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FINITENESS OF ORBIT STRUCTURE FOR REAL FLAG MANIFOLDS

ABSTRACT. Let G be a reductive real Lie group, σ an involutive automorphism of G, and $L = G^\sigma$ the fixed point set of σ. It is shown that G has only finitely many L-conjugacy classes of parabolic subgroups, so if P is a parabolic subgroup of G then there are only finitely many L-orbits on the real flag manifold G/P. This is done by showing that G has only finitely many L-conjugacy classes of σ-stable Cartan subgroups. These results extend known facts for the case where G is a complex group and L is a real form of G.

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1. INTRODUCTION

One knows [4] that there are only finitely many conjugacy classes of Cartan subalgebras in a reductive real Lie algebra. If G is a complex reductive Lie group and L is a real form, it follows [8] that there are only finitely many L-conjugacy classes of parabolic subgroups of G. In particular, if $X = G/P$ is a complex flag manifold of G, then [8] there are only finitely many L-orbits on X. Here we extend these results to the case where G is a reductive real Lie group and L is the fixed point set of an involutive automorphism, and we indicate the scope of applicability of the extension.

2. CONJUGACY OF CARTAN SUBALGEBRAS

If G is a Lie group then G_0 denotes its identity component, $\mathfrak{g}$ denotes its Lie algebra, and $\mathrm{Int}(\mathfrak{g})$ denotes the inner automorphism group $\{\mathrm{Ad}(g): g \in G_0)$ of $\mathfrak{g}$.

THEOREM 1. *Let $\mathfrak{g}$ be a reductive real Lie algebra, σ an involutive automorphism of $\mathfrak{g}$, and $\mathfrak{l} = \mathfrak{g}^\sigma$ its fixed point set. Let L_0 denote the analytic subgroup of* $\mathrm{Int}(\mathfrak{g})$ *for* $\mathfrak{l}$. *Then there are only finitely many L_0-conjugacy classes of σ-stable Cartan subalgebras of* $\mathfrak{g}$.

Proof. It suffices to consider the case where $\mathfrak{g}$ is semisimple and has no proper σ-stable ideal. We now assume that, and we fix the notation $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ where $\mathfrak{m} = \{x \in \mathfrak{g} : \sigma(x) = -x\}$.

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Case 1: $\mathrm{Int}(\mathfrak{g})$ *is compact.* This case follows directly from Kostant's proof [4] that $\mathfrak{g}^* = \mathfrak{l} + i\mathfrak{m}$ has only finitely many conjugacy classes of Cartan subalgebras. Kostant shows that every Cartan subalgebra of $\mathfrak{g}^*$ is conjugate to a σ-stable one and that $\mathfrak{g}^*$ has only finitely many L_0-conjugacy classes of σ-stable Cartan subalgebras $\mathfrak{h}^*$. Then $\mathfrak{h}^* \leftrightarrow \mathfrak{h} = (\mathfrak{h}^* \cap \mathfrak{l}) + i(\mathfrak{h}^* \cap i\mathfrak{m})$ gives our finiteness assertion.

Case 2: $\mathrm{Int}(\mathfrak{g})$ *is complex and* σ *is conjugate-linear.* Then $\mathfrak{l}$ is a real form of $\mathfrak{g}$, and our assertion is Kostant's result [4] on finiteness of the number of conjugacy classes of Cartan subalgebras of $\mathfrak{l}$.

Case 3: $\mathrm{Int}(\mathfrak{g})$ *is complex and* σ *is complex-linear.* This is the nontrivial case. Fix a Cartan involution θ of $\mathfrak{g}$ that commutes with σ. Its fixed point set $\mathfrak{g}^\theta$ is a σ-stable compact real form of $\mathfrak{g}$, so

$$\mathfrak{g}^\theta = \mathfrak{l}^\theta + \mathfrak{m}^\theta \quad \text{where} \quad \mathfrak{l}^\theta = \mathfrak{l} \cap \mathfrak{g}^\theta \quad \text{and} \quad \mathfrak{m}^\theta = \mathfrak{m} \cap \mathfrak{g}^\theta .$$

