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PARTIAL SPIN STRUCTURES AND INDUCED REPRESENTATIONS OF LIE GROUPS

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In this paper I will sketch a geometric setting for induced representations. So far, the setting has been useful in the study of unitary representations of reductive Lie groups. The square integrable representations of those groups can be obtained by examining geometrically defined differential operators on appropriate vector bundles. The other representations involved in harmonic analysis on the group are then constructed by inducing square integrable representations of certain subgroups. The setting can be viewed as a sort of measurable version of variation of certain types of geometric structures.

1. Measurable families. We start with the notion of *measurable family of C^∞ manifolds*. This consists of (i) a locally trivial Borel fibre space $p: X \to Z$ where X and Z are analytic Borel spaces, (ii) the structure of C^∞ n-manifold on each fibre $Y_z = p^{-1}(z)$, and the compatibility condition (iii) X induces the intrinsic Borel structure on each Y_z. Then *partially C^∞ bundle* over X means a locally trivial Borel fibre space $\mathscr{W} \to X$ such that the $\mathscr{W}|_{Y_z} \to Y_z$ all are C^∞ bundles with the same structure group, and *partially C^∞ section* means a Borel section C^∞ over each Y_z.

EXAMPLE 1. *Partially C^∞ function* on X means a Borel function C^∞ on each Y_z.

EXAMPLE 2. The *partial tangent bundle* $\mathscr{T} \to X$ is the union of the tangent bundles $\mathscr{T}_z \to Y_z$, with Borel structure as follows. A partially C^∞ function $f: X \to R$ has *differential* $df: \mathscr{T} \to R$ given by $(df)|_{\mathscr{T}_z} = d(f|_{\mathscr{T}_z})$, and $\mathscr{T}$ has Borel structure defined by all these df. Then $\mathscr{T} \to X$ is a partially C^∞ vector bundle; its partially C^∞

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sections are the Borel families of C^∞ vector fields on the Y_z.

Let $\mathfrak{S}$ be a geometric structure, e.g. riemannian. By *measurable family of $\mathfrak{S}$-manifolds* we mean a measurable family $p: X \to Z$ of C^∞ manifolds, and a collection of partially C^∞ bundles and sections over X whose restrictions to each Y_z define an $\mathfrak{S}$-structure there. For example, a measurable family of riemannian manifolds comes from the partial tangent bundle $\mathcal{T} \to X$ and an appropriate partially C^∞ section of $\mathcal{T}^* \otimes \mathcal{T}^*$.

2. Example: partially harmonic spinors. Fix a measurable family $p: X \to Z$ of oriented riemannian n-manifolds and a Lie group homomorphism $\alpha: U \to SO(n)$ that factors $U \to^{\tilde{\alpha}} \mathrm{Spin}(n) \to SO(n)$. As above, differentials of partially C^∞ functions specify the *partial oriented orthonormal frame bundle* $\pi: \mathcal{F} \to X$. Suppose that we have a partially C^∞ principal U-bundle $\pi_U: \mathcal{F}_U \to X$, a bundle map $\tilde{\alpha}: \mathcal{F}_U \to \mathcal{F}$ given by α on each fibre, and a partially C^∞ connection on $\mathcal{F}_U$ whose Y_z-restrictions map under $\tilde{\alpha}$ to the riemannian connections on the $\mathcal{F}|_{Y_z}$.

Let μ be a finite-dimensional unitary representation of U, say on V_μ. We have the associated partially C^∞ vector bundle $\mathcal{V}_\mu = \mathcal{F}_U \times_U V_\mu \to X$. If s is the spin representation of $\mathrm{Spin}(n)$ this gives the *partial spin bundle* $\mathcal{S} = \mathcal{V}_{s \cdot \alpha} \to X$ and the bundle $\mathcal{S} \otimes \mathcal{V}_\mu \to X$ of $\mathcal{V}_\mu$-valued partial spinors. These bundles are hermitian.

