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HEAT EQUATION, PROPORTIONALITY PRINCIPLE, AND VOLUME OF FUNDAMENTAL DOMAINS

1. Introduction

In this note, we extend the Hirzebruch proportionality principle to the coefficients in the asymptotic expansions for the Laplacians on differential forms with values in homogeneous vector bundles over symmetric spaces. The case zero-forms and the trivial line bundle is a proportionality principle for the trace of the heat kernel. For $2m$-dimensional manifolds, the case of m-th order terms of some asymptotic expansions is a proportionality principle for the indices of certain elliptic complexes. These results have implications for the volumes of fundamental domains of discrete subgroups, and in refining these implications we also develop a proportionality principle for equivariant characteristic classes.

Hirzebruch's original work [9] on proportionality studied ordinary characteristic classes on hermitian symmetric spaces, and Serre [10] studied the Euler class in a general setting. N. Wallach tells us that his student Miatelo obtained a result similar to Corollary 2.8 below, by methods of harmonic analysis on semisimple Lie groups. And of course many mathematicians, starting with C. L. Siegel [11], looked for lower bounds on the volume of fundamental domains for discrete groups.

Our result on the volumes of fundamental domains is a qualitative improvement on previous work. The latter was a matter of positive lower bounds for the volume, while we show that the volume is an integral multiple of a certain number.

2. Asymptotic expansions

Let G/K be a symmetric space of noncompact type, and let $M' = G'/K$ denote the compact dual symmetric space. Here we take G and G' to be analytic subgroups of a complex group G_C and $K = G \cap G'$. Let Γ

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denote a discrete, co-compact, torsion-free subgroup of G, and denote $M = \Gamma\backslash G/K$ the compact locally symmetric space with fundamental group Γ and universal cover G/K. We view M and M' as riemannian manifolds with invariant metric arising from the Cartan-Killing form on the complexification of the Lie algebra of G.

Let $\pi : K \to GL(V)$ be a (finite dimensional, unitary) representation of K. We consider the associated hermitian vector bundles.

(2.1) $$E'_\pi : G' \times_K V \to M' \quad \text{and} \quad E_\pi : (\Gamma\backslash G) \times_K V \to M$$

with typical fibre V. The corresponding second order elliptic operators

(2.2) $$\Delta'_{\pi,p} : \text{Laplacian on } \Lambda^p(M') \otimes E'_\pi \quad \text{and}$$
$$\Delta_{\pi,p} : \text{Laplacian on } \Lambda^p(M) \otimes E_\pi$$

define zeta functions (trace heat kernel)

(2.3) $$\zeta_{M',\pi,p}(t) = \sum e^{-\lambda' t} \quad \text{and} \quad \zeta_{M,\pi,p}(t) = \sum e^{-\lambda t},$$

where the summations run over the eigenvalues (with multiplicities) of $\Delta'_{\pi,p}$ and $\Delta_{\pi,p}$. If $d = \dim M' = \dim M$, then these zeta functions have asymptotic expansions at $t = 0$ (see Gilkey [8]),

(2.4a) $$\zeta_{M',\pi,p}(t) \sim (4\pi t)^{-d/2}(a'_{0,p,\pi} + a'_{1,p,\pi}t + \cdots + a'_{n,p,\pi}t^n + 0(t^{n+1}))$$

and

(2.4b) $$\zeta_{M,\pi,p}(t) \sim (4\pi t)^{-d/2}(a_{0,p,\pi} + a_{1,p,\pi}t + \cdots + a_{n,p,\pi}t^n + 0(t^{n+1})).$$

Our main result is the observation.

2.5. THEOREM. $a_{k,\pi,p} = (-1)^k \dfrac{\text{Vol}(M)}{\text{Vol}(M')} a'_{k,\pi,p}$.

Proof. $a_{k,\pi,p}$ is a local invariant of M, that is $a_{k,\pi,p} = \int_M P_{k,\pi,p}(x)\,dx$, where dx is the volume element and $P_{k,\pi,p}(x)$ is a polynomial in the curvatures of M and E at x and their covariant derivatives. All those covariant derivatives vanish here, so $P_{k,\pi,p}(x)$ is homogeneous of degree k in the curvatures. Since M is locally homogeneous, now $a_{k,\pi,p} = P_{k,\pi,p}(x) \cdot \text{Vol}(M)$. If $y \in M'$, the same considerations hold, and the curvatures are the negatives of those at $x \in M$, so $a'_{k,\pi,p} = P_{k,\pi,p}(y) \cdot \text{Vol}(M') = (-1)^k P_{k,\pi,p}(x) \cdot \text{Vol}(M')$. Q.E.D.

