# Zeta Functions and Their Asmptotic Expansions for Compact Symmetric Spaces of Rank One

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# Zeta Functions and Their Asymptotic Expansions for Compact Symmetric Spaces of Rank One

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## §0. Introduction

In this paper we apply É. Cartan's theory of class 1 representations [3] to derive explicit formulae for the $\zeta$-functions of the compact riemannian symmetric spaces of strictly positive curvature. We then combine those formulae with an asymptotic expansion of Mulholland [5] and evaluate the coefficients in the Minakshisundaram asymptotic expansion (see [1]) of the $\zeta$-function.

## §1. Generalities on Compact Symmetric Spaces

We assemble the basic facts required to discuss $\zeta$-functions of compact symmetric spaces from the representation-theoretic viewpoint. In principle, everything here in §1 is contained in Garth Warner's book [6], and we refer to Warner [6] and Helgason [4] for the original sources (of which Cartan [3] is the principal one).

Fix a compact riemannian symmetric space $M$ and let $G$ be the largest connected group of isometries. Thus $G$ is a compact connected Lie group with an involutive automorphism $\sigma$, and $M=G/K$ where $K$ is an open subgroup of $G^\sigma=\{g\in G:\sigma(g)=g\}$, and the riemannian metric on $M$ derives from a positive definite invariant bilinear form on the Lie algebra of $G$.

$\hat{G}$ denotes the set of all equivalence classes $[\pi]$ of irreducible unitary representations $\pi$ of $G$. Given $[\pi]$, $V_\pi$ denotes the (finite dimensional complex Hilbert) space on which $\pi$ represents $G$. A class $[\pi]\in\hat{G}$ is of *class 1* relative to $K$ if there exists

$$0\neq v\in V_\pi \quad \text{such that} \quad \pi(k)\,v=v \quad \text{for all} \quad k\in K,$$

that is if $V_\pi$ has a nonzero $K$-fixed vector. Let us write

$$\hat{G}_K=\{[\pi]\in\hat{G}:\ [\pi] \text{ is of class 1 relative to } K\}. \tag{1.1}$$

$G$ acts on $L_2(M)$ through its left regular representation, that is

$$[l(g)f](x)=f(g^{-1}x) \quad \text{for } f\in L_2(M), \quad g\in G \quad \text{and } x\in M=G/K.$$

This action decomposes over $\hat{G}_K$ as follows.

1.2. THEOREM (É. Cartan [3]. *$L_2(M)=\sum_{[\pi]\in\hat{G}_K} V_\pi$ as unitary left $G$-module.*

*Proof.* $L_2(G)=\sum_{\hat{G}} V_\pi\otimes V_{\pi*}$ according to the Peter-Weyl Theorem. Here $V_\pi\otimes V_{\pi*}$ is identified with the space of all matrix coefficient functions

$$f_{v,w}(g)=\langle v, \pi(g)\, w\rangle \quad \text{for } v, w\in V_\pi \quad \text{and } g\in G$$

of $[\pi]$. The left and right actions of $G$ on $L_2(G)$ are

$$[l(g_1)\otimes r(g_2)f](x)=f(g_1^{-1}\, xg_2),$$

so the action on coefficients of $[\pi]$ is

$$\{l(g_1)\otimes r(g_2)\} f_{v,w}+ f_{\pi(g_1)v,\pi(g_2)w},$$

which is $\pi\otimes\pi^*$.

View $L_2(M=G/K)$ as $\{f\in L_2(G): f(gk)=f(g)$ for $g\in G$ and $k\in K\}$. Writing superscripts for invariants and $1_K$ for the trivial 1-dimensional representation of $K$, now

$$\begin{aligned} L_2(M)=L_2(G)^{r(K)}&=\textstyle\sum_{\hat{G}} V_\pi\otimes V_{\pi*}^K\\ &=\textstyle\sum_{\hat{G}} \operatorname{mult}(1_K, \pi^*\mid_K)\; V_\pi=\sum_{\hat{G}} \operatorname{mult}(1_K, \pi\mid_K)\; V_\pi\\ &=\textstyle\sum_{\hat{G}_K} \operatorname{mult}(1_K, \pi\mid_K)\; V_\pi \end{aligned}$$

as unitary left $G$-module. The latter multiplicities all $=1$; for example see Helgason [4, p. 408] for a proof of Gelfand's theorem that a certain algebra $C^\natural(G)$, which is $W^*$-dense in the commuting algebra of $l(G)$ on $L_2(M)$, is abelian. *q.e.d.*

Now let $\mathfrak{g}$ denote the Lie algebra of $G$, $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}_C$, and $\mathfrak{Z}$ the center of $\mathfrak{G}$. Every class $[\pi]\in\hat{G}$ maps every element of $\mathfrak{Z}$ to a scalar, giving an associative algebra homormorphism that we denote

$$\pi\colon \mathfrak{Z}\to C, \quad \text{infinitesimal character of } [\pi].$$

Recall that the riemannian metric on $M$ is derived from an invariant positive definite inner product on $\mathfrak{g}$. If $\{x_1, \ldots, x_n\}$ is an orthonormal basis then $\sum x_i^2\in\mathfrak{Z}$ and depends only on the inner product, and as differential operator

$$-\sum x_i^2 = \Delta, \quad \text{the Laplace-Beltrami operator on } M. \tag{1.3}$$

If we use the negative of the Cartan-Killing form of $\mathfrak{g}$ for the inner product, then $-\sum x_i^2 = \Omega$, the Casimir element of $\mathfrak{G}$, and so $\Delta = l(\Omega)$ on $L_2(M)$.

Define $\zeta_M(t) = \sum_\lambda e^{-\lambda t}$ where $\lambda$ ranges over the eigenvalues (with multiplicity) of the Laplace-Beltrami operator (1.3). This is the trace of the heat kernel. The Minakshisundaram-Pleijel zeta function $\sum_\lambda \lambda^{-}s$ is related to $\zeta_M$ by a Mellin transform.

1.4. COROLLARY. *If the riemannian metric on M is defined by the negative of the Cartan-Killing form of* $\mathfrak{g}$ *then M has* $\zeta$*-function given by*

$$\zeta(t) = \sum_{[\pi]\in \hat{G}_K} \quad (\textit{degree of } \pi)\, e^{-t\pi(\Omega)}$$

*where* $\Omega \in \mathfrak{Z}$ *is the Casimir element of* $\mathfrak{G}$.

To specify the $\zeta$-function of $M$ we now have to describe $\hat{G}_K$, and specify degree $(\pi)$ and $\pi(\Omega)$ for every class $[\pi] \in \hat{G}_K$.

