IL NUOVO CIMENTO VOL. 28 A, N. 2 21 Luglio 1975

Charge Conjugation and Segal's Cosmology.

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(ricevuto il 26 Marzo 1975)

Summary. — We show that the operation of charge conjugation (and hence also CPT) can be realized as an actual geometric transformation within the framework of Segal's chronogeometric theory. We also discuss some geometric questions connected to the foundations of the chronogeometric theory.

1. – In a recent series of papers (1) SEGAL has developed a theory of the cosmos based on the fifteen-dimensional conformal group. This is the group of (locally defined) transformations of Minkowski space M which preserve the Lorentz metric up to a scalar multiple, or, what amounts to the same thing, which preserve the light-cones. This group can also be characterized as the group of (locally defined) transformations which preserve Maxwell's equations. (It is very easy to see this fact, known since the turn of the century, using exterior algebra: Let V be a vector space endowed with a nondegenerate scalar product of arbitrary signature. The $*$-operator, cf. (2), maps $\wedge^k V^* \to \wedge^{n-k} V^*$ where $n = \dim V$, and depends on the choice of scalar product. If we modify the scalar product by multiplying it by a nonzero scalar, the various $*$-operators get multiplied by powers of that scalar. It is easy to check that if $n = 2m$

(1) I. SEGAL: *Astronomy and Astrophys.*, **18**, 143 (1972). Also *A variant of special relativity and extragalactic astronomy*, to appear.
(2) L. H. LOOMIS and S. STERNBERG: *Advanced Calculus* (Reading, Mass., 1966).

and we take $k = m$, the map $*$: $\Lambda^m V^* \to \Lambda^m V^*$ is unchanged. The equations $\mathrm{d}\omega = 0$, $\mathrm{d} * \omega = 0$ are thus conformally invariant, if ω is an exterior form of degree m on a $2m$-dimensional pseudo-Riemannian manifold. If we take $m = 1$ and $\omega = u\,\mathrm{d}x + v\,\mathrm{d}y$ on the Euclidean plane, we get the Cauchy-Riemann equations. If we take ω to be the two-form giving the electromagnetic field on Minkowski space, we get the Maxwell equations.)

Special relativity asserts that the group of symmetries of nature is the subgroup of the conformal group which consists of those transformations which also preserve the class of « inertial frames ». (We shall elaborate on this point below.) This group is then the eleven-dimensional group, which we shall denote by G_{11}, consisting of the (globally defined) Lorentz transformations, the scale transformations (*i.e.* dilatations of M) and the translations. Let g_{11} denote the Lie algebra of G_{11} and let g_{15} denote the Lie algebra of conformal vector fields on M. Thus each element of g_{15} is a globally defined vector field on M, but not every element of g_{11} can be exponentiated to a global transformation on M. The Lie algebra g_{11} is a subalgebra of g_{15} (and consists precisely of those vector fields which can be exponentiated). Let G_{15} be a Lie group whose Lie algebra is g_{15}, so that G_{11} is a closed subgroup of G_{15}. (We could choose G_{15} to be the simply connected group. However, we shall prefer to make a different choice; as we shall indicate in the next Section, we shall take G_{15} to be the group isomorphic to $SO_{2,4}$.) If we wish to regard G_{15} as the « group of physics », then it becomes natural to « complete » the Minkowski space M to obtain a manifold $\tilde{M}$ on which the entire group G_{15} acts. Let us examine what is involved in this procedure of « conformal completion ».

We are looking for a manifold $\tilde{M}$ which is a homogeneous space for the group G_{15} together with a smooth map, $f: M \to \tilde{M}$, such that

$$f(gm) = \tilde{g}f(m) \qquad \text{for all } g \in G_{11}$$

and

$$df_m(\xi_m) = \tilde{\xi}_{f(m)} \qquad \text{for all } \xi \in g_{15},$$

where ξ_m deotes the value of the vector field ξ at $m \in M$, where $\tilde{\xi}$ is the vector field on $\tilde{M}$ given by the action of G_{15} on $\tilde{M}$, and where we denote the action of $g \in G$ on $x \in \tilde{M}$ by $\tilde{g}x$. Now this problem, as it stands, does not admit a unique solution. Indeed, suppose that $(\tilde{M}, f)$ is such a solution, and let k be any element of G_{11} and let us define a new action of G_{15} on $\tilde{M}$, by setting

$$\tilde{g}' x = (kgk^{-1})^{\sim} x,$$

and a new map f' by

$$f'(m) = f(km).$$

Then it is easy to see that the new action and the new map so defined are again a solution to our problem.

On the other hand, a solution to our problem does exist. Indeed, let us pick some point $x_0 \in M$. Let g^0_{15} consist of the vector fields in g_{15} which vanish at x_0, and let $g^0_{11} = g_{11} \cap g^0_{15}$. As we shall see below, the algebra g^0_{15} generates a closed subgroup, call it H, of G_{15}. Then $H \cap G_{11}$ is the subgroup generated by $g_{11} \cap g^0_{15}$ and coincides with the isotropy group of x_0 in G_{11}. We now define $\tilde{M} = G_{15}/H$ and map $M = G/(H \cap G_{11})$ into $\tilde{M}$ by setting

$$f(g(H \cap G_{11})) = gH .$$

This gives a solution to our problem.

To see the structure of g^0_{15}, it is convenient to give a general description of the conformal algebra of any vector space V possessing a nondegenerate scalar product. We can identify vector fields on V with V-valued functions, using the linear structure on V. If ξ is any V-valued function, its differential $d\xi$ can be thought of as a $V \otimes V^*$-valued function, *i.e.* as a $\mathrm{Hom}(V, V)$-valued function. The condition that a vector field ξ is conformal can then be written as

$$d\xi \in o(V) \oplus \mathbf{R} \qquad \text{at all points} ,$$

where $o(V)$ denotes the orthogonal algebra of V and $\mathbf{R}$ the scalar multiples of the identity on V. It is well known that, if $\dim V \geqslant 3$, the only solutions to the above equations must be polynomial vector fields of degree at most two. Furthermore, if we break up any conformal vector field into its homogeneous components, this corresponds to a decomposition of the conformal algebra (if $\dim V \geqslant 3$) into a vector-space direct sum