Calculation of the $a'_{k,\pi,p}$ is an algebraic problem with the Peter-Weyl Theorem for G', Cartan's highest weight theory for the representations of G', Frobenius' Reciprocity Theorem, and the decomposition of a representation of G' under restriction to K. See Cahn-Wolf [5, §1]. Theorem 2.5 carries the result from the compact symmetric space M' to the compact locally symmetric space M.

In case π is trivial and $p = 0$, the operators (2.2) reduce to

$$\Delta' : \text{Laplacian on } L_2(M') \quad \text{and} \quad \Delta : \text{Laplacian on } L_2(M). \tag{2.6}$$

Then (2.3) reduce to the ordinary zeta functions $\zeta_{M'}$ and ζ_M, and (2.4) becomes

$$\zeta_{M'}(t) \sim (4\pi t)^{-d/2}(a'_0 + a'_1 t + \cdots + a'_n t^n + 0(t^{n+1})) \quad \text{as } t \downarrow 0 \tag{2.7a}$$

and

$$\zeta_M(t) \sim (4\pi t)^{-d/2}(a_0 + a_1 t + \cdots + a_n t^n + 0(t^{n+1})) \quad \text{as } t \downarrow 0. \tag{2.7b}$$

Theorem 2.5 now specializes to

2.8. COROLLARY. $a_k = (-1)^k \dfrac{\text{Vol}(M)}{\text{Vol}(M')} a'_k.$

In this case, the a'_k can be calculated explicitly (see Cahn-Wolf [5, 6]) and that gives the a_k.

In the hermitian symmetric case, Theorem 2.5 and Corollary 2.8 hold for the complex Laplacians. If M and M' are spin manifolds in a consistent way, then they hold for the Dirac Laplacians (see Wolf [12, 13]).

3. Elliptic complexes

We now assume that M' and M have even dimension $d = 2m$. Let $\mathcal{S}'_\pi$ and $\mathcal{S}_\pi$ denote their respective twisted signature complexes (Gilkey [8]) using forms with values in E'_π and E_π. Then

$$\text{Index}(\mathcal{S}'_\pi) = \sum (-1)^p a'_{m,\pi,p} \quad \text{and} \tag{3.1}$$
$$\text{Index}(\mathcal{S}_\pi) = \sum (-1)^p a_{m,\pi,p}.$$

Using this, Theorem 2.5 gives us

$$\text{Index}(\mathcal{S}_\pi) \cdot \text{Vol}(M') = (-1)^m \,\text{Index}(\mathcal{S}'_\pi) \cdot \text{Vol}(M). \tag{3.2}$$

More generally, if $\mathscr{C}'$ is any natural elliptic complex over M', and $\mathscr{C}$ is the corresponding complex over M, then our considerations give

3.3. THEOREM. $\operatorname{Index}(\mathscr{C}) \cdot \operatorname{Vol}(M') = (-1)^m \operatorname{Index}(\mathscr{C}') \cdot \operatorname{Vol}(M)$.

Of course Theorem 3.3 can also be derived from the Atiyah-Singer Index Theorem.

The point of Theorem 3.3 is that $\operatorname{Index}(\mathscr{C}')$ can often be calculated directly. Here are a few examples.

de Rham Complex. The index of the de Rham complex is the Euler-Poincaré characteristic χ. If $\operatorname{rank} K < \operatorname{rank} G$ then $\chi(M') = 0$. If $\operatorname{rank} K = \operatorname{rank} G$ then $\chi(M') = |W(G')|/|W(K)|$, quotient of the orders of the Weyl groups. Now Theorem 3.3 gives

$$\chi(M) = 0 \textit{ if } \operatorname{rank} K < \operatorname{rank} G, \textit{ and} \tag{3.4}$$
$$\chi(M) = (-1)^m \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(M')} \frac{|W(G')|}{|W(K)|} \textit{ if } \operatorname{rank} K = \operatorname{rank} G.$$

Dolbeault complex. The index of the Dolbeault complex is the arithmetic genus A. In the hermitian case, $A(M') = 1$, e.g. by the Bott-Borel-Weil Theorem [3]. So Theorem 3.3 gives

$$A(M) = (-1)^m \operatorname{Vol}(M)/\operatorname{Vol}(M') \quad \textit{in the hermitian case.} \tag{3.5}$$

This is equivalent to Hirzebruch's [9, Satz 4].