The Lie algebra $\mathfrak{g}$ decomposes under the automorphism $\sigma$ as $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ where $\mathfrak{k}$ is the $(+1)$-eigenspace and $\mathfrak{s}$ is the $(-1)$-eigenspace. Of course, $\mathfrak{k}$ is the Lie algebra of $K$. Choose

$$\mathfrak{a}: \text{ maximal abelian subspace of } \mathfrak{s}, \tag{1.5a}$$

and

$$\Sigma_{\mathfrak{a}}^{+} \text{ positive } \mathfrak{a}_{\mathbf{C}}\text{-root system on } \mathfrak{g}_{\mathbf{C}}. \tag{1.5b}$$

Define $\mathfrak{m}' = \{x \in \mathfrak{k} : [x, \mathfrak{a}] = 0\}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Then

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \text{ is a Cartan subalgebra of } \mathfrak{g}. \tag{1.6a}$$

Any choice of positive $\mathfrak{t}_{\mathbf{C}}$-root system on $\mathfrak{m}_{\mathbf{C}}$ specifies a choice of

$$\begin{aligned} &\Sigma^{+}: \text{ positive } \mathfrak{h}_{\mathbf{C}}\text{-root system on } \mathfrak{g}_{\mathbf{C}} \text{ such that} \\ &\quad \Sigma_{\mathfrak{a}}^{+} = \{\phi|_{\mathfrak{a}} : \phi \in \Sigma^{+} \text{ and } \phi|_{\mathfrak{a}} \neq 0\}. \end{aligned} \tag{1.6b}$$

Each class $[\pi] \in \hat{G}$ is specified by its highest weight relative to $(\mathfrak{h}, \Sigma^{+})$, and the class 1 representations have a certain remarkable property.

1.7. THEOREM (É. Cartan [3]). *If* $[\pi]\in\hat{G}_K$ *has highest weight* $\lambda$ *relative to* $(\mathfrak{h}, \Sigma^+)$, *then* $\lambda(\mathfrak{t})=0$, *that is* $\lambda\in i\mathfrak{a}^*$.

*Proof.* The noncompact dual $\tilde{\mathfrak{g}}=\mathfrak{k}+i\mathfrak{s}$ of $\mathfrak{g}$ has Iwasawa decomposition $\tilde{\mathfrak{g}}=\mathfrak{n}+i\mathfrak{a}+\mathfrak{k}$ where $\mathfrak{n}$ is the sum of its $\Sigma_\mathfrak{a}^+$-negative $(i\mathfrak{a})$-root spaces. Writing capital German letters for universal enveloping algebras of complexifications, now $\mathfrak{G}=\mathfrak{N}\mathfrak{A}\mathfrak{K}$. Decompose

$$V_\pi=\sum V_{\pi,\nu} \text{ sum of weight spaces.}$$

Let $w$ be a nonzero $K$-fixed vector and decompose

$$w=\sum w_\nu \quad \text{where } w_\nu\in V_{\pi,\nu}.$$

Then

$$\begin{aligned} V_\pi&=\pi(\mathfrak{G})\,w=\pi(\mathfrak{N})\,\pi(\mathfrak{A})\,\pi(\mathfrak{K})\,w=\pi(\mathfrak{N})\,\pi(\mathfrak{A})\,w \\ &\subset\pi(\mathfrak{N})\sum\nolimits_{w_\nu\neq 0} V_{\pi,\nu}\subset\sum\nolimits_{\mu\leqslant\nu}\sum\nolimits_{w_\nu\neq 0} V_{\pi,\mu}. \end{aligned}$$

We conclude that $w_\lambda\neq 0$. As $\pi(\mathfrak{t})\,w=0$ now $\lambda(\mathfrak{t})=0$. As the weights are in $i\mathfrak{h}^*$now $\lambda\in i\mathfrak{a}^*$. *q.e.d.*

1.8 THEOREM (É. Cartan [3]; S. Helgason [4], [7]). *Define*

$$\Lambda^+=\{\lambda\in i\mathfrak{a}^*\colon \langle\lambda,\psi\rangle/\langle\psi,\psi\rangle \textit{ integer } \geqslant 0 \quad \textit{for all } \psi\in\Sigma_\mathfrak{a}^+\} \tag{1.9}$$

*where* $\langle\ ,\ \rangle$ *is the Cartan-Killing form. Then*

$$[\pi]\to \textit{highest weight relative to } (\mathfrak{h}, \Sigma_\mathfrak{a}^+)$$

*is an injective map from* $\hat{G}_K$ *into* $\Lambda^+$. *If K is connected and G is simply connected then it is a bijection.*

The proof is technical and we refer to Chapter III of Warner [6].

1.10. COROLLARY. *If M is simply connected and if its riemannian metric derives from the negative of the Cartan-Killing form of* $\mathfrak{g}$, *then M has* $\zeta$*-function given by*

$$\zeta_M(t)=\sum_{\lambda\in\Lambda^+} P(\lambda)\,e^{-tq(\lambda)} \tag{1.11}$$

*where*

$$\varrho=\tfrac{1}{2}\sum_{\phi\in\Sigma^+}\phi \quad \textit{and} \quad \varrho_\mathfrak{a}=\varrho|_\mathfrak{a}; \tag{1.12a}$$

$$P(\lambda)=\prod_{\phi\in\Sigma^+}\frac{\langle\lambda+\varrho,\phi\rangle}{\langle\varrho,\phi\rangle}; \tag{1.12b}$$

*and*

$$q(\lambda)=\|\lambda+\varrho\|^2-\|\varrho\|^2=\|\lambda+\varrho_{\mathfrak{a}}\|^2-\|\varrho_{\mathfrak{a}}\|^2. \tag{1.12c}$$

*Proof.* Write $[\pi_\lambda]$ for the class with highest weight $\lambda$. Simple connectivity and Theorem 1.8 insure that

$$\hat{G}_K\ni[\pi_\lambda]\to\lambda\in\Lambda^+$$

is bijective. The Hermann Weyl Degree Formula says that $[\pi_\lambda]$ has degree $P(\lambda)$ as in (1.12), and it is standard that $\pi_\lambda$ acts on the Casimir element by

$$\pi_\lambda(\Omega)=\|\lambda+\varrho\|^2-\|\varrho\|^2 \quad \text{for all } [\pi_\lambda]\in\hat{G}.$$

Here $\varrho=\varrho_{\mathfrak{a}}+\varrho_{\mathfrak{t}}$ with $\langle\lambda,\varrho_{\mathfrak{t}}\rangle=0$ by Theorem 1.7, so also

$$\pi_\lambda(\Omega)=\|\lambda+\varrho_{\mathfrak{a}}\|^2-\|\varrho_{\mathfrak{a}}\|^2.$$

Now $\pi_\lambda(\Omega)=q(\lambda)$ as in (1.12) and our formula for $\zeta(t)$ follows from Corollary 1.4. *q.e.d.*

In the sequel we will explicitly calculate the ingredients (1.12) for symmetric spaces of rank 1 (that is, where $\dim\mathfrak{a}=1$), obtaining explicit formulae for their $\zeta$-functions, and then study the asymptotic behaviour of these $\zeta$-functions.