$$V + (o(V) + \mathbf{R}) + V' ,$$

where V consists of the constant vector fields, $(o(V) + \mathbf{R})$ consists of the linear vector fields, and V' consists of the quadratic vector fields. The subspaces V and V' are nonsingularly paired under the Lie bracket into $o(V) + \mathbf{R}$, and in fact into the $\mathbf{R}$-component if we project onto the centre. In fact the structure of the conformal algebra can be most succinctly summarized as follows. We can, using the scalar product on V, identify $o(V)$ with $\wedge^2(V)$. Here the element $u \wedge v$ is identified with the linear transformation sending w into $(v, w)u - (u, w)v$. Let us construct a new vector space W two dimensions greater, obtained by adjoining two isotropic vectors f_{-1} and f_4. The scalar product on W is defined by keeping the old scalar product between elements of V and setting

$$(f_{-1}, v) = (f_4, v) = (f_{-1}, f_{-1}) = (f_4, f_4) = 0 \quad \text{and} \quad (f_{-1}, f_4) = 1 .$$

Then the *conformal algebra is isomorphic to the orthogonal algebra* $o(W)$, *with the constant vector fields being identified with elements of the form* $f_{-1}\wedge v$, *the quadratic fields with elements of the form* $f_4\wedge v$, and the infinitesimal scale transformation identified with the element $f_{-1}\wedge f_4$ (and $o(V)$ with $\wedge^2(V)\subset\wedge^2 W$). We refer to (3) for a description of these facts. In particular, the algebra g_{15} can be identified with the algebra $o_{2,4}$. If we take x_0 to be the origin, then g_{15}^0 consists of the linear and quadratic vector fields. By the above description, this is the eleven-dimensional subalgebra of g_{15} isomorphic to g_{11}, and, indeed, conjugate to g_{11} by an element of the adjoint group of g_{15}. Thus g_{15}^0 does indeed generate a closed subgroup of G_{15} and we obtain a comformal completion $\tilde{M}$ as indicated above. It is not difficult to see that $\tilde{M}$ admits a conformal structure invariant under G_{15} and that our embedding of M into $\tilde{M}$ is conformal.

Notice that, once we have picked a point x_0 and specified its image point $f(x_0)$ in $\tilde{M}$, then the question of conformal completion has a unique solution. This is because the algebra $g_{15}^0\cap g_{11}$ is the reductive subalgebra $o_{1,3}+\mathbf{R}$ which acts completely reducibly on g_{15}. It breaks g_{15} up into three inequivalent subspaces, and hence there is only one way of enlarging the subalgebra $o_{1,3}+\mathbf{R}$ to a subalgebra, g_{15}^0 of g_{15} with $g_{15}^0\cap g_{11}=o_{1,3}+\mathbf{R}$. There is thus only one candidate for H. Hence, if we specifiy the image of x_0, we can identify $\tilde{M}$ with G_{15}/H.

It is clear from the above discussion that g_{11} can be characterized as the normalizer, in g_{15}, of the subalgebra consisting of the constant vector fields. We can think of the concept of a « family of inertial frames » as being the same as some subalgebra of g_{15} acting as constant vector fields, *i.e.* as « infinitesimal translations ». It is in this sense that we can regard the group G_{11} as the group preserving both Maxwell's equations and the notion of inertial frame.

2. – In order to proceed further, it will be convenient for us to have an explicit model for $\tilde{M}$. In what follows we shall take G_{15} to be the group $SO_{2,4}$, the identity component of the group of orthogonal transformations of $\mathbf{R}^{2,4}$, where $\mathbf{R}^{2,4}$ is the six-dimensional real space endowed with a metric of signature $++----$. Following SEGAL, we will let $\overline{M}$ denote the projective null quadric, *i.e.* a point x in $\overline{M}$ is a null line in $\mathbf{R}^{2,4}$. Let us choose some point $x_\infty\in\overline{M}$, and let P_∞ denote the isotropy group of x_∞. In view of the discussion of Sect. 1, we know that P_∞ is an eleven-dimensional group of $SO_{2,4}$ which is isomorphic to G_{11}. Let us set

$$\Omega=\{x\in\overline{M},\ x \text{ not orthogonal to } x_\infty\}$$

(3) I. M. SINGER and S. STERNBERG: *The infinite groups of Lie and Cartan*, in *Journal d'Analyse Mathématique* (1965).

and

$$\Xi = \{x \in \overline{M},\ x \text{ orthogonal to } x_\infty \text{ but } x \neq x_\infty\}.$$

It is clear that the four-dimensional submanifold Ω, the three-dimensional submanifold Ξ and the zero-dimensional submanifold $\{x_\infty\}$ are all stable under P_∞. We claim that P_∞ acts transitively on each of these, so that they provide the orbit decomposition of $\overline{M}$. Indeed, let us choose some null vector $f_{-1} \in x_\infty$. Then, if x_0 is some point in Ω, we can choose some $f_4 \in x_0$ with $(f_{-1}, f_4) = 1$. If we choose some other $x \in \Omega$, then we can find a vector f, lying in x, with $(f_{-1}, f) = 1$. By standard linear algebra (Witt's theorem) we can find an element of $SO_{2,4}$ which carries the pair f_{-1}, f_4 into the pair f_{-1}, f. In particular, it carries x_∞ into x_∞ (and so lies in P_∞) and maps x_0 into x. Thus P_∞ acts transitively on Ω. A similar argument shows that P_∞ acts transitively on Ξ. We have thus proved

Proposition 2.1. – *The isotropy group of a point $x_\infty \in \overline{M}$ is an eleven-dimensional subgroup P_∞ of $SO_{2,4}$ isomorphic to the Poincaré group plus scale transformations, the group which we denoted by G_{11} in the preceding Section. Under P_∞ the manifold $\overline{M}$ decomposes into three orbits: the open (four-dimensional) orbit Ω, the three-dimensional orbit Ξ and the zero-dimensional orbit $\{x_\infty\}$.*