Signature complex. When the dimension $d = 2m$ is divisible by 4, the signature τ is defined to be the signature of the symmetric bilinear form $H^m \times H^m \to H^{2m}$ (cup product). It is the index of the ordinary (trivial line bundle) signature complex. Evidently $\tau(M') = \pm 1$ if $\dim H^m(M'; R) = 1$, so Theorem 3.3 and the fact [14, Theorem 6] that $\tau(M) \geqq 0$ give

$$\tau(M) = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(M')} \quad \textit{for } m \textit{ even and} \quad \dim H^m(M'; R) = 1. \tag{3.6}$$

That applies, for example, to the even quaternion projective spaces $\operatorname{Sp}(2l+1)/\operatorname{Sp}(2l) \times \operatorname{Sp}(1)$.

At this point we tabulate some information for the $M' = G'/K$ where $\operatorname{rank} K = \operatorname{rank} G'$ and G' is an exceptional group. Note the curious fact: if $\dim H^m(M'; R) \neq 0$ then $\dim H^m(M'; R)$ divides $\chi(M')$. This is also the case for many classical M', but not all, e.g. not $\operatorname{Sp}(8)/\operatorname{SU}(8)$.

(3.7)	$M' = G'/K$	$2m$	$\chi(M')$	$\dim H^m(M'; R)$	$\tau(M')$
(i)	$G_2/SO(4)$	8	3	1	1
(ii)	$F_4/\mathrm{Spin}(9)$	16	3	1	1
(iii)	F_4/A_1C_3	28	12	0	0
(iv)	E_6/T_1D_5	32	27	3	3
(v)	E_6/A_1A_5	40	36	4	4
(vi)	E_7/T_1E_6	54	56	0	0
(vii)	E_7/A_1D_6	64	63	7	7
(viii)	E_7/A_7	70	72	0	0
(ix)	E_8/A_1E_7	112	120	8	8
(x)	E_8/D_8	128	135	9	7

So (3.6) holds directly for $G_2/SO(4)$ and the Cayley plane $F_4/\mathrm{Spin}(9)$, and holds after possible division by a small number for E_6/T_1D_5, E_6/A_1A_5, E_7/A_1D_6, E_8/A_1E_7, and E_8/D_8.

The information in (3.7) comes out of the Hirsch Formula [0, §26], the primitive invariants for the various simple groups (Borel-Chevalley [2]), and the recent results of Shaw Mong [14].

4. VOLUME OF FUNDAMENTAL DOMAIN

In the appropriate normalization of Haar measures, Theorem 3.3 says

4.1. THEOREM. Let $\mathscr{C}'$ be a natural elliptic complex over M'. If $\mathrm{Index}(\mathscr{C}') \neq 0$ then

$$\mathrm{Vol}(\Gamma\backslash G) = |\mathrm{Index}(\mathscr{C})| \cdot \mathrm{Vol}(G')/|\mathrm{Index}(\mathscr{C}')|.$$

{See Cahn [4] for $\mathrm{Vol}(G')$ here, and note that $\mathscr{C}'$ exists with $\mathrm{Index}(\mathscr{C}') \neq 0$ precisely when rank K = rank G.}

We interpret Theorem 4.1 as saying: if rank K = rank G then there is a number $\mathrm{Vol}(G')/|\mathrm{Index}(\mathscr{C}')| > 0$ such that $\mathrm{Vol}(\Gamma\backslash G)$ is a positive integral multiple of that number. At worst, one can use the de Rham complex and conclude

$$(4.2) \qquad \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G') \cdot |W(K)|/|W(G')|, \qquad n \textit{ positive integer.}$$

But that could be derived directly from Chern's Gauss-Bonnet Theorem [7]; see Serre [10, §3]. Evidently, the information contained in Theorem 4.1, under this interpretation, is optimized by minimizing $|\mathrm{Index}(\mathscr{C}')|$. Here are a few examples.