## §2. Odd Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the $\zeta$-functions of the spheres and real projective spaces of odd dimension $2n-1$,

$$S^{2n-1}=SO(2n)/SO(2n-1), \qquad n\geqslant 1, \tag{2.1a}$$

and

$$P^{2n-1}(R)=S^{2n-1}/\{\pm I\}=SO(2n)/O(2n-1). \tag{2.1b}$$

If $n=1$, both are circles $S^1=\{z\in\mathbf{C}:|z|=1\}=\{e^{i\theta}:\theta \text{ real}\}$. $L_2(S^1)=\sum_{-\infty}^{\infty}V_m$ where $V_m$ is the 1-dimensional span of

$$f_m(e^{i\theta})=e^{im\theta}, \qquad m \text{ integer}.$$

Normalize the riemannian metric so that the circle has length $l$. Then the metric is $ds^2=(l/2\pi)^2\, d\theta^2$, so the circle has Laplace-Beltrami operator

$$\Delta=-(2\pi/l)^2 \frac{\partial^2}{\partial\theta^2}:\ f_m \mapsto (2\pi/l)^2\, m^2 f_m .$$

We conclude that the circle of length $l$ has $\zeta$-function

$$\zeta_{S^1}(t)=1+2\sum_{m=1}^{\infty} e^{-t\{2\pi m/l\}^2} \tag{2.2}$$

If $n=2$ then $G=SO(4)$ has Dynkin diagram $D_2$: $\underset{\alpha_1}{\circ}\ \underset{\alpha_2}{\circ}$. Then $\Sigma^+=\{\alpha_1, \alpha_2\}$ and $\Sigma_{\mathfrak{a}}^+=\{\alpha\}$ where $\varrho=\varrho_{\mathfrak{a}}=\alpha=\frac{1}{2}(\alpha_1+\alpha_2)$, so $\Lambda^+=\{m\alpha: m\geqslant 0 \text{ integer}\}$ and $q(m\alpha)=\|m\alpha+\varrho_{\mathfrak{a}}\|^2-\|\varrho_{\mathfrak{a}}\|^2=(m^2+2m)\, \|\varrho_{\mathfrak{a}}\|^2$ and we calculate

$$P(m\alpha)=\frac{\langle m\alpha+\varrho, \alpha_1\rangle}{\langle\varrho, \alpha_1\rangle}\cdot\frac{\langle m\alpha+\varrho, \alpha_2\rangle}{\langle\varrho, \alpha_2\rangle}=(m+1)^2 .$$

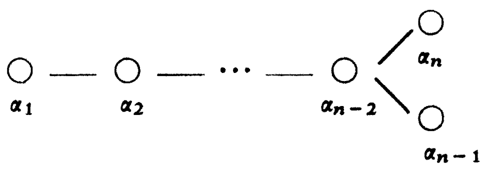
Using the negative of the Killing form to specify the riemannian metrics of $S^3$ and $P^3(R)$, the tables at the end of Bourbaki [2] show $\langle\alpha_i, \alpha_i\rangle=\frac{1}{2}$, so $\|\varrho_{\mathfrak{a}}\|^2=\frac{1}{4}\langle\alpha_1+\alpha_2, \alpha_1+\alpha_2\rangle=\frac{1}{4}$, and

$$\zeta_{S^3}(t)=\sum_{m=0}^{\infty} (m+1)^2\, e^{-t\,(m^2+2m)/4} . \tag{2.3a}$$

It is classical that $\pi_{m\alpha}(-I)=1$ just when $m$ is even, so also

$$\zeta_{P^3(R)}(t)=\sum_{r=0}^{\infty} (2r+1)^2\, e^{-t\,(r^2+r)} . \tag{2.3b}$$

Now we assume $n\geqslant 3$ in (2.1) so that $G=SO(2n)$ is a simple group of type $D_n$, and denote its Dynkin diagram




with

$$\Sigma_{\mathfrak{a}}^+=\{\alpha\},\quad \alpha_1\big|_{\mathfrak{a}}=\alpha \quad\text{and}\quad \alpha_i\big|_{\mathfrak{a}}=0 \quad\text{for } i>1 .$$

Relative to an appropriate positive multiple of the Cartan-Killing form, $i\mathfrak{h}^*$ has orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ such that

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 1 \leqslant i < n \quad \text{and} \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n .$$

Thus $\Sigma^+$ consists of the roots $\varepsilon_i \pm \varepsilon_j$ for $1 \leqslant i < j \leqslant n$, and so $\alpha = \varepsilon_1$ and $\varepsilon_1 \pm \varepsilon_j$ $(1 < j \leqslant n)$ are the roots that restrict to $\alpha$.
Now

$$\Lambda^+ = \{m\varepsilon_1 : m \geqslant 0 \text{ integer}\}, \qquad \varrho_\alpha = (n-1)\,\varepsilon_1 \quad \text{and} \quad \varrho = \sum_{j=1}^{n-1} (n-j)\,\varepsilon_j .$$

If $1 \leqslant i < j \leqslant n$ then $\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle = \{(n-i) \pm (n-j)\} \, \|\varepsilon_1\|^2$, so

$$\frac{\langle m\varepsilon_1 + \varrho, \varepsilon_i \pm \varepsilon_j \rangle}{\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle} = 1 \quad \text{if } i > 1, = \frac{m + (n-1) \pm (n-j)}{(n-1) \pm (n-j)} \quad \text{if } i = 1 .$$

That gives us

$$P(m\varepsilon_1) = \prod_{j=2}^{n} \frac{m + 2n - j - 1}{2n - j - 1} \cdot \frac{m - 1 + j}{j - 1} = \frac{m + n - 1}{n - 1} \prod_{k=1}^{2n-3} \frac{m+k}{k} .$$

Recall from the tables at the end of Bourbaki [2] that $\|\varepsilon_1\|^2 = 1/4\,(n-1)$. Now

$$q(m\varepsilon_1) = \|m\varepsilon_1 + \varrho_\alpha\|^2 - \|\varrho_\alpha\|^2 = \{m^2 + 2m(n-1)\}/4(n-1) .$$

Now Corollary 1.10 gives us

2.4. THEOREM. *Let $S^{2n-1}$ denote the sphere of odd dimension $2n-1$, $n \geqslant 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has $\zeta$-function*

$$\zeta_{S^{2n-1}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{m+n-1}{n-1} \cdot \prod_{k=1}^{2n-3} \frac{m+k}{k} \right\} e^{-t\,\{m^2 + 2m(n-1)\}/4\,(n-1)} . \tag{2.5}$$

The real projective space $P^{2n-1}(R) = S^{2n-1}/\{\pm I\}$ has $\zeta$-function given by summing the summands of (2.5) whose representations $[\pi_{m\varepsilon_1}]$ occur in $L_2(P^{2n-1}(R))$, that is the ones with a vector fixed under the subgroup $SO(2n-1) \cup (-I_{2n}) \cdot SO(2n-1)$. These are the $[\pi_{m\varepsilon_1}]$ whose kernel contains $-I_{2n}$, which are easily seen to be the ones for which $m$ is even.