Let M denote the nilradical of P_∞, so that M is a four-dimensional commutative (vector) group. The group P_∞ is the semi-direct product of P_∞/M with M, and the group P_∞/M acts on M as Lorentz transformations followed by dilatations. Thus M has a Minkowski metric determined only up to scale, *i.e.* the Minkowski « angle » is well defined. If we choose an « origin » $x_0 \in \Omega$, then we get a map $f_0: M \to \Omega$ given by $f_0(v) = v \cdot x_0$ for $v \in M$. The subgroup preserving both the « antipode » x_∞ and the origin x_0 is a seven-dimensional group G_7, isomorphic to $\mathbf{R}^+ \times SO_{1,3}$. We let $S(= S(x_0, x_\infty))$ denote the one-parameter subgroup of dilatations in G_7. Thus S is the centre of G_7 and consists of the dilatations in P_∞. It follows that *the set of Minkowski metrics on M, and hence on Ω, is a homogeneous space of S.*

Let us choose x_∞ and x_0 as above, and let U be the two-dimensional space that they span. Since x_∞ and x_0 are nonorthogonal null lines in U, it follows that the restriction of the metric of $\mathbf{R}^{2,4}$ to U is nondegenerate and, in fact, has signature $+-$. Let V be the orthogonal complement of U in $\mathbf{R}^{2,4}$, so that V is a four-dimensional subspace carrying an induced metric of signature $+---$. An element of S acts by multiplying a vector on the line x_∞ by some positive number r and on a vector in the line x_0 by multiplication by r^{-1}, and so as a hyperbolic transformation on U, and acts trivially on V. (Indeed, if we choose some vector $f_{-1} \in x_\infty$ and a vector $f_4 \in x_0$, the element $f_{-1} \wedge f_4$ is an infinitesimal generator of S.) The semi-simple part of G_7 acts as $SO_{1,3}$ on V and trivially on U (and is generated by linear combinations of $v_1 \wedge v_2$ with $v_i \in V$). Each choice of $f_{-1} \in x_\infty$ determines a linear map of V

into M, by sending $v \in V$ into $f_{-1} \wedge v \in M \subset o_{2,4}$ and this map is equivariant with respect to the action of $SO_{1,3}$. Replacing f_{-1} by rf_{-1} means that we now identify v with $rf_{-1} \wedge v$, which means that the same element of $\bar{M}$ is identified with a vector in V which is r^{-1} as large. Thus replacing the vector f_{-1} by rf_{-1} has the effect of multiplying the Lorentz metric by r^{-2}.

The group S also acts transitively on the space of positive-definite lines in U (and also on the space of negative-definite lines). If $f_{-1} \in x_\infty$ and $f_4 \in x_0$ are chosen so that $(f_{-1}, f_4) = 1$, then we can parametrize the set of all positive-definite lines in U by the unit vectors

$$e_+ = af_{-1} + bf_4 , \qquad a > 0,\ 2ab = 1 ,$$

and this vector is transformed into

$$raf_{-1} + r^{-1}bf_4 ,$$

with the negative-definite lines parametrized similarly, except that $2ab = -1$.

Suppose that we now choose a space-time splitting $V = V_0 + V_3$ of V. This is the same as choosing a space-time splitting of M, or as choosing a timelike direction in M. If we have also chosen a positive line $\{e_{-1}\}$ in U, then e_{-1} together with V_0 span a two-dimensional positive-definite plane, call it F_2, in $\mathbf{R}^{2,4}$, and its orthogonal space F_4 is a negative-definite four-space spanned by V_3 and e_4, where e_4 is the negative-definite vector (of length -1) orthogonal to e_{-1}. Given F_2 and F_4 we obtain a subgroup, isomorphic to $SO_2 \times SO_4$, preserving these two subspaces. The element $e_{-1} \wedge e_0$ is clearly an infinitesimal generator for the SO_2, where e_0 is the unit vector in V_0 (chosen so as to correspond to a forward direction of time). It is this element that SEGAL proposes to use as the energy generator instead of element $f_{-1} \wedge e_0$, which is Minkowski time translation. The subgroup $SO_2 \times SO_4$ acts transitively on $\bar{M}$, and we obtain the global space-time splitting $\bar{M} = S^1 \times S^3/Z_2$, where $S^1 = SO_2 \cdot x_0$ and $S^3 = SO_4 \cdot x_0$. SEGAL calls this splitting an « observer » and uses the geometry of S^3 for various computations in cosmology. Both the « chronogeometric energy generator » and the notion of « observer » depend, in addition to the choice of x_∞ (which is equivalent to the choice of « inertial frames »), to the choice of origin x_0 to the local space-time splitting $V = V_0 + V_3$, and to the choice of vector $e_{-1} \in U$. This last choice is somewhat unfamiliar, and it becomes of importance to relate it to other similar choices and to understand the physical and operational significance of making such a choice. In the following Section we shall see that the same choice is involved in the selection of the operator of « charge conjugation ». In the present Section we shall investigate how the geometrical picture changes under the action of the scale group S.

When we act with a scale transformation, we change e_{-1}, SO_2 and SO_4, and also S^1 and S^3, and the metric on M. The point x_0 is unchanged, as is the splitting $V = V_0 + V_3$ and the corresponding splitting $M = M_0 + M_3$. All the spheres S^3 are tangent to M_3 at x_0. We propose to show that under a scale transformation the sphere S^3 changes in such a way that it has the same radius relative to the new metric on M. Thus we have the amusing situation that if we try to make a scale transformation so as to normalize the radius of the sphere, we find that the sphere has moved in just such a way as to keep the same numerical value of its radius in the new units. To do this computation we make use of the following observation (which is a consequence of the discussion in the preceding Section): *Let X and Y be two elements of the Lie algebra $o_{2,4}$, and suppose that $Xf_4 = Yf_4 \bmod x_0$. Then if we consider X and Y as vector fields on $\bar{M}$, they take on the same value at x_0.*

We may identify M with the tangent space to $\bar{M}$ at x_0. Let ξ be some vector in M_3. According to our original choice of f_{-1}, we can write $\xi = f_{-1}\wedge v$ for some $v \in V_3$, and, in our original choice of metric, ξ has the same length as v, which we might as well take to be -1. There is then a unique o_4 generator determined by our choice of $e_{-1} = af_{-1} + bf_4$ which is tangent to ξ. It is given by $a^{-1}e_4\wedge v = \eta$, where

$$e_4 = af_{-1} - bf_4 .$$

The circumference of the sphere is the period of the circle generated by η, multiplied by the absolute value of the length of the tangent vector ξ. Here η generates a circle with period $2\pi a$, so that a is the radius of the sphere. We apply a scale transformation, and so get a new f_{-1}, a new v, a new η and compare. For simplicity we organize the computation according to the following table:

