$ positive eigenvalues and $q$ negative eigenvalues. Let $\{\Sigma_\nu\}_{\nu \in \mathbf{N}}$ be a sequence of compact subvarieties of $Z$, each of pure dimension $q$, and having $x$ as a limit point. Suppose there exists an Hermitian metric for which $\sup_{\nu \in N} \operatorname{vol}(\Sigma_\nu) < +\infty$. Then there exists a $\xi \in H^q(B, \Omega^q)$ such that*

$$\sup_{\nu \in \mathbf{N}} \left| \int_{\Sigma_\nu} \xi \right| = +\infty .$$

The following lemma describes the behavior of the Levi form of $\varphi$ restricted to $\partial D_c$ near limit points of compact subvarieties.

**2.5.14.** LEMMA. *Let $x_0 \in \partial D_c$ be a limit point of $q$-dimensional compact submanifolds of $D_c$, and suppose $d\varphi(x_0) \neq 0$. Then the Levi form $L(\varphi) = i\partial\bar{\partial}\varphi$ restricted to $T_{x_0}(\partial D_c)$ has $n - q - 1$ positive eigenvalues and $q$ negative eigenvalues.*

*Proof.* Since $L(\varphi)$ is nondegenerate on $T_{x_0}(D)$, and $L(\varphi)$ has $n - q$ positive eigenvalues and $q$ negative eigenvalues, it follows that $L(\varphi)|_{T_{x_0}(\partial D)}$ has either $n - q - 1$ positive eigenvalues and $q$ negative eigenvalues or $n - q$ positive eigenvalues and $q - 1$ negative eigenvalues. We will assume the latter situation in conjunction with the hypothesis of the lemma to derive a contradiction.

Thus assume that $L(\varphi)|_{T_{x_0}(\partial D_c)}$ has $n - q$ positive eigenvalues and $q - 1$ negative eigenvalues. The following construction is due to Andreotti-Norguet [2, I, pp. 225–226]. Choose coordinates $(z_1, \cdots, z_n)$ defined in $U \ni x_0$, so that $(0, \cdots, 0) = x_0$ and

$$\varphi(z) = \varphi(0) + \operatorname{Re} z_1 - \sum_{j=1}^{q} a_j |z_j|^2 + \sum_{j=q+1}^{n} a_j |z_j|^2 + O(|z|^3) ,$$

where $a_j > 0$, $j = 1, \cdots, n$. Here our assumption that $L(\varphi)|_{T_{x_0}(\partial D_c)}$ has $q - 1$ negative eigenvalues is used in making $\operatorname{Re} z_1$ the normal coordinate to $\partial D_c$. Let

$$\xi = (z_1, \cdots, z_q),\ \eta = (z_{q+1}, \cdots, z_n),\ |\xi|^2 = |z_1|^2 + \cdots + |z_q|^2,$$
$$|\eta|^2 = |z_{q+1}|^2 + \cdots + |z_n|^2$$

and thus $|z|^2 = |\xi|^2 + |\eta|^2$. Let $E = \{\xi = 0\} \cap U$, and find an $\varepsilon > 0$ so that

$$\varphi|_E(z) - \varphi(0) = \sum_{j=q-1}^{n} a_j |z_j|^2 + O(|z|^3) > 0$$

for $0 < |\eta|^2 < \varepsilon^2$. Letting $D' = D_c \cap U$, we set

$$P = \{z \in U: |\xi|^2 < \sigma^2,\ |\eta|^2 < \varepsilon^2\}$$

for $\sigma$ small and fixed, and

$$\dot{P} = \{z \in U: |\xi|^2 < \sigma^2,\ |\eta|^2 = \varepsilon^2\}$$

(part of the boundary of $P$). It follows that

$$\bar{P} \subset U, \dot{P} \subset U \cap (D - \bar{D}') .$$

Let $Q = \{z \in P: \eta = 0\}$ and let $\pi: P \to Q$ be the natural projection $\pi(\xi, \eta) = \eta$. Note that $E \cap D' = \emptyset$, and from this it follows that the image of $\pi$ restricted to any subset of $D' \cap P$ cannot contain the origin in $Q$. Now let $K$ be any compact subset of $D_c$ and let $S = K \cap P$, then it follows that $\pi|_S$ is a proper mapping into $Q$. Indeed, if $\pi(S) = S_0$, consider

$$\pi^{-1}\big(S_0 \cap [|\xi|^2 \leqq \sigma_1^2]\big) \cap S, \sigma_1 < \sigma ,$$

and suppose this set is not compact in $S$. Then there must be a point in this inverse image with coordinates $(\xi, \eta)$ where $|\eta|^2 = \varepsilon^2$ since $|\xi|^2 \leqq \sigma_1^2 < \sigma^2$. But then $(\xi, \eta) \in \dot{P}$ and $\dot{P} \cap D' = \dot{P} \cap D_c = \emptyset$. Thus $\pi|_S$ is proper.

Suppose now that $\Sigma \subset D_c$ is a compact subvariety of dimension $l$ and $\Sigma \cap P \neq \emptyset$. Then $\pi|_{\Sigma \cap P}$ is proper and holomorphic. It follows from the Remmert proper mapping theorem (Gunning and Rossi [17]) that $\pi(\Sigma \cap P)$ is a closed subvariety of $Q$ of dimension $l$ (the fact that dimension $\pi(\Sigma \cap P)$ is $l$ is due to the fact that the fibres of $\pi|_{\Sigma \cap P}$ have to be compact complex subvarieties of $P$, and since $P$ is Stein, these fibres are zero dimensional). By the hypothesis of the lemma there is a sequence $\{\Sigma_\nu\}$ of compact subvarieties of $D_c$ of dimension $q$ which have $x_0$ as a limit point. Choose a $\Sigma_\nu$ in this sequence whose intersection with $P$ is non-empty. Then, by the above construction, $\pi(\Sigma_\nu \cap P)$ will be a closed $q$-dimensional subvariety of $Q$ which does not contain the origin. But $Q$ is a ball of dimension $q$, and the only closed $q$-dimensional subvariety of $Q$ is $Q$ itself which does contain the origin. Thus we have a contradiction. Q.E.D.

We can now prove the principal lemma in this section.

**2.5.15.** LEMMA. *Suppose $c$ is such that $d\varphi \neq 0$, for all $x \in \partial D_c$, then $M_c$ is Stein.*

*Proof.* Since $M_c$ is an open subset of the Stein manifold $M_X = G_{\mathbf{C}}/L$ (Lemma 2.4.8), it suffices to verify that $M_c$ is holomorphically convex. Recalling that $M_c$ is relatively compact in $M$ (Lemma 2.5.1), we see that it suffices to find, for any discrete sequence $\{\lambda_\nu\}$ of points in $M_c$ which converge to a boundary point $\lambda_0 \in \partial M_c$, a holomorphic function $f \in \mathcal{O}(M_c)$ such that

$$\sup_\nu |f(\lambda_\nu)| = +\infty .$$

Let $\Sigma_\nu = \pi^{-1}(\lambda_\nu)$, and let us observe that, since $\lambda_\nu \to \lambda_0 \in \partial M_c$, it follows from Lemma 2.5.1 that the compact subvarieties $\{\Sigma_\nu\}$ have a limit point $x_0 \in \partial D_c$. By Lemma 2.5.14 we have that $L(\varphi)|_{T_{x_0}(\partial D_c)}$ has $n - q - 1$ positive eigenvalues and $q$ negative eigenvalues. Moreover, the domain $D_c$ is strongly

$q$-pseudoconvex. Since $X$ is projective algebraic (Proposition 2.1.5) it is equipped with a Kähler form $\omega$, and we see that the volume of the fibres in $\mathcal{Y}_X$ is a constant. That is, for $\lambda \in M_X$, we let

$$\operatorname{vol}(Y_\lambda) = \int_{Y_\lambda} \omega^q = \int_{Y_\lambda} \underbrace{\omega \wedge \cdots \wedge \omega}_{q\text{-factors}} .$$

Then letting $v(\lambda) = \operatorname{vol}(Y_\lambda)$, we see that $v = \pi_*(\tau^*\omega^q)$, and thus $dv = d(\pi_*\tau^*(\omega^q)) = \pi_*\tau^*(d(\omega^q)) = 0$, since $d\omega = 0$, by the Kähler property. Thus $v$ is a constant on the connected manifold $M_X = G_{\mathbf{C}}/L$.

Thus we have verified all of the hypotheses of Theorem 2.5.13, and can then conclude that there exists a Dolbeault cohomology class $\xi$ of bidegree $(q, q)$ on $D_c$ satisfying

$$\sup_{\nu \in \mathbf{N}} \left| \int_{\Sigma_\nu} \xi \right| = +\infty .$$

Now $f = \pi_*\tau^*\xi$ is a holomorphic function with

$$\sup_{\nu \in \mathbf{N}} |f(dv)| = +\infty ,$$

and the lemma is proved. Q.E.D.

To finish the proof of Theorem 2.5.6, we note that the family of open submanifolds $[M_c \subset M]_{c \in \mathbf{R}}$ satisfies the following conditions of Docquier-Grauert [11]:

i) $M_c$ is Stein for a dense subset of $\mathbf{R}$ (Lemma 2.5.14 and Sard's lemma showing that $\partial D_c$ is smooth for almost all $c$);

ii) $\bigcup_{c \in \mathbf{R}} M_c = M$;

iii) $M_{c_1} \subset M_{c_2}$ if $c_1 < c_2$;

iv) $\bigcup_{-\infty < c < c_0} M_c$ is the union of the connected components of $M_{c_0}$, $c_0 \leqq \infty$;

v) $M_{c_0}$ is the union of the connected components of the interior of $\bigcap_{c_0 < c < \infty} M_c$, $-\infty < c_0 < \infty$.

By the main theorem in Docquier-Grauert [11], it then follows that $M$ itself is Stein, and this concludes the proof of Theorem 2.5.6. Q.E.D.

## 3. Homogeneous vector bundles

### 3.1. Homogeneous holomorphic vector bundles

In Chapter 2 we discussed in some detail the geometry of the flag domains $D = G/V$ contained in a complex flag manifold $X = G_{\mathbf{C}}/P$, where $G_{\mathbf{C}}$ is a complex semisimple Lie group and $P$ is a parabolic subgroup. In this chapter we turn our attention to homogeneous holomorphic vector bundles $\mathbf{E} \to D$. They are the holomorphic vector bundles on $D$ with the property that the action of $G$ on $D$ lifts to an action on the total space, by holomorphic bundle maps. Examples of homogeneous vector bundles on $D$ are the holomorphic tangent bundle $T_D$, and the canonical bundle $\mathbf{K}_G = \bigwedge^{\dim_{\mathbf{C}} D}\mathbf{T}_D^*$, and these are described in more detail below in terms of the homogeneous structure of $D$. In general for a homogeneous space of the form $S = G/Q$, where $G$ is a Lie group and $Q$ a closed subgroup, we associate to each continuous representation $\mu$ of $Q$ on a complex vector space $E_\mu$, the homogeneous $C^\infty$ complex vector bundle $\mathbf{E}_\mu = G \times_\mu E_\mu \to S$ defined by the equivalence relation $(g, z) \sim (g\cdot q, \mu(q)\cdot z)$ in $G \times E_\mu$ (cf. Bott [10]).

Using the notation of Chapter 2 we now describe in detail the basic properties of homogeneous holomorphic vector bundles $\mathbf{E} \to D$, where $D = G(x_0)$ is a flag domain in a complex flag manifold $X = G_{\mathbf{C}}/P$. If necessary, replace $P$ by a conjugate so that $x_0 = 1\cdot P \in G_{\mathbf{C}}/P = X$.

**3.1.1.** LEMMA. *Let $\mu$ be a continuous representation of the isotropy group $G \cap P = \{g \in G\colon g(x_0) = x_0\}$ on a complex vector space $E_\mu$. Let*

$$\mathbf{E}_\mu = G \times_\mu E_\mu \longrightarrow G/G \cap P = D$$

*denote the associated homogeneous complex vector bundle. Suppose that $D$ carries a $G$-invariant Radon measure, that is* [44, Thm. 6.3] *that $P = (G \cap P)_{\mathbf{C}}\cdot P^n$ where $P^n$ is its nilradical. Then $\mathbf{E}_\mu \to D$ is a $G$-homogeneous holomorphic vector bundle in such a way that the holomorphic sections over an open set $U \subset D$ are represented by the functions*

$$f\colon \tilde{U} = \{g \in G\colon g(x_0) \in U\} \longrightarrow E_\mu$$

*such that $f(gp) = \mu(p)^{-1}f(g)$ for $g \in G$ and $p \in G \cap P$, and $\xi(f) = 0$ for $\xi \in \mathfrak{p}^n$, where every $\xi \in \mathfrak{g}_{\mathbf{C}}$ is viewed as a left invariant complex vector field on $G$.*

*Note.* The hypothesis holds whenever $G$ has a compact Cartan subgroup, in particular when $G \cap P$ is compact. See [34] for the corresponding result where $G$ is replaced by an arbitrary extension and the equations for sections are used to define the complex structure on $\mathbf{E}_\mu$.

*Proof.* Since $G \cap P$ is connected, $\mu$ extends uniquely to a holomorphic representation $\mu'$ of $P$ on $E_\mu$ such that $\mu'(P^n) = 1$. Let $\mathbf{E}'_\mu = G_{\mathbf{C}} \times_{\mu'} E_\mu \to G_{\mathbf{C}}/P = X$ denote the corresponding $G_{\mathbf{C}}$-homogeneous holomorphic vector bundle. Note that $\mathbf{E}_\mu$ is the underlying real structure of $\mathbf{E}'_\mu|_D$ and that $\mathbf{E}'_\mu|_D$ has local holomorphic sections as described, and observe that these sections specify the complex structure on $\mathbf{E}'_\mu|_D$. Now give $\mathbf{E}_\mu$ the structure of $\mathbf{E}'_\mu|_D$. Q.E.D.

We fix the notation

(3.1.2) $\qquad \mathbf{T}_D \longrightarrow D$ is the holomorphic tangent bundle

so that $\bigwedge^p(\mathbf{T}_D^*) \otimes \bigwedge^q(\bar{\mathbf{T}}_D^*) \to D$ is the bundle of $(p, q)$-forms on $D$. If $\mathbf{E} \to D$ is a holomorphic vector bundle we denote

$$(3.1.3) \qquad \mathcal{E}^{p,q}(D; \mathbf{E})\colon C^\infty \text{ sections of } \mathbf{E} \otimes \textstyle\bigwedge^p(\mathbf{T}_D^*) \otimes \bigwedge^q(\bar{\mathbf{T}}_D^*) \longrightarrow D\,.$$

The elements of $\mathcal{E}^{p,q}(D; \mathbf{E})$ are called *$(p, q)$-forms on $D$ with values in* $\mathbf{E}$. The $(0, 1)$-component of exterior differentiation is well defined there and gives us the first order operator

$$(3.1.4) \qquad \bar\partial\colon \mathcal{E}^{p,q}(D; \mathbf{E}) \longrightarrow \mathcal{E}^{p,q+1}(D; \mathbf{E})\,.$$

A form $\omega \in \mathcal{E}^{p,q}(D; \mathbf{E})$ is called *$\bar\partial$-closed* if $\bar\partial\omega = 0$, called *$\bar\partial$-exact* if $\omega \in \bar\partial\mathcal{E}^{p,q-1}(D; \mathbf{E})$. Since $\bar\partial^2 = 0$, exact forms are closed and one has the *Dolbeault cohomology* spaces

$$(3.1.5) \qquad H^{p,q}(D; \mathbf{E}) = \{\omega \in \mathcal{E}^{p,q}(D; \mathbf{E})\colon \bar\partial\omega = 0\}/\bar\partial\mathcal{E}^{p,q-1}(D; \mathbf{E})\,.$$

Uniform convergence of derivatives on compact sets defines a Fréchet space structure on each $\mathcal{E}^{p,q}(D; \mathbf{E})$. Evidently $\bar\partial$ is continuous. If $\mathbf{E} \to D$ is homogeneous under $G$, then $G$ acts naturally on $\mathcal{E}^{p,q}(D; \mathbf{E})$, and the action

$$G \times \mathcal{E}^{p,q}(D; \mathbf{E}) \longrightarrow \mathcal{E}^{p,q}(D; \mathbf{E})$$

is continuous. If $\bar\partial$ has closed range, then $H^{p,q}(D; \mathbf{E})$ inherits a Fréchet structure, and, in the homogeneous case, the action $G \times H^{p,q}(D; \mathbf{E}) \to H^{p,q}(D; \mathbf{E})$ is continuous.

All this applies equally well to the maximal compact subvariety $Y = K(x_0) \subset D$. By a theorem of Serre we know that $\bar\partial$ always has closed range, so the $H^{p,q}(Y, \mathbf{E})$ all are Fréchet spaces (cf. Wells [41]).