Dolbeault Complex. If $\mathscr{C}'$ is the Dolbeault complex then $\mathrm{Index}(\mathscr{C}') = A(M') = 1$, so

$$(4.3) \qquad \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G'), \qquad n \textit{ positive integer, in the hermitian case.}$$

That applies whenever G is a connected linear group locally isomorphic to a product whose factors are of the form $\mathrm{SU}(p, q)$, $\mathrm{SO}(2, l)$, $\mathrm{SO}^*(2l)$, $\mathrm{Sp}(l; R)$, E_{6,T_1D_5} and E_{7,T_1E_6}.

Signature complex. If the dimension $d = 2m$ is divisible by 4, then Theorem 4.1 says

$$(4.4) \qquad \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')/\tau(M'), \quad n \textit{ integer, if } \tau(M') \neq 0.$$

As noted in (3.6) and (3.7), this applies to

$\mathrm{Sp}(2l, 1)$, G_{2,A_1A_1} and F_{4,B_4} with $\tau(M') = 1$;
E_{6,T_1D_5} with $\tau = 3$,
E_{7,A_1D_6} and E_{8,D_8} with $\tau = 7$;
E_{6,A_1A_5} with $\tau = 4$, E_{8,A_1E_7} with $\tau = 8$.

5. K-Characteristic classes

Recall the definition of the K-characteristic ring of M'. If B_K is the classifying space for K and $\sigma : M' \to B_K$ induces the principal K-bundle $G \to M'$, then $\sigma^* H^*(B_K; Z)$ is the K-characteristic subring of $H^*(M'; Z)$. Its elements, the K-characteristic classes, are obtained modulo torsion as follows. Let $\mathfrak{K}$ be the Lie algebra of K, $\mathscr{C}(\mathfrak{K})$ the graded associative algebra of Ad_K-invariant polynomials $\mathfrak{K} \to C$. Fix a K-connection on $G' \to M'$ and let Ω' denote its curvature form. If $c \in \mathscr{C}(\mathfrak{K})$, then the de Rham class $[c(\Omega')] \in H^*(M'; C)$ is independent of the connection. We say that c is integral if, in this way, it gives an integral class on B_K. The coefficient homomorphism $H^*(M'; Z) \to H^*(M'; C)$ maps the K-characteristic ring onto $\{[c(\Omega')] : c \in \mathscr{C}(\mathfrak{K})$ and c is integral.$\}$ Now we have a mild variation on Hirzebruch Proportionality [9]:

5.1. THEOREM. Suppose that M and M' have even dimension $d = 2m$ and let $c \in \mathscr{C}(\mathfrak{K})$ be integral and of degree m. Then the K-characteristic numbers

$$c[M'] = \int_{M'} c(\Omega') \quad \text{and} \quad c[M] = \int_M c(\Omega)$$

are integers, and

$$c[M'] \cdot \mathrm{Vol}(M) = (-1)^m c[M] \cdot \mathrm{Vol}(M').$$

{As in the proof of Theorem 2.5, this is a matter of local homogeneity and $\Omega = -\Omega'$.}

5.2. COROLLARY. If rank $K =$ rank G, so $d = 2m$ and $\sigma^* H^{2m}(B_K; Z)$ has finite index r in $H^{2m}(M'; Z)$, then $\mathrm{Vol}(\Gamma \backslash G)$ is an integral multiple of $\mathrm{Vol}(G')/r$.

{For then we have $c \in \mathscr{C}(\mathfrak{K})$ of degree m with $c[M'] = r$.}

As an application of Corollary 5.2, suppose that

(5.3a) M' is a product of complex and quaternionic grassmannians.

In other words, suppose that

(5.3b) G is covered by a group $\Pi\, \mathrm{SU}(p_i, q_i) \times \Pi\, \mathrm{Sp}(l_j, m_j)$.