Old	New
f_{-1}	rf_{-1}
$\xi = f_{-1}\wedge v$	$\xi = (rf_{-1})\wedge r^{-1}v$
length of ξ is -1	length of ξ is $-r^{-1}$
$\eta = e_4\wedge a^{-1}v = (af_{-1} - bf_4)\wedge a^{-1}v$	$\eta^r = (raf_{-1} - r^{-1}bf_4)\wedge r^{-1}a^{-1}v = r^{-1}A\,\mathrm{d}C_r\,\eta$, where C_r denotes the scale transformation with parameter r
period of circle is $2\pi a$	period of circle is $2\pi ra$ since $A\,\mathrm{d}C_r$ is an automorphism
circumference is $2\pi a$	circumference is $2\pi a$

Thus the radius of the new sphere, relative to the new metric, is the same as the radius of the old sphere, relative to the old metric, when we act by a scale transformation.

3. – In this Section we show how the choice of e_{-1} is related to charge conjugation. The connection is via the so-called «spin conformal algebra» introduced in (4) and its connection with the geometry of the bounded domains as developed in (5). We begin with a quick review of concept of charge conjugation.

The group $Sl_{2,\mathbf{C}}$ is the universal covering group of the group $SO_{1,3}$. We can explicitly see the relation between these two groups as follows: let us write the vector $p = (p_0, p_1, p_2, p_3)$ in $\mathbf{R}^{1,3}$ as the two-by-two Hermitian matrix

$$P = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix},$$

so that $\det P = p_0^2 - p_1^2 - p_2^2 - p_3^2$ gives the Minkowski metric. For any $g \in Sl_{2,\mathbf{C}}$ the matrix gPg^* is again Hermitian and the map $P \rightsquigarrow gPg^*$ gives a linear representation of $Sl_{2,\mathbf{C}}$ acting as Lorentz transformations. It is not difficult to see that this gives a surjective homomorphism of $Sl_{2,\mathbf{C}}$ onto $SO_{1,3}$ with kernel $\{I, -I\}$.

The group $Sl_{2,\mathbf{C}}$ has two inequivalent irreducible representations on a complex two-dimensional space, known as the spin representations of type $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. They can be given by $u \rightsquigarrow gu$ and $u \rightsquigarrow g^{*-1}u$, or, at the Lie algebra level, by

$$u \rightsquigarrow Au \quad \text{and} \quad u \rightsquigarrow -A^*u, \qquad A \in sl_{2,\mathbf{C}},\ u \in \mathbf{C}^2.$$

These two representations are clearly not equivalent over the complex numbers, since the first representation is holomorphic and the second is antiholomorphic. However they are conjugate via an antilinear map. Indeed, define the antilinear map $\star : \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \bar{u}_2 \\ -\bar{u}_1 \end{pmatrix},$$

and a direct computation shows that

$$\star Au = -A^* \star u \qquad \text{for } A \in sl_{2,\mathbf{C}}.$$

The meaning of the $\star$-operator is the following. Let Z be a complex vector space equipped with a nondegenerate Hermitian scalar product, and with a preferred orientation, *i.e.* a choice of basis of $\wedge^n Z$, where $n = \dim Z$. There is an induced Hermitian scalar product on each of the spaces $\wedge^k Z$ and the

(4) L. Corwin, Y. Neeman and S. Sternberg: *Graded Lie algebras in mathematics and physics*, to appear in *Rev. Mod. Phys.*

(5) S. Sternberg and J. A. Wolf: *Graded Lie algebras and bounded homogeneous domains*, to appear in *Transactions of the American Mathematical Society*.

antilinear map $\star$: $\wedge^k Z \to \wedge^{n-k} Z$ is defined by

$$(v, \star u) = v \wedge u, \qquad v \in \wedge^{n-k} Z,\ u \in \wedge^k Z,$$

where we have identified $\wedge^n Z$ with $\mathbf{C}$. (For $n = 2$ and $k = 1$ and the standard Hermitian form on $\mathbf{C}^2$ we get the $\star$-operator written above.) For any linear transformation A on $\wedge^k Z$, we obtain a linear transformation A^a on $\wedge^{n-k} Z$ defined by

$$A^a v \wedge u = v \wedge A u .$$

It then follows that

$$\star A u = A^{a*} \star u .$$

For a two-by-two matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$A^a = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so that for $A \in sl_{2,\mathbf{C}}$ we have $A^a = -A$, which is the reason the $\star$ intertwines $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

There is a representation of the Clifford algebra associated to the Lorentz metric on the direct sum $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The images of the elements of degree one in the Clifford algebra are known as Dirac matrices. The Dirac matrices act irreducibly on the space $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, and leave invariant (infinitesimally) a nondegenerate Hermitian form β and an antisymmetric bilinear form α. These together (just as above) define an antilinear map of $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ into itself. It commutes with the action of $Sl_{2,\mathbf{C}}$ and interchanges the two components. This map is charge conjugation. On each component it must be some multiple of $\star$. There is a certain ambiguity in the choice of this multiple, and we shall see that it is related to the choice of vector e_{-1} in the preceding Section. To see this we must fit the Dirac matrices into the framework of the conformal algebra. Before doing so it might be instructive to record the following fact. Let C_Q be the Clifford algebra associated to a real quadratic form of signature (p, q). Then the terms of degree one and two in C_Q form, under commutation, a Lie algebra isomorphic to $o_{p,q+1}$. (The terms of degree two form a Lie algebra isomorphic to $o_{p,q}$ and adding the terms of degree one extends this algebra to $o_{p,q+1}$ as can easily be checked by examining generators.) In the case at hand we get a subalgebra $o_{1,4}$. This will sit as the subalgebra of $o_{2,4}$ which stabilizes e_{-1}.

