ON ISOCLINIC CLOSURE:
CORRECTION TO "ELLiptic SpACES IN
GRASSmann MANIFOLDS"

by

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Let $F$ be a field $R$ (real), $C$ (complex) or $K$ (quaternion), $F^k$ the positive definite left unitary vector space of dimension $k$ over $F$, and $G_{n,k}(F)$ the Grassmann manifold of $n$-dimensional $F$-subspaces of $F^k$ with its usual structure as a Riemannian symmetric space. If $W$ is a subspace of $F^k$ then $\pi_W$ denotes the orthogonal projection $F^k \to W$. Subspaces $B, B'$ of the same dimension in $F^k$ are iso-clinic if $\pi_B: B' \to B$ is an $F$-unitary similarity. For example, the connected totally geodesic submanifolds $B$ in $G_{n,k}(F)$ such that any two distinct elements of $B$ have zero intersection as subspaces of $F^k$, have the property that the elements of $B$ are pairwise iso-clinic [2, Theorems 2 and 4], [3, Theorem 2]. In [2] and [3] one finds a complete classification of all such submanifolds $B$.

After writing out that classification, Wolf considered an arbitrary subset $A$ of $G_{n,k}(F)$ whose elements are pairwise iso-clinic, and in [3, Section 7] he claimed to define an operation of "iso-clinic closure" enlarging $A$ to a totally geodesic submanifold $A_* = B \subset G_{n,k}(F)$ of the type described above. That iso-clinic closure operation depended in an essential manner on the following property:

(*) If $X, B, B' \in G_{n,k}(F)$ are pairwise iso-clinic with $B \neq B'$ then $Z = \pi_{B \oplus B}(X)$ either is 0 or is $n$-dimensional and iso-clinic to $X$.

If $k = 2n$ then (*) follows from the Hurwitz equations; see [2, Theorem 1]. Professor Y.-C. Wong [4; Chapter III, Section 11] and Daniel Shapiro (unpublished) gave examples showing that (*) fails for $(n, k) = (2, 6)$, and thus fails whenever $n$ is even and $k > 3n$.

Here is a counterexample to (*) for $k = 2n + 1$, thus for all $(n, k)$ with $k > 2n$. Let $\{e_1, \ldots, e_n; f_1, \ldots, f_n; u\}$ be an orthonormal basis of $F^{2n+1}$ and define the linear spans

$$B = \{e_1, \ldots, e_n\}_F \quad \text{and} \quad B' = \{e_1 + f_1, \ldots, e_n + f_n\}_F,$$

and

$$X = \{x_1, \ldots, x_n\}_F \quad \text{where} \quad x_1 = e_1 + f_1 + 2\sqrt{2}u$$

and

$$x_j = e_j - 3f_j \quad \text{for} \quad 2 \leq j \leq n.$$
Then \( B \) and \( B' \) are clearly isoclinic, and the \( F \)-hermitian inner products of the \( x_1 \) are

\[
x_1 \cdot x_1 = 1 + 1 + 8 = 10 = 1 + 9 = x_j \cdot x_j \quad \text{for} \ 2 \leq j \leq n
\]

and

\[
x_i \cdot x_j = 0 \quad \text{for} \ i \neq j.
\]

Since \( e_j - 3f_j = -(e_j + f_j) + 2(e_j - f_j) \), the orthogonal projection \( \pi_B : X \to B \) sends \( x_1 \) to \( e_1 \) and \( x_j \) to \( e_j \) for \( 2 \leq j \leq n \), and the orthogonal projection \( \pi_{B'} : X \to B' \) sends \( x_1 \) to \( e_1 + f_1 \) and \( x_j \) to \( -(e_j + f_j) \) for \( 2 \leq j \leq n \). These are similarities, so \( X, B \) and \( B' \) are pairwise isoclinic, while \( \pi_{B \oplus B'} : X \to B \oplus B' \) sends \( x_1 \) to \( e_1 + f_1 \) and \( x_j \) to \( e_j - 3f_j \) for \( 2 \leq j \leq n \), which is not a similarity.

While we no longer have the isoclinic closure operation, we can still define isoclinically closed set as a subset \( A \subset G_{n,k}(F) \) whose elements are pairwise isoclinic and satisfy the following:

If \( B, B' \in A \) with \( B \neq B' \), then \( A \) contains the isoclinic sphere on \( B \oplus B' \) constructed in [2, Chapter I] from \( \{ X \in A : X \subset B \oplus B' \} \).

This agrees with the definition of isoclinically closed set in [3, Section 7] when the latter happens to make sense. Except for that definition we must throw out everything in [3, Section 7]. The notions of reducibility and support in [3, Section 8] remain valid, and [3, Lemma 10] is true for isoclinically closed sets:

Let \( B \) be an isoclinically closed set of pairwise isoclinic \( n \)-dimensional subspaces of \( F^k \). Then \( B = B^1 \cup \cdots \cup B^m \) where the \( B^i \) are isoclinically closed and irreducible, and \( \text{supp} B^i \perp \text{supp} B^j \) for \( i \neq j \). In the topology induced from \( G_{n,k}(F) \), the \( B^i \) are the topological components of \( B \).

Thus the main result of [3, Chapter II], which is Theorem 4 in Section 9 there, is correct as stated and proved.

**REFERENCES**

1. D. B. Shapiro, Similarities and isoclinic planes, to appear,
4. Y.-C. Wong, Geometry of \( n \)-planes in Euclidean and pseudo-Euclidean spaces and differential geometry of Grassmann manifolds and Cartan domains,