If $\mathcal{S} \to D$ is a sheaf, then $H^q(D; \mathcal{S})$ denotes the sheaf cohomology groups. The case of interest is

$$(3.1.6) \qquad \mathcal{O}(\mathbf{E}) \longrightarrow D\,,$$

the sheaf of germs of holomorphic sections of $\mathbf{E} \to D$, where $\mathbf{E} \to D$ is a holomorphic vector bundle. Dolbeault's theorem says that

$$(3.1.7) \qquad H^q\big(D; \mathcal{O}(\mathbf{E})\big) \text{ is naturally isomorphic to } H^{0,q}(D; \mathbf{E})\,.$$

Now let us be more specific. We assume that

$$(3.1.8) \qquad G \cap P \text{ is compact; that is } G \cap P = V$$

and we have a compact Cartan subgroup $H$ of $G$ and a maximal compact subgroup $K$ of $G$ such that

$$(3.1.9) \qquad H \subset V \subset K\,.$$

Further, we have a positive $\mathfrak{h}_C$-root system $\Delta^+$ on $\mathfrak{g}_C$, and a subset $\Phi$ of the simple roots, such that

$$\mathfrak{p} = \mathfrak{p}^r + \mathfrak{p}^n$$

with

(3.1.10) $$\mathfrak{p}^r = \mathfrak{v}_C = \mathfrak{h}_C + \sum_{\langle\Phi\rangle} \mathfrak{g}_C^\beta + \mathfrak{g}_C^{-\beta} \text{ and } \mathfrak{p}^n = \sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}^{-\alpha}$$

where $\langle\Phi\rangle = \{\alpha \in \Delta^+ : \alpha$ is a linear combination from $\Phi\}$.

The root system $\Delta$ decomposes into the set $\Delta_K$ of compact roots and the set $\Delta_S$ of noncompact roots. If $\theta$ is the Cartan involution of $G$ with fixed point set $K$, so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ is the Cartan decomposition into eigenspaces of $\theta$, then

$$\mathfrak{k}_C = \mathfrak{h}_C + \sum_{\Delta_K} \mathfrak{g}_C^\alpha \text{ and } \mathfrak{s}_C = \sum_{\Delta_S} \mathfrak{g}_C^\gamma .$$

We write $\Delta_K^+$ for $\Delta_K \cap \Delta^+$ and $\Delta_S^+$ for $\Delta_S \cap \Delta^+$. $G \cap P = V \subset K$ says $\Phi \subset \Delta_K^+$, so $\langle\Phi\rangle$, which in effect is $\Delta_V^+$, is a subset of $\Delta_K^+$. Let us fix the notation

(3.1.11) $$\rho = \rho_G = \frac{1}{2}\sum_{\Delta^+} \beta,\ \rho_K = \frac{1}{2}\sum_{\Delta_K^+} \beta \text{ and } \rho_V = \frac{1}{2}\sum_{\langle\Phi\rangle} \beta .$$

Let $\Psi$ denote the simple root system for $(\mathfrak{g}_C, \Delta^+)$, and enumerate

(3.1.12) $$\Phi = \{\varphi_1, \cdots, \varphi_r\} \subset \{\varphi_1, \cdots, \varphi_l\} = \Psi .$$

Given $\lambda \in \mathfrak{h}_C^*$, and letting $\langle , \rangle$ be the Killing form on $\mathfrak{g}_C$, we denote

(3.1.13) $$n_i(\lambda) = 2\langle\lambda, \varphi_i\rangle / \langle\varphi_i, \varphi_i\rangle .$$

$\lambda$ is called *integral* if all the $n_i(\lambda)$ are integers. If $G_C$ is simply connected, then $\lambda$ is integral if and only if

$$e^\lambda : \exp(\xi) \longmapsto e^{\lambda(\xi)} , \qquad \xi \in \mathfrak{h} ,$$

is a well defined character on the torus group $H$. For example, $\rho$ is integral with $n_i(\rho) = 1$ for all $i$, and $\rho_V$ is integral within $V$ with $n_i(\rho_V) = 1$ for $i \leq r$.

If $\mu$ is an irreducible representation of $V$, then $\mu|_H$ is a finite sum of characters $e^\lambda$ where the $\lambda$ are integral elements of $\mathfrak{h}_C^*$ called the *weights* of $\mu$. If we impose a lexicographic order on the real span $i\mathfrak{h}^*$ of the roots such that $\Delta^+$ consists of positive elements, then $\mu$ has a unique highest weight and it does not depend on the choice of the ordering. That highest weight determines $\mu$ up to equivalence, and we write

(3.1.14)
- $\mu_\lambda$: irreducible representation of $V$ with highest weight $\lambda$;
- $E_\lambda$: representation space of $\mu_\lambda$;
- $\mathbf{E}_\lambda \to D$: holomorphic vector bundle associated by Lemma 3.1.3;
- $\mathcal{E}_\lambda \to D$: sheaf of germs of holomorphic sections of $\mathbf{E}_\lambda \to D$ (i.e., $\mathcal{O}(\mathbf{E}_\lambda)$, cf. 3.1.6).*)

*) $\mathcal{E}_\lambda$ here should not be confused with the notation for $C^\infty(p, q)$ forms $\mathcal{E}^{p,q}$, and this will be clear from the context.

If $G_{\mathbf{C}}$ is simply connected, the highest weights of representations of $V$ are exactly the linear functionals $\lambda \in \mathfrak{h}_{\mathbf{C}}^*$ such that

$$n_i(\lambda) \text{ is a non-negative integer for } 1 \leqq i \leqq r\,, \quad \text{an arbitrary integer for } r < i \leqq l\,. \tag{3.1.15}$$

In any case, the highest weights satisfy (3.1.15). We note that the degree $\dim E_\lambda$ of $\mu_\lambda$ is 1 just when $n_i(\lambda) = 0$ for $1 \leqq i \leqq r$, so the homogeneous holomorphic line bundles over $D$ are the $\mathbf{E}_\lambda \to D$ with $n_i(\lambda) = 0$ for $1 \leqq i \leqq r$, $n_i(\lambda)$ an arbitrary integer for $r < i \leqq l$.

The holomorphic tangent bundle $\mathbf{T}_D \to D$ is the restriction of the holomorphic tangent bundle $\mathbf{T}_X \to X$, and the latter is the homogeneous holomorphic vector bundle over $X = G_{\mathbf{C}}/P$ associated to the representation of $P$ on $T = \mathfrak{g}_{\mathbf{C}}/\mathfrak{p} \cong \sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^\gamma$. Similarly the bundle $\mathbf{T}_D^* \to D$ of (1, 0)-forms is based on the dual space $T^*$ of $T$. As the Killing form pairs $\sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^\gamma$ to $\sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{-\gamma} = \mathfrak{p}^n$, we identify $T^*$ with $\mathfrak{g}_{\mathbf{C}}/(\mathfrak{p}^r + \sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^\gamma) \cong \mathfrak{p}^{-n}$. Let $n = \dim_{\mathbf{C}} D$, so the canonical bundles $\{(n, 0)\text{-forms}\}$ are $\mathbf{K}_D = \bigwedge^n \mathbf{T}_D^* \to D$ and $\mathbf{K}_X = \bigwedge^n \mathbf{T}_X^* \to X$. Note that $P^n$ acts trivially on $\bigwedge^n \mathbf{T}^*$, and that $V$ acts on $\bigwedge^n \mathbf{T}^*$ by the character $e^\lambda$ where $\lambda = -2(\rho - \rho_V)$. This shows that $\mathbf{K}_D \to D$ is the homogeneous holomorphic line bundle $\mathbf{E}_{2(\rho_V - \rho)} \to D$.

Similarly, the holomorphic tangent bundle $\mathbf{T}_Y \to Y$ is based on $\mathfrak{k}_{\mathbf{C}}/\mathfrak{p} \cap \mathfrak{k}_{\mathbf{C}} \cong \sum_{\Delta_K^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^\gamma$, and if $s = \dim_{\mathbf{C}} Y$ then $\mathbf{K}_Y = \bigwedge^s \mathbf{T}_Y^* = \mathbf{E}_{2(\rho_V - \rho_K)}|_Y$.

The irreducible representations of $K$ similarly are described by highest weight relative to $\Delta_K^+$. Here we must be careful because the simple root system for $(\mathfrak{k}_{\mathbf{C}}, \Delta_K^+)$ usually is not contained in $\Psi$.

We end these preliminaries by recalling the Bott-Borel-Weil theorem for our maximal compact subvariety $Y = K/V = K_{\mathbf{C}}/P \cap K_{\mathbf{C}}$ (Bott [10], cf. Griffiths-Schmid [14], Kostant [26]). First, it says

$$\text{if } \langle \lambda + \rho_K, \beta \rangle = 0 \text{ for some } \beta \in \Delta_K \text{ then } H^q(Y; \mathcal{E}_\lambda) = 0 \text{ for all } q\,. \tag{3.1.16}$$

If $\langle \lambda + \rho_K, \beta \rangle \neq 0$ for all $\beta \in \Delta_K$, then we denote

$$q_K(\lambda + \rho_K) = \left|\{\beta \in \Delta_K^+ : \langle \lambda + \rho_K, \beta \rangle < 0\}\right|$$

and we take the element $w$ of the Weyl group $W(K, H)$ such that $\langle w(\lambda + \rho_K), \beta \rangle > 0$ for all $\beta \in \Delta_K^+$. Now the second and main part of the Bott-Borel-Weil theorem says that

$$\begin{aligned} &H^q(Y; \mathcal{E}_\lambda) = 0 \quad \text{for} \quad q \neq q_K(\lambda + \rho_K), \text{ and} \\ &H^{q_K(\lambda+\rho_K)}(Y, \mathcal{E}_\lambda) \text{ is the } K\text{-module of} \\ &\text{highest weight } \; w(\lambda + \rho_K) - \rho_K\,. \end{aligned} \tag{3.1.17}$$

For example, let us apply this to the sheaf $\mathcal{K}_Y \to Y$ of germs of holomorphic sections of the canonical bundle $\mathbf{K}_Y \to Y$. We saw $\mathbf{K}_Y = \mathbf{E}_{2(\rho_V - \rho_K)}|_Y$. Now $\langle \rho_V - \rho_K, \beta \rangle = 0$ for every $\beta \in \Delta_V$, and it is easy to check that $\langle \rho_K - \rho_V, \beta \rangle > 0$ for all $\beta \in \Delta_V^+ \backslash \langle \Phi \rangle$. It follows that

$$\langle 2(\rho_V - \rho_K) + \rho_K, \beta \rangle > 0 \text{ for } \beta \in \langle \Phi \rangle, < 0 \text{ for } \beta \in \Delta_K^+ \backslash \langle \Phi \rangle$$

so (3.1.17) applies and the cohomology occurs in dimension $|\Delta_K^+ \backslash \langle \Phi \rangle| = \dim_{\mathbf{C}} Y = s$.

## 3.2. Schmid's Identity Theorem and some consequences

In this section we state a variation on the Identity Theorem [28, Corollary 6.5] of Wilfried Schmid and we indicate consequences used in the sequel. The Identity Theorem is proved in Section 3.3.

Retain the notation and conventions of Section 3.1 and consider a homogeneous holomorphic vector bundle

$$\mathbf{E}_\lambda \longrightarrow D \text{ for a representation } \mu_\lambda \text{ of } V \text{ with highest weight } \lambda \in i\mathfrak{h}^* .$$

We will assume that $\mathbf{E}_\lambda \to D$ is *nondegenerate* in the sense that

$$(3.2.1) \quad \begin{cases} \text{if } \beta_1, \cdots, \beta_l \text{ are distinct noncompact positive } \mathfrak{h}_{\mathbf{C}}\text{-roots then} \\ \qquad \langle \lambda + \rho_V + \beta_1 + \cdots + \beta_l, \alpha \rangle > 0 \text{ for all } \alpha \in \langle \Phi \rangle = \Delta_V^+ \\ \text{and} \\ \qquad \langle \lambda + \rho_K + \beta_1 + \cdots + \beta_l, \gamma \rangle < 0 \text{ for all } \gamma \in \Delta_K^+ \backslash \langle \Phi \rangle . \end{cases}$$

Two remarks on (3.2.1): First, $\langle \rho_V, \alpha \rangle = \langle \rho_K, \alpha \rangle$ for all $\alpha \in \Delta_V$, so the second condition of (3.2.1) is equivalent to: $\langle \lambda + \rho_K + \beta_1 + \cdots + \beta_l, \alpha \rangle > 0$ for all $\alpha \in \langle \Phi \rangle$. Second, in the case where $V$ is reduced to the compact Cartan subgroup $H$, $\Delta_V$ is empty and (3.2.1) reduces to:

$$\langle \lambda + \rho_K + \beta_1 + \cdots + \beta_l, \gamma \rangle < 0 \text{ for all } \gamma \in \Delta_K^+ .$$

The Identity Theorem is

**3.2.2. THEOREM.** *Suppose that* $\mathbf{E}_\lambda \to D$ *is nondegenerate* (3.2.1) *and that* $c \in H^s(D; \mathcal{E}_\lambda)$, $s = \dim_{\mathbf{C}} Y$, *such that* $c$ *restricts to the zero cohomology class on every fibre* $gY$ *of* $D \to G/K$. *Then* $c = 0$.

The proof will yield the complementary result

**3.2.3. THEOREM.** *If* $\mathbf{E}_\lambda \to D$ *is nondegenerate* (3.2.1) *then* $H^q(D; \mathcal{E}_\lambda) = 0$ *for* $q \neq s$.

One important and immediate consequence of the Identity Theorem is

**3.2.4. THEOREM.** *If* $\mathbf{E}_\lambda \to D$ *is nondegenerate* (3.2.1), *then* $H^s(D; \mathcal{E}_\lambda)$ *is a Fréchet space on which the natural action of* $G$ *is a continuous representation.*

This is just a matter of proving Fréchet, and that follows from

**3.2.5.** LEMMA. *The maps* $\bar\partial\colon \mathcal{E}^{0,q-1}(D; \mathbf{E}_\lambda) \to \mathcal{E}^{0,q}(D; \mathbf{E}_\lambda)$ *have closed range.*

*Proof* (W. Schmid). Let $\{\bar\partial\varphi_n\} \to \omega$ in $\mathcal{E}^{0,q}(D; \mathbf{E}_\lambda)$. As $\bar\partial$ is continuous, $\bar\partial\omega = 0$, so $\omega$ represents a cohomology class $c \in H^q(D; \mathcal{E}_\lambda)$. We must show $c = 0$; then $\omega$ will be in the range of $\bar\partial$.

If $q \neq s$ then $c = 0$ by Theorem 3.2.3.

Let $q = s$. Let $g \in G$ so $gY$ is a fibre of $D \to G/K$. As $Y$ is compact, $\bar\partial\colon \mathcal{E}^{0,q-1}(gY; \mathbf{E}_\lambda) \to \mathcal{E}^{0,q}(gY; \mathbf{E}_\lambda)$ has closed range. The topologies being compatible, now $\{\bar\partial\varphi_n|_{gY}\} \to \omega|_{gY}$ shows $\omega|_{gY}$ to be $\bar\partial$-exact, so $c|_{gY} = 0$. Thus the Identity Theorem says $c = 0$. Q.E.D.

Lemma 3.2.5 allows us to apply Serre duality. Writing $\mathbf{K}_D \to D$ for the canonical bundle, and then glancing back at Theorem 3.2.3, we conclude (letting $H_c^*(\ )$ denote cohomology with compact supports)

**3.2.6.** COROLLARY. *Let* $\mathbf{E}_\lambda \to D$ *be nondegenerate,* $n = \dim_{\mathbf{C}} D$ *and* $s = \dim_{\mathbf{C}} Y$. *Then* $H_c^{n-q}(D; \mathcal{E}_\lambda^* \otimes \mathcal{K}_D) = 0$ *for* $q \neq s$ *and* $H_c^{n-s}(D; \mathcal{E}_\lambda^* \otimes \mathcal{K}_D)$ *is the strong dual of* $H^s(D; \mathcal{E}_\lambda)$.

We close Section 3.2 with some comments on the existence of vector bundles satisfying the nondegeneracy condition (3.2.1).

First we note that nondegenerate bundles $\mathbf{E}_\lambda \to D$ always exist. At the end of Section 3.1 we saw that

$$\langle 2\rho_V - \rho_K, \alpha\rangle > 0 \quad \text{for} \quad \alpha \in \langle\Phi\rangle$$

and

$$\langle 2\rho_V - \rho_K, \gamma\rangle < 0 \quad \text{for} \quad \gamma \in \Delta_K^+ \backslash \langle\Phi\rangle\ .$$

It follows that

$$\lambda = a\{(b+1)\rho_V - b\rho_K\},\ a \text{ and } b \text{ positive integers, } a \gg 0\ ,$$

satisfies the nondegeneracy conditions (3.2.1). More generally, to construct $\lambda$ satisfying (3.2.1) in terms of the integers $n_i = 2\langle\lambda, \varphi_i\rangle/\langle\varphi_i, \varphi_i\rangle$, we first choose integers

$$n_i \geqq \max\left\{\frac{-2\langle\beta_1 + \cdots + \beta_t, \varphi_i\rangle}{\langle\varphi_i, \varphi_i\rangle}\colon \beta_1, \cdots, \beta_t \in \Delta_s^+ \text{ are distinct}\right\}$$

for $1 \leqq i \leqq r$, and then choose $n_{r+1}, \cdots, n_l$ sufficiently large negative so that the $\langle\lambda + \beta_1 + \cdots + \beta_t, \gamma\rangle < \langle -\rho_K, \gamma\rangle$ for all $\gamma \in \Delta_K^+ \backslash \langle\Phi\rangle$.