Let dz be a positive σ-finite Borel measure on Z. If γ, δ are Borel sections of $\mathcal{S} \otimes \mathcal{V}_\mu$ their global inner product is $\langle \gamma, \delta \rangle = \int_Z (\int_{Y_z} \langle \gamma_y, \delta_y \rangle \, dy) \, dz$. This defines a Hilbert space $L_2(\mathcal{S} \otimes \mathcal{V}_\mu) = \{\gamma$ Borel section of $\mathcal{S} \otimes \mathcal{V}_\mu : \langle \gamma, \gamma \rangle < \infty\}$, the *square integrable $\mathcal{V}_\mu$-valued partial spinors* on X. Evidently it is the direct integral $\int_Z L_2((\mathcal{S} \otimes \mathcal{V}_\mu)|_{Y_z}) \, dz$ of the spaces of square integrable sections over the Y_z.

If n is even, $\mathcal{S}$ splits as direct sum of the partial half-spin bundles $\mathcal{S}^\pm$, and we have the usual Dirac operators $D_z^\pm: C^\infty(\mathcal{S}^\pm \otimes \mathcal{V}_\mu|_{Y_z}) \to C^\infty(\mathcal{S}^\mp \otimes \mathcal{V}_\mu|_{Y_z})$ on sections C^∞ over Y_z. These fit together to form the *partial Dirac operators* $D^\pm$: $C^\infty(\mathcal{S}^\pm \otimes \mathcal{V}_\mu) \to C^\infty(\mathcal{S}^\mp \otimes \mathcal{V}_\mu)$ on partially C^∞ $\mathcal{V}_\mu$-valued spinors. If the riemannian manifold Y_z is complete then $D_z = D_z^+ \oplus D_z^-$ is essentially selfadjoint on $L_2(\mathcal{S} \otimes \mathcal{V}_\mu|_{Y_z})$ from domain consisting of the compactly supported C^∞ sections. Thus, if Y_z is complete a.e. (Z, dz), then $D = D^+ \oplus D^-$ is essentially selfadjoint, and we have Hilbert spaces

$$H_2(\mathcal{V}_\mu) = H_2^+(\mathcal{V}_\mu) \oplus H_2^-(\mathcal{V}_\mu)$$

where $H_2^\pm(\mathcal{V}_\mu) = \{\omega \in L_2(\mathcal{S}^\pm \otimes \mathcal{V}_\mu): D^\pm \omega = 0\} = \int_Z H_2^\pm(\mathcal{V}_\mu|_{Y_z}) \, dz$. These spaces are closed in L_2 and consist of partially C^∞ sections. We refer to the elements of $H_2(\mathcal{V}_\mu)$ as the *square integrable $\mathcal{V}_\mu$-valued partially harmonic spinors* on X.

3. Application to symmetric spaces. Let G be a reductive Lie group, $\mathfrak{g}$ its Lie algebra. We assume that $\mathrm{ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$ whenever $g \in G$. G^0 denotes the identity component of G, $Z_G(G^0)$ its G-centralizer, and $G^\dagger = Z_G(G^0)G^0$. We assume that $Z_G(G^0)$ has a closed abelian subgroup Z with G/ZG^0 finite.

Let H be a Cartan subgroup of G, θ a Cartan involution with $\theta(H) = H$, and K the fixed point set of θ. Split $H = T \times A$ where $T = H \cap K$. Choose a positive

$\mathfrak{a}$-root system $\Sigma_{\mathfrak{a}}^{+}$ on $\mathfrak{g}$, let $\mathfrak{n}$ be the sum of the negative $\mathfrak{a}$-root spaces, and define $N = \exp(\mathfrak{n})$. Then the normalizer of N in G is a cuspidal parabolic subgroup $P = MAN$ of G, where $Z_G(A) = M \times A$. The H-series of unitary equivalence classes of representations of G consists of the $[\pi_{\eta,\sigma}] = [\mathrm{Ind}_{P\uparrow G}(\eta \otimes e^{i\sigma})]$ where $[\eta]$ is a square integrable representation class of M and $\sigma \in \mathfrak{a}^*$. Plancherel measure on the unitary dual of G is concentrated on the union of the various H-series.