Taking G_C simply connected,

$$G' = \Pi\, \mathrm{SU}(p_i + q_i) \times \Pi\, \mathrm{Sp}(l_j + m_j)$$

and

$$K = \Pi S(U(p_i) \times U(q_i)) \times \Pi\, \mathrm{Sp}(l_j) \times \mathrm{Sp}(m_j),$$

so

$$H^*(G'; Z) \quad \text{and} \quad H^*(K; Z) \quad \text{are torsion free.}$$

Since it is now known that a compact connected group modulo a maximal torus is torsion free in cohomology, a result of Borel [0; Prop. 30.2] says $H^*(M'; Z) = \sigma^* H^*(B_K; Z)$, and in particular $r = 1$ in Corollary 5.2. In summary,

(5.4) $\quad \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')$, n *positive integer, in the case* (5.3)

Borel's result [0; Prop. 30.2] in fact shows that the number r in Corollary 5.2 is a product of powers of primes p for which $H^*(G'; Z)$ or $H^*(K; Z)$ has p-torsion. In view of (3.7), this sharpens (4.2) in the cases where M' is E_7/A_1D_6, E_8/D_8 or a real grassmannian of even dimension, telling us

(5.5a) $\quad$ if $G = E_{7,A_1D_6}$ then $\mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')$, some integer n;

(5.5b) $\quad$ if $G = E_{8,D_8}$ then $\mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')$, some integer n;

(5.6) $\quad \begin{cases} \text{if } G \text{ is locally isomorphic to } \mathrm{SO}(2u, 2v) \text{ or } \mathrm{SO}(2u, 2v+1) \\ \text{and if } 2^c \text{ is the highest power of 2 that divides } 2\binom{u+v}{v}, \\ \text{then } \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')/2^c \text{ for some integer } n. \end{cases}$

6. Real grassmannians

We now make some calculations to improve (5.6). The result is

6.1. THEOREM. The even dimensional oriented real Grassman manifolds $M' = G'/K = \mathrm{SO}(2u+l)/\mathrm{SO}(2u) \times \mathrm{SO}(l)$ have 2 as a K-characteristic number.

As an immediate consequence we will have

(6.2) $\quad \begin{cases} \text{if } G \text{ is locally isomorphic to } \mathrm{SO}(2u, 2v) \text{ or } \mathrm{SO}(2u, 2v+1) \\ \text{then } \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')/2 \text{ for some positive integer } n. \end{cases}$

6.3. LEMMA. Express the non-oriented real Grassmann manifold as $M'' = G''/K'' = O(k+l)/O(k) \times O(l)$. Then $H^*(M''; Z)$ is equal to its K''-characteristic subring, that is, is the image of the cohomology $H^*(B_{K''}; Z)$ of the classifying space.

Proof. Here $M' = \mathrm{SO}(k+l)/\mathrm{SO}(k) \times \mathrm{SO}(l) = G'/K$ and $\xi : M' \to M''$ denotes the standard 2-sheeted covering. Consider the diagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & B_K \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ M'' & \xrightarrow{\beta} & B_{K''} \end{array}$$

where the horizontal maps to classifying spaces induce the principal bundles $G' \to M'$ and $G'' \to M''$, and where η is induced by $K \subset K''$. We must prove β^* surjective on integral cohomology.

Borel [1; Theorem 11.1] proved that η^* is bijective and β^* is surjective for Z_2-cohomology.

Let p be an odd prime. The coverings ξ and η are of order 2 and 4, and the spaces in the diagram have no cohomology p-torsion, so ξ^* and η^* are injective for Z_p-cohomology. Also, from the Stieffel-Whitney classes, $B_{SO(k)} \to B_{O(k)}$ and $B_{SO(l)} \to B_{O(l)}$ induce surjective maps for Z_p-cohomology, so η^* is surjective as well. Borel's result [0; Prop. 30.2] shows α^* surjective for Z_p-cohomology. We conclude that β^* is surjective.

Now $\beta^*: H^*(B_{K''}; Z_p) \to H^*(M''; Z_p)$ is surjective for all p, and that proves the lemma. Q.E.D.

6.4. LEMMA. Let $k = 2u$ and $l = 2v$, even, so that M'' has a G'-invariant orientation. Give M'' the riemannian metric such that $M' \to M''$ is a local isometry and let Ω'' be the curvature form. Then there is an integral, Ad(K'')-invariant, polynomial c on $\mathfrak{K}$ such that

$$\int_{M''} c(\Omega'') = 1 \quad \text{and} \quad \int_{M'} c(\Omega') = 2.$$

{This proves Theorem 6.1 for l even.}

Proof. Lemma 6.3 gives us c, as required, with $\int_{M''} c(\Omega'') = 1$. Since $\xi: M' \to M''$ is riemannian, $\Omega' = \xi^*\Omega$, and so $\int_{M'} c(\Omega') = 2\int_{M''} c(\Omega'') = 2$ because ξ is 2-sheeted. Q.E.D.