Second, we note that a homogeneous holomorphic line bundle can satisfy (3.2.1) only under special conditions:

**3.2.7.** PROPOSITION. *If G is simple, and if a homogeneous holomorphic* *line bundle* $\mathbf{L}_\lambda \to D$ *is nondegenerate* (3.2.1), *then either* $V = H$ *and* $P$ *is a Borel subgroup of* $G_{\mathbf{C}}$, *or* $V = K$ *and* $D$ *is a Hermitian symmetric space* $G/K$ *of noncompact type.*

*Proof.* Let $\mathbf{L}_\lambda \to D$ be a homogeneous holomorphic line bundle. Then the representation $\mu_\lambda$ of $V$ with highest weight $\lambda$ must have degree 1. This says $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Delta_V$.

Suppose $H \neq V \neq K$. Then $\Delta_V$ is non-empty and $\sum_{\Delta_K^+ \setminus \Delta_V^+} \mathfrak{g}_{\mathbf{C}}^{\gamma}$ contains a nontrivial irreducible $V$-module $E$. Let $\beta \in \Delta_K^+ \setminus \langle \Phi \rangle$ denote the lowest weight of $V$ on $E$. Then $\langle \beta, \alpha \rangle < 0$ for some simple root $\alpha$ of $\mathfrak{v}_{\mathbf{C}}$, and we calculate

$$\frac{2\langle \lambda + \rho_V + \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \rho_V, \alpha \rangle}{\langle \alpha, \alpha \rangle} + \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 1 + \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \leqq 0 .$$

This contradicts (3.2.1).

Now either $V = H$ or $V = K$. If $V = H$ then $P$ is reduced to its Borel subgroup because $P$ has reductive part $V_{\mathbf{C}}$. If $V = K$, then $G/K = D$, so $G/K$ has invariant complex structure and thus $D$ is the Hermitian symmetric space $G/K$. Q.E.D.

In the Borel case $V = H$, one obtains nondegenerate line bundles $\mathbf{L}_\lambda \to D$ by taking $\lambda$ to be a large negative multiple of $\rho_K$. In the Hermitian symmetric case $V = K$ one obtains nondegenerate line bundles $\mathbf{L}_\lambda \to D$ by taking $\lambda$ to be a large positive multiple of $\rho_K$.

Of course if $G = G_1 \times \cdots \times G_t$ with $G_i$ simple, then $D = D_1 \times \cdots \times D_t$ accordingly, and the homogeneous holomorphic line bundles $\mathbf{L}_\lambda \to D$ are just tensor products of line bundles $\mathbf{L}_{\lambda_i} \to D_i$ with $\lambda = \lambda_1 + \cdots + \lambda_t$. $\mathbf{L} \to D$ is nondegenerate just when all the $\mathbf{L}_{\lambda_i} \to D_i$ are nondegenerate, so Proposition 3.2.7 settles the question of whether $D$ admits nondegenerate homogeneous holomorphic line bundles.

## 3.3. Proof of the Identity Theorem

The Identity Theorem was proved by Wilfried Schmid [28] in the case where $V$ is reduced to the compact Cartan subgroup $H$. Our proof is a combination of Schmid's arguments with a Leray spectral sequence argument. The spectral sequence comes from a holomorphic projection

(3.3.1) $$f: \tilde{X} = G_{\mathbf{C}}/B \longrightarrow G_{\mathbf{C}}/P = X \text{ by } f(gB) = gP$$

where $B$ is the Borel subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} + \sum_{\Delta^+} \mathfrak{g}_{\mathbf{C}}^{-\alpha}$. Take $\tilde{x}_0 = 1 \cdot B \in \tilde{X}$ as base point sitting over $x_0 = 1 \cdot P \in X$ and denote

$$\tilde{D} = G(\tilde{x}_0) \cong G/H, \text{ open } G\text{-orbit in } \tilde{X} ,$$

and

$$\tilde{Y} = K(\tilde{x}_0) \cong K/H, \text{ maximal compact subvariety of } \tilde{D}.$$

Then $f$ restricts to holomorphic fibrations,

$$f\colon \tilde{D} \longrightarrow D \text{ and } \tilde{Y} \longrightarrow Y \text{ with fibre } V/H. \tag{3.3.2}$$

We apply Schmid's arguments to $\tilde{D}$ and carry the results down to $D$ by the Leray sequence. The nondegeneracy hypothesis (3.2.1) combines with the Bott-Borel-Weil theorem to collapse the spectral sequence.

The holomorphic normal bundles to $Y$ in $D$ and to $\tilde{Y}$ in $\tilde{D}$ are the $K_{\mathbf{C}}$-homogeneous holomorphic vector bundles

$$\begin{cases} \mathbf{N} \longrightarrow Y \text{ and } \tilde{\mathbf{N}} \longrightarrow \tilde{Y} \text{ associated to the adjoint} \\ \text{actions of } K_{\mathbf{C}} \cap P \text{ and of } K_{\mathbf{C}} \cap B \text{ on} \\ N = \sum_{\Delta_S^+} \mathfrak{g}_{\mathbf{C}}^{\beta}. \end{cases} \tag{3.3.3}$$

Notice that $\tilde{\mathbf{N}} = f^*\mathbf{N}$ where $f$ is given by (3.3.2).

**3.3.4.** Lemma. *If $\mathbf{E}_\lambda \to D$ is nondegenerate* (3.2.1), $l \geqq 0$ *and* $q \neq \dim_{\mathbf{C}} Y$, *then* $H^q(Y; \mathcal{E}_\lambda \otimes \Lambda^l \mathfrak{N}) = 0$.

*Proof.* Let $\mathbf{L}_\lambda \to \tilde{Y}$ denote the homogeneous holomorphic line bundle for the character $e^\lambda$ on $K_{\mathbf{C}} \cap B$ and write $\mathbf{L}_{\lambda,l}$ for $\mathbf{L}_\lambda \otimes \Lambda^l \tilde{\mathbf{N}}$. As $K_{\mathbf{C}} \cap B$ is solvable, it is triangulable on the fibre of $\mathbf{L}_{\lambda,l}$. That gives a sequence

$$0 = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_r = \mathbf{L}_{\lambda,l}$$

of $K_{\mathbf{C}}$-homogeneous holomorphic vector bundles over $\tilde{Y}$, such that each $\mathbf{F}_i/\mathbf{F}_{i-1}$ is a homogeneous line bundle $\mathbf{L}_{\lambda_i}$ for a character $e^{\lambda_i}$ on $K_{\mathbf{C}} \cap B$. Here $\lambda_i$ is of the form $\lambda + \beta_1 + \cdots + \beta_l$ where the $\beta_j \in \Delta_S^+$ are distinct. So (3.2.1) ensures

$$\langle \lambda_i + \rho_K, \gamma \rangle < 0 \text{ for all } \gamma \in \Delta_K^+ \backslash \langle \Phi \rangle.$$

Now the Bott-Borel-Weil theorem (3.1.17) says

$$H^q(\tilde{Y}; \mathcal{L}_{\lambda_i}) = 0 \text{ for } q < |\Delta_K^+ \backslash \langle \Phi \rangle| = \dim_{\mathbf{C}} Y.$$

In consequence, the exact sequences

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{L}_{\lambda_i} \longrightarrow 0$$

give us

$$H^q(\tilde{Y}; \mathcal{F}_{i-1}) \cong H^q(\tilde{Y}; \mathcal{F}_i) \text{ for } q < \dim_{\mathbf{C}} Y.$$

Iterating this and using $\mathcal{F}_1 = \mathcal{L}_{\lambda_1}$, we conclude

$$H^q(\tilde{Y}; \mathcal{L}_{\lambda,l}) = 0 \text{ for } q < \dim_{\mathbf{C}} Y. \tag{3.3.5}$$

In case $V = H$, that completes the proof of the lemma.

The Leray spectral sequence (Godement [12]) is based on some direct image sheaves $f_*^p(\mathfrak{L}_{\lambda,l}) \to Y$ constructed as follows. If $p$ is an integer $\geqq 0$, then $f_*^p(\mathfrak{L}_{\lambda,l})$ is associated to the presheaf that assigns the abelian group $H^p(\widetilde{Y} \cap f^{-1}U; \mathfrak{L}_{\lambda,l})$ to an open set $U \subset Y$. As $f\colon \widetilde{Y} \to Y$ is a holomorphic fibre bundle, $f_*^p(\mathfrak{L}_{\lambda,l})$ is the sheaf of germs of holomorphic sections of the bundle over $Y$ with fibre $H^p(\widetilde{Y} \cap \pi^{-1}x_0; \mathfrak{L}_{\lambda,l})$. Restrict the bundles $\mathbf{F}_i$ and $\mathbf{L}_{\lambda_i}$ of the proof of (3.3.5) to $\widetilde{Y} \cap f^{-1}(x_0) \cong V_{\mathbf{C}}/B \cap V_{\mathbf{C}}$, without change of notation. From (3.2.1), each

$$\langle \lambda_i + \rho_V, \alpha \rangle > 0 \quad \text{for all } \alpha \in \langle \Phi \rangle .$$

Now the Bott-Borel-Weil theorem says

$$H^p(\widetilde{Y} \cap f^{-1}x_0; \mathfrak{L}_{\lambda_i}) = 0 \quad \text{for } p > 0$$

and $H^0(\widetilde{Y} \cap f^{-1}x_0; \mathfrak{L}_{\lambda_i})$ is the $V_{\mathbf{C}}$-module $E_{\lambda_i}$ of highest weight $\lambda_i$. Using $\widetilde{\mathbf{N}} = f^*\mathbf{N}$ we conclude

$$f_*^p(\mathfrak{L}_{\lambda,l}) = 0 \quad \text{for } p > 0$$

(3.3.6)

and

$$f_*^0(\mathfrak{L}_{\lambda,l}) = \mathcal{E}_\lambda \otimes \Lambda^l \mathfrak{N} .$$

In view of (3.3.6), the Leray spectral sequence collapses and gives us

$$H^q(\widetilde{Y}; \mathfrak{L}_{\lambda,l}) \cong H^q(Y; \mathcal{E}_\lambda \otimes \Lambda^l \mathfrak{N}) .$$

That combines with (3.3.5) to give the statement of the lemma. Q.E.D.

For every root $\alpha \in \Delta$ choose nonzero $e_\alpha \in \mathfrak{g}_{\mathbf{C}}^\alpha$ and define $h_\alpha \in \mathfrak{h}_{\mathbf{C}}$ by $\langle h_\alpha, h \rangle = \alpha(h)$ for all $h \in \mathfrak{h}_{\mathbf{C}}$. Then constants $n_{\alpha,\beta}$ are defined by

$$[e_\alpha, e_\beta] = n_{\alpha,\beta} e_{\alpha+\beta} \quad \text{if } \alpha, \beta, \alpha + \beta \in \Delta .$$

It is standard that the $e_\alpha$ may be chosen so that

$$\text{(3.3.7)} \qquad \begin{cases} \langle e_\alpha, e_\beta \rangle = \delta_{\alpha,-\beta} \text{ and } [e_\alpha, e_{-\alpha}] = h_\alpha; \\ n_{\alpha,\beta} \text{ is real and } n_{-\alpha,-\beta} = -n_{\alpha,\beta} \text{ for } \alpha, \beta, \alpha + \beta \in \Delta; \\ n_{\alpha,\beta} = n_{\beta,\gamma} = n_{\gamma,\alpha} \text{ for } \alpha, \beta, \gamma \in \Delta \text{ with } \alpha + \beta + \gamma = 0; \\ \tau e_\alpha = -e_{-\alpha} \text{ where } \tau \text{ is conjugation of } \mathfrak{g}_{\mathbf{C}} \text{ over } \mathfrak{g}_u . \end{cases}$$

Then the conjugation $\sigma$ of $\mathfrak{g}_{\mathbf{C}}$ over $\mathfrak{g}$, and the Cartan involution $\theta = \sigma\tau = \tau\sigma$, satisfy

$$\sigma(e_\alpha) = \varepsilon_\alpha e_{-\alpha} = -\theta(e_\alpha)$$

where $\varepsilon_\alpha = 1$ for $\alpha \in \Delta_S$, $-1$ for $\alpha \in \Delta_K$; and thus also

$$\varepsilon_\alpha \varepsilon_\beta \varepsilon_\gamma = -1 \quad \text{whenever } \alpha, \beta, \gamma \in \Delta \text{ with } \alpha + \beta + \gamma = 0 .$$

The dual complex-valued linear differential forms on $G$ are given by

$$\omega^\alpha(e_\beta) = \delta_\beta^\alpha \quad \text{and} \quad \omega^\alpha(\mathfrak{h}_{\mathbf{C}}) = 0 .$$

Calculating

$$d\omega^{\alpha}(e_{\beta}, e_{\gamma}) = \frac{1}{2}\{e_{\beta}\cdot\omega^{\alpha}(e_{\gamma}) - e_{\gamma}\cdot\omega^{\alpha}(e_{\beta}) - \omega^{\alpha}[e_{\beta}, e_{\gamma}]\}$$

one has the Maurer-Cartan equations

$$d\omega^{\alpha} = -\frac{1}{2}\sum\nolimits_{\beta+\gamma=\alpha} n_{\beta,\gamma}\omega^{\beta}\wedge\omega^{\gamma} + \text{terms involving } \mathfrak{h}^* .$$

Let $\omega$ belong to the space $\mathcal{E}^{0,q}(D; \mathbf{E}_{\lambda})$ of $\mathbf{E}_{\lambda}$-valued $(0, q)$-forms on $D$. Then $\omega$ has lift to $G$ of the form

$$\tilde{\omega} = \sum\nolimits_{|B|=q} f_B\omega^{-B} ,$$

$$B = \{\beta_1, \cdots, \beta_q\} \subset \Delta^+\backslash\langle\Phi\rangle,\ \omega^{-B} = \omega^{-\beta_1}\wedge\cdots\wedge\omega^{-\beta_q}$$

where the $f_B: G \to E_{\lambda}$ represent sections of $\mathbf{E}_{\lambda}$. Then the Maurer-Cartan equations above give

$$\begin{aligned}(\bar\partial\omega)^{\sim} &= \sum\nolimits_{\alpha\in\Delta^+\backslash\langle\Phi\rangle}\sum\nolimits_{|B|=q} e_{-\alpha}(f_B)\omega^{-\alpha}\wedge\omega^{-B} \\ &\quad + \frac{1}{2}\sum\nolimits_{1\leqq i\leqq q}\sum\nolimits_{\substack{\alpha,\beta\in\Delta^+\backslash\langle\Phi\rangle\\ \alpha+\beta=\beta_i}}\sum\nolimits_{|B|=q}(-1)^{i-1}n_{\alpha,\beta}f_B\omega^{-\alpha}\wedge\omega^{-\beta}\wedge\omega^{-(B\backslash\beta_i)} .\end{aligned} \tag{3.3.8}$$

We will need this for a calculation below.

The typical fibre $N = \sum_{\Delta_S^+}\mathfrak{g}^{\alpha}$ of the holomorphic normal bundle $\mathbf{N}\to Y$ carries an $\mathrm{Ad}(V)$-invariant positive definite Hermitian inner product $(e, f) = -\langle e, \tau f\rangle$. If $\Delta_S^+ = \{\alpha_1, \cdots, \alpha_t\}$ and $e_i = e_{\alpha_i}\in\mathfrak{g}_{\mathbf{C}}^{\alpha_i}$ as above, then (3.3.7) shows that $\{e_1, \cdots, e_t\}$ is an orthonormal basis of $N$. Thus $\Lambda^l(N)$ has orthonormal basis consisting of the $e_I = e_{i_1}\wedge\cdots\wedge e_{i_l}$ where $1\leqq i_1 < \cdots < i_l \leqq t$. Now we have $K$-invariant Hermitian metrics on the $\Lambda^l\mathbf{N}\to Y$, and the $K$-invariant tensor fields

$$h^l = \sum\nolimits_{|I|=l}\bar{e}_I\otimes e_I \tag{3.3.8a}$$

combine with those metrics to give the complex conjugation operations $\Lambda^l\mathbf{N}\to\Lambda^l\bar{\mathbf{N}}$.

Let $\mathcal{E}^{0,q}_{(r)}(D; \mathbf{E}_{\lambda})$ denote the space of all $\mathbf{E}_{\lambda}$-valued $(0, q)$-forms on $D$ that vanish to order $\geqq r$ on the fibres of $D\to G/K$. Thus $\mathcal{E}^{0,q}_{(r)}(D; \mathbf{E}_{\lambda})$ consists of all $\omega\in\mathcal{E}^{0,q}(D; \mathbf{E}_{\lambda})$ whose lift to $G$ has the form

$$\tilde{\omega} = \sum f_B\omega^{-B} \text{ where } f_B = 0 \text{ whenever } |B\cap\Delta_S^+| < r .$$

If $\alpha, \beta\in\Delta^+$ with $\alpha+\beta\in\Delta_S^+$, then one of $\alpha, \beta$ must be noncompact, so (3.3.8) gives us

$$\bar\partial\mathcal{E}^{0,q}_{(r)}(D; \mathbf{E}_{\lambda}) \subset \mathcal{E}^{0,q+1}_{(r)}(D; \mathbf{E}_{\lambda}) . \tag{3.3.9}$$

Recall the $K$-invariant sections $h^l$ of $\Lambda^l\bar{\mathbf{N}}\otimes\Lambda^l\mathbf{N}\to Y$ given by (3.3.8a) and define

$$\mathcal{R}_r\colon \mathcal{E}^{0,q}_{(r)}(D;\mathbf{E}_\lambda)\longrightarrow \mathcal{E}^{0,q-r}(Y;\mathbf{E}_\lambda\otimes \textstyle\bigwedge^r\mathbf{N})$$

by restriction to $Y$ and contraction with $h^r$. In other words, if $\xi_1,\cdots,\xi_{q-r}$ are tangent to $Y$ at a point $x\in Y$,

(3.3.10) $$(\mathcal{R}_r\omega)_x(\xi_1,\cdots,\xi_{q-r})=\textstyle\sum_{|I|=r}\omega_x(\bar e_I;\xi_1,\cdots,\xi_{q-r})\otimes e_I\ .$$

Comparing definitions we have

(3.3.11) $$\mathcal{E}^{0,q}_{(r+1)}(D;\mathbf{E}_\lambda)=\{\omega\in\mathcal{E}^{0,q}_{(r)}(D;\mathbf{E}_\lambda)\colon \mathcal{R}_r(g^*\omega)=0 \text{ for all } g\in G\}\ .$$

In particular, $\mathcal{E}^{0,q}_{(r+1)}$ is in the kernel of $\mathcal{R}_r$.