2.6. COROLLARY. *Let $P^{2n-1}(R)$ denote the real projective space of odd dimension $2n-1$, $n \geqslant 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has $\zeta$-function*

$$\zeta_{P^{2n-1}(R)}(t) = \sum_{r=0}^{\infty} \left\{ \frac{2r+n-1}{n-1} \cdot \prod_{k=1}^{2n-3} \frac{2r+k}{k} \right\} e^{-t\{r^2+r(n-1)\}/(n-1)}. \tag{2.7}$$

## §3. Even Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the $\zeta$-functions of the spheres and real projective spaces of even dimension $2n$,

$$S^{2n} = SO(2n+1)/SO(2n), \quad n \geqslant 1, \tag{3.1a}$$

and

$$P^{2n}(R) = S^{2n}/\{\pm I\} = SO(2n+1)/SO(2n) \times O(1). \tag{3.1b}$$

$G = SO(2n+1)$ has Dynkin diagram

$$B_n: \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} = \underset{\alpha_n}{\bullet}$$

with

$$\Sigma_{\mathfrak{a}}^{+} = \{\alpha\}, \quad \alpha_1\big|_{\mathfrak{a}} = \alpha \quad \text{and} \quad \alpha_i\big|_{\mathfrak{a}} = 0 \quad \text{for } i > 1.$$

Arguing as in §2 one proves

3.2. THEOREM. *Let $S^{2n}$ denote the sphere of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n+1)$. It has $\zeta$-function*

$$\zeta_{S^{2n}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} \right\} e^{-t\{m^2+m(2n-1)\}/(4n-2)}. \tag{3.3}$$

and

3.4. COROLLARY. *Let $P^{2n}(R)$ denote the real projective space of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the*

*Cartan-Killing form $SO(2n+1)$. It has $\zeta$-function*

$$\zeta_{P^{2n}(R)}(t)=\sum_{r=0}^{\infty}\left\{\frac{4r+2n-1}{2n-1}\prod_{k=1}^{2n-2}\frac{2r+k}{k}\right\}e^{-t\{2r^2+r(2n-1)\}/(2n-1)}. \tag{3.5}$$

## §4. Complex Projective Spaces

We state the formula for the $\zeta$-function of the complex projective spaces

$$P^n(C)=U(n+1)/U(n)\times U(1)=SU(n+1)/S(U(n)\times U(1)) \tag{4.1}$$

of complex dimension $n$, real dimension $2n$. Since $P^1(C)$ is the sphere $S^2$, already considered in §3, we will work under the hypothesis $n>1$. Then a glance at the case $n=1$ of (3.3) will show our conclusion valid in general.

$G=SU(n+1)/\{e^{2\pi ik/(n+1)}I\}$ has Dynkin diagram

$$A_n:\ \underset{\alpha_1}{\circ}-\underset{\alpha_2}{\circ}-\cdots-\underset{\alpha_n}{\circ}$$

with

$$\Sigma^+=\{\alpha, 2\alpha\},\qquad \alpha_1|_{\mathfrak{a}}=\alpha=\alpha_n|_{\mathfrak{a}},\qquad \alpha_i|_{\mathfrak{a}}=0 \quad\text{for } 1<i<n.$$

Arguing as before and using the case $n=1$ of Theorem 3.2,

4.2. THEOREM. *Let $P^n(C)$ denote the complex projective $n$-space with riemannian metric induced by the negative of the Cartan-Killing form of $SU(n+1)$. It has $\zeta$-function*

$$\zeta_{P^n(C)}(t)=\sum_{m=0}^{\infty}\left\{\frac{2m+n}{n}\prod_{k=0}^{n-1}\left(\frac{m+k}{k}\right)^2\right\}e^{-t\{m^2+mn\}/(n+1)} \tag{4.3}$$

## §5. Quaternionic Projective Spaces

Here is the formula for the $\zeta$-functions of the quaternionic projective spaces

$$P^{n-1}(Q)=Sp(n)/Sp(n-1)\times Sp(1),\qquad n\geqslant 2, \tag{5.1}$$

of real dimension $4(n-1)$. Here note that $P^1(Q)=S^4$.

$G=Sp(n)/\{\pm I\}$ has Dynkin diagram

$$C_n:\ \underset{\alpha_1}{\bullet}-\underset{\alpha_2}{\bullet}-\cdots-\underset{\alpha_{n-1}}{\bullet}=\underset{\alpha_n}{\circ}$$

with

$$\Sigma_{\mathfrak{a}}^{+}=\{\alpha, 2\alpha\}, \quad \alpha_2|_{\mathfrak{a}}=\alpha, \ \alpha_i|_{\mathfrak{a}}=0 \quad \text{for } i\neq 2.$$

An argument similar to that of §2 gives

5.2. THEOREM. *Let $P^{n-1}(Q)$ denote the quaternionic projective $n-1$ space, with riemannian metric induced by the negative of the Cartan-Killing form of $Sp(n)$. It has $\zeta$-function*

$$\zeta_{P^{n-1}(Q)}(t)=\sum_{m=0}^{\infty}\left\{\frac{2m+2n-1}{2n-1}\cdot\prod_{r=2}^{2n-2}\frac{m+r}{r}\cdot\prod_{s=1}^{2n-3}\frac{m+s}{s}\right\}e^{-t(m^2+2mn-m)/2(n+1)}.$$

Notice that the case $n=2$ is $P^1(Q)=S^4$, where both (3.3) and (5.3) provide the same $\zeta$-function

$$\zeta_{S^4}(t)=\sum_{m=0}^{\infty}\left\{\frac{2m+3}{3}\cdot\frac{m+2}{2}\cdot\frac{m+1}{1}\right\}e^{-t(m^2+3m)/6} \tag{5.4}$$

## §6. The Cayley Projective Plane

Finally, we work out the $\zeta$-function for the Cayley projective plane

$$P^2(\mathrm{Cay})=F_4/\mathrm{Spin}(9), \quad \text{real dimension } 16. \tag{6.1}$$