Let L_0^θ denote the analytic subgroup of L_0 for $\mathfrak{l}^\theta$. Case 1 tells us that $\mathfrak{g}^\theta$ has only finitely many L_0^θ-conjugacy classes of σ-stable Cartan subalgebras. Since $\mathfrak{g}^\theta$ is a real form of $\mathfrak{g}$, this says that $\mathfrak{g}$ has only finitely many L_0^θ-conjugacy classes of θ-stable σ-stable Cartan subalgebras. L_0^θ-conjugacy implies L_0-conjugacy. Now we need only show that every L_0-conjugacy class of σ-stable Cartan subalgebras of $\mathfrak{g}$ contains a θ-stable algebra.

Let $\mathscr{H}$ be an L_0-conjugacy class of σ-stable Cartan subalgebras of $\mathfrak{g}$. We first consider the situation

$$\text{if} \quad \mathfrak{h} \in \mathscr{H} \quad \text{then} \quad \mathfrak{h} \subset \mathfrak{m} \quad \text{i.e.} \quad \mathfrak{h} \cap \mathfrak{l} = 0 .$$

Fix $\mathfrak{h} \in \mathscr{H}$ and let $\mathfrak{h}^\#$ be its real form spanned by the roots. If ϕ is an $\mathfrak{h}$-root then $\sigma^*\phi = -\phi$ because $\mathfrak{h} \subset \mathfrak{m}$. Thus $\mathfrak{g}$ has a σ-stable compact real form $\mathfrak{g}'$ with $i\mathfrak{h}^\# \subset \mathfrak{g}'$. Let θ' denote the Cartan involution of $\mathfrak{g}$ that is complex conjugation over $\mathfrak{g}'$. Evidently $\theta'(\mathfrak{h}) = \mathfrak{h}$. Thus, to exhibit a θ-stable element of $\mathscr{H}$ it suffices to prove that θ and θ' are L_0-conjugate, and for this we need only check that $\mathfrak{g}' = \mathfrak{g}^{\theta'}$ is L_0-conjugate to $\mathfrak{g}^\theta$.

As $\mathfrak{g}'$ is σ-stable, $\mathfrak{l}^{\theta'} = \mathfrak{l} \cap \mathfrak{g}'$ is a compact real form of $\mathfrak{l}$, hence is L_0-conjugate to $\mathfrak{l}^\theta$. Thus we may assume $\mathfrak{l}^{\theta'} = \mathfrak{l}^\theta$. That assumption made, $\mathfrak{g}' = \mathfrak{l}^\theta + \mathfrak{m}^{\theta'}$ just as $\mathfrak{g}^\theta = \mathfrak{l}^\theta + \mathfrak{m}^\theta$. Since $\mathfrak{l}^\theta$ is a real form of $\mathfrak{l}$, $\mathfrak{l}^\theta \subset \mathfrak{g}^{\theta\theta'}$ and $\theta\theta'$ is complex linear, we now have $\mathfrak{l} \subset \mathfrak{g}^{\theta\theta'}$.

$\zeta = \theta'^{-1} \cdot \theta$ is a complex linear automorphism of $\mathfrak{g}$ and trivial on $\mathfrak{l}$, so $\zeta|_\mathfrak{m}$ commutes with every element of $L_0|_\mathfrak{m}$. If $\zeta|_\mathfrak{m} = \pm 1$ then $\theta = \pm\theta'$ on $\mathfrak{m}$, so $\mathfrak{m}^{\theta'} = \mathfrak{m}^\theta$ or $\mathfrak{m}^{\theta'} = i\mathfrak{m}^\theta$. If $\mathfrak{m}^{\theta'} = \mathfrak{m}^\theta$ then $\mathfrak{g}' = \mathfrak{g}^\theta$ as desired. If $\mathfrak{m}^{\theta'} = i\mathfrak{m}^\theta$ then $\mathfrak{g}' = \mathfrak{l}^\theta + i\mathfrak{m}^\theta$, which is noncompact, contradicting the fact that θ' is a Cartan involution. Now we assume $\zeta|_\mathfrak{m} \neq \pm 1$. $\mathfrak{g}$ is simple because it has no proper