**3.3.12.** LEMMA (W. Schmid [28]). *If $\omega\in\mathcal{E}^{0,q}_{(r)}(D;\mathbf{E}_\lambda)$, then $\mathcal{R}_r(\bar\partial\omega)$ is well defined by* (3.3.9), *and*

$$\bar\partial\mathcal{R}_r(\omega)=(-1)^r\mathcal{R}_r(\bar\partial\omega)\ .$$

*Proof.* Split $\omega=\psi+\varphi$ where $\varphi\in\mathcal{E}^{0,q}_{(r+1)}(D;\mathbf{E}_\lambda)$ and $\psi$ lifts to $G$ as

$$\tilde\psi=\textstyle\sum f_{BC}\,\omega^{-B}\wedge\omega^{-C}\ ,$$
$$B=\{\beta_1,\cdots,\beta_r\}\subset\Delta_S^+\ ,$$
$$C=\{\gamma_1,\cdots,\gamma_{q-r}\}\subset\Delta_K^+\ .$$

Now $(\mathcal{R}_r\psi)^\sim=\sum f_{BC}\,\omega^{-C}\otimes e_B$ and so, using (3.3.8) restricted to $K$, and calculating $e_{-\alpha}(f_{BC}e_B)$ by the product rule, we have

(3.3.13) $$\begin{aligned}(\bar\partial\mathcal{R}_r\psi)^\sim&=\textstyle\sum_{\alpha\in\Delta_K^+\setminus\langle\Phi\rangle}\sum_{B,C}e_{-\alpha}(f_{BC})\omega^{-\alpha}\wedge\omega^{-C}\otimes e_B\\&+\frac12\textstyle\sum_{1\le i\le q-r}\sum_{\substack{\alpha_1,\alpha_2\in\Delta_K^+\setminus\langle\Phi\rangle\\ \alpha_1+\alpha_2=\gamma_i\\ B,C}}(-1)^{i-1}n_{\alpha_1,\alpha_2}f_{BC}\,\omega^{-\alpha_1}\wedge\omega^{-\alpha_2}\wedge\omega^{-(C\setminus\gamma_i)}\otimes e_B\\&+\textstyle\sum_{\alpha\in\Delta_K^+\setminus\langle\Phi\rangle}\sum_{B,C}f_{BC}\,\omega^{-\alpha}\wedge\omega^{-C}\otimes\bigwedge^r\big(\mathrm{ad}(e_{-\alpha})\big)\cdot e_B\ .\end{aligned}$$

Since

$$\textstyle\bigwedge^r\big(\mathrm{ad}(e_{-\alpha})\big)\cdot e_B=\sum_{i=1}^r e_{\beta_1}\wedge\cdots\wedge e_{\beta_{i-1}}\wedge[e_{-\alpha},e_{\beta_i}]\wedge e_{\beta_{i+1}}\wedge\cdots\wedge e_r$$

the third term of the right hand side of (3.3.13) is

$$\textstyle\sum_{1\le i\le r}\sum_{\substack{\alpha\in\Delta_K^+\setminus\langle\Phi\rangle\\ \beta_i-\alpha\in\Delta_K^+}}\sum_{B,C}(-1)^{i-1}n_{-\alpha,\beta_i}f_{BC}\,\omega^{-\alpha}\wedge\omega^{-C}\otimes e_{\beta_i-\alpha}\wedge e_{B\setminus\beta_i}\ .$$

Similarly

(3.3.14) $$\begin{aligned}(-1)^r(\bar\partial\psi)^\sim&=\textstyle\sum_{\alpha\in\Delta_K^+\setminus\langle\Phi\rangle}\sum_{B,C}e_{-\alpha}(f_{BC})\omega^{-B}\wedge\omega^{-\alpha}\wedge\omega^{-C}\\&+\frac12\textstyle\sum_{1\le i\le q-r}\sum_{\substack{\alpha_1,\alpha_2\in\Delta_K^+\setminus\langle\Phi\rangle\\ \alpha_1+\alpha_2=\gamma_i\\ B,C}}(-1)^{i-1}n_{\alpha_1,\alpha_2}f_{BC}\,\omega^{-B}\wedge\omega^{-\alpha_1}\wedge\omega^{-\alpha_2}\wedge\omega^{-(C\setminus\gamma_i)}\\&+\frac12\textstyle\sum_{1\le i\le r}\sum_{\substack{\alpha_1,\alpha_2\in\Delta^+\setminus\langle\Phi\rangle\\ \alpha_1+\alpha_2=\beta_i\\ B,C}}(-1)^{r+i-1}n_{\alpha_1,\alpha_2}f_{BC}\,\omega^{-\alpha_1}\wedge\omega^{-\alpha_2}\wedge\omega^{-(B\setminus\beta_i)}\wedge\omega^{-C}\\&+\text{lifts of terms in }\mathcal{E}^{0,q+1}_{(r+1)}(D;\mathbf{E}_\lambda)\ .\end{aligned}$$

In the third term of the right hand side of (3.3.14), $\alpha_1 + \alpha_2 = \beta_i \in \Delta_S^+$ forces one of the $\alpha_1, \alpha_2$ to be noncompact and the other to be compact. So the third term becomes

$$\sum_{1\leq i\leq r} \sum_{\substack{\alpha\in\Delta_K^+\setminus\langle\Phi\rangle \\ \beta_i-\alpha\in\Delta_S^+}} \sum_{B,C} (-1)^{i-1} n_{\alpha,\beta_i-\alpha} f_{BC}\omega^{-\beta_i+\alpha} \wedge \omega^{-(B\setminus\beta_i)} \wedge \omega^{-\alpha} \wedge \omega^{-C} .$$

As $n_{\alpha,\beta_i-\alpha} = n_{-\alpha,\beta_i}$ from (3.3.7), we conclude $\bar\partial\mathcal{R}_r(\psi) = (-1)^r\mathcal{R}_r(\bar\partial\psi)$. Since $\omega - \psi \in \mathcal{E}^{0,q}_{(r+1)}(D; \mathbf{E}_\lambda)$, the lemma follows from (3.3.9) and (3.3.11). Q.E.D.

**3.3.15.** LEMMA (W. Schmid [28]). *Let* $\omega \in \mathcal{E}^{0,q}_{(r)}(D; \mathbf{E}_\lambda)$ *be such that* $\bar\partial\omega \in \mathcal{E}^{0,q+1}_{(r+1)}(D; \mathbf{E}_\lambda)$, *so* (3.3.11) *and Lemma* 3.3.12 *tell us that* $\bar\partial\mathcal{R}_r(g^*\omega) = 0$ *for all* $g \in G$. *Suppose further that every* $\mathcal{R}_r(g^*\omega)$ *is* $\bar\partial$-*exact. Then there exists* $\psi \in \mathcal{E}^{0,q-1}_{(r)}(D; \mathbf{E}_\lambda)$ *with* $\omega - \bar\partial\psi \in \mathcal{E}^{0,q}_{(r+1)}(D; \mathbf{E}_\lambda)$.

*Proof.* Every $g \in G$ has unique factorization $\exp(\xi)k$, $\xi \in \mathfrak{s}$ and $k \in K$, and $\exp(\xi)K \mapsto \exp(\xi)$ is a $C^\infty$ section to $G \to G/K$. Now

(3.3.15′) $\quad G/K \longrightarrow \mathcal{E}^{0,q-r}(Y; \mathbf{E}_\lambda \otimes \Lambda^r\mathbf{N})$ by $\exp(\xi)K \longmapsto \mathcal{R}_r(\exp(\xi)^*\omega)$

is a $C^\infty$ map from $G/K$ to the $\bar\partial$-exact forms in the Fréchet space $\mathcal{E}^{0,q-r}(Y; \mathbf{E}_\lambda \otimes \Lambda^r\mathbf{N})$. As $Y$ is compact, $\bar\partial$ has closed range [41], so

$$0 \longrightarrow \{\psi \in \mathcal{E}^{0,q-r-1}\colon \bar\partial\psi = 0\} \longrightarrow \mathcal{E}^{0,q-r-1} \longrightarrow \bar\partial\mathcal{E}^{0,q-r-1} \longrightarrow 0$$

is an exact sequence of Fréchet spaces. Thus [1] the $C^\infty$ map (3.3.15′) lifts to a $C^\infty$ map

$G/K \longrightarrow \mathcal{E}^{0,q-r-1}(Y; \mathbf{E}_\lambda \otimes \Lambda^r\mathbf{N})$ by $\exp(\xi)K \longmapsto \varphi_\xi$ with $\bar\partial\varphi_\xi = \mathcal{R}_r(\exp(\xi)^*\omega)$.

That gives $\psi \in \mathcal{E}^{0,q-1}_{(r)}(D; \mathbf{E}_\lambda)$ with $\varphi_\xi = (-1)^r\mathcal{R}_r(\exp(\xi)^*\psi)$. If $g \in G$, say $g = pk$ with $p = \exp(\xi)$, $\xi \in \mathfrak{s}$, and $k \in K$, then

$$\mathcal{R}_r(g^*(\omega - \bar\partial\psi)) = k^*\{\mathcal{R}_r(p^*\omega) - (-1)^r\bar\partial\,\mathcal{R}_r(p^*\psi)\} = 0 .$$

Now $\omega - \bar\partial\psi \in \mathcal{E}^{0,q}_{(r+1)}(D; \mathbf{E}_\lambda)$ by (3.3.11). Q.E.D.

Now we prove the Identity Theorem and Theorem 3.2.3 in their Dolbeault cohomology formulation.

**3.3.16.** THEOREM. *Let* $\mathbf{E}_\lambda \to D$ *be nondegenerate* (3.2.1) *and let* $\omega \in \mathcal{E}^{0,q}(D; \mathbf{E}_\lambda)$ *be* $\bar\partial$-*closed.*

1. *If* $q \neq s = \dim_{\mathbf{C}} Y$ *then* $\omega$ *is* $\bar\partial$-*exact.*
2. *If* $q = s$ *and if every* $\omega|_{gY}$ *is* $\bar\partial$-*exact, then* $\omega$ *is* $\bar\partial$-*exact.*

*Proof* (Schmid [28]). $D$ is $(s + 1)$-complete by Theorem 2.5.9, so $H^q(D; \mathcal{E}_\lambda) = 0$ for $q > s$, proving $\omega$ is $\bar\partial$-exact if $q > s$, [1].

Now assume $q \leq s$. Each $\mathcal{R}_0(g^*\omega) = \omega|_{gY} \in \mathcal{E}^{0,q}(Y; \mathbf{E}_\lambda)$ is $\bar\partial$-exact by hypothesis if $q = s$, by Lemma 3.3.4 if $q < s$. Now Lemma 3.3.15 provides

$$\psi_1 \in \mathcal{E}^{0,q-1}(D; \mathbf{E}_\lambda) \text{ such that } \omega - \bar{\partial}\psi_1 \in \mathcal{E}^{0,q}_{(1)}(D; \mathbf{E}_\lambda) .$$

$\omega - \bar{\partial}\psi_1$ is $\bar{\partial}$-closed, and each $\mathcal{R}_1(g^*(\omega - \bar{\partial}\psi_1)) \in \mathcal{E}^{0,q-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{N})$ is $\bar{\partial}$-exact by Lemma 3.3.4, so Lemma 3.3.15 provides

$$\psi_2 \in \mathcal{E}^{0,q-1}_{(1)}(D; \mathbf{E}_\lambda) \text{ such that } \omega - \bar{\partial}(\psi_1 + \psi_2) \in \mathcal{E}^{0,q}_{(2)}(D; \mathbf{E}_\lambda) .$$

Continuing, we obtain $\psi_j \in \mathcal{E}^{0,q-1}_{(j-1)}$, $3 \leqq j \leqq q$, such that $\omega - \bar{\partial}(\psi_1 + \cdots + \psi_q) \in \mathcal{E}^{0,q}_{(q)}(D; \mathbf{E}_\lambda)$. Every $\bar{\partial}$-closed form in $\mathcal{E}^{0,q}_{(q)}(D; \mathbf{E}_\lambda)$ is zero because $H^0(Y; \mathbf{E}_\lambda \otimes \bigwedge^q \mathbf{N}) = 0$, so now $\omega = \bar{\partial}(\psi_1 + \cdots + \psi_q)$. Q.E.D.

## 3.4. The Main Cohomology Representation Theorem

Let $D$ be an open orbit of a complex flag manifold $X$, and let $\mathbf{E}_\lambda \to D$ be a homogeneous vector bundle over $D$, using the same notation as in the previous sections of this chapter. In the classical case where, say, $D$ is a bounded symmetric domain, and $\mathbf{E}_\lambda$ is the canonical bundle, $H^0(D, \mathcal{E}_{m\lambda}) \cong \Gamma(D, \mathcal{O})$ is an infinite dimensional Fréchet space. Also, if $\Gamma$ is a discrete subgroup of $G$ acting on $D$, then the *invariant sections*, which we denote by $H^0_\Gamma(D, \mathcal{E}_{m\lambda})$, are the classical *automorphic forms on $D$ of weight $m$* (cf. Borel [8]; in the one-dimensional case one must add an additional growth condition), and are a finite dimensional subspace of $H^0(D, \mathcal{E}_{m\lambda})$ when $\Gamma$ is an arithmetic subgroup of $G$. The vanishing theorem of Schmid (Theorem 3.3.16) shows that in the general (non-Hermitian) case when $D$ has nontrivial compact complex fibres of dimension $s > 0$, the vector space $H^0(D, \mathcal{E}_\lambda) = 0$ for nondegenerate holomorphic vector bundles, and thus $H^0_\Gamma(D, \mathcal{E}_\lambda) = 0$, and there are no classical automorphic forms. However, Theorem 3.2.4 tells us that $H^s(D, \mathcal{E}_\lambda)$ is a Fréchet space, which we shall see later is infinite dimensional (cf. Schmid [28]). Thus we let $H^s_\Gamma(D, \mathcal{E}_\lambda)$ be the $\Gamma$-invariant cohomology classes in $H^s(D, \mathcal{E}_\lambda)$ and, following Griffiths [15], we call $H^s_\Gamma(D, \mathcal{E}_\lambda)$ the vector space of *automorphic cohomology classes* on $D$ (with respect to the particular nondegenerate vector bundle $\mathbf{E}_\lambda \to D$). At present it is unknown whether this vector space of automorphic cohomology classes is finite dimensional or not. Nevertheless, we are able to represent automorphic cohomology classes as sections of an associated homogeneous vector bundle over the parameter space for the fibres, $M_D$, and this we carry out in this section, utilizing the previous results.

Consider the diagram

(3.4.1)
$$\begin{array}{ccc} \mathcal{Y}_D & \xrightarrow{\tau} & D \\ \downarrow{\scriptstyle\pi} & & \\ M = M_D & & \end{array}$$

as in (2.5.2). Let $\mathbf{E}_\lambda \to D$ be a homogeneous holomorphic vector bundle over $D$, whose sheaf of holomorphic sections over $D$ is again denoted by $\mathcal{E}_\lambda$. Let $\mathbf{F}_\lambda \to \mathcal{Y}_D$ be the pullback bundle $\mathbf{F}_\lambda = \tau^*\mathbf{E}_\lambda$, and let $\mathcal{F}_\lambda$ be the corresponding sheaf of holomorphic sections. There is a natural mapping

$$\tau^*\colon H^q(D, \mathcal{E}_\lambda) \longrightarrow H^q(\mathcal{Y}_D, \mathcal{F}_\lambda)\ , \tag{3.4.2}$$

which can, for example, be represented by the pullback of $\bar\partial$-closed $\mathbf{E}_\lambda$-valued $(0, q)$-forms on $D$, by Dolbeault's theorem.

Suppose now that $\mathcal{F}$ is any coherent analytic sheaf on $\mathcal{Y}_D$, then, as in Section 3.3 we let $\pi^p_*\mathcal{F}$ denote the direct image sheaves on $M$. The Leray spectral sequence for this fibration $\pi$ has the form

$$H^q(M, \pi^p_*\mathcal{F}) \Longrightarrow H^r(\mathcal{Y}_D, \mathcal{F})\ . \tag{3.4.3}$$

Let $H^r(\mathcal{Y}_D, \mathcal{F})$ and $H^q(M, \pi^p_*\mathcal{F})$ be equipped with their natural topologies induced by uniform convergence of holomorphic functions on compact subsets.

