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POINCARÉ THETA SERIES AND L_1 COHOMOLOGY

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1. Introduction. Ninety-five years ago, Poincaré revolutionized the theory of automorphic forms by introducing the method of summing over a discontinuous group. In modern language and somewhat greater generality, one has

D: a bounded symmetric domain in $\mathbf{C}^n$;

$\mathbf{K}$: the canonical line bundle (of $(n, 0)$-forms) over D; and

Γ: a discontinuous group of analytic automorphisms of D.

One considers holomorphic sections φ of powers $\mathbf{K}^m \to D$, for example $(dz^1 \wedge \cdots \wedge dz^n)^m$, and forms the *Poincaré theta series*

$$\theta(\varphi) = \sum_{\gamma\in\Gamma} \gamma^*(\varphi) \equiv \sum_{\gamma\in\Gamma} \varphi \circ \gamma^{-1}.$$

$\mathbf{K}^m$ carries a natural Γ-invariant hermitian metric, and if m is sufficiently large ($m \geqq 2$ for the unit disc in $\mathbf{C}$), then $\mathbf{K}^m \to D$ has absolutely integrable holomorphic sections; in fact $(dz^1 \wedge \cdots \wedge dz^n)^m$ is L_1. When φ is L_1, the series $\theta(\varphi)$ is absolutely convergent, uniformly on compact subsets of D, and represents a Γ-invariant holomorphic section of $\mathbf{K}^m \to D$. The Γ-invariant holomorphic sections of $\mathbf{K}^m \to D$ are the *Γ-automorphic forms of weight m on D*. See Borel [4] for a systematic discussion.

Poincaré's construction is the primary source of automorphic forms on D. The automorphic forms of a given weight m form a finite-dimensional space $H^0_\Gamma(D; \mathcal{O}(\mathbf{K}^m))$. For m sufficiently large, the corresponding map of $\Gamma\backslash D$ is a quasi-projective embedding, i.e., the quotients of elements of $H^0_\Gamma(D; \mathcal{O}(\mathbf{K}^m))$ generate the function field of $\Gamma\backslash D$.

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An important aspect of automorphic function theory in several variables is the special case

$$D = \{p \times p \text{ complex matrices } Z : Z = {}^tZ \text{ and } I - ZZ^* \gg 0\},$$

which is analytically equivalent to the "Siegel upper half-space"

$$H_p = \{p \times p \text{ complex matrices } Z : Z = {}^tZ \text{ and } \operatorname{Im} Z \gg 0\}$$

of degree p. It has complex dimension $p(p + 1)/2$, and is the space of normalized Riemann period matrices of degree p. For appropriate choice of Γ, the equivalence classes of period matrices of Riemann surfaces of genus p sit in $\Gamma\backslash D$.

When Griffiths studied periods of integrals on algebraic manifolds [8], [9], he saw that generally the corresponding period matrix domains D are not bounded symmetric domains. In fact [20], they carry no nonconstant holomorphic functions. These period matrix domains belong to a well-understood [12], [20] class of open homogeneous complex manifolds that we call *flag domains*. Here the first difficulty (see Schmid [12], [13]) is that one cannot expect to find sections of line bundles, or even vector bundles, but must look to cohomology of degree $s = \dim_C Y$ where Y is a maximal compact subvariety of D. In particular there are no automorphic forms in the classical sense on D, and one is led to the ***automorphic cohomology space***

$$H^s_\Gamma(D; \mathcal{O}(E)) = \{\Gamma\text{-invariant classes in } H^s(D; \mathcal{O}(E))\}$$

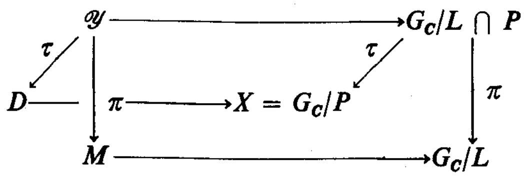
where $E \to D$ is a "nondegenerate" homogeneous holomorphic vector bundle.

At present, very little is known about automorphic cohomology, especially when $\Gamma\backslash D$ is noncompact. For example, even in the Griffiths period domain case one does not know whether $H^s_\Gamma(D; \mathcal{O}(E))$ is finite dimensional, nor does one know how to relate it to function theory on $\Gamma\backslash D$. Recently, however, we constructed absolutely integrable cohomology classes $\varphi \in H^s(D; \mathcal{O}(E))$ for a certain specific class of bundles $E \to D$, and we showed that the Poincaré series $\theta(\varphi) = \sum_{\gamma\in\Gamma} \gamma^*(\varphi)$ always converges to an automorphic cohomology class. That is what we describe below.