Following ([5]), we consider the Lie algebra $su_{2,3}$, the algebra of all five-by-five complex matrices Q which satisfy

$$QJ + JQ^* = 0 \quad \text{and} \quad \operatorname{Tr} Q = 0,$$

where

$$J = \begin{pmatrix} 0 & 0 & I \\ 0 & 1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

where I is the two-by-two identity matrix. Then $su_{2,3} = g$ decomposes as $g = g_{-2} + g_{-1} + g_0 + g_1 + g_2$, where

g_2 consists of all matrices $\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X \in u_2$,

g_1 consists of all matrices $\begin{pmatrix} 0 & u & 0 \\ 0 & 0 & -u^* \\ 0 & 0 & 0 \end{pmatrix}$, $u \in \mathbf{C}^2$,

g_0 consists of all matrices $\begin{pmatrix} A & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -A^* \end{pmatrix}$, $A \in Gl_{2,\mathbf{C}}$, $\lambda = 2 \operatorname{Im} \operatorname{Tr} A$,

g_{-1} consists of all matrices $\begin{pmatrix} 0 & 0 & 0 \\ -v^* & 0 & 0 \\ 0 & v & 0 \end{pmatrix}$, $v \in \mathbf{C}^2$,

and

g_{-2} consists of all matrices $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix}$, $Y \in u_2$.

We have $[g_i, g_j] \subset g_{i+j}$.

The algebra $g_{\text{even}} = g_{-2} + g_0 + g_2$ is isomorphic to $o_{2,4} \oplus \mathbf{R}$, with the projection onto the centre, $\mathbf{R}$ being given by $2 \operatorname{Im} \operatorname{Tr} A$. The grading on g_{even} is consistent with the grading we introduced in the preceding Sections for $o_{2,4}$ given by the choice of x_∞ and x_0.

Under the action of $A \in sl_{2,\mathbf{C}}$ regarded as an element of g_0, the space g_1 transforms as $(\frac{1}{2}, 0)$ and the space g_{-1} transforms as $(0, \frac{1}{2})$. The space g_{odd} is irreducible under the action of g_{even} (and, in fact, is a spin representation for $o_{2,4}$). We may identify the complexification of g_{even}, the vector space $g^{\mathbf{C}}_{\text{even}}$, as consisting of all matrices of the form

$$\begin{pmatrix} A & 0 & B \\ 0 & d^* & 0 \\ C & 0 & D \end{pmatrix},$$

where A, B, C and D are arbitrary complex two-by-two matrices and $d \in \mathbf{C}$.

The complex conjugation $^-$ of g_{even} over g_{even} is then given by

$$\begin{pmatrix} A & 0 & B \\ 0 & d^* & 0 \\ C & 0 & D \end{pmatrix}^- = \begin{pmatrix} -D^* & 0 & -B^* \\ 0 & -d & 0 \\ -C^* & 0 & -A^* \end{pmatrix}.$$

There is an equivariant anti-Hermitian form, $H\colon g_{\text{odd}} \times g_{\text{odd}} \to g^{\mathbf{C}}_{\text{even}}$ given by

$$H(w, w') = -2 \begin{pmatrix} uv'^* & 0 & uu'^* \\ 0 & ((u, v') + (v, u'))^* & \\ vv'^* & 0 & vu'^* \end{pmatrix},$$

where

$$w = \begin{pmatrix} 0 & u & 0 \\ -v^* & 0 & -u^* \\ 0 & v & 0 \end{pmatrix} \quad \text{and} \quad w' = \begin{pmatrix} 0 & u' & 0 \\ -v'^* & 0 & -u'^* \\ 0 & v' & 0 \end{pmatrix}.$$

The Lie bracket is given by $[w, w'] = \operatorname{Re} H(w, w')$, and we obtain a graded Lie-algebra structure, cf. ([4]) or ([5]) by taking $\operatorname{Im} H$. If we let π denote projection of $g^{\mathbf{C}}_{\text{even}}$ onto its centre, then πH is an invariant $\mathbf{C}$-valued anti-Hermitian form. Thus (after dividing by $2i$) we see that the $\mathbf{C}$-valued Hermitian form

$$\beta(w, w') = (u, v') + (v, u')$$

is invariant under the entire algebra g_{even}.

We are now going to write down an involutive automorphism, $Q \to Q^c$, which will have the following properties:

i) $g_i^c = g_{-i}$,

ii) $A = A^c$ for $A \in sl_{2,\mathbf{C}} \subset g_0$,

iii) $w \to w^c$ is antilinear

and

iv) $H(w^c, w'^c) = H(w, w')^{-c}$,

where, on the right-hand side of iv) we have extended ^{c} so as to be defined on $g^{\mathbf{C}}_{\text{even}}$. Notice that since g_{odd} generates g, the automorphism is determined once we know it on g_{odd}. On the other hand, i), ii) and iii) imply that the map of $g_1 \to g_{-1}$ intertwines the action of $sl_{2.\mathbf{C}}$ and hence must be some multiple of $\star$, and similarly the map from g_{-1} to g_1 must be some other multiple of $\star$. The fact that the map $w \to w^c$ is involutive implies that the maps must be of the form $a\star\colon g_1 \to g_{-1}$ and $b\star\colon g_{-1} \to g_1$, where $ab = -1$. Thus ^{c} is determined up to the nonzero complex number a. On the other hand, if we start with some c satisfying all our conditions, we can conjugate by any element of the centre of the group generated by g_0. This group is $S \times U_1$, where S is the group of scale transformations, and the effect of the action of $S \times U_1$ will be to multiply a by an arbitrary nonzero complex number. Thus we need only exhibit one value of a which works. Let us take $a = i$ and explicitly write

$$\begin{pmatrix} A & u & X \\ -v^* & -i\lambda & u^* \\ Y & v & -A^* \end{pmatrix}^c = \begin{pmatrix} -A^a & i\star v & -Y^a \\ -(i\star u)^* & i\lambda & -(i\star v)^* \\ -X^a & i\star u & A^{a*} \end{pmatrix}.$$

A direct verification shows that i)-iv) do indeed hold for this choice of « charge conjugation ». Notice that the elements in g_{even} which are left fixed by c consist of matrices

$$\begin{pmatrix} A & 0 & X \\ 0 & 0 & 0 \\ -X^a & 0 & -A^* \end{pmatrix}, \qquad A \in sl_{2.\mathbf{C}},\ X \in u_2 .$$

It is easy to check that this subalgebra of $o_{2.4}$ consists of those elements which infinitesimally preserve the vector $e_{-1} = (1/\sqrt{2})(f_{-1} + f_4)$ relative to a fixed identification. Indeed, one can verify that charge conjugation, when restricted to $o_{2,4}$, is given by conjugation by the element of $O_{2,4}$, which consists of reflection through a positive-definite line in U, and the identity on V, where U and V are the two subspaces of $\mathbf{R}^{2,4}$ described in the preceding Section. Thus a choice of charge conjugation does indeed determine a positive-definite line in U, and hence a choice of charge conjugation together with a space-time splitting determines Segal's energy generator.