$G=F_4$ has Dynkin diagram


with $\Sigma^+=\{\alpha, 2\alpha\}$ where $\alpha_4|_{\mathfrak{a}}=\alpha$ and the other three $\alpha_i|_{\mathfrak{a}}=0$. Relative to an appropriate multiple of the Cartan-Killing form, $i\mathfrak{h}^*$ has orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ with

$$\alpha_1=\varepsilon_2-\varepsilon_3, \ \alpha_2=\varepsilon_3-\varepsilon_4, \ \alpha_3=\varepsilon_4 \quad \text{and} \quad \alpha_4=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4).$$

Thus $\Sigma^+$ consists of the roots

$$\varepsilon_i(1\leqslant i\leqslant 4), \ \varepsilon_i\pm\varepsilon_j(1\leqslant i<j\leqslant 4), \quad \tfrac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4).$$

Now $\alpha=\frac{1}{2}\varepsilon_1$ and the $\frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)$ are the roots restricting to $\alpha$, $2\alpha=\varepsilon_1$ and the roots restricting to it are $\varepsilon_1$, $\varepsilon_1\pm\varepsilon_2$, $\varepsilon_1\pm\varepsilon_3$, $\varepsilon_1\pm\varepsilon_4$ .

Thus

$$\Lambda^+ = \{m\varepsilon_1 : m \geqslant 0 \text{ integer}\}, \quad \varrho_a = 11\alpha = \tfrac{11}{2}\varepsilon_1, \quad \varrho = \tfrac{1}{2}(11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4).$$

Now calculate

$$P(m\varepsilon_1) = \frac{2m+11}{11} \cdot \prod_{q=1}^{3} \frac{m+q}{q} \prod_{r=4}^{7} \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{16} \frac{m+s}{s}.$$

From the tables at the end of Bourbaki [2], $\|\varepsilon_1\|^2 = 1/18$, so $\|\alpha\|^2 = 1/72$, and thus

$$q(m\varepsilon_1) = \|(2m+11)\,\alpha\|^2 - \|11\alpha\|^2 = (m^2 + 11m)/18.$$

Now Corollary 1.10 says

6.2. THEOREM. *Let $P^2$* (Cay) *denote the Cayley projective plane with riemannian metric induced by the negative of the Cartan-Killing form of $F_4$. It has $\zeta$-function*

$$\zeta_{P^2\,(\mathrm{Cay})}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+11}{11} \cdot \prod_{q=1}^{3} \frac{m+q}{q} \cdot \prod_{r=4}^{7} \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{10} \frac{m+s}{s} \right\} e^{-t(m^2+11m)/18}. \tag{6.3}$$

## § 7. The Asymptotic Expansion for Compact Riemannian Manifolds

We wish to give a brief account of the properties of the eigenvalues of the Laplacian, $\Delta$, of a compact riemannian manifold. We will assume $(M, g)$ is a compact riemannian manifold without boundary of dimension $d$. Then $\Delta$ will be a self-adjoint elliptic operator with eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. It is known that these eigenvalues contain a great deal of geometric information about $(M, g)$ and a tool to recover some of this information is the zeta-function, $\zeta_M(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}$. The interest of the zeta-function comes from the following theorem.

7.1. THEOREM (Minakshisundaram). *If $(M, g)$ is a compact riemannian manifold without boundary of dimension d then there exist constants $a_n$, $n \geqslant 0$, such that*

$$\zeta_M(t) = (4\pi t)^{-d/2}\,(a_0 + a_1 t + \cdots + a_k t\text{-} + 0(t^{k+1}))$$

as $t \downarrow 0$.

*Proof.* See Berger [1].

In the remainder of this paper we will compute the coefficients, $a_m$, of the asymptotic expansion of $\zeta_M(t)$ when $M$ is a symmetric space of rank one.

## §8. Summation Lemmas

We wish to analyze the zeta-functions derived in the first part of this paper using certain classical summation formulas. The formulas we need will be gathered together in this section.

8.1. LEMMA. *Let* $f(t)=\sum_{s\in\mathbf{Z}} e^{-s^2t}$. *Then* $f(t)=\pi^{1/2}t^{-1/2}+0(e^{-1/t})$ as $t\downarrow 0$ *and* $(-1)^k f^{(k)}=\sum_{s\in\mathbf{Z}} s^{2k} e^{-s^2t}=(\frac{1}{2})(\frac{3}{2})\cdots(2k-1)/2\,\pi^{1/2}\,t^{-(2k+1)/2}+0(e^{-1/t})$ *as* $t\downarrow 0$.

*Proof.* If $r(x)=e^{-x^2t}$ we may apply the Poisson summation formula to derive

$$\sum_{s\in\mathbf{Z}} e^{-s^2t}=\pi^{1/2}t^{-1/2}\sum_{s\in\mathbf{Z}} e^{-\pi^2s^2/t}. \tag{8.2}$$

The first part of the Lemma now follows by noting that $\sum_{s\neq 0} e^{-\pi^2s^2/t}$ is $0(e^{-1/t})$. We may now take derivatives with respect to $t$ to derive the second formula. *q.e.d.*

For the remainder of the paper we will define $b_0=1$, $b_k=(\frac{1}{2})(\frac{3}{2})\cdots((2k-1)/2)$, $k\geqslant 1$. Note $\pi^{1/2}b_k=\Gamma((2k+1)/2)$.

8.3. LEMMA. *Let* $g(t)=\sum_{j=0}^{\infty}(2j+1)\,e^{-(j+1/2)^2t}$. *Then*

$$g(t)=\frac{1}{t}+c_0+c_1t+\frac{c_2}{2!}t^2+\cdots+\frac{c_n}{n!}t^n+0(t^{n+1})$$

*and*

$$g^{(k)}(t)=\frac{(-1)^k k!}{t^{k+1}}+c_k+c_{k+1}t+\cdots+\frac{c_{k+n}}{n!}t^n+0(t^{n+1})$$

*as* $t\downarrow 0$, *where*

$$c_n=\frac{(-1)^n}{(n+1)}B_{2n+2}(1-2^{-2n-1}),$$

$B_n$ *is the nth Bernoulli number.*

*Proof.* See Mulholland [5].

Before proceeding we wish to note that $\sum_{n=0}^{\infty}(c_n/n!)\,t^n$ is not convergent. Therefore Lemma 8.3 gives an asymptotic series.

8.4. LEMMA. *Let* $g_1(t)=\sum_{j=0}^{\infty}(4j+1)\,e^{-(2j+1/2)^2t}$ *and* $g_2(t)=\sum_{j=0}^{\infty}(4j+3)\,e^{-(2j+3/2)^2t}$. *Then* $g_1(t)+g_2(t)=g(t)$ *and*

$$g_i^{(k)}(t)=\tfrac{1}{2}\left(\frac{1}{t}+c_0+c_1t+\cdots+\frac{c_n}{n!}t^n+0(t^{n+1})\right) \qquad i=1,2$$

$$g_i^{(k)}(t)=\tfrac{1}{2}\left(\frac{(-1)^k k!}{t^{k+1}}+c_k+c_{k+1}t+\cdots+\frac{c_{k+n}}{n!}t^n+0(t^{n+1})\right) \qquad i=1,2$$

*as* $t\downarrow 0$.