Choose a positive $\mathfrak{t}_C$-root system $\Sigma_{\mathfrak{t}}^{+}$ on $\mathfrak{m}_C$ and let $2\rho_{\mathfrak{t}} = \sum_{\beta\in\Sigma^+}\beta$. If $\nu \in i\mathfrak{t}^*$ is integral and $\mathfrak{m}_C$-regular let $[\eta_\nu]$ denote the corresponding square integrable representation class of M^0; its restriction to $Z_{M^0} = Z_M(M^0) \cap M^0$ is $e^{\nu-\rho_{\mathfrak{t}}}$. If χ is an irreducible unitary representation of $Z_M(M^0)$ with same Z_{M^0}-restriction then $\chi \otimes \eta$ is a square integrable representation of $M = Z_M(M^0)M^0$, and $\eta_{\chi,\nu} = \mathrm{Ind}_{M^\dagger\uparrow M}(\chi \otimes \eta_\nu)$ is an irreducible square integrable representation of M. All the unitary equivalence classes of irreducible square integrable representations of M are of that form $[\eta_{\chi,\nu}]$, so the H-series of G consists of the $[\pi_{\chi,\nu,\sigma}] = [\pi_{\eta_{\chi,\nu},\sigma}]$.

Retain the notation above, define $U = K \cap M^\dagger$, and consider $X = G/UAN \to G/M^\dagger AN = Z$. Replacing G by a Z_2-extension if necessary, the linear isotropy action of U on the tangent space of $M^\dagger/U$ factors through $\mathrm{Spin}(n)$, $n = \dim M^\dagger/U$, and we have the setup of § 2 with $\mathscr{F}_U = G/AN \to G/UAN = X$. If μ is a finite-dimensional unitary representation of U, say on V_μ, we view $\mathscr{V}_\mu \to G/UAN = X$ as the G-homogeneous bundle for the representation of UAN given by $uan \to e^{\rho_{\mathfrak{a}}+i\sigma}(a)\mu(u)$ where $2\rho_{\mathfrak{a}} = \sum_{\alpha\in\Sigma_{\mathfrak{a}}^+}(\dim \mathfrak{g}^\alpha)\alpha$. Then G acts on $H_{\frac{1}{2}}^{\pm}(\mathscr{V}_\mu)$ by a unitary representation $\pi_{\mu,\sigma}^{\pm}$.

T is a Cartan subgroup of U; let $\rho_{\mathfrak{t},\mathfrak{u}}$ be half the sum of the positive $\mathfrak{t}_C$-roots of $\mathfrak{u}_C$. $U = Z_M(M^0)U^0$. If μ is irreducible now $\mu = \chi \otimes \mu^0$ and we express the highest weight of μ^0 as $\nu - \rho_{\mathfrak{t}} + \rho_{\mathfrak{t},\mathfrak{u}}$. Suppose that $\nu + \rho_{\mathfrak{t}}$ is $\mathfrak{m}$-regular and $\sigma \in \mathfrak{a}^*$. Then the distribution characters of $\pi_{\mu,\sigma}^{\pm}$ and $\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}$ are related by a difference formula

$$\Theta_{\pi_{\mu,\sigma}^{+}} - \Theta_{\pi_{\mu,\sigma}^{-}} = \Theta_{\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}}.$$

In particular, $\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}$ occurs as a subrepresentation in one of $\pi_{\mu,\sigma}^{\pm}$. Incidentally, every H-series class is of this form $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$. Further, under a mild additional condition on $\nu + \rho_{\mathfrak{t}}$, one of the Hilbert spaces $H_{\frac{1}{2}}^{\pm}(\mathscr{V}_\mu)$ is reduced to zero, and so the other $[\pi_{\mu,\sigma}^{\mp}]$ is equal to $[\pi_{\chi,\nu+\rho_{\mathfrak{t}}}]$.

In summary, the action of G on the Hilbert spaces $H_{\frac{1}{2}}^{\pm}(\mathscr{V}_\mu)$ of square integrable partially harmonic spinors provides geometric realizations for a family of unitary representation classes that carries the Plancherel measure of G.

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