For $p = (p_0, p_1, p_2, p_3)$ let us define the element $\gamma(p) \in o_{1,4}$ by

$$\gamma(p) = \begin{pmatrix} 0 & 0 & iP \\ 0 & 0 & 0 \\ -iP^a & 0 & 0 \end{pmatrix}, \qquad \text{where } P = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}.$$

Then it is easy to check that

$$[\gamma(p), [\gamma(p), w]] = \|p\|^2 w$$

for any $w \in g_{\text{odd}}$. Thus the $\gamma(p)$ are indeed Dirac matrices.

Finally we remark that charge conjugation is realized geometrically as conjugation of $SU_{2,3}$ by a certain antiholomorphic isometry of $D_{2,3,C}$. In effect, the isometry group of $D_{2,3,C}$ has two topological components, the identity component consisting of the holomorphic isometries and the other component consisting of the antiholomorphic ones (see (6), p. 264, for this fact due to CARTAN). The holomorphic isometries are the $P \to g(P)$, $g \in SU_{2,3}$. The anti-holomorphic ones are the $P \to g(\bar{P})$, $g \in SU_{2,3}$ and $P \to \bar{P}$ induced by complex conjugation of $C^{2,3}$ and $R^{2,3}$. Conjugation by the latter induces every outer automorphism on $SU_{2,3}$.

4. – The Segal cosmos appears as the set of boundary components of a certain type in the Grassmann manifold of negative-definite 4-planes in $R^{2,4}$, and the choice of $SO_2 \times SO_4$ in $SO_{2,4}$ is the choice of an interior point in that Grassmannian. Here we discuss the general concept of boundary components of Grassmann manifolds, cf. (7) and (8), show how these components are related by interior geodesics, and elucidate the role of the scale transformation. The calculation is the same in the general case as in the case (negative-definite 4-planes in $R^{2,4}$) of the Segal cosmos.

In what follows F will denote the real or complex numbers or the algebra of real quarternions, $F^{p,q}$ will denote the space of $(p+q)$-tuplets endowed with the symmetric (if real) or Hermitian (if complex or quarternion) form

$$H(z) = z_1\bar{z}_1 + \ldots + z_p\bar{z}_p - z_{p+1}\bar{z}_{p+1} - \ldots - z_{p+q}\bar{z}_{p+q},$$

where $^-$ denotes conjugation of F over the reals, and so is to be ignored if F consists of the reals. We let $D_{p,q,F}$ denote the open subset of the Grassman manifold of q-planes in $F^{p,q}$ consisting of those q-planes for which the restriction of H is negative definite. Let P be such a negative-definite plane, and let W denote the p-dimensional (positive-definite) subspace spanned by the first p standard basis vectors of $F^{p,q}$. Then we clearly must have $P \cap W = \{0\}$, and hence the projection (along W) of P onto the space spanned by the last q standard basis vectors $e_{p+1}, \ldots, e_{p+q}$ is a bijection. Thus P has a unique basis of the form $w_1 + e_{p+1}, \ldots, w_q + e_{p+q}$, where the $w_i \in W$. If we

(6) J. A. WOLF: *Spaces of Constant Curvature*, 3rd edition (Boston, Mass., 1973).
(7) F. I. KARPELEVIČ: *Trans. Moscow Math. Soc.*, **14**. 48 (1965).
(8) J. A. WOLF and A. KORÁNYI: *Amer. Journ. Math.*, **87**, 899 (1965).

write these vectors out as column vectors, we obtain the matrix

$$\begin{pmatrix} Z \\ I \end{pmatrix},$$

where Z is a matrix with p rows and q columns, and I is the $q\times q$ identity matrix. The condition that P be negative definite becomes the requirement that the matrix $I - Z^*Z$ be positive definite.

We let $U_{p,q} = U_{p,q;F}$ denote the group of linear transformations of $F^{p,q}$ which preserve H. In the quarternion case we agree to put scalar multiplication on the right so that linear transformations act on the left, as usual. This does not introduce any complications. The group $U_{p,q}$ acts transitively on the space of k-planes having a specified signature under the restriction of H. In particular it acts transitively on $D_{p,q,F}$. The closure of $D_{p,q,F}$ in the variety of all q-planes consists of those q-planes for which the restriction of H is negative semi-definite. We denote by $\partial_i D_{p,q,F}$ the set of those negative semi-definite q-planes on which H has nullity i. Thus $\partial_0 D_{p,q,F} = D_{p,q,F}$, while the boundary of $D_{p,q,F}$ is the union from one to q of the $\partial_i D_{p,q,F}$. Each of the $\partial_i D_{p,q,F}$ is also an orbit for the action of $U_{p,q}$. Let E be some fixed isotropic subspace of dimension r. We let $\partial_E D$ denote the set of all q-dimensional semi-definite subspaces b such that $b \cap b^{\perp} = E$. Thus

$$\partial_r D = \bigcup_E \partial_E(D) \qquad \text{with} \qquad \dim E = r.$$

If x is any point of $D_{p,q,F}$, then the isotropy group of x is clearly isomorphic to $U_p \times U_q$, which is a maximal compact subgroup of $U_{p,q}$. The Killing form on $U_{p,q}$ induces a Riemann metric on $D_{p,q,F}$, making $D_{p,q,F}$ into a symmetric space. If σ is the involution fixing x, then σ breaks up the Lie algebra of $U_{p,q}$ into the decomposition $k + p$, where p can be identified with the tangent space to x. The geodesics through x are then the curves of the form $\gamma_\xi(t) = \exp[t\xi]x$, $\xi \in p$.

Notice that the groups $U_{2,2,C}$ and $SO_{2,4,R}$ are related by a homomorphism $U_{2,2,C} \to SO_{2,4,R}$ whose kernel consists of scalar matrices, so that the spaces $D_{2,2,C}$ and $D_{2,4,R}$ are isometric.