*Proof. See Mulholland* [5].

The last lemma of this section is similar to Lemma 8.3.

8.5. LEMMA. *Let* $h(t)=\sum_{j=0}^{\infty} 2j\, e^{-j^2t}$. *Then*

$$h(t)=\frac{1}{t}+d_0+d_1t+\frac{d_2}{2!}t^2+\cdots+\frac{d_n}{n!}t^n+0(t^{n+1})$$

*and*

$$h^{(k)}(t)=\frac{(-1)^k k!}{t^{k+1}}+d_k+d_{k+1}t+\cdots+\frac{d_{n+k}}{n!}t^n+0(t^{n+1})$$

*as* $t\downarrow 0$ *with* $d_n=[(-1)^n/(n+1)]\, B_{2n+2}$.

*Proof.* A slight modification of Mulholland's method gives the desired result.

## §9. The Asymptotic Expansion for Odd Dimensional Spheres and Real Projective Spaces

Starting in this section we will analyze the zeta-functions developed in §1 through §6 using the results in the previous section. The goal of this section is to calculate the coefficients, $a_n$, for the symmetric spaces analyzed in §2.

$M=S^1$. $\zeta_M(t)=f((4\pi^2/l^2)\,t)$. Consequently

$$\zeta_M(t)=\pi^{1/2}\left(\frac{4\pi^2}{l^2}t\right)^{-1/2}+0(e^{-1/t})$$
$$=l(4\pi t)^{-1/2}+0(e^{-1/t}).$$

Thus we conclude $a_0=l$ and $a_m=0$, $m\geqslant 1$.

$M=S^3$. $\zeta_M(t)=\sum_{m=0}^{\infty}(m+1)^2\, e^{-t(m^2+2m)/4}$

$$= \sum_{p=0}^{\infty} p^2 e^{-t(p^2-1)/4} \qquad \text{(where } p=m+1\text{)}$$

$$= \tfrac{1}{2} e^{t/4} \sum_{p=-\infty}^{\infty} p^2 e^{-p^2 t/4}$$

$$= \tfrac{1}{2} e^{t/4} (-1) f'\left(\frac{t}{4}\right)$$

$$= 2\pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{16\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES,$$

therefore $a_m = 16\pi^2/4^m m!$. $ES$ is an error which is exponentially small as $t \downarrow 0$.

$M = P^3(\mathbf{R})$.

$$\zeta_M(t) = \sum_{r=0}^{\infty} (2r+1)^2 e^{-(r^2+r)t}$$

$$= \sum_{r=0}^{\infty} (2r+1)^2 e^{-[(2r+1)^2-1]t/4}$$

$$= \tfrac{1}{2} e^{t/4} \left[ \sum_{s \in \mathbf{Z}} s^2 e^{-s^2 t/4} - \sum_{s \in 2\mathbf{Z}} s^2 e^{-s^2 t/4} \right]$$

$$= \tfrac{1}{2} e^{t/4} \left[ (-1) f'\left(\frac{t}{4}\right) + 4 f'(t) \right]$$

$$= \frac{\pi^{1/2}}{2} e^{t/4} [4t^{-3/2} - 2t^{-3/2}] + ES$$

$$= \pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{8\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES$$

so $a_m = 8\pi^2/4^m m!$.

$M = S^{2a-1}$, $n \geqslant 3$. At this point we wish to make a general comment about the procedure we will employ in this and following sections. Though we have established a bijection between the representations of $\hat{G}_K$ and functionals $\lambda \in \Lambda^+$ the proper variable to use is not $\lambda$ but $\lambda + \varrho_\alpha$. This is the guiding principle behind all changes of variable which are used in this and following sections.

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{m+n-1}{n-1} \prod_{k=0}^{2n-3} \frac{m+k}{k} \right\} e^{-t\{m^2+2m(n-1)\}/4(n-1)}.$$

We will let $s = m+n-1$. Then

$$\zeta_M(t)=\frac{1}{(n-1)(2n-3)!}\sum_{s=n-1}^{\infty}\left\{\prod_{j=0}^{n-2}(s^2-j^2)\right\}e^{-\{s^2-(n-1)^2\}t/4(n-1)}$$

$$=\frac{e^{((n-1)/4)t}}{2(n-1)(2n-3)!}\sum_{s\in\mathbf{Z}}\left\{\prod_{j=0}^{n-2}(s^2-j^2)\right\}e^{-s^2t/4(n-1)} \tag{9.1}$$

We now define $\alpha_{k,n}$ by

$$\prod_{j=0}^{n-2}(s^2-j^2)=\sum_{k=0}^{n-1}\alpha_{k,n}s^{2k}.$$

Then

$$\zeta_M(t)=\frac{e^{(n-1)t/4}}{(2n-2)!}\sum_{s\in\mathbf{Z}}\sum_{k=0}^{n-1}\alpha_{k,n}s^{2k}\,e^{-s^2(t/4(n-1))}$$

$$=\frac{e^{(n-1)t/4}}{(2n-2)!}\sum_{k=0}^{n-1}\alpha_{k,n}(-1)^k f^{(k)}\left(\frac{t}{4(n-1)}\right)$$

$$=\frac{\pi^{1/2}\,e^{(n-1)t/4}}{(2n-2)!}\sum_{k=0}^{n-1}\alpha_{k,n}b_k\left(\frac{t}{4(n-1)}\right)^{-(2k+1)/2}+ES.$$

Now by convolving the series we conclude that

$$a_m=\frac{2^{2n-1}\pi^n}{(2n-2)!}\sum_{k=0}^{m}\frac{(n-1)^{n-1/2}}{k!}\alpha_{n-1-k,n}b_{n-1-k}4^{n-1/2-2k}\quad\text{if } m<n$$

and

$$a_m=\frac{2^{2n-1}\pi^n}{(2n-2)!}\sum_{k=m-n}^{m}\frac{(n-1)^{m-1/2}}{k!}\alpha_{m-k-1,n}b_{m-k-1}4^{m-1/2-2k}$$

if $m\geqslant n$.