**3.4.4.** THEOREM. *The topological vector space $H^r(\mathcal{Y}_D, \mathcal{F})$ is a Fréchet space which is topologically isomorphic to $H^0(M, \pi^r_*\mathcal{F})$.*

*Proof.* Since $M$ is Stein, by Theorem 2.5.1, we have by Cartan's Theorem B that $H^r(M, \pi^p_*\mathcal{F}) = 0$ for $r > 0$, since, by Grauert's direct image theorem (Grauert [13]), the direct image sheaves are coherent. Thus the spectral sequence (3.4.3) is completely degenerate and we have algebraically $H^0(M, \pi^q_*\mathcal{F}) \cong H^q(\mathcal{Y}_D, \mathcal{F})$. Moreover, the natural continuous mapping in the spectral sequence giving the above algebraic isomorphism is the edge homomorphism

$$e\colon H^q(\mathcal{Y}_D, \mathcal{F}) \longrightarrow H^0(M, \pi^q_*\mathcal{F})\ . \tag{3.4.5}$$

Since $e^{-1}(0)$ is a closed set, it follows that $H^q(\mathcal{Y}_D, \mathcal{F})$ is Hausdorff and thus a Fréchet space. Moreover, $H^0(M, \pi^q_*\mathcal{F})$ is a Fréchet space, and thus $e$ in (3.4.5) is a bijective continuous mapping of Fréchet spaces. By the open mapping theorem we conclude that $e$ is a topological isomorphism. Q.E.D.

We now return to the pullback mapping (3.4.2). Recall that

$$s = \dim_{\mathbf{C}} K/V = \dim_{\mathbf{C}} Y \subset D.$$

**3.4.6.** THEOREM. *Suppose that $\mathbf{E}_\lambda \to D$ is a non-degenerate homogeneous vector bundle. The mapping*

$$\tau^* H^s(D, \mathcal{E}_\lambda) \longrightarrow H^s(\mathcal{Y}_D, \mathcal{F}_\lambda)$$

*is a topological injection of Fréchet spaces.*

*Proof.* We know by Theorem 3.2.4 that $H^s(D, \mathcal{E}_\lambda)$ is a Fréchet space.

By Theorem 3.4.4, $H^s(\mathcal{Y}_D, \mathcal{F}_\lambda)$ is also Fréchet. We need to show two things: first, that $\tau^*$ is algebraically injective, and, second, that $\tau^*$ has closed range. The fact that $\tau^*$ has closed range follows from the fact that $\mathcal{Y}_D \xrightarrow{\tau} D$ is a holomorphic fibre bundle. One uses a family of semi-norms on $\mathcal{Y}_D$ which are compatible via the projection $\tau$ with a family of semi-norms on $D$, which is simple to construct. Then a sequence of forms $\tau^*\varphi_\nu$ which converge on $\mathcal{Y}_D$ to $\psi$ implies that $\varphi_\nu$ will converge on $D$ to an element $\varphi \in \mathcal{E}^{0,q}(D, \mathbf{E}_\lambda)$, and $\tau^*\varphi = \psi$. Since $\tau^*$ commutes with $\bar\partial$, we see that $\tau^*(H^s(D, \mathcal{E}_\lambda))$ is closed in $H^s(\mathcal{Y}_D, \mathcal{F}_\lambda)$. To see that $\tau^*$ is injective, suppose that we represent $\xi \in H^s(D, \mathcal{E}_\lambda)$ by a $\bar\partial$-closed form $\varphi$ with coefficients in $\mathbf{E}_\lambda$. Then suppose that $\tau^*\xi = 0$ in $H^s(\mathcal{Y}_D, \mathcal{F}_\lambda)$, i.e., $\tau^*\varphi$ is $\bar\partial$-exact. This implies that $\tau^*\varphi$ is $\bar\partial$-exact on the $G$-translates of the principal fibre $Y$, i.e., on the fibres of $D$. Thus we have that $\varphi$ restricted to the fibres of $D$ is $\bar\partial$-exact on those fibres, and hence by Theorem 3.3.16 we find that $\varphi$ is $\bar\partial$-exact on $D$. Thus $\xi = 0$, and injectivity is proved. Q.E.D.

We can now state and easily prove our principal representation theorem.

3.4.7. THEOREM. *Let $\mathbf{E}_\lambda$ be a nondegenerate homogeneous vector bundle over $D$. The composition of the mappings* (3.4.2) *and* (3.4.5) *is a topological injection of Fréchet spaces*

$$\sigma\colon H^s(D, \mathcal{E}_\lambda) \longrightarrow H^0(M, \pi^s_*\mathcal{F}_\lambda) ,$$

*which is equivariant with respect to the action of $G$.*

*Proof.* The first part of the theorem follows immediately from Theorems 3.4.4 and 3.4.6. We merely note that the action of $G$ on $\mathbf{E}_\lambda$ and on the homogeneous space $D = G/V$ induces an action on $H^s(D, \mathcal{E}_\lambda)$. This action is compatible with the mappings $\tau$ and $\pi$ since the elements of $G$ map fibres of $D$ to fibres of $D$ inducing an action on $\mathcal{Y}_D$ and on $M$. We then see that the action of $G$ on $H^s(\mathcal{Y}_D, \mathcal{E}_\lambda)$ induces an action on $H^0(M, \pi^s_*\mathcal{F}_\lambda)$ and thus the theorem is proved. Q.E.D.

*Remark.* The sheaf $\pi^s_*\mathcal{F}_\lambda$ is a locally free sheaf of rank $= \dim_{\mathbf{C}} H^s(Y, \mathbf{E}_\lambda)$ (since $\pi$ is of maximal rank and $\mathcal{F}_\lambda$ is a locally free sheaf on $\mathcal{Y}_D$). So Theorem 3.4.7 gives a representation of cohomology on $D$ (with coefficients in $\mathbf{E}_\lambda$) by sections of a holomorphic vector bundle $\widetilde{\mathbf{E}}_\lambda \to M$ whose sheaf of sections is $\pi^s_*\mathcal{F}_\lambda$. Note that even if $\mathbf{E}_\lambda$ were a homogeneous line bundle on $D$, it would not follow that $\widetilde{\mathbf{E}}_\lambda$ would be a line bundle. Also, we see that the action of $G$ on $M$ induces an action of $G$ on the sections of $\widetilde{\mathbf{E}}_\lambda \to M$. Thus we could say that $\widetilde{\mathbf{E}}_\lambda$ is a homogeneous vector bundle in the general sense. However, $M$ can be homogeneous only in the "Hermitian case":

**3.4.8. PROPOSITION.** *$M$ is homogeneous under a subgroup of $G_{\mathbf{C}}$ if, and only if, $G/K$ is Hermitian symmetric with $D \to G/K$ holomorphic or anti-holomorphic along each irreducible factor.*

*Proof.* The maximal subgroup of $G_{\mathbf{C}}$ preserving $M$ is $Q = \{g \in G_{\mathbf{C}}: g'Y \subset D \Rightarrow gg'Y \subset D\}$, which contains $G$ but cannot contain a simple factor of $G_{\mathbf{C}}$ just when it is homogeneous under $G$, in which case $M \cong G/K$. That is the situation in the "Hermitian case." In the "non-Hermitian case," Proposition 2.3.5 shows $\dim M = \dim_{\mathbf{R}} G_{\mathbf{C}} - \dim_{\mathbf{R}} L > \dim G/K$, so $M$ cannot be homogeneous. Q.E.D.

In the notation of Section 2.3, $M$ is a subset of $G_{\mathbf{C}}/L$, and $\widetilde{\mathbf{E}}_\lambda \to M$ is the restriction of a $G_{\mathbf{C}}$-homogeneous holomorphic bundle $\widetilde{\mathbf{E}}'_\lambda \to G_{\mathbf{C}}/L$. The representation of $L$ involved here is seen from the proof of Theorem 3.4.4, the Bott-Borel-Weil theorem and Proposition 2.3.5.

## 4. Poincaré series and integrability

### 4.1. The Poincaré series of an absolutely integrable cohomology class

In this section we formulate the basic result on convergence of the Poincaré series associated to an absolutely integrable cohomology class $c \in H^s(D; \mathcal{E}_\lambda)$ and a discrete subgroup $\Gamma \subset G$. The convergence follows directly from the results of Sections 2 and 3 and a theorem of Griffiths [15]. Then in the remainder of Section 4 we show, under appropriate conditions on $\lambda$, that $H^s(D; \mathcal{E}_\lambda)$ contains nonzero absolutely integrable classes.

Let $D = G(x_0) \subset X = G_{\mathbf{C}}/P$ as before, so $D \cong G/G \cap P$. From now on we assume that $V = G \cap P$ is compact. Thus we have $G$-invariant Hermitian metrics on (the fibres of) the holomorphic tangent bundle $\mathbf{T}_D \to D$, and more generally on every homogeneous holomorphic vector bundle. This gives us a pointwise inner product on $\mathbf{E}$-valued $(p, q)$-forms and gives us a $G$-invariant volume element on $D$. For every real number $r \geqq 1$ we denote

$$\text{(4.1.1)} \qquad \mathcal{E}_r^{p,q}(D; \mathbf{E}) = \left\{\varphi \in \mathcal{E}^{p,q}: \|\varphi\|_r = \left(\int_D \|\varphi(x)\|^r dx\right)^{1/r} < \infty\right\}^{*)}$$

where $dx$ is the invariant volume element and $\|\varphi(x)\| = \langle\varphi(x), \varphi(x)\rangle^{1/2}$ is the norm associated to the inner product on the fibre of $\mathbf{E} \otimes \bigwedge^p \bar{\mathbf{T}}_D^* \otimes \bigwedge^q \bar{\mathbf{T}}_D^*$ over $x \in D$. We denote

$$\text{(4.1.2)} \qquad L_r^{p,q}(D; \mathbf{E}): \text{Banach space completion of } \mathcal{E}_r^{p,q}(D; \mathbf{E})$$

and observe that $L_2^{p,q}(D; \mathbf{E})$ is a Hilbert space with inner product

*) $\mathcal{E}_r^{p,q}$ should not be confused with $\mathcal{E}_{(r)}^{p,q}$ used in Section 3.3 in the proof of Schmid's Identity Theorem.

$$(\varphi, \psi)_D = \int_D (\varphi, \psi)_x dx = \int_D \varphi \barwedge \bar{*}_{\mathbf{E}} \psi \; . \tag{4.1.3}$$

Here we are using the Hodge-Kodaira orthocomplementation operators

$$\mathbf{E} \otimes \Lambda^p \mathbf{T}_D^* \otimes \Lambda^q \bar{\mathbf{T}}_D^* \underset{\bar{*}_{\mathbf{E}}^{-1}}{\overset{\bar{*}_{\mathbf{E}}}{\rightleftarrows}} \mathbf{E}^* \otimes \Lambda^{n-p} \mathbf{T}_D^* \otimes \Lambda^{n-q} \bar{\mathbf{T}}_D^*$$

for our inner products, and $\barwedge$ is exterior product followed by contraction of $\mathbf{E}$ with $\mathbf{E}^*$ (cf. Wells [41], p. 175). The compactly supported $C^\infty$ $(p, q)$ forms are in each $\mathcal{E}_r^{p,q}$ and dense in $L_r^{p,q}$, so the latter may be viewed as their Banach space completion.

We say that a Dolbeault cohomology class $[\varphi] \in H^{p,q}(D; \mathbf{E})$ is of Lebesgue class $L_r$ if it is represented by a form $\varphi \in L_r^{p,q}(D; \mathbf{E})$, and we denote

$$\begin{cases} H_r^{p,q}(D; \mathbf{E})\text{: the } L_r \text{ classes in } H^{p,q}(D; \mathbf{E}) \text{ and} \\ H_r^q(D; \mathcal{E}) = \{c \in H^q(D; \mathcal{E})\text{: } c \text{ corresponds to a class in } H_r^{0,q}(D; \mathbf{E})\} \; . \end{cases} \tag{4.1.4}$$

Later in this chapter we will give sufficient conditions for the existence of integrable ($L_1$) and square integrable ($L_2$) cohomology classes. We are interested in the integrable classes, but their existence depends on representation-theoretic facts about the square integrable classes.

Let $\Gamma$ be a discrete subgroup of $G$. In other words, since $D \cong G/V$ with $V$ compact, $\Gamma$ is a subgroup of $G$ whose action on $D$ is properly discontinuous:

$$\text{if } Z \subset D \text{ is compact then } \{\gamma \in \Gamma\text{: } \gamma Z \text{ meets } Z\} \text{ is finite} \; .$$

If $c \in H^q(D; \mathcal{E})$ we form the *Poincaré series*

$$\theta(c) = \textstyle\sum_{\gamma \in \Gamma} \gamma^*(c) \; . \tag{4.1.5}$$

If $s = \dim_{\mathbf{C}} Y$ as before, our result on convergence of these series is

**4.1.6.** THEOREM. *Let* $\mathbf{E}_\lambda \to D$ *be a nondegenerate* (3.2.1) *homogeneous holomorphic vector bundle,* $\Gamma$ *a discrete subgroup of* $G$*, and* $c \in H_1^s(D; \mathcal{E}_\lambda)$. *Then the Poincaré series* $\theta(c) = \sum_\Gamma \gamma^*(c)$ *converges, in the Fréchet topology of* $H^s(D; \mathcal{E}_\lambda)$*, to a* $\Gamma$*-invariant class.*

Convergence of the Poincaré series for $L_1$ classes in dimension $s$ was conjectured by Griffiths [15, p. 616] for the case where $\mathbf{E}_\lambda$ is a high power of the canonical bundle. In view of Proposition 3.2.7, a high power of the canonical bundle is nondegenerate only when $P$ is a Borel subgroup of $G_{\mathbf{C}}$, that is, only when $V$ is reduced to $H$. The result that Griffiths proves in [15] is a weak form of Theorem 4.1.7 below in which nondegeneracy is not required (except perhaps implicitly for the existence of $L_1$ classes), but the fibres $gY$ of $D \to G/K$ are required to be complete intersections in $D$ (which may be true in general but is only known in a few particular cases).

These results are related to some theorems of Godement, Harish-Chandra, and Borel (see [8], § 9) which are proved by methods of harmonic analysis on $G$. Those theorems apply to the case where

$$c \text{ is } K\text{-finite, i.e., } \{k^*c\colon k \in K\} \text{ has finite dimensional span}$$

and

$$c \text{ is } \mathfrak{Z}\text{-finite, where } \mathfrak{Z} \text{ is the center of the enveloping algebra of } \mathfrak{g}_{\mathbf{C}}\,.$$

We will see below that $\mathfrak{Z}$-finiteness is not a serious restriction, but $K$-finiteness essentially says that $c$ has a finite Fourier series. At any rate, in this case one has by their methods convergence of $\theta(c)$, and also the result that $\theta(c)$ has a bounded $\Gamma$-invariant Dolbeault representative.

Let $\sigma\colon H^s(D; \mathcal{E}_\lambda) \to H^0(M; \pi^s_*\tau^*\mathcal{E}_\lambda)$ be the $G$-equivariant Fréchet injection of Theorem 3.4.7. Then we can form the Poincaré series for $\sigma(c)$ and $\Gamma$, and prove

4.1.7. THEOREM. *Let* $\mathbf{E}_\lambda \to D$ *be a nondegenerate* (3.2.1) *homogeneous holomorphic vector bundle,* $\Gamma$ *a discrete subgroup of* $G$, *and* $c \in H^s_1(D; \mathcal{E}_\lambda)$. *Then the Poincaré series on* $M$

$$\theta(\sigma(c)) = \textstyle\sum_{\gamma \in \Gamma} \gamma^*(\sigma(c))$$

*converges in the Fréchet topology to a* $\Gamma$-*invariant section in* $H^0(M; \pi^s_*\tau^*\mathcal{E}_\lambda)$.

Theorem 4.1.6 is a consequence of Theorems 3.4.7 and 4.1.7.

The remainder of Section 4.1 is a proof of Theorem 4.1.7 following Griffiths' line of argument [15, pp. 619–623], but using nondegeneracy to simplify matters, clarify some technical points, and avoid the restriction that the fibres of $D \to G/K$ be complete intersections.

Fix a $\bar\partial$-closed $L_1$ form $\varphi \in \mathcal{E}^{0,s}_1(D; \mathbf{E}_\lambda)$ whose Dolbeault class $[\varphi] \in H^{0,s}_1(D; \mathbf{E}_\lambda)$ corresponds to $c$. If $Z$ is a compact subset of $D$, let $b(Z)$ denote the (finite) number of elements $\gamma \in \Gamma$ such that $\gamma(Z)$ meets $Z$. Evidently

$$(4.1.8) \qquad b\,||\varphi||_1 = b\int_D ||\varphi(x)||\,dx \geqq \textstyle\sum_{\gamma \in F} \displaystyle\int_{\gamma Z} ||\varphi(x)||\,dx, \quad b = b(Z)\,,$$

for every finite subset $F \subset \Gamma$.