The detailed proof of the theorems discussed in this paper appear in [18]. Some of these results had been announced previously by one of us in a preliminary fashion in [17].

2. Flag domains. A *complex flag manifold* is a compact complex homogeneous space $X = G_C/P$ where G_C is a complex semisimple Lie group and P is a parabolic subgroup. Fix a noncompact real form G of G_C. Then G has only finitely many orbits on X, so in particular there are open orbits. A *flag domain* is a (necessarily open) orbit $G(x) \subset X$ on which the isotropy subgroups of G are compact. Replacing P by a conjugate, the flag domains have the form $D = G(x_0) \cong G/V$ where $x_0 = 1 \cdot P$ and $V = G \cap P$ is compact. Then V contains a compact Cartan subgroup H of G, so it sits in a unique maximal compact subgroup K of G, and we have $Y = [K(x_0) \cong K/V$: maximal compact subvariety of $D]$. All this is classical [20].

We now consider the "linear deformation space" $\pi : \mathcal{Y} \to M$ of Y, given as follows. M is the set of all gY, $g \in G_C$, such that $gY \subset D$, and $\mathcal{Y}$ is the disjoint union of these gY with $\pi(gY) = \{gY\}$. More precisely, let $L = \{g \in G_C : gY = Y\}$. Then L is a complex subgroup, $K_C \subset L$, and we have



where the horizontal arrows are inclusions of open subsets. In particular $\pi : \mathcal{Y} \to M$ is a holomorphic mapping of maximal rank. We prove

THEOREM 1. *M is a Stein manifold.*

This had earlier been conjectured by Griffiths [8], and one of us [16] had checked the case $D = SO(2h, 1)/U(h)$. The principal tools in the proof are a clear understanding of the group L, Schmid's exhaustion function for D [12], the Andreotti-Norguet solution to the generalized Levi problem for analytic cycles on q-convex manifolds [1], [2], and the Docquier-Grauert exhaustion principle for Stein manifolds [7].

3. Homogeneous vector bundles. As above, $D = G(x_0) \cong G/V$ is a flag domain, $Y = K(x_0) \cong K/V$ is a maximal compact subvariety, and their dimensions are $n = \dim_C D$ and $s = \dim_C Y$.

If μ is a unitary representation of V then E_μ will denote the representation space, and $\mathbf{E}_\mu = G \times_\mu E_\mu \to G/V = D$ will denote the associated homogeneous hermitian C^∞ vector bundle. Any extension $\tilde{\mu}$ of μ to a holomorphic representation of P on E_μ defines a holomorphic vector bundle $\tilde{\mathbf{E}}_\mu \to G_C/P = X$ such that $\mathbf{E}_\mu = \tilde{\mathbf{E}}_\mu |_D$, and thus imposes a holomorphic vector bundle structure on $\mathbf{E}_\mu \to D$. If μ is irreducible, there is exactly one extension $\tilde{\mu}$, and so we may view $\mathbf{E}_\mu \to D$ as a G-homogeneous holomorphic vector bundle in a unique way.

Recall the compact Cartan subgroup H of G with $H \subset V \subset K$ and consider a positive $\mathfrak{h}_C$-root system Δ^+ on $\mathfrak{g}_C$ such that $\mathfrak{p} = \mathfrak{p}^r + \mathfrak{p}^n$ is the sum of reductive part and nilradical where, for some subset Φ of the simple roots,

$$\mathfrak{p}^r = \mathfrak{b}_C = \mathfrak{h}_C + \sum_{\langle\Phi\rangle} \mathfrak{g}_C^{\beta} + \mathfrak{g}_C^{-\beta} \quad \text{and} \quad \mathfrak{p}^n = \sum_{\Delta^+ \setminus \langle\Phi\rangle} \mathfrak{g}_C^{-\alpha}.$$

Here $\langle\Phi\rangle = \{\alpha \in \Delta^+ : \alpha \text{ is a linear combination from } \Phi\}$ is the positive $\mathfrak{h}_C$-root system on $\mathfrak{b}_C$. In these orderings, we denote

μ_λ : irreducible representation of V with highest weight λ,
E_λ : representation space of μ_λ,
$\mathbf{E}_\lambda$: associated hermitian holomorphic vector bundle on D,
$\mathcal{E}_\lambda$: sheaf $\mathcal{O}(\mathbf{E}_\lambda)$ of germs of holomorphic sections.