We now describe the geodesics of $D_{p,q,F}$ and their asymptotic behaviour at the boundary.

Theorem 4.1. *Let x be a point of $D_{p,q,F}$ and let E be a totally isotropic r-dimensional subspace of $F^{p,q}$. Then there exist geodesics passing through x*

$$g_t(x) = \exp[t\xi]x$$

tending to $\partial_E D$ as $t \to \infty$. The set of all such geodesics is parametrized by the cone of $r\times r$ positive-definite Hermitian matrices over F, and all such geodesics *tend to the same boundary point*

$$E \oplus (x \cap E^{\perp}) \in \partial_E D$$

as $t \to +\infty$.

Proof. We may, by an appropriate choice of basis, assume that x is spanned by $\{e_{p+1}, \ldots, e_{p+q}\}$ and that E is spanned by $\{e_1 + e_{p+1}, \ldots, e_r + e_{p+r}\}$. In this basis $\{e_1, \ldots, e_{p+q}\}$ of $F^{p,q}$, the Lie algebra of $U_{p,q}$ consists of all matrices of the form

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $A + A^* = 0$ $(A \in u_p)$, $C + C^* = 0$ $(C \in u_q)$ and B is an arbitrary $p \times q$ matrix over F. The matrix $\begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$ belongs to $U_{p,q}$, has x as its only fixed point in D, and is an involution. Thus conjugation by this matrix is the desired involution σ. The positive eigenspace of σ is k, which thus consists of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \qquad A \in u_p \text{ and } C \in u_q,$$

and the isotropy group of x is $U_p \times U_q$. The negative eigenspace of σ is p and consists of all matrices of the form

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \qquad \text{where } B \text{ is } p \times q.$$

We must now investigate the conditions imposed on B by the requirement $g_t(x) \to \partial_E D$, as $t \to \infty$. Let b be a limit point of $g_t(x)$. Then, for any t, $g_t(b)$ is also a limit point, so $g_t(b) = b$. Now also $g_t(b^{\perp}) = b^{\perp}$, so $E = b \cap b^{\perp} = g(b \cap b^{\perp}) = g_t(E)$. Differentiating the equation $g_t(E) = E$ with respect to t yields

$$\xi E \subset E,$$

or

$$\xi(e_i + e_{p+i}) = \sum_{j=1}^{r} b_i^j (e_j + e_{p+j}), \qquad i = 1, \ldots, r.$$

From the form of ξ we conclude that the matrix B has the block decomposition

$$\begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \qquad B'^* = B',$$

where B' is the $r\times r$ matrix $B'=(b_i^j)$ and B'' is some matrix with $p-r$ rows and $q-r$ columns. We now propose to bring B to a more convenient normal form by conjugation by a suitable element in $(U_{p,q})_x \cap (U_{p,q})_E$, *i.e.* the subgroup which simultaneously fixes x and E. This subgroup consists of all matrices of the form

$$\begin{pmatrix} A & 0 & & \\ 0 & A' & & \\ & & A & 0 \\ & & 0 & A'' \end{pmatrix},$$

where $A\in U_r$, $A'\in U_{p-r}$, and $A''\in U_{q-r}$.

Conjugation of $\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$ by such a matrix has the effect of replacing it by $\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$, where

$$C=\begin{pmatrix} AB'A^{-1} & 0 \\ 0 & A'B''A''^{-1} \end{pmatrix}.$$

Since B' is self-adjoint, we can find some unitary operator A which diagonalizes it. Furthermore, by suitable choices of A' and A'' we can also arrange that $A'B''A''^{-1}$ is real diagonal. (Indeed, by preliminary pre- and post-multiplica tion by unitary matrices we can arrange that B'' becomes a square matrix surrounded by zeros, and that the square matrix is nonsingular. By polar decomposition we may assume that this matrix is of the form PU. We can then pre- and post-multiply by unitaries to eliminate the U and to diagonalize P.) We may thus assume that $A'B''A''^{-1}$ consists of all zeros with the possible exception of some positive entries running down the main diagonal. We propose to show that all the diagonal entries of $AB'A^{-1}$ are positive and all the entries of $A'B''A''^{-1}$ are zero. After the conjugation ξ is given by

$$\xi=\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \qquad B=\begin{pmatrix} b_1 & & & & \\ & \ddots & & & \\ & & b_m & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \qquad B^*=\begin{pmatrix} b_1 & & & & \\ & \ddots & & & \\ & & b_m & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

so

$$g_t=\exp[t\xi]=\begin{pmatrix} U_t & V_t \\ V_t^* & W_t \end{pmatrix} \quad \text{with } U_t=\begin{pmatrix} \cosh(tb_1) & & & & \\ & \ddots & & & \\ & & \cosh(tb_m) & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix},$$

$$W_t = \begin{pmatrix} \cosh(tb_1) & & & & \\ & \ddots & & & \\ & & \cosh(tb_m) & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix},$$

$$V_t = \begin{pmatrix} \sinh(tb_1) & & & & \\ & \ddots & & & \\ & & \sinh(tb_m) & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

Thus

$$g_t e_{p+i} = \sinh(tb_i) e_i + \cosh(tb_i) e_{p+i} .$$

As $t \to \infty$ the line spanned by the vector on the right tends to the line through $e_i + e_{p+i}$ if $b_i > 0$, to the line through e_{p+i} if $b_i = 0$ and to the line through $e_i - e_{p+i}$ if $b_i < 0$. Since E is contained in the limit set of $g_t x$, we conclude that $b_i > 0$ for $i \leqslant r$. Since E is the maximal isotropic space contained in the limit set we conclude that all the remaining b_i must be zero. We thus see that the limit of $g_t(x)$ consists precisely of the space spanned by $e_1 + e_{p+1}, \ldots, e_r + e_{p+r}, e_{p+r+1}, \ldots, e_{p+q}$, which is exactly $E \oplus (x \cap E^\perp)$.

Now the conjugated ξ is

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \quad \text{with} \quad B = \begin{pmatrix} b_1 & & & & \\ & \ddots & & & \\ & & b_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

$b_i > 0$, and so the original ξ was $\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$, where B is an arbitrary $\begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$ with B' $r \times r$ Hermitian positive definite. This completes the proof of Theorem 4.1.