The inclusion $gY \to D$ induces the restriction $r_g\colon H^s(D; \mathcal{E}_\lambda) \to H^s(gY; \mathcal{E}_\lambda)$. Note that $H^s(gY; \mathcal{E}_\lambda)$ is the fibre over $gY \in M = M_D$ of the vector bundle whose sheaf of germs of holomorphic sections is $\pi^s_*\tau^*\mathcal{E}_\lambda \to M$. Thus

$$(4.1.9) \qquad \sigma(c)(gY) = r_g(\varphi), \quad g \in G_{\mathbf{C}} \text{ with } gY \subset D\,.$$

Using Corollary 3.2.6 and Serre duality on $gY$, we denote the map on dual spaces induced from $r_g$ by

$$(4.1.10) \qquad r_g^*\colon H^0(gY; \mathcal{E}_\lambda^* \otimes \mathcal{K}_{gY}) \longrightarrow H_c^{n-s}(D; \mathcal{E}_\lambda^* \otimes \mathcal{K}_D) .$$

Given $\psi \in H^0(gY; \mathcal{E}_\lambda \otimes \mathcal{K}_{gY}) = H^{s,0}(gY; \mathbf{E}_\lambda)$, and $c = [\varphi] \in H^s(D; \mathcal{E}_\lambda)$ we have

$$\langle \sigma(c)(gY), \psi \rangle = \langle r_g(\varphi), \psi \rangle = \langle \varphi, r_g^*\psi \rangle = \int_D \varphi \barwedge r_g^*\psi .$$

We will use this to calculate

4.1.11. LEMMA. *Let $\psi \in H^0(gY; \mathcal{E}_\lambda^* \otimes \mathcal{K}_{gY})$ and let $F$ be a finite subset of $\Gamma$. Then there is a constant $a = a(\psi; F) > 0$ such that*

$$\textstyle\sum_{\gamma \in F} |\langle \gamma^*\sigma(c)(gY), \psi \rangle| \leqq a\, ||\varphi||_1 .$$

*Proof.* $r_g^*\psi$ has compact support $Z \subset D$. Now denote $a = b(Z)\cdot \sup_{x \in Z} ||r_g^*(\psi)(x)|| = b(Z)\cdot \sup_{x \in \gamma Z} ||(\gamma^{-1})^* r_g^*\psi(x)||$. Observe that

$$\langle \gamma^*\sigma(c)(gY), \psi \rangle = \langle r_g\gamma^*\varphi, \psi \rangle = \langle \gamma^*\varphi, r_g^*\psi \rangle =$$
$$\int_Z \gamma^*\varphi \barwedge r_g^*\psi = \int_Z \gamma^*(\varphi \barwedge (\gamma^{-1})^* r_g^*\psi) = \int_{\gamma Z} \varphi \barwedge (\gamma^{-1})^* r_g^*\psi .$$

That gives

$$|\langle \gamma^*\sigma(c)(gY), \psi \rangle| \leqq \int_{\gamma Z} ||\varphi(x)||\cdot||(\gamma^{-1})^* r_g^*\psi(x)||\, dx \leqq \frac{a}{b(Z)} \int_{\gamma Z} ||\varphi(x)||\, dx .$$

The assertion now follows from (4.1.8). Q.E.D.

Now we must be more precise about the support of $r_g^*\psi$ as cohomology class:

4.1.12. LEMMA. *If $\psi \in H^0(gY; \mathcal{E}_\lambda^* \otimes \mathcal{K}_{gY})$ then the compactly supported class $r_g^*\psi$, viewed as a linear functional on $H^s(D; \mathcal{E}_\lambda)$, has support in $gY$. In particular, if $U$ is a neighborhood of $gY$ in $D$, then $r_g^*\psi$ is represented by a form in $\mathcal{E}^{0,s}(D; \mathbf{E}_\lambda)$ with support in $U$.*

*Proof.* In the duality

$$H^{0,q}(D, \mathbf{E}_\lambda) \longleftrightarrow H_c^{0,n-q}(D, \mathbf{K}_D \otimes \mathbf{E}_\lambda^*)$$

we have used $C^\infty$ forms above to represent the pairing by integration, but to compute compact supports it is easier to represent $H_c^{0,n-q}(D, \mathbf{K}_D \otimes \mathbf{E}^*)$ by currents. First we note that

$$H_c^{0,n-q}(D, \mathbf{K}_D \otimes \mathbf{E}_\lambda^*) \cong H_c^{n,n-q}(D, \mathbf{E}_\lambda^*) ,$$

and if $T \in \mathcal{K}^{n,n-q}(D, \mathbf{E}_\lambda^*)$ is a current of type $(n, n-q)$ with coefficients in $\mathbf{E}_\lambda^*$ and with compact support, then the current pairing $\langle T, \xi \rangle$ is well defined for $\xi \in \mathcal{E}^{0,q}(D, \mathbf{E}_\lambda)$ (currents in $\mathcal{K}^{n,n-q}(D, \mathbf{E}^*)$ with compact support can be defined as the dual space to $\mathcal{E}^{0,q}(D, \mathbf{E})$, cf. e.g., Serre [33], Wells [42], Harvey

[22]). Thus we have

$$\begin{array}{ccc} H^s(D, \mathbf{E}_\lambda) & \xrightarrow{r_g} & H^s(gY, \mathbf{E}_\lambda) \\ \updownarrow & & \updownarrow \\ H^{n,n-s}(D, \mathbf{E}_\lambda^*) & \xleftarrow{r_g^*} & H^{s,0}(Y, \mathbf{E}_\lambda^*) \, . \end{array}$$

Define the current $T_g$ acting on forms $\xi \in \mathcal{E}^{0,s}(D, \mathbf{E}_\lambda)$ by

$$\langle T_g, \xi\rangle = \int_{gY} \psi_g \wedge \xi = \int_{gY} \psi_g \wedge r_g(\xi)$$

where $\psi_g$ given above is an element of $H^0(gY, K_{gY} \otimes \mathbf{E}_\lambda^*)$, i.e., an $(s, 0)$-form which is $\bar\partial$-closed with coefficients in $\mathbf{E}_\lambda^*$. Thus $\psi_g \wedge \xi$ is a scalar $(s, s)$-form on $gY$ and the integration makes sense, giving a well defined current. Moreover $\bar\partial T_g = 0$, since

$$\begin{aligned} \langle \bar\partial T_g, \xi\rangle &= \langle T_g, \bar\partial\xi\rangle \\ &= \int_{gY} \psi_g \wedge \bar\partial\xi \\ &= \int_{gY} \bar\partial\psi_g \wedge \xi = 0 \, , \end{aligned}$$

after integrating by parts once. Thus $T_g$ is a current representing a cohomology class in $H^{n,n-s}(D, \mathbf{E}_\lambda^*)$ with support equal to $gY$. We now show that the class represented by $T_g$ is indeed $r_g^*(\psi_g)$, but this is clear since

$$\begin{aligned} \langle r_g^*(\psi_g), \xi\rangle &= \langle \psi_g, r_g(\xi)\rangle \\ &= \int_{gY} \psi_g \wedge r_g(\xi) \end{aligned}$$

which is the definition of the action of $T_g$ on $\xi$. Q.E.D.

**4.1.13.** LEMMA. *Let $x_0 \in M$. Then there exists a compact neighborhood $U$ of $x_0$ in $M$ such that, given $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ with*

$$\sum_{\gamma\in\Gamma-F} |\langle \gamma^*(\sigma(\varphi))_g, \psi_g\rangle| < \varepsilon$$

*for $gL \in U$.*

*Proof.* Choose $U$ to be a compact neighborhood of $x_0$ such that $\bigcup_{gL\in U} gY$ is compact in $D$. Let $\psi_0(g) \in H_c^{0,n-q}(D, \mathbf{K}_D \otimes \mathbf{E}^*)$ be defined by $r_g^*(\psi_g)$. By Lemma 4.1.12, $\operatorname{supp}\psi_0(g) = gY$. We can choose a $C^\infty$ representative $\psi(g)$ for $\psi_0(g)$ with support close to $gY$ (but not equal to $gY$, which is impossible), and we can choose these representatives $\psi(g)$ with supports close enough to $gY$ so that

$$\bigcup_{\lambda\in U} \operatorname{supp}\psi(g) = C$$

is compact in $D$. Let

$$\alpha = \sup_{\substack{x \in C \\ \lambda \in U}} |\psi(g)|_x \,.$$

Then $\alpha < \infty$. Now $\varphi$ is given by hypothesis to be in $L^1$, so we can choose $\tilde{C}$ large and compact in $D$ so that

$$\int_{D-\tilde{C}} |\varphi| \, d\mu < \frac{\varepsilon}{\alpha b(C)} \,.$$

Let

$$F = \{\gamma \in \Gamma: \gamma C \cap \tilde{C} \neq \varnothing\} \,,$$

which is a finite set by proper discontinuity of $\Gamma$. For $gL \in U$, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma - F} |\langle \gamma^*(\sigma(\varphi))_g, \psi_g \rangle| &\leqq \sum_{\gamma \in \Gamma - F} \int_{\gamma C} |\varphi| \, |\gamma^{*-1}\psi(g)| \, dx \\ &\leqq \alpha \sum_{\gamma \in \Gamma - F} \int_{\gamma C} |\varphi| \, dx \\ &\leqq \alpha b(C) \int_{D - \tilde{C}} |\varphi| \, dx < \varepsilon \,. \end{aligned}$$ Q.E.D.

In conclusion we see that Lemma 4.1.11 and Lemma 4.1.13 combine to give the convergence of

$$\sum_{\gamma \in \Gamma} \langle \gamma^*(\sigma(\varphi))_g, \psi_g \rangle$$

uniformly on compact subsets of $M$. But since the fibre of the vector bundle associated with the locally free sheaf $\pi_*^s \mathcal{O}(\tau^* \mathbf{E}_\lambda)$ is given by $H^s(gY, \mathbf{E}_\lambda)$, and $\psi_g$ is an element of the dual space of this fibre for each $\lambda \in M$, we can conclude that the Poincaré series

$$\theta(\sigma(\varphi)) = \sum_{\gamma \in \Gamma} \gamma^*(\sigma(\varphi))$$

converges uniformly on compact subsets of $M$. This completes the proof of Theorem 4.1.8.

## 4.2. Square integrable cohomology

In order to obtain the absolutely integrable cohomology classes that we need for the Poincaré series in Theorems 4.1.6 and 4.1.7, we apply some unitary representation theory to the cohomology groups based on square integrable forms. In this section we discuss those square integrable cohomology spaces.

As before, $D = G(x_0) \subset X = G_{\mathbf{C}}/P$, so $D \cong G/G \cap P$ and we assume that $V = G \cap P$ is compact. $\mathbf{E} \to D$ is a homogeneous holomorphic vector bundle with a $G$-invariant hermitian metric; that is, $\mathbf{E} \to D = G/V$ is associated to a *unitary* representation of $V$. We also have a $G$-invariant Hermitian metric on the holomorphic tangent bundle $\mathbf{T}_D \to D$, and thus a $G$-invariant volume element $dx$ on $D$. The operator

$$\bar{\partial}: \mathcal{E}^{p,q}(D; \mathbf{E}) \longrightarrow \mathcal{E}^{p,q+1}(D; \mathbf{E})$$

has formal adjoint $\bar{\partial}^* = -\bar{*}_E * \bar{\partial} \bar{*}_E$, where $\bar{*}_E *$ and $\bar{*}_E$ are the Hodge-Kodaira orthocomplementation operators as in Section 4.1 (cf. Wells [41], p. 177). Let

$$\bar{\square} = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : \tag{4.2.1}$$

be the Kodaira-Hodge-Laplace operator acting on $\mathcal{E}^{p,q}(D; \mathbf{E})$. View $\bar{\square}$ as an operator on $L_2^{p,q}(D; \mathbf{E})$ with dense domain consisting of the compactly supported $C^\infty$ $\mathbf{E}$-valued $(p, q)$-forms on $D$. The work [4] of Andreotti-Vesentini shows that $\bar{\square}$ is essentially self-adjoint; that is, $\bar{\square}$ has a unique self-adjoint extension, which is its closure. We also write $\bar{\square}$ for the closure. Andreotti and Vesentini also show that the Hilbert space $L_2^{p,q}(D; \mathbf{E})$ is an orthogonal direct sum

$$L_2^{p,q}(D; \mathbf{E}) = \mathcal{H}^{p,q}(D; \mathbf{E}) \oplus \mathrm{cl}[\bar{\partial} L_2^{p,q-1}(D; \mathbf{E})] \oplus \mathrm{cl}[\bar{\partial}^* L_2^{p,q+1}(D; \mathbf{E})] \tag{4.2.2}$$

where

$$\begin{aligned} &\bar{\square} \text{ has kernel } \mathcal{H}^{p,q}(D; \mathbf{E}) \subset \mathcal{E}_2^{p,q}(D; \mathbf{E}) , \\ &\bar{\partial} \text{ has kernel } \mathcal{H}^{p,q}(D; \mathbf{E}) \oplus \mathrm{cl}[\bar{\partial} L_2^{p,q-1}(D; \mathbf{E})] , \\ &\bar{\partial}^* \text{ has kernel } \mathcal{H}^{p,q}(D; \mathbf{E}) \oplus \mathrm{cl}[\bar{\partial}^* L_2^{p,q+1}(D; \mathbf{E})] . \end{aligned} \tag{4.2.3}$$

A form $\varphi \in L_2^{p,q}(D; \mathbf{E})$ is called *harmonic* if $\bar{\square}\varphi = 0$. From (4.2.2) and (4.2.3), we see that the Hilbert space

$\mathcal{H}^{p,q}(D; \mathbf{E})$: square integrable harmonic $\mathbf{E}$-valued $(p, q)$-forms on $D$

is the analogue of Dolbeault cohomology where one only uses square integrable forms, and so we call it the *square integrable cohomology group*.

The action of $G$ commutes with $\bar{\partial}$, $\bar{*}_E$ and $\bar{*}_E *$, thus also with $\bar{\partial}^*$. Now $G$ commutes with $\bar{\square}$ and acts on $\mathcal{H}^{p,q}(D; \mathbf{E})$. It is easy to see that this action is a unitary representation. If $\mu$ is an irreducible representation of $V$, we denote

$$\pi_\mu^q: \text{unitary representations of } G \text{ on } \mathcal{H}^{0,q}(D; \mathbf{E}_\mu) . \tag{4.2.4}$$

The representations $\pi_\mu^q$ now are completely understood ([30], [45], [32]), and we proceed to describe them.

Let $\hat{G}$ denote the set of all equivalence classes $[\pi]$ of irreducible unitary representations $\pi$ of $G$. The *discrete series* of $G$ is

$$\hat{G}_{\mathrm{disc}} = \{[\pi] \in \hat{G}: [\pi] \text{ is a subrepresentation of the regular representation}\} .$$

If $[\pi] \in \hat{G}$, then $H_\pi$ denotes its representation space, and the *coefficients* of $[\pi]$ are the functions

$$f_{u,v}: G \longrightarrow \mathbf{C} \ \text{ by } f_{u,v}(g) = (u, \pi(g)v); \ \ u, v \in H_\pi .$$

If $[\pi] \in \hat{G}$ then the following conditions are equivalent:

$$(4.2.5)\qquad \begin{array}{rl}\text{(i)} & [\pi]\in\hat{G}_{\rm disc};\\ \text{(ii)} & \text{there exist nonzero } u,\ v\in H_\pi \text{ with } f_{u,v}\in L_2(G);\\ \text{(iii)} & \text{whenever } u,\ v\in H_\pi \text{ the coefficient } f_{u,v}\in L_2(G)\,.\end{array}$$

In view of (4.2.5), the discrete series classes are often called *square integrable.* We say that a vector $v \in H_\pi$ is *K-finite* if $\{\pi(k)v\colon k\in K\}$ is contained in a finite dimensional subspace of $H_\pi$. Restricting $\pi$ to $K$ one sees that $K$-finite vectors are dense in $H_\pi$. The $L_1$ analog of (4.2.5) is: A class $[\pi]\in\hat{G}$ is

$$(4.2.6)\qquad \textit{integrable} \text{ if } f_{u,v}\in L_1(G) \text{ for all } K\text{-finite } u,\ v\in H_\pi\,.$$

Since $|f_{u,v}(g)| \leqq ||u||\cdot||v||$, $L_1$ implies $L_2$, and so integrable classes are square integrable. The converse is false.