Proof of Theorem. Write $K = K_1 \times K_2$ where $K_1 = SO(2u)$ and $K_2 = SO(l)$, and let $l = 2v$ or $2v + 1$.

The usual (vector) representation τ_1 of K_1 has weight system $\{a_1, \ldots, a_u, -a_1, \ldots, -a_u\}$ and has Pfaffian $\mathrm{Pf}(\tau_1) = a_1 a_2 \ldots a_u \in \mathscr{C}(\mathfrak{K}_1)$. The vector representation τ_2 of K_2 has weight system

$\{b_1, \dots, b_v, -b_1, \dots, -b_v, ((0))\}$, where (( )) indicates that the term occurs just when $l = 2v + 1$. Now the representation of K on the tangent space of M' is $\tau_1 \otimes \tau_2$, which has weight system $\{\pm(a_j + b_k), \pm(a_j - b_k), ((\pm a_j))\}$, $1 \leq j \leq u$ and $1 \leq k \leq v$. That has Pfaffian

$$E = ((a_1 a_2 \dots a_u)) \cdot \prod_{j,k} (a_j^2 - b_k^2) \in \mathscr{C}(\mathfrak{K}). \tag{6.5}$$

and $E(\Omega')$ is the Euler class on M'.

We apply Lemma 6.4 to the case $l = 2v$. The fact that c is integral and K''-invariant, says that, on the Lie algebra of the maximal torus, c is an integral polynomial in the elementary symmetric functions $\sigma_s(a_1^2, \dots, a_u^2)$ and the $\sigma_t(b_1^2, \dots, b_v^2)$. Since $c(\Omega')[M'] = 2$ and

$E(\Omega')[M'] = \chi(M') = 2\binom{u+v}{v}$, (6.5) tells us that

$$\prod_{j,k} (a_j^2 - b_k^2) = \binom{u+v}{v} c + \sum p_i q_i, \tag{6.6}$$

where the p_i are integral polynomials in the $\sigma_s(a_1^2, \dots, a_u^2)$ and the $\sigma_t(b_1^2, \dots, b_v^2)$, and the q_i are integral polynomials in the $\sigma_w(a_1^2, \dots, b_v^2)$.

Now let $l = 2v + 1$. Multiplying (6.6) by $a_1 a_2 \dots a_u$, we see from (6.5) that

$$E = \binom{u+v}{v} c' + \sum p'_i q_i, \tag{6.7}$$

where $c' = (a_1 a_2 \dots a_u)c$ and the $p'_i = (a_1 a_2 \dots a_u)p_i$ are integral polynomials in the

$$\sigma_s(a_1^2, \dots, a_u^2), \quad \mathrm{Pf}(\tau_1) \quad \text{and} \quad \sigma_t(b_1^2, \dots, b_v^2),$$

and where the q_i are integral polynomials in the $\sigma_w(a_1^2, \dots, b_v^2)$. Now c', p_i and $q_i \in \mathscr{C}(\mathfrak{K})$, all integral, with the q_i invariant by conjugation from G'. Thus (6.7) says that

$$c'(\Omega')[M'] = \binom{u+v}{v}^{-1} \cdot E(\Omega')[M'] = 2$$

is a K-characteristic number of M'. Q.E.D.

7. Summary for fundamental domains

We proved a number of integrality statements for volumes of fundamental domains, and here we assemble them for the case of simple groups. Remember, that the 2-sphere is hermitian and that the 4-sphere is the quaternion projective line.

7.1. THEOREM. Let G be a connected linear simple Lie group with rank K = rank G. If Γ is a torsion free discrete subgroup with $\Gamma\backslash G$ compact, then

(7.2) $\quad \mathrm{Vol}(\Gamma\backslash G) = n \cdot \mathrm{Vol}(G')/r \quad$ for some integer $n > 0$

where r depends only on G, and where $r = 1$ except possibly in the cases

(7.3)	G (to local isomorphism)	r is a divisor of
	$SO(2u, l)$, $(1, 1) \neq (u, l) \neq (2, 1)$	2
	F_{4,A_1C_3}	12
	E_{6,A_1A_5}	4
	E_{7,A_7}	72
	E_{8,A_1E_7}	8

It seems clear that a closer look at the exceptional groups will improve the result on the value of r, especially for E_{7,A_7}.

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