If G_C is simply connected, which we may assume without loss of generality, and if $\Phi = \{\varphi_1, \cdots, \varphi_r\} \subset \{\varphi_1, \cdots, \varphi_l\} = \Psi$ is a simple system for $(\mathfrak{g}_C, \Delta^+)$, then the possibilities for λ are given by

$$\frac{2\langle\lambda, \varphi_i\rangle}{\langle\varphi_i, \varphi_i\rangle} \quad \begin{array}{l} \text{is an integer for } 1 \leqq i \leqq l \\ \text{and is} \geqq 0 \text{ for } 1 \leqq i \leqq r. \end{array}$$

Further, let Δ_K^+ denote the set of compact positive roots and Δ_S^+ the noncompact positive roots, so $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ with

$$\mathfrak{k}_C = \mathfrak{h}_C + \sum_{\Delta_K^+} (\mathfrak{g}_C^{\beta} + \mathfrak{g}_C^{-\beta}) \quad \text{and} \quad \mathfrak{s}_C = \sum_{\Delta_S^+} (\mathfrak{g}_C^{\gamma} + \mathfrak{g}_C^{-\gamma}).$$

Finally define $2\rho = 2\rho_G = \sum_{\Delta^+} \gamma$, $2\rho_K = \sum_{\Delta_K^+} \beta$ and $2\rho_V = \sum_{\langle\Phi\rangle} \alpha$.

A homogeneous holomorphic vector bundle $E_\lambda \to D$ is *nondegenerate* if

$$\langle \lambda + \rho_K + \beta_1 + \cdots + \beta_l, \alpha \rangle > 0 \quad \text{for all } \alpha \in \langle \Phi \rangle \text{ and}$$
$$\langle \lambda + \rho_K + \beta_1 + \cdots + \beta_l, \gamma \rangle < 0 \quad \text{for all } \gamma \in \Delta_K^+ \backslash \langle \Phi \rangle$$

whenever $\{\beta_1, \cdots, \beta_l\} \subset \Delta_S^+$ are distinct. This is just what one needs to apply the Borel-Weil-Bott theorem [6] to conclude: The sheaf cohomology $H^q(Y, \mathcal{O}(E_\lambda \otimes \bigwedge^l N)) = 0$ for $0 \leqq q < s$ and all l, where $N \to Y$ is the holomorphic normal bundle of Y in D. Then a variation on Schmid's identity theorem [12, Corollary 6.5] says

PROPOSITION 1. *If $E_\lambda \to D$ is nondegenerate then $H^q(D; \mathcal{E}_\lambda) = 0$ for $q \neq s$, and if $c \in H^s(D; \mathcal{E}_\lambda)$ with $c|_{gY} = 0$ for all $g \in G$ then $c = 0$. Further $H^s(D; \mathcal{E}_\lambda)$ is an infinite-dimensional Fréchet space on which G acts by a continuous representation.*

Recall the linear deformation space of §2. The maps $D \xleftarrow{\tau} \mathcal{Y} \xrightarrow{\pi} M$ are holomorphic, maximal rank and G-equivariant. First, that gives $F_\lambda = \tau^* E_\lambda \to \mathcal{Y}$ is a pull-back bundle. Second, it gives us $\pi^s_* \mathcal{F}_\lambda \to M$, sth direct image sheaf, where $\mathcal{F}_\lambda = \mathcal{O}(F_\lambda)$. Using the identity theorem one sees that $\tau^*: H^s(D; \mathcal{E}_\lambda) \to H^s(\mathcal{Y}; \mathcal{F}_\lambda)$ is a G-equivariant topological injection of Fréchet spaces. Since M is Stein, Cartan's Theorem B and the Grauert direct image theorem show that the edge homomorphism

$$e: H^s(\mathcal{Y}; \mathcal{F}_\lambda) \to H^0(M; \pi^s_* \mathcal{F}_\lambda)$$

of the Leray spectral sequence is a topological isomorphism. This establishes our principal representation theorem.