$M=P^{2n-1}(\mathbf{R})$. To compute the asymptotic expansion for the projective spaces we take $s=r+(n-1)/2$. Then

$$\frac{2r+n-1}{n-1}\prod_{j=1}^{2n-3}\frac{2r+k}{k}=\frac{2}{(2n-2)!}\prod_{j=0}^{n-2}4(s^2-(j/2)^2)=\frac{4^{n-1/2}}{(2n-2)!}\sum_{j=0}^{n-1}\alpha'_{j,n}s^{2j}.$$

$$\zeta_M(t)=\frac{4^{n-1}}{(2n-2)!}e^{(n-1)t/4}\sum_{s\in 1/2\mathbf{Z}}\sum_{j=0}^{n-1}\alpha'_{j,n}s^{2j}\,e^{-s^2t/n-1}$$

$$=\frac{4^{n-1}}{(2n-2)!}e^{(n-1)t/4}\sum_{j=0}^{n-1}\alpha'_{j,n}4^{-j}f^{(j)}\left(\frac{t}{4(n-1)}\right)$$
$$=\frac{4^{n-1}\pi^{1/2}}{(2n-2)!}e^{(n-1)t/4}\sum_{j=0}^{n-1}\alpha'_{j,n}b_j4^{-j}\left(\frac{t}{4(n-1)}\right)^{-(2j+1)/2}+ES.$$

Therefore

$$a_m=\frac{4^{n-1}\pi^n}{(2n-2)!}\sum_{j=0}^{m}\frac{(n-1)^{n-1/2}}{j!}\alpha'_{n-1-j,n}b_{n-1-j}4^{-j-1/2}\quad\text{if } m<n$$

and

$$a_m=\frac{4^{n-1}\pi^n}{(2n-2)!}\sum_{j=m-n}^{m}\frac{(n-1)^{m-1/2}}{j!}\alpha'_{m-j-1,n}b_{m-j-1}4^{-j-1/2}\quad\text{if } m\geqslant n.$$

## §10. The Asymptotic Expansion for Even Dimensional Spheres and Real Projective Spaces

$M=S^{2n}$. As in the preceding section we will change variables to utilize the lemmas of §8. Recall that

$$\zeta_M(t)=\sum_{m=0}^{\infty}\left\{\frac{2m+2n-1}{2n-1}\prod_{k=1}^{2n-2}\frac{m+k}{k}\right\}e^{-t\{m^2+m(2n-1)\}/4n-2}.$$

We will let $s=m+(n-1/2)$. Then

$$\frac{2m+2n-1}{2n-1}\prod_{k=1}^{2n-2}\frac{m+k}{k}=\frac{2s}{(2n-1)!}\prod_{j=1/2}^{n-3/2}(s^2-j^2)=\frac{2s}{(2n-1)!}\sum_{j=0}^{n-1}\beta_{j,n}s^{2j},$$

where the product runs through the half-integers which are not integers. Also $t\{m^2+m(2n-1)\}/4n-2=t\{s^2-(n-1/2)^2\}/4n-2$. Therefore

$$\zeta_M(t)=\frac{e^{(n-1/2)t/4}}{(2n-1)!}\sum_{s\geqslant 1/2}\sum_{j=0}^{n-1}\beta_{j,n}2s^{2j+1}e^{-s^2t/4(n-1/2)}$$
$$=\frac{e^{(n-1/2)t/4}}{(2n-1)!}\sum_{j=0}^{n-1}\beta_{j,n}(-1)^j g^{(j)}\left(\frac{t}{4(n-1/2)}\right)$$

Thus

$$a_m=\frac{(4\pi)^2}{(2n-1)!}\sum_{k=0}^{m}\frac{(n-1-m+k)!}{k!}4^{n-m-1}(n-1/2)^{m-n+2k+1}\beta_{n-1-m+k,n}\quad\text{if } m<n$$

and

$$a_m = \frac{(4\pi)^n}{(2n-1)!}\left(\sum_{k=0}^{n-1} \frac{k!}{(m-n+k+1)!}(n-1/2)^{m-n+2k+2}\,\beta_{k,n}4^{n-m} + \sum_{k=0}^{m-n}\sum_{j=0}^{n-1} \frac{(-1)^j c_{j+k}\beta_{j,n}(n-1/2)^{m-n-2k}}{k!(m-n-k)!}4^{n-m}\right)$$

if $m \geqslant n$.

$M = P^{2n}(\mathbf{R})$. Recall that

$$\zeta_M(t) = \sum_{r=0}^{\infty}\left\{\frac{4r+2n-1}{2n-1}\prod_{k=0}^{2n-2}\frac{2r+k}{k}\right\} e^{-t\{2r^2+r(2n-1)\}/2n-1}.$$

We will let $s = r + ((2n-1)/4)$. Then

$$\frac{4r+2n-1}{2n-1}\prod_{k=1}^{2n-2}\frac{2r+k}{k} = \frac{4s}{(2n-1)!}\prod_{j=1/2}^{n-3/2}\left(4s^2 - \frac{j^2}{4}\right) = \frac{4s}{(2n-1)!}\sum_{j=0}^{n-1}\beta'_{j,n}(2s)^{2j}$$

and

$$t\{2r^2 + r(2n-1)\}/(2n-1) = t\left(2s^2 - 2\left(\frac{2n-1}{4}\right)^2\right)/(2n-1).$$

Then

$$\begin{aligned}\zeta_M(t) &= \frac{e^{t((2n-1)/8)}}{(2n-1)!}\sum_{s\geqslant 1/2}\sum_{j=0}^{n-1}\beta'_{j,n}\,4s(2s)^{2j}\,e^{-2s^2t/2n-1}\\ &= \frac{e^{t(n-1/2)/4}}{(2n-1)!}\sum_{j=0}^{n-1}\beta'_{j,n}(-1)^j g_i^{(j)}\left(\frac{t}{2(2n-1)}\right)\end{aligned}$$

where $i = 1$ if $n$ is odd and $i = 2$ if $n$ is even.
Thus

$$a_m = \frac{(4\pi)^n}{2(2n-1)!}\sum_{k=0}^{m}\frac{(n-1-m+k)!}{k!}4^{n-m-1}(n-1/2)^{n-m+2k+1}\,\beta'_{n-1-m+k,n}$$

if $m < n$ and

$$a_m = \frac{(4\pi)^m}{2(2n-1)!}\left(\sum_{k=0}^{n-1}\frac{k!}{(m-n+k+1)!}(n-1/2)^{m-n+2k+2}\,\beta'_{k,n}4^{n-m} + \sum_{k=0}^{m-n}\sum_{j=0}^{n-1}\frac{(-1)^j c_{j+k}\beta'_{j,n}(n-1/2)^{m-n-2k}}{k!(m-n-k)!}4^{n-m}\right) \quad \text{if } m \geqslant n.$$

## §11. The Asymptotic Expansion for Complex Projective Spaces

We will have to treat $P^n(\mathbf{C})$ with 2 separate arguments according to whether $n$ is odd or even. The reason for this division is that when $n$ is even $\varrho_a \in \Lambda^+$ while when $n$ is odd $\varrho \notin \Lambda^+$. The treatment of the 2 cases will then differ only in that when $n$ is even we will use Lemma 8.3 while for $n$ odd we will use Lemma 8.5.