The space of compactly supported $C^\infty$ functions $G\rightarrow \mathbf{C}$ is denoted by $C_c^\infty(G)$, and we view it with the standard locally convex topology. If $[\pi]\in\hat{G}$ then $\pi(f)=\int_G f(g)\pi(g)dg$ is a trace class operator on $H_\pi$ for every $f\in C_c^\infty(G)$, and

$$(4.2.7)\qquad \Theta_\pi\colon C_c^\infty(G)\longrightarrow \mathbf{C}\ \text{ by } \Theta_\pi(f)=\operatorname{trace}\pi(f)$$

is continuous, i.e., is a Schwartz distribution on $G$. The distribution $\Theta_\pi$ is called the *global* or *distribution character* of $[\pi]$, and it specifies $[\pi]$ within $\hat{G}$. In addition, $\Theta_\pi$ is *invariant* under conjugation by elements of $G$, and it is an *eigendistribution* of

$$\mathfrak{Z}\colon \text{algebra of bi-invariant differential operators on } G\,.$$

The eigenvalues define the *infinitesimal character* of $[\pi]$

$$(4.2.8)\qquad \chi_\pi\colon \mathfrak{Z}\longrightarrow\mathbf{C} \text{ homomorphism by } z\Theta_\pi=\chi_\pi(z)\Theta_\pi\,.$$

$G$ has a dense open subset, the *regular set*, given by

$$G'=\{g\in G\colon \{\xi\in\mathfrak{g}\colon \mathrm{Ad}(g)\xi=\xi\} \text{ is a Cartan subalgebra of } \mathfrak{g}\}\,.$$

Harish-Chandra proved

$$(4.2.9)\qquad \Theta_\pi \text{ is a locally } L_1 \text{ function on } G, \text{ analytic on } G'\,.$$

Now we can state Harish-Chandra's description ([20], [21], [45]) of $\hat{G}_{\rm disc}$ in our notation. For convenience, replace $G_{\mathbf{C}}$ by a finite cover if necessary so that $\rho=\rho_G$ exponentiates to a well defined character $e^\rho$ on the compact Cartan subgroup $H$ of $G$. Then

$$\Delta_{G,H}=\textstyle\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$$

is well defined on $H$ and nonzero on $H\cap G'$. Let

$$\Lambda'=\{\lambda\in i\mathfrak{h}^*\colon e^\lambda \text{ defined and } \tilde{\omega}(\lambda)\neq 0\}\qquad \text{where } \tilde{\omega}(\lambda)=\textstyle\prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle\,.$$

If $\lambda\in\Lambda'$, Harish-Chandra associates a class $[\pi_\lambda]\in\hat{G}_{\rm disc}$, characterized by

$$\Theta_{\pi_\lambda}|_{H\cap G'} = c_G(-1)^{q(\lambda)}(1/\Delta_{G,H})\sum \det_{\mathfrak{h}}(w)e^{w\lambda} \tag{4.2.10}$$

where

$$q(\lambda) = |\{\alpha \in \Delta_K^+ : \langle \lambda, \alpha\rangle < 0\}| + |\{\gamma \in \Delta_S^+ : \langle \lambda, \gamma\rangle > 0\}| \tag{4.2.11}$$

and the summation runs over the Weyl group $W_{G,H} = W_{K,H}$. Every class in $\hat{G}_{\mathrm{disc}}$ is one of the $[\pi_\lambda]$ just specified, and discrete series classes $[\pi_\lambda] = [\pi_{\lambda'}]$ if and only if $\lambda' \in W_{G,H}(\lambda)$.

Now we can specify the representations $\pi_\mu^q$ of $G$ on the square integrable cohomology spaces $\mathcal{H}^{0,q}(D; \mathbf{E}_\mu)$. Let $\lambda$ be the highest weight of $\mu$, so $[\mu] = [\mu_\lambda] \in \hat{V}$ and $\mathbf{E}_\mu = \mathbf{E}_\lambda$. Combining [45, Theorem 7.2.3] with the recent vanishing theorems of Schmid [32] (cf. [31], [37]), we have

$$\begin{cases} \text{(i) if } \lambda + \rho \notin \Lambda' \text{ then every } \mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) = 0\,, \\ \text{(ii) if } \lambda + \rho \in \Lambda' \text{ and } q \neq q(\lambda + \rho) \text{ then } \mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) = 0\,, \\ \text{(iii) if } \lambda + \rho \in \Lambda' \text{ and } q = q(\lambda + \rho) \text{ then } \pi_\mu^q \text{ belongs to the discrete} \\ \qquad \text{series class } [\pi_{\lambda+\rho}]. \end{cases} \tag{4.2.12}$$

See Section 4.3 below for a sketchy indication of the proof.

We will say that $\mathbf{E}_\lambda \to D$ is *$L_1$-nonsingular* if

$$\lambda + \rho \in \Lambda', \text{ and } |\langle \lambda + \rho, \beta\rangle| > \frac{1}{2}\sum\nolimits_{\alpha\in\Delta^+} |\langle \alpha, \beta\rangle| \text{ for all } \beta \in \Delta_S^+\,. \tag{4.2.13}$$

Trombi and Varadarajan [36] proved that (4.2.13) is a necessary condition for the square integrable class $[\pi_{\lambda+\rho}]$ to be integrable, and recently Hecht and Schmid [23, 24] completed the proof that (4.2.13) is sufficient for $[\pi_{\lambda+\rho}]$ to be integrable. Thus (4.2.12) specializes to

**4.2.14.** PROPOSITION. *Let $\mathbf{E}_\lambda \to D$ be an $L_1$-nonsingular* (4.2.13) *homogeneous holomorphic vector bundle. The $G$ acts on $\mathcal{H}^{0,q(\lambda+\rho)}(D; \mathbf{E}_\lambda)$ by the integrable discrete series representation $[\pi_{\lambda+\rho}]$.*

Recall (3.1.12) the simple root system $\Psi = \{\varphi_1, \cdots, \varphi_l\}$ for $(\mathfrak{g}_{\mathbf{C}}, \Delta^+)$ such that $\Phi = \{\varphi_1, \cdots, \varphi_r\}$. Consider $\lambda$ defined by the integers $n_i = 2\langle\lambda, \varphi_i\rangle / \langle\varphi_i, \varphi_i\rangle$ where

$$\begin{cases} \text{(i) if } i \leqq r \text{ then } n_i \geqq \max\left\{0, \dfrac{-2\langle\beta_1 + \cdots + \beta_t, \varphi_i\rangle}{\langle\varphi_i, \varphi_i\rangle} \text{ with} \right. \\ \qquad \left. \beta_1, \cdots, \beta_t \in \Delta_S^+ \text{ distinct}\right\} \text{ and} \\ \text{(ii) if } r < i \text{ then } n_i \text{ is sufficiently large negative so that} \\ \qquad \text{(a) } \langle\lambda + \beta_1 + \cdots + \beta_t, \gamma\rangle < -\langle\rho_K, \lambda\rangle \text{ for } \gamma \in \Delta_K^+ \backslash \langle\Phi\rangle \text{ and} \\ \qquad\qquad \beta_j \in \Delta_S^+ \text{ distinct.} \\ \qquad \text{(b) } \langle\lambda + \rho, \beta\rangle < -\dfrac{1}{2}\sum\nolimits_{\alpha\in\Delta^+} |\langle\alpha, \beta\rangle| \text{ for all } \beta \in \Delta_S^+. \end{cases} \tag{4.2.15}$$

Then (i) without the 0 and (iia) say that $\mathbf{E}_\lambda \to D$ is nondegenerate (3.2.1). The 0 in (i) says $\langle \lambda+\rho, \alpha\rangle > 0$ for all $\alpha \in \langle \Phi\rangle$, (iia) with $t=0$ says $\langle\lambda+\rho, \gamma\rangle<0$ for all $\gamma \in \Delta_K^+\backslash\langle\Phi\rangle$ and (iib) implies $\langle\lambda+\rho, \beta\rangle<0$ for all $\beta\in\Delta_S^+$; so $\lambda+\rho\in\Lambda'$ with $q(\lambda+\rho) = |\Delta_K^+\backslash\langle\Phi\rangle| = s$. Glancing back at (4.2.13) we now see

**4.2.16.** PROPOSITION. *A homogeneous holomorphic vector bundle* $\mathbf{E}_\lambda \to D$ *satisfies* (4.2.15) *if and only if it is nondegenerate* (3.2.1) *and* $L_1$*-nonsingular* (4.2.13) *with* $q(\lambda+\rho) = s$. *In that case,* $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) = 0$ *for* $q \neq s$, *and* $G$ *acts on* $\mathcal{H}^{0,s}(D; \mathbf{E}_\lambda)$ *by the integrable discrete series representation* $[\pi_{\lambda+\rho}]$.

## 4.3. Absolute integrability of $K$-finite cohomology

Retain the setup of Section 4.2. Every square integrable harmonic form is $C^\infty$ and $\bar\partial$-closed, and so defines a Dolbeault cohomology class. That gives us a $G$-equivariant homomorphism

$$\mathcal{H}^{0,q}(D; \mathbf{E}) \longrightarrow H^{0,q}(D; \mathbf{E}) \text{ by } \omega \longmapsto \omega + \bar\partial\mathcal{E}^{0,q-1}(D; \mathbf{E}) \tag{4.3.1}$$

of our square integrable cohomology space to the subspace $H_2^{0,q}(D; \mathbf{E})$ represented by square integrable forms. In this section we examine the isomorphism of (4.2.12 (iii)) to see, for $\mathbf{E} = \mathbf{E}_\lambda$ satisfying (4.2.15), that (4.3.1) gives a $G$-isomorphism of $\mathcal{H}^{0,s}(D, \mathbf{E})$ onto $H_2^{0,s}(D; \mathbf{E})$ which maps the dense subspace of $K$-finite classes into $H_1^{0,s}(D; \mathbf{E})$. That provides the $L_1$ cohomology classes which we can sum in the Poincaré series of Theorems 4.1.6 and 4.1.7.

Several comments are in order before we proceed. First, the natural map (4.3.1) is not injective in general; Theorem 3.2.3 and Proposition 4.2.14 give situations in which $\mathcal{H}^{0,q}(D; \mathbf{E})$ is infinite dimensional and $H^{0,q}(D; \mathbf{E}) = 0$; for example, $q = 1$ with any positive power of the holomorphic tangent bundle over the unit disc. Second, given (4.2.15), a class in $\mathcal{H}^{0,s}(D; \mathbf{E}_\lambda)$ does not have to be $K$-finite to map into $H_1^{0,s}(D; \mathbf{E}_\lambda)$; one only needs that its $K$-isotypic components go to zero fast enough. Third, one can prove the results of this section using the methods of [28] and [29] together with some $L_1$ a priori estimates based on [36]—that was done in the original version of this paper—but here we take a shorter route using recent results ([31], [32]) of W. Schmid.

Fix $[\mu] \in \hat V$, say with highest weight $\lambda$, such that $\lambda+\rho\in\Lambda'$, and let $q = q(\lambda+\rho)$. We need Schmid's equivalence [32] of $\pi_\mu^q$ with the discrete series representation $\pi_{\lambda+\rho}$ of $G$.

Whenever $[\pi] \in \hat G$, we write $H_\pi^\infty$ for the space of $K$-finite vectors in the representation space $H_\pi$. It consists of analytic vectors, and is a module for the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}_{\mathbf{C}}$. Let $\Omega_K \in \mathfrak{G}$ denote the "Casimir" element that is the sum of the squares of a basis of $\mathfrak{k}_{\mathbf{C}}$ orthonormal with respect to the Killing form of $\mathfrak{g}_C$. Each $d\pi(\Omega_K)^n$ is a symmetric operator on $H_\pi^\infty$, and we set

$$H_\pi^\omega = \bigcap_{n=0}^\infty \left(\text{domain of } H_\pi\text{-closure of } d\pi(\Omega_K)^n|_{H_\pi^\infty}\right) .$$

It is equal to the intersection of the closures of the $d\pi(\Xi)$, $\Xi \in \mathfrak{G}$, from $H_\pi^\infty$; and of course $H_\pi^\infty \subset H_\pi^\omega \subset H_\pi$.

Let $\delta: H_\pi^\infty \otimes \bigwedge(\mathfrak{p}^n)^* \to H_\pi^\infty \otimes \bigwedge(\mathfrak{p}^n)^*$ denote the coboundary operator for Lie algebra cohomology of the $\mathfrak{p}^n$-module $H_\pi^\infty$. We proceed as in [30, § 3] and [32, § 3], using $\mathfrak{p}^n$ in place of $\Sigma_{\Delta+}\mathfrak{g}_C^{-\alpha}$ and $V = G \cap P$ in place of the compact Cartan subgroup $H$, to examine the cohomologies $H^p(\mathfrak{p}^n; H_\pi^\infty)$. The usual $\mathrm{Ad}_G(K)$-invariant positive Hermitian inner product $\langle \xi, \eta\rangle = -(\xi, \theta(\bar{\eta}))$ on $\mathfrak{g}_C$ gives a Hilbert space structure to $\mathfrak{p}^n$ and thus also to $H_\pi \otimes \bigwedge(\mathfrak{p}^n)^*$. As in [30] and [32], $\delta + \delta^*$ is essentially self adjoint there from the domain $H_\pi^\infty \otimes \bigwedge(\mathfrak{p}^n)^*$, so each "harmonic space" $\mathcal{H}^p(\pi)$ $\big(=$kernel of closure of $(\delta + \delta^*)|_{H_\pi \otimes \bigwedge^p(\mathfrak{p}^n)^*}\big)$ is a closed subspace. Let $\Delta$ denote the square of the closure $\delta + \delta^*$, so that $\mathcal{H}^p(\pi)$ is the kernel of $\Delta$ on $H_\pi \otimes \bigwedge^p(\mathfrak{p}^n)^*$. Then, as in [32, Lemma 3.6], $\varphi \mapsto$ (Lie algebra cohomology class of $\varphi$) defines a $V$-module isomorphism $\mathcal{H}^p(\pi) \to H^p(\mathfrak{p}^n; H_\pi^\omega)$, and as in [32, Lemma 3.21] $H_\pi^\infty \hookrightarrow H_\pi^\omega$ defines a $V$-module isomorphism $H^p(\mathfrak{p}^n; H_\pi^\infty) \to H^p(\mathfrak{p}^n; H_\pi^\omega)$. Therefore, as in [32, Thm. 3.1],

$$\text{(4.3.2)} \qquad \mathcal{H}^p(\pi) \text{ is } V\text{-module isomorphic to } H^p(\mathfrak{p}^n; H_\pi^\infty) .$$

If $M$ is a $V$-module then $M^V$ denotes the subspace of $V$-fixed vectors. If $[\mu_\lambda] \in \hat{V}$ we write $M_{-\lambda}$ for the $\mu_\lambda^*$-isotopic component of $M$, and so $\dim(M \otimes E_\lambda)^V = \dim(M_{-\lambda} \otimes E_\lambda)^V$ is the multiple of $\mu_\lambda^*$ by which $V$ acts on $M_{-\lambda}$.

We use Harish-Chandra's notation for infinitesimal characters of classes $[\pi] \in \hat{G}$. Thus, if $\lambda \in \Lambda'$ the discrete series class $[\pi_\lambda]$ has infinitesimal character $\chi_\lambda$.

Fix $[\mu_\lambda] \in \hat{V}$. Schmid's result [30, Lemma 6] holds in our situation, in the form

$$\text{(4.3.3)} \qquad \mathcal{H}^{0,p}(D; \mathbf{E}_\lambda) \cong \int_{\hat{G}} H_\pi \otimes \{\mathcal{H}^p(\pi^*) \otimes \mathbf{E}_\lambda\}^V d\pi$$

as unitary $G$-module, where $d\pi$ is Plancherel measure. If we write $\tau^p$ for the action of $V$ on $\bigwedge^p(\mathfrak{p}^n)^*$ induced by $\mathrm{Ad}_G$, then this isomorphism comes from

$$\begin{aligned}\mathcal{H}^{0,p}(D; \mathbf{E}_\lambda) \hookrightarrow \{f: G \longrightarrow \textstyle\bigwedge^p(\mathfrak{p}^n)^* \otimes E_\lambda: \|f\| \in L_2,\ f(gv) = (\tau^p \otimes \mu_\lambda)(v)^{-1}f(g)\} \\ \cong \{F \in L_2(G) \otimes \textstyle\bigwedge^p(\mathfrak{p}^n)^* \otimes E_\lambda: (r \otimes \tau^p \otimes \mu_\lambda)(v)F = F\} \\ = \int_{\hat{G}} H_\pi \otimes \{H_\pi^* \otimes \textstyle\bigwedge^p(\mathfrak{p}^n)^* \otimes E_\lambda\}^V d\pi\end{aligned}$$

and comparison of $\Delta$ with the Hodge-Kodaira-Laplace operator (4.2.1). A result [47] of Casselman and Osborne also holds here: if $H^p(\mathfrak{p}^n; H^\infty_{\pi^*})_{-\lambda} \neq 0$ then $\pi$ has infinitesimal character $\chi_{\lambda+\rho}$. Note that (4.2.12(i)) follows using (4.3.2). Now, as in [32, Cor. 3.22 and 3.23],

(4.3.4a) $\qquad \pi^p_{\mu_\lambda}$ is a sum of discrete series representations of $G$

and

(4.3.4b) $\qquad [\pi] \in \hat{G}_{\mathrm{disc}}$ has multiplicity $\dim\{\mathcal{H}^p(\pi^*) \otimes E_\lambda\}^V$ in $\pi^p_{\mu_\lambda}$.

The remainder of Schmid's proof [32] of (4.2.12) consists of combining [26] and [31] with a close look at the Hochschild-Serre spectral sequence for the $\mathfrak{p}^n$-cohomology of an $(H^*_{\pi_\nu})^\infty$ relative to the subalgebra $\mathfrak{p}^n \cap \mathfrak{k}_C$; it gives us: if $[\pi_\nu] \in \hat{G}_{\mathrm{disc}}$ where $\nu - \rho$ is $V$-dominant, then

$$(4.3.5) \quad \begin{cases} \dim\{H^p(\mathfrak{p}^n; H^\infty_{\pi^*_\nu}) \otimes E_\lambda\}^V = 1 \text{ if } \nu = \lambda + \rho \text{ and } p = q(\lambda + \rho)\,, \\ \qquad\qquad\qquad\qquad\qquad\quad = 0 \text{ otherwise}\,. \end{cases}$$

Now (4.2.12) follows from (4.3.2), (4.3.4) and (4.3.5).