THEOREM 2. *If $E_\lambda \to D$ is nondegenerate then $e \circ \tau^*$ is a G-equivariant topological injection*

$$\sigma: H^s(D; \mathcal{E}_\lambda) \to H^0(M; \pi^s_* \mathcal{F}_\lambda)$$

of Fréchet spaces.

We note that $\pi^s_* \mathcal{F}_\lambda \to M$ is locally free. In fact it is $\mathcal{O}(\tilde{E}_\lambda)$ where $\tilde{E}_\lambda \to M$ is the holomorphic vector bundle obtained by restriction from the G_C-homogeneous bundle $\tilde{E}'_\lambda \to G_C/L$ associated to the L-module $H^s(Y; \mathcal{E}_\lambda)$. Thus the theorem represents s-cohomology on the flag domain D by sections of a holomorphic vector bundle over the Stein manifold M.

The principal representation theorem is the exact statement of a theorem conjectured by Griffiths [8], [9] and announced by one of us [18].

4. Poincaré series. Since V is compact, the flag domain $D \cong G/V$ has a G-invariant hermitian metric, and so we can speak of the pointwise norm of differential forms with values in a hermitian vector bundle $E \to D$. That gives us the Lebesgue classes

$$\mathcal{E}_r^{p,q}(D; E) = \left\{ E\text{-valued } (p, q)\text{-forms } \varphi: \int_D \|\varphi(x)\|^r < \infty \right\}.$$

We say that a sheaf cohomology class $c \in H^q(D; \mathcal{O}(E))$ is of Lebesgue class L_r if it has a Dolbeault representative in $\mathcal{E}_r^{0,q}(D; E)$, and $H_r^q(D; \mathcal{O}(E))$ denotes the set of all such classes.

THEOREM 3. *Let $E_\lambda \to D$ be nondegenerate, let $c \in H_1^s(D; \mathcal{E}_\lambda)$, and let Γ be a discrete subgroup of G. Then the Poincaré series $\theta(c) = \sum_{\gamma \in \Gamma} \gamma^* c$ converges, in the Fréchet space topology of $H^s(D; \mathcal{E}_\lambda)$, to a Γ-invariant class.*

A weaker version of this theorem was given by Griffiths in [8].

The idea is to use the principal representation theorem and reduce to

THEOREM 4. *Let $E_\lambda \to D$ be nondegenerate, let $c \in H_1^s(D; \mathcal{E}_\lambda)$, let Γ be a discrete subgroup of G, and recall the Fréchet injection $\sigma : H^s(D; \mathcal{E}_\lambda) \to H^0(M; \pi_*^s \mathcal{F}_\lambda)$. The Poincaré series $\theta(\sigma(c)) = \sum_{\gamma \in \Gamma} \gamma^*(\sigma(c))$ converges in the Fréchet topology to a Γ-invariant section of $\pi_*^s \mathcal{F}_\lambda$.*

Theorem 4 is a variation on a result of Griffiths [8], and our proof follows the classical pattern, as amplified by Griffiths, but modified to take into account the nondegeneracy of E_λ. The result is related to some theorems of Godement, Harish-Chandra and Borel (see [5, §9]) which are proved by methods of harmonic analysis on G. Those theorems apply to the case where

(1) c is K-finite, i.e., $\{k^* c : k \in K\}$ has finite-dimensional span, and

(2) c is $\mathfrak{Z}$-finite where $\mathfrak{Z}$ is the center of the enveloping algebra of $\mathfrak{g}_C$.

We will see below that $\mathfrak{Z}$-finiteness is not a serious restriction, but K-finiteness essentially says that c has finite Fourier series. At any rate, this gives convergence of $\theta(c)$, and also gives the result that $\theta(c)$ has a bounded Γ-invariant Dolbeault representative.

5. Square-integrable cohomology. In order to produce L_1 cohomology classes for the Poincaré series of Theorems 3 and 4, we must first digress and discuss L_2 cohomology and unitary representations.