$M = P^n(\mathbf{C})$, *n odd*

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+n}{n} \prod_{k=1}^{n-1} \left( \frac{m+k}{k} \right)^2 \right\} e^{-t\{m^2+mn\}/(n+1)}.$$

Let $s = m + (n/2)$. Then

$$\frac{2m+n}{n} \prod_{k=1}^{n-1} \left( \frac{m+k}{k} \right)^2 = \frac{2s}{n!(n-1)!} \prod_{j=1/2}^{n/2-1} (s^2 - j^2)^2 = \frac{2s}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}$$

and $t\{m^2+mn\}/n+1 = t\{s^2-(n/2)^2\}/n+1$. Therefore

$$\begin{aligned}\zeta_M(t) &= \frac{e^{tn^2/4(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s \geqslant 1/2} 2s^{2k+1} e^{-s^2 t/(n+1)} \\ &= \frac{e^{tn^2/4(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} g^{(k)}(t/(n+1)).\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \sum_{k=0}^{m} (n+1)^{n-1-m} \left( \frac{n}{2} \right)^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n}$$

if $m < n-1$ and

$$\begin{aligned}a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} &\left( \sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} \left( \frac{n}{2} \right)^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right. \\ &\left. + \sum_{k=0}^{m-n+1} \left( \frac{n^2}{4(n+1)} \right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} c_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right)\end{aligned}$$

if $m \geqslant n-1$.

$M = P^n(\mathbf{C})$, *n even.* Since $n$ is even we will let $n = 2n_0$. If $s = m + n_0$ then we will write

$$\prod_{k=1}^{n-1} (m+k)^2 = \prod_{k=0}^{n_0-1} (s^2 - j^2)^2 = \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}.$$

Then

$$\zeta_M(t) = \frac{e^{tno^2/(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s\geqslant 0} 2s^{2k+1}\, e^{-s^2t/(n+1)}$$
$$= \frac{e^{tno^2/(n+1)}}{n!\,n-1!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} h^{(k)}(t/(n+1)).$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \sum_{k=0}^{m} (n+1)^{n-1-m}\, n_0^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n} \quad \text{if } m<n-1$$

and

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \left( \sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} n_0^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right.$$
$$\left. + \sum_{k=0}^{m-n-1} \left(\frac{n_0^2}{n+1}\right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} d_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right) \quad \text{if } m\geqslant n-1.$$

## § 12. The Asymptotic Expansion of Quaternionic Projective Spaces

If $M = P^{n-1}(Q)$, $n \geqslant 2$ then recall that

$$\zeta_M(t) = \sum_{m=0}^{\infty} \frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{P=1}^{2n-2} \frac{m+P}{P} e^{-t(m^2+2mn-m)/2(n+1)}.$$

We will let $s = m + (n-1/2)$. Then

$$\frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{P=1}^{2n-3} \frac{m+P}{P} = \frac{2s}{(2n-1)!(2n-3)!} \prod_{j=1/2}^{n-3/2} (s^2-j^2) \prod_{j=1/2}^{n-5/2} (s^2-j^2)$$
$$= \frac{2s}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} s^{2k}$$

$$t(m^2+2mn-m)/2(n+1) = t(s^2-(n-1/2)^2)/2(n+1).$$

Therefore

$$\zeta_M(t) = \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \sum_{s\geqslant 1/2} \delta_{k,n} 2s^{2k+1}\, e^{-s^2}$$
$$= \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} (-1)^k g^{(k)}(t)$$

Thus

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!\,(2n-3)!} \sum_{k=0}^{m} \left( \frac{(n-1/2)^2}{2(n+1)} \right)^k \frac{(2n-3-m+k)!}{k!} \delta_{2n-3-m+k,\,n} \qquad \text{if } m < 2n-2$$

and

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!\,(2n-3)!} \left( \sum_{k=0}^{2n-3} \left( \frac{(n-1/2)^2}{2(n+1)} \right)^{2(m+2n-3-k)} \frac{k!}{(m+2n-3-k)!} \delta_{k,n} + \sum_{k=0}^{m-2n+2} \frac{(n-1/2)^{2k}}{2^k (n+1)^k k!} \sum_{j=0}^{2n-3} \frac{(-1)^j \delta_{j,n} c_{j+m-k}}{(m-k)!} \right) \quad \text{if } m \geqslant 2n-2 .$$

## §13. The Asymptotic Expansion of the Cayley Projective Plane

We will deal with the Cayley projective plane by letting $s = m + 11/2$. Then

$$P(m\varepsilon_1) = \frac{3!}{11!7!} 2s(s^2-(1/2)^2)^2 (s^2-(3/2)^2)^2 (s^2-(5/2)^2)(s^2-(7/2)^2)(s^2-(9/2)^2)$$
$$= \frac{3!}{11!7!} 2s \sum_{j=0}^{7} \eta_j s^{2j}$$

with

$$\eta_7 = 1, \quad \eta_6 = -\frac{170}{4}, \quad \eta_5 = \frac{10{,}437}{16}, \quad \eta_4 = -\frac{262{,}075}{64}, \quad \eta_3 = \frac{2{,}858{,}418}{256},$$
$$\eta_2 = -\frac{13{,}020{,}525}{1024}, \quad \eta_1 = \frac{18{,}455{,}239}{4096}, \quad \eta_0 = -\frac{8{,}037{,}225}{16{,}384}.$$

$t(m^2+11m)/18 = t(s^2-(11/2)^2)/18$ so

$$\zeta_{P^2(\mathrm{Cay})} = \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^{7} \sum_{s \geqq 1/2} \eta_j 2s^{2j+1} e^{-s^2 t}$$
$$= \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^{7} \eta_j (-1)^j g^{(j)}(t).$$

Therefore

$$a_m = \frac{3!}{7!11!} (4\pi)^8 \sum_{k=0}^{m} \left( \frac{121}{72} \right)^k \eta_{7-m+k} \frac{(7-m+k)!}{k!} \quad \text{if } m \leqslant 7$$

$$a_m = \frac{3!}{7!11!} (4\pi)^8 \left( \sum_{k=0}^{7} \left( \frac{121}{72} \right)^{(m+7-k)} \eta_k \frac{k!}{(m+7-k)!} + \sum_{k=0}^{m-8} \frac{(121/72)^k}{k!} \sum_{j=0}^{7} \frac{(-1)^j \eta_j c_{j+m-k}}{(m-k)!} \right), \quad \text{if } m > 7 .$$

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