The proof of Theorem 4.1 tells us when and how two boundary components can be joined by interior geodesics.

Theorem 4.2. – *Let E and E' be nonzero totally isotropic subspaces of $F^{p,q}$. Then the boundary components $\partial_E(D)$ and $\partial_{E'}(D)$ are joined by a geodesic in $D = D_{p,q,F}$ if and only if*

i) *E and E' have the same dimension, say r,*

and

ii) *H restricts to a form of signature (r, r) on $E + E'$.*

Under these circumstances the geodesics joining $\partial_E(D)$ and $\partial_E(D')$ are just *the* $\gamma(t) = \gamma_1(t) \oplus y$, *where*

$\gamma_1(t)$ *is a geodesic «joining» E and E' in the* $D_{r,r,F}$ *based on* $E \oplus E'$ *and*

y *is a negative definite* $(q-r)$*-plane, element of the* $D_{p-r,q-r,F}$ *based on* $(E \oplus E')^{\perp}$.

The union of all such geodesics is a totally geodesic submanifold of $D_{p,q,F}$ *isometric to* $D_{r,r,F} \times D_{p-r,q-r,F}$.

Proof. – In the proof of Theorem 4.1 $\lim_{t \to -\infty} g_t(x)$ is the space spanned by $\{e_1 - e_{p+1}, \ldots, e_r - e_{p+r}; e_{p-r+1}, \ldots, e_{p+q}\}$, which belongs to $\partial_{E'}(D)$ where E' is spanned by $\{e_1 - e_{p+1}, \ldots, e_r - e_{p+r}\}$. Thus $\dim E' = r = \dim E$, and $E + E'$ is the span of $\{e_1, \ldots, e_r; e_{p+1}, \ldots, e_{p+r}\}$, space on which H has signature (r, r).

Conversely, let E and E' be totally isotropic r-dimensional subspaces of $F^{p,q}$ such that H has signature (r, r) on $E + E'$. Then $F^{p,q}$ has an orthonormal basis $\{e_1, \ldots, e_{p+q}\}$ such that

$$E \text{ is spanned by } \{e_1 + e_{p+1}, \ldots, e_r + e_{p+r}\}$$

and

$$E' \text{ is spanned by } \{e_1 - e_{p+1}, \ldots, e_r - e_{p+r}\}.$$

The proof of Theorem 4.1 gives geodesics «joining» $\partial_E(D)$ and $\partial_{E'}(D)$.

Recall that the arbitrary geodesic ray from $x \in D = D_{p,q,F}$ to $E + (x \cap E^{\perp}) \in \partial_E(D)$ was of the form $g_t(x) = g_t(z) + y$ where $\{e_{p+1}, \ldots, e_{p+q}\}$ spanned x, z is the span of $\{e_{p+1}, \ldots, e_{p+r}\}$, and $\{e_{p+1+r}, \ldots, e_{p+q}\}$ spans y. There g_t was the identity on y, and $g_t(z)$ was the span of the $\sinh(tb_i)\, e_i + \cosh(tb_i)\, e_{p+i}$, $1 \leqslant i \leqslant r$, so $t \to g_t(z)$ was a geodesic joining E to E' in the $D_{r,r,F}$ based on $E + E'$. Intrinsically, $z = x \cap (E + E')^{\perp}$ and $y = x \cap E^{\perp} = x \cap E'^{\perp}$. We can replace y by any y' in the $D_{p-r,q-r,F}$ based on $(E + E')^{\perp}$, thus replacing $x = z \oplus y$ by $x' = z \oplus y'$, replacing the geodesic $g_t(x)$ by $g_t(x') = g_t(z) \oplus y'$, and replacing the limit points $E \oplus y$ and $E' \oplus y$ by $E \oplus y'$ and $E' \oplus y'$. This completes the characterization of the interior geodesic joining $\partial_E(D)$ and $\partial_{E'}(D)$, and it describes the union of all such geodesics as $\{z \oplus y : y \in D_{r,r,F}$ based on $E \oplus E'$ and $z \in D_{p-r,q-r,F}$ based on $(E \oplus E')^{\perp}\}$.

Since it is stable under the symmetry to D at each of its points, that set forms a totally geodesic submanifold.

Theorem 4.2 is proved.

In the case $r = 1$ and $F = R$, $D_{r,r,F}$ is a line, and $D_{p-r,q-r,F} = D_{p-1,r-1,R}$ cuts each geodesic $\partial_E(D) \to \partial_{E'}(D)$ in a single point. If further $p = 2$, then $D_{p-1,q-1,R}$ is a real hyperbolic $(q-1)$-space. We reformulate this case as

Corollary 4.3. – *Let E be an isotropic line in* $R^{2,q}$ *and* $x \in D = D_{2,q,R}$. *Then there is a unique (up to change of parameter) geodesic ray in* $D_{2,q,R}$ *from x to the boundary component* $\partial_E(D)$.

Let E' be another isotropic line in $R^{2,q}$. Then $\partial_E(D)$ and $\partial_{E'}(D)$ are joined by an interior geodesic if and only if E' is not orthogonal to E. In that case, all such geodesics are given by

$$t \to \left(\exp[tb]f_{-1} + \exp[-tb]f_4\right)R \oplus y\,,$$

where $b \neq 0$, $E = f_{-1}R$, $E' = f_4R$, and y belongs to the real hyperbolic $(q-1)$-space $D_{1,q-1,R}$ based on $(E \oplus E')^{\perp}$.

● RIASSUNTO (*)

Si dimostra che l'operazione di coniugazione della carica (e quindi anche la *CPT*) si puó realizzare con un'effettiva trasformazione geometrica reale entro la struttura della teoria cronogeometrica di Segal. Si discutono anche alcune questioni geometriche connesse con i fondamenti della teoria cronogeometrica.

(*) *Traduzione a cura della Redazione.*

Зарядовое сопряжение и космология Сегала.

Резюме (*). — Показывается, что операция зарядового сопряжения (и, следовательно, *CPT*) может быть реализована, как реальное геометрическое преобразование в рамках хроногеометрической теории Сегала. Мы также обсуждаем некоторые геометрические вопросы, связанные с обоснованием хроногеометрической теории.

(*) *Переведено редакцией.*