Now we reverse the map that gives (4.3.3). Fix a discrete series class $[\pi_{\lambda+\rho}]$. Using the Weyl group $W(G, H)$ we may suppose that $\lambda$ is $V$-dominant, i.e., that we have $[\mu_\lambda] \in \hat{V}$. Using (4.3.2) and (4.3.5) with $q = q(\lambda + \rho)$, the isomorphism $H_{\pi_{\lambda+\rho}} \to \mathcal{H}^{0,q}(D; \mathbf{E}_\lambda)$ is given by

$$(4.3.6) \quad \begin{aligned} H_{\pi_{\lambda+\rho}} &\cong H_{\pi_{\lambda+\rho}} \otimes \{\mathcal{H}^q(\pi^*_{\lambda+\rho}) \otimes E_\lambda\}^V \\ &\hookrightarrow H_{\pi_{\lambda+\rho}} \otimes H^*_{\pi_{\lambda+\rho}} \otimes \textstyle\bigwedge^q(\mathfrak{p}^n)^* \otimes E_\lambda \hookrightarrow L_2(G) \otimes \bigwedge^q(\mathfrak{p}^n)^* \otimes E_\lambda\,. \end{aligned}$$

As in [32, Lemma 3.4], $\mathcal{H}^q(\pi^*_{\lambda+\rho}) \subset (H^*_{\pi_{\lambda+\rho}})^\omega \otimes \bigwedge^q(\mathfrak{p}^n)^*$. Also, if $[\pi] \in \hat{G}_{\mathrm{disc}}$ then $H_\pi \otimes (H^*_\pi)^\omega \hookrightarrow L_2(G)$ has image in $C^\infty(G)$. Since the

$$\begin{array}{ccc} H_{\pi_{\lambda+\rho}} \otimes [\{(H^*_{\pi_{\lambda+\rho}})^\omega \otimes \bigwedge^p(\mathfrak{p}^n)^*\} \otimes E_\lambda]^V & \hookrightarrow & \mathcal{E}^{0,p}(D; \mathbf{E}_\lambda) \\ \Big\downarrow 1\otimes\delta\otimes 1 & & \Big\downarrow \bar\partial \\ H_{\pi_{\lambda+\rho}} \otimes [\{(H^*_{\pi_{\lambda+\rho}})^\omega \otimes \bigwedge^{p+1}(\mathfrak{p}^n)^*\} \otimes E_\lambda]^V & \hookrightarrow & \mathcal{E}^{0,p+1}(D; \mathbf{E}_\lambda) \end{array}$$

commute, and since every $\delta$-cocycle in $(H^*_{\pi_{\lambda+\rho}})^\omega \otimes \bigwedge^q(\mathfrak{p}^n)^*$ is cohomologous to one in $(H^*_{\pi_{\lambda+\rho}})^\infty \otimes \bigwedge^q(\mathfrak{p}^n)^*$, now every form in the image of (4.3.6) is $\bar\partial$-cohomologous to one in the image of

$$(4.3.7) \quad H_{\pi_{\lambda+\rho}} \otimes (H^*_{\pi_{\lambda+\rho}})^\infty \otimes \textstyle\bigwedge^q(\mathfrak{p}^n)^* \otimes E_\lambda \hookrightarrow \{L_2(G) \cap C^\infty(G)\} \otimes \bigwedge^q(\mathfrak{p}^n)^* \otimes E_\lambda\,.$$

In particular, every form in the image of $H^\infty_{\pi_{\lambda+\rho}}$ under (4.3.6) is $\bar\partial$-cohomologous to a form in the image of $H^\infty_{\pi_{\lambda+\rho}} \otimes (H^*_{\pi_{\lambda+\rho}})^\infty \otimes \bigwedge^q(\mathfrak{p}^n)^* \otimes E_\lambda$ under (4.3.7). That gives us

**4.3.8.** THEOREM. *If $\mathbf{E}_\lambda \to D$ is $L_1$-nonsingular and $q = q(\lambda + \rho)$, then the natural map* (4.3.1) *sends every K-finite element of $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda)$ into $H^{0,q}_1(D; \mathbf{E}_\lambda)$.*

Of course, Theorem 4.3.8 is not very useful unless the natural map (4.3.1) is nontrivial. For that, we will prove

**4.3.9.** THEOREM. *Let* $\mathbf{E}_\lambda \to D$ *be a nondegenerate* (3.2.1) *homogeneous holomorphic vector bundle. Then the natural map* $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) \to H^{0,q}(D; \mathbf{E}_\lambda)$ *is a topological injection, with image* $H_2^{0,q}(D; \mathbf{E}_\lambda)$, *for* $0 \leqq q \leqq s$.

We start the proof with two technical lemmas. Recall, from Section 3.3, the spaces $\mathcal{E}_{(r)}^{0,p}(D; \mathbf{E}_\lambda)$ of all $\mathbf{E}_\lambda$-valued $(0, p)$-forms on $D$ that vanish to order $\geqq r$ on every fibre $gY$ of $D \to G/K$. Also recall the holomorphic normal bundle $\mathbf{N} \to Y$ to $Y$ in $\mathfrak{D}$ and the maps $\mathcal{R}_r$: $\mathcal{E}_{(r)}^{0,p}(D; \mathbf{E}_\lambda) \to \mathcal{E}^{0,p-r}(Y, \mathbf{E}_\lambda \otimes \Lambda^r\mathbf{N})$.

**4.3.10.** LEMMA. *Let*

$$\varphi \in L_2^{0,q}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r)}^{0,q}(D; \mathbf{E}_\lambda) \cap \bar\partial\mathcal{E}^{0,q-1}(D; \mathbf{E}_\lambda)$$

*with* $0 \leqq r \leqq q$ *and* $0 < q \leqq s$. *Then there exists* $\psi \in L_2^{0,q-1}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r)}^{0,q-1}(D; \mathbf{E}_\lambda)$ *with* $\varphi - \bar\partial\psi \in \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)$.

*Proof*. First, suppose that $\{\varphi_{gY}\} \subset \mathcal{E}^{0,p}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N})$ is a $C^\infty$ family of $\bar\partial$-exact forms parameterized by $G/K$, where $0 \neq p \neq s+1$. We show that there is another $C^\infty$ family $\{\psi_{gY}\} \subset \mathcal{E}^{0,p-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N})$ with $\bar\partial\psi_{gY} = \varphi_{gY}$ and $L_2$-norms over $Y$ satisfying $||\psi_{gY}||_Y^2 \leqq c\,||\varphi_{gY}||_Y^2$ for some $c > 0$. For let $\{\zeta_{gY}\} \subset \mathcal{E}^{0,p-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N})$ be any $C^\infty$ family with $\bar\partial\zeta_{gY} = \varphi_{gY}$. Lemma 3.3.4 says $H^{0,p-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N}) = 0$, and $Y$ is compact, so the Laplacian $\bar\Box_Y$ for $\mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N} \to Y$ has bounded inverse $\mathcal{G}$ on $L_2^{0,p-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N})$, say $||\mathcal{G}|| \leqq c$. Set $\psi_{gY} = \bar\partial^*\bar\partial\mathcal{G}\zeta_{gY}$. That gives a $C^\infty$ family, $\bar\partial\psi_{gY} = \bar\partial\bar\Box_Y\mathcal{G}\zeta_{gY} = \bar\partial\zeta_{gY} = \varphi_{gY}$, and

$$||\varphi_{gY}||_Y^2 = ||\bar\partial\psi_{gY}||_Y^2 = (\bar\Box_Y\psi_{gY}, \psi_{gY})_Y \geqq c^{-1}\,||\psi_{gY}||_Y^2\ ,$$

as claimed.

Now let $S$ be the image of a $C^\infty$ section to $G \to G/K$ and set $\varphi_{gY} = \mathcal{R}_r(g^*\varphi)$ for $g \in S$. That is a $C^\infty$ family in $\mathcal{E}^{0,q-r}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N})$. Each $\varphi_{gY}$ is $\bar\partial$-exact: if $q - r < s$, $\varphi_{gY}$ is $\bar\partial$-closed by Lemma 3.3.12 and so $\bar\partial$-exact by Lemma 3.3.4; if $q - r = s$ then $r = 0$ and, if $\varphi = \bar\partial\zeta$, $\varphi_{gY} = (g^*\bar\partial\zeta)|_Y = \bar\partial((g^*\zeta)|_Y)$. The paragraph above, gives us a $C^\infty$ family

$$\{\psi_{gY}\} \subset \mathcal{E}^{0,q-r-1}(Y; \mathbf{E}_\lambda \otimes \mathbf{\Lambda}^r\mathbf{N}) \text{ with } \bar\partial\psi_{gY} = \varphi_{gY} \text{ and } ||\psi_{gY}||_Y^2 \leqq c\,||\varphi_{gY}||_Y^2\ .$$

The formula for $\mathcal{R}_r$ in the first few lines of the proof of Lemma 3.3.12 provides $\psi \in \mathcal{E}_{(r)}^{0,q-1}(D; \mathbf{E}_\lambda)$ with $\mathcal{R}_r(g^*\psi) = \psi_{gY}$ and $||g^*\psi||_Y^2 = ||\psi_{gY}||_Y^2$ for $g \in S$. Now the $L_2$-norm of $\psi$ over $D$ is finite:

$$\begin{aligned} ||\psi||_D^2 &= \int_{G/K} ||g^*\psi||_Y^2 d(gK) = \int_{G/K} ||\psi_{gY}||_Y^2 d(gK) \\ &\leqq c\int_{G/K} ||\varphi_{gY}||_Y^2 d(gK) \leqq c\int_{G/K} ||g^*\varphi||_Y^2 d(gK) = c\,||\varphi||_D^2 < \infty\ . \end{aligned}$$

So $\psi \in L_2^{0,q-1}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r)}^{0,q-1}(D; \mathbf{E}_\lambda)$. Each $\mathcal{R}_r(g^*(\varphi - \bar\partial\psi)) = 0$ by construction of $\psi$, which says $\varphi - \bar\partial\psi \in \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)$. Q.E.D.

**4.3.11.** LEMMA. *In Lemma 4.3.10, if $\varphi$ is $K$-finite and $\mathfrak{Z}$-finite then $\psi$ can be chosen so that $\varphi - \bar\partial\psi$ is $K$-finite, $\mathfrak{Z}$-finite and square integrable.*

*Proof.* Let $F \subset \hat{K}$ be a finite subset such that $\varphi$ transforms under $F$. In other words, if $\bar\alpha_F$ is the sum of the normalized characters $\deg(\kappa)$ trace $\kappa$, $\kappa \in F$, then $\alpha_F * \varphi = \int_K \alpha_F(k) k^*(\varphi) dk$ is equal to $\varphi$. Start with $\psi$ given by Lemma 4.3.10 and define $\psi' = \alpha_F * \psi$. Then $\psi' \in L_2^{0,q-1}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r)}^{0,q-1}(D; \mathbf{E}_\lambda)$, $\psi'$ is $K$-finite, and $\varphi - \bar\partial\psi' = \alpha_F*(\varphi - \bar\partial\psi) \in \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)$.

As $\varphi$ is $K$-finite and $\mathfrak{Z}$-finite, it transforms under $G$ by some finite subset $J \subset \hat{G}_{\text{disc}}$. Let $\bar\beta_J$ be the sum of the normalized characters $\deg(\pi)\Theta_\pi$, $\deg(\pi)$ meaning formal degree, and set $\psi'' = \beta_J * \psi' = \int_G \beta_J(g) g^* \psi' dg$. $\psi''$ remains $K$-finite, in $L_2^{0,q-1}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r)}^{0,q-1}(D; \mathbf{E}_\lambda)$, and such that $\varphi - \bar\partial\psi'' = \beta_J*(\varphi - \bar\partial\psi'') \in \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)$. But $\psi''$ also is $\mathfrak{Z}$-finite.

Recall the Casimir operators $\Omega_K, \Omega_G \in \mathfrak{G}$. Express $\psi'' = \sum f_{BC}\omega^{-B} \wedge \omega^{-C}$ where $B \subset \Delta_S^+$, $C \subset \Delta_K^+ \setminus \Phi$ and $|B| = r$. As $\psi''$ is $L_2$ and $\mathfrak{Z}$-finite, each $\Omega_G(f_{B,C}) \in L_2(G)$. As $\psi''$ is $K$-finite, each $\Omega_K(f_{B,C}) \in L_2(G)$. Taking a linear combination, we see that each $\sum_{\alpha \in \Delta_S^+} e_\alpha(e_{-\alpha}(f_{B,C})) \in L_2(G)$. Integrating $(\sum_{\Delta_S^+} e_\alpha e_{-\alpha} f_{B,C}, f_{B,C})$ by parts, we get $e_{-\alpha}(f_{B,C}) \in L_2(G)$ for every $\alpha \in \Delta_S^+$.

We use (3.3.8) to calculate $\bar\partial\psi'' = (\bar\partial\psi'')_r + (\bar\partial\psi'')_{r+1} + (\bar\partial\psi'')_{r+2}$ where subscript denotes exact order of vanishing on the fibres of $D \to G/K$. Here

$$(\bar\partial\psi'')_{r+2} = \frac{1}{2}\sum_{\gamma \in C} \sum_{\substack{\alpha,\beta \in \Delta_S^+ \\ \alpha+\beta=\gamma}} \sum_{B,C} (\pm n_{\alpha,\beta}) f_{B,C}\omega^{-\alpha} \wedge \omega^{-\beta} \wedge \omega^{-B} \wedge \omega^{C \setminus \{\gamma\}}$$

is square integrable because each $f_{B,C} \in L_2(G)$,

$$(\bar\partial\psi'')_{r+1} = \sum_{\alpha \in \Delta_S^+} \sum_{B,C} e_{-\alpha}(f_{B,C})\omega^{-\alpha} \wedge \omega^{-B} \wedge \omega^{-C}$$

is square integrable because each $e_{-\alpha}(f_{B,C}) \in L_2(G)$, and $(\bar\partial\psi'')_r = \varphi_r$, because $\varphi - \bar\partial\psi'' \in \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)$, which gives its square integrability. So now

$$\varphi - \bar\partial\psi'' \in L_2^{0,q}(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(r+1)}^{0,q}(D; \mathbf{E}_\lambda)\,.$$

But $\bar\partial$ commutes with the action of $K$ and of $\mathfrak{Z}$, so $\bar\partial\psi''$ inherits $K$-finiteness and $\mathfrak{Z}$-finiteness from $\psi''$, and now also $\varphi - \bar\partial\psi''$ is $K$-finite and $\mathfrak{Z}$-finite. Q.E.D.

*Proof of Theorem* 4.3.9. We first prove injectivity on $K$-finite elements of $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda)$.

Let $\varphi \in \mathcal{H}^{0,q}(D; \mathbf{E}_\lambda)$ be $K$-finite and $\bar\partial$-exact. Note that $\varphi$ is $\mathfrak{Z}$-finite because $G$ is irreducible on $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda)$. Lemmas 4.3.10 and 4.3.11 give us $\psi_1 \in L_2^{0,q-1}(D; \mathbf{E}_\lambda)$ such that $\varphi - \bar\partial\psi_1$ is $K$-finite, $\mathfrak{Z}$-finite, $L_2$ and in $\mathcal{E}_{(1)}^{0,q}(D; \mathbf{E}_\lambda)$.

Now apply Lemmas 4.3.10 and 4.3.11 to $\varphi - \bar{\partial}\psi_1$ with $r = 1$ to get $\psi_2 \in L_2^{0,q-1}(D; \mathbf{E}_\lambda)$ such that $\varphi - \bar{\partial}(\psi_1 + \psi_2)$ is $K$-finite, $\mathfrak{Z}$-finite, $L_2$ and in $\mathcal{E}_{(2)}^{0,q}(D; \mathbf{E}_\lambda)$. Iterating, we have $\{\psi_1, \cdots, \psi_q\} \subset L_2^{0,q-1}(D; \mathbf{E}_\lambda)$ such that $\varphi - \bar{\partial}(\psi_1 + \cdots + \psi_q) \in L_2^q(D; \mathbf{E}_\lambda) \cap \mathcal{E}_{(q)}^{0,q}(D; \mathbf{E}_\lambda) = 0$. So now $\varphi = \bar{\partial}(\psi_1 + \cdots + \psi_q)$ lies in $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) \cap \bar{\partial} L_2^{0,q-1}(D; \mathbf{E}_\lambda) = 0$.

We have just seen $\mathcal{H}^{0,q}(D; \mathbf{E}_\lambda) \to H^{0,q}(D; \mathbf{E}_\lambda)$ injective on the dense subspace of $K$-finite vectors. It is continuous, so now it must be injective. Q.E.D.

Finally, we combine Theorems 4.3.8 and 4.3.9 to see

4.3.12. THEOREM. *Let $\mathbf{E}_\lambda \to D$ be a nondegenerate* (3.2.1) *$L_1$-nonsingular* (4.2.13) *homogeneous holomorphic vector bundle with $q(\lambda + \rho) = s$. Then $G$ acts on $H_2^s(D; \mathbf{E}_\lambda)$ by the integrable discrete series representation $[\pi_{\lambda+\rho}]$, and every $K$-finite class $c \in H_2^s(D; \mathbf{E}_\lambda)$ is absolutely integrable,* i.e. *is in $H_1^s(D; \mathbf{E}_\lambda)$.*

This provides the promised abundance of $L_1$ cohomology classes that we can sum in the Poincaré series of Theorems 4.1.6 and 4.1.7.

RICE UNIVERSITY, HOUSTON, TEXAS
UNIVERSITÄT GÖTTINGEN
UNIVERSITY OF CALIFORNIA AT BERKELEY
THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL

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