If $E \to D$ is a G-homogeneous hermitian holomorphic vector bundle, then one has the Kodaira-Hodge-Laplace operator $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on the spaces $\mathcal{E}^{p,q}(D; E)$ of smooth E-valued (p, q)-forms. $\square$ defines a selfadjoint operator $\square$ on the Hilbert space completion of $\mathcal{E}_2^{p,q}(D; E)$, whose kernel

$$\mathcal{H}^{p,q}(D; E) : L_2 \text{ harmonic } E\text{-valued } (p, q)\text{-forms}$$

is a closed subspace consisting of C^∞ forms. That gives

$$\pi_\mu^q : \text{unitary representation of } G \text{ on } \mathcal{H}^{0,q}(D; E_\mu).$$

Recently Schmid [14] settled the "Langlands conjecture," completing the identification of the π_μ^q as follows. Let $\Lambda' = \{\nu \in i\mathfrak{h}^* : e^\nu \text{ defined on } H \text{ and } \langle \nu, \alpha \rangle \neq 0 \text{ for all } \alpha \in \Delta^+\}$. Given $\nu \in \Lambda'$, Let

$$q(\nu) = \left|\{\alpha \in \Delta_K^+ : \langle \nu, \alpha \rangle < 0\}\right| + \left|\{\gamma \in \Delta_S^+ : \langle \nu, \gamma \rangle > 0\}\right|$$

and

$[\pi_\nu] = \omega(\nu)$: Harish-Chandra's discrete series representation class for G parametrized by ν (see [10]).

Then, if λ is the highest weight of μ,

(1) *if* $\lambda + \rho \notin \Lambda'$ *then every* $\mathscr{H}^{0,q}(D; E_\mu) = 0$;
(2) *if* $\lambda + \rho \in \Lambda'$ *and* $q \neq q(\lambda + \rho)$ *then* $\mathscr{H}^{0,q}(D; E_\mu) = 0$;
(3) *if* $\lambda + \rho \in \Lambda'$ *and* $q = q(\lambda + \rho)$ *then* $\pi_\lambda^q \in [\pi_{\lambda+\rho}]$.

One of the first consequences of this is

THEOREM 5. *If* $E_\lambda \to D$ *is nondegenerate, then the natural surjective map*

$$\mathscr{H}^{0,s}(D; E_\lambda) \ni \omega \mapsto (\textit{Dolbeault class}) \in H_2^s(D; \mathscr{E}_\lambda)$$

is injective. If $\lambda + \rho \in \Lambda'$ *with* $q(\lambda + \rho) = s$, *then G acts on the image by the discrete series representation* $[\pi_{\lambda+\rho}]$.

6. Absolutely integrable cohomology. An irreducible unitary representation π of G is *integrable* if the coefficient $f_{u,v}(g) = \langle u, \pi(g)v\rangle \in L_1(G)$ whenever u and v are K-finite. As $|f_{u,v}(g)| \leqq \|u\| \cdot \|v\|$, then $f_{u,v} \in L_2(G)$, so $[\pi]$ is in the discrete series.

We will say that the homogeneous holomorphic vector bundle $E_\lambda \to D$ is *L_1-nonsingular* if $\lambda + \rho \in \Lambda'$, and $|\langle \lambda + \rho, \beta\rangle| > \frac{1}{2}\sum_{\alpha\in\Delta^+} |\langle \alpha, \beta\rangle|$ for all $\beta \in \Delta_s^+$. That is a necessary (Trombi and Varadarajan [15]) and sufficient (Hecht and Schmid [11]) condition for the discrete series class $[\pi_{\lambda+\rho}]$ to be integrable.

THEOREM 6. *Let* $E_\lambda \to D$ *be nondegenerate and* L_1*-nonsingular with* $q(\lambda + \rho) = s$. *Then G acts on* $H_2^s(D; \mathscr{E}_\lambda)$ *by the integrable discrete series representation* $[\pi_{\lambda+\rho}]$, *and every K-finite class* $c \in H_2^s(D; \mathscr{E}_\lambda)$ *is absolutely integrable, i.e., is in* $H_1^s(D; \mathscr{E}_\lambda)$.

Since the K-finite elements are dense in the infinite-dimensional Hilbert space $H_2^s(D; \mathscr{E}_\lambda)$, this provides an abundance of L_1 cohomology classes that we can sum in Poincaré series to obtain automorphic cohomology.

The proof of Theorem 6 uses a direct image construction of Schmid [12] and follows a route suggested by him to one of us.

Fix $E_\lambda \to D$ nondegenerate and denote $U_\lambda = H^s(Y; \mathscr{E}_\lambda)$ and $W_\lambda = U_\lambda \otimes \mathfrak{s}_C$. Then K acts irreducibly on U_λ with *lowest* weight $\nu = w(\lambda + \rho_K) - \rho_K$ for a certain element w of the Weyl group, and we have a K-invariant $W_\lambda = W_\lambda^+ \oplus W_\lambda^-$ where $W_\lambda^\pm$ is a sum of K-modules of lowest weight $\nu \pm \beta$, $\beta \in \Delta_s^+$. Writing

$$U_\lambda \to G/K, \quad W_\lambda^\pm \to G/K \quad \text{and} \quad W = W^+ \oplus W^- \to G/K$$

for the associated G-homogeneous vector bundles, we have an exact sequence of Fréchet space maps

$$0 \to H^s(D; \mathscr{E}_\lambda) \xrightarrow{\zeta} C^\infty(U_\lambda) \xrightarrow{\mathscr{D}} C^\infty(W_\lambda^+)$$

where $C^\infty(\,\cdot\,)$ denotes the Fréchet space of C^∞ sections viewed as a subspace of $C^\infty(G) \otimes U_\lambda$ or $C^\infty(G) \otimes W_\lambda^+$. It is given by

$$\zeta(c)(g) = (g^*c)|_Y \in U_\lambda$$

and

$$\mathscr{D}(F) = \text{projection}_{(W_\lambda \to W_\lambda^+)} \left\{ \sum_{\beta\in\Delta_s^+} e_\beta(F) \otimes e_{-\beta} \right\}$$

for a certain normalization of root vectors $e_\gamma \in \mathfrak{g}_C^s$. In other words, the direct image map ζ is a G-equivariant Fréchet isomorphism of $H^s(D; \mathscr{E}_\lambda)$ onto the kernel $C^\infty(U_\lambda)_{\mathscr{D}}$ of $\mathscr{D}$.

Using our knowledge of the representation of G on $H_2^s(D; \mathscr{E}_\lambda)$, one can follow square-integrability through the direct image map ζ and see that it maps $H_2^s(D; \mathscr{E}_\lambda)$ onto

$$^0L_2(U_\lambda)_{\mathscr{D}} : L_2 \text{ closure of } \{F \in C^\infty(U_\lambda)_{\mathscr{D}} : F \text{ is } L_2 \text{ and } \mathfrak{Z}\text{-finite}\}.$$

If $E_\lambda \to D$ is L_1-nonsingular with $q(\lambda + \rho) = s$, one can further see that *if* $F \in {}^0L_2(U_\lambda)_{\mathscr{D}}$ *is K-finite then* $F: G \to U_\lambda$ *is* L_1.

The identity theorem (Proposition 1) is proved by a careful examination of the order of vanishing of differential forms along the fibres of $D \to G/K$. Standard methods of harmonic analysis on semisimple Lie groups allow one to carry square-integrability through those considerations and obtain both Theorem 5 and the above characterization of $\zeta \cdot H_2^s(D; \mathscr{E}_\lambda)$. To carry absolute integrability we make use of an estimate as follows.

LEMMA. *Let* $f \in L_p(G)$ *where* $1 \leqq p \leqq 2$. *Let* ξ *belong to the universal enveloping algebra* $\mathscr{G}$ *so that both* f *and* $\xi(f)$ *are* $\mathfrak{Z}$*-finite, left K-finite and* L_2. *Then* $\xi(f) \in L_p(G)$.

Using the lemma, we obtain the L_1 version of the technique used to prove the identity theorem, and that tells us that *if* $F \in {}^0L_2(U_\lambda)_{\mathscr{D}}$ *is K-finite and* L_1 *then* $\zeta^{-1}(F) \in H_1^s(D; \mathscr{E}_\lambda)$. Theorem 6 follows.

7. Some questions. Some obvious questions come to mind at this point.

(1) Which Poincaré series $\theta(c)$ are nonzero?

(2) Is $H_f^s(D; \mathscr{E}_\lambda)$ finite dimensional, as in the classical cases?

(3) What is the dimension of the space of Poincaré series arising from a given $E_\lambda \to D$? How does that space compare with the full automorphic cohomology space $H_f^s(D; \mathscr{E}_\lambda)$?

(4) How does one obtain quasi-projective embeddings from automorphic cohomology?

(5) Can one construct meromorphic functions on $\Gamma\backslash D$ using holomorphic arc components of boundary orbits [20] in the way that Bailey and Borel [3] use boundary components [19]? How would such Eisenstein series be related to our Poincaré series?

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