σ-stable ideal, so now $(\mathfrak{g}^\theta, \mathfrak{l}^\theta)$ is an irreducible hermitian symmetric pair, and $\mathfrak{m}=\mathfrak{m}_+ + \mathfrak{m}_-$ direct sum of $\mathfrak{l}$-invariant $\mathfrak{l}$-irreducible subspaces that are interchanged by θ and also by θ'. Thus we have a complex number α with

$$\zeta|_{\mathfrak{m}_+} = \alpha \quad \text{and} \quad \zeta|_{\mathfrak{m}_-} = \bar{\alpha}.$$

Let $x_\pm \in \mathfrak{m}_\pm$ and compute

$$\begin{aligned} x_+ + x_- &= \theta^2(x_+ + x_-) \\ &= (\theta'\zeta)^2(x_+ + x_-) \\ &= \theta'\zeta(\theta'\bar{\alpha}x_- + \theta'\alpha x_+) \\ &= \theta'(\bar{\alpha}\theta'\alpha x_+) + \theta'(\alpha\theta'\bar{\alpha}x_-). \end{aligned}$$

Thus $\alpha\bar{\alpha}=1$ and $\zeta = \mathrm{Ad}(z)$ where z is central in L_0^θ. Now $\theta\mathfrak{m}^{\theta'} = \theta' \mathrm{Ad}(z)\mathfrak{m}^{\theta'} = \theta'\mathfrak{m}^{\theta'} = \mathfrak{m}^{\theta'}$, so

$$\mathfrak{m}^{\theta'} = (\mathfrak{m}^{\theta'} \cap \mathfrak{m}^\theta) + (\mathfrak{m}^{\theta'} \cap i\mathfrak{m}^\theta).$$

The second summand vanishes because $\mathfrak{g}'$ is compact. We conclude $\mathfrak{g}' = \mathfrak{g}^\theta$.

In summary: if $\mathscr{H}$ is an L_0-conjugacy class of σ-stable Cartan subalgebras of $\mathfrak{g}$, and if $\mathfrak{h} \in \mathscr{H}$ implies $\mathfrak{h} \subset \mathfrak{m}$, we have shown that $\mathscr{H}$ contains a θ-stable algebra.

Now let $\mathscr{H}$ be any L_0-conjugacy class of σ-stable Cartan subalgebras of $\mathfrak{g}$. Let $\mathfrak{h} \in \mathscr{H}$ and split $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) + (\mathfrak{h} \cap \mathfrak{m})$. Passing to the derived algebra of the $\mathfrak{g}$-centralizer of $\mathfrak{h} \cap \mathfrak{l}$, we reduce to the case just considered, obtaining a θ-stable L_0-conjugate of $\mathfrak{h}$. This completes the argument for Case 3.

Case 4: the general case. Let $n = \dim \mathfrak{g}$, $r = \operatorname{rank} \mathfrak{g}$ and $N = \binom{n}{r} - 1$. Use Plücker coordinates to view the Grassmannian of r-dimensional subspaces of $\mathfrak{g}^C$ as a subvariety of the complex projective space $P^N(C)$. Let L_0^C denote the analytic subgroup of $\mathrm{Int}(\mathfrak{g}^C)$ with Lie algebra $\mathfrak{l}^C$. Now every L_0^C-conjugacy class of Cartan subalgebras of $\mathfrak{g}^C$ is an L_0^C-orbit on $P^N(C)$ under the rational representation $\Lambda^r(\mathrm{Ad}_{\mathrm{Int}(\mathfrak{g}^C)}|_{L_0^C})$. If $\mathcal{O}$ is such an orbit, then [6, Lemma 1.1] $\mathcal{O} \cap P^N(R)$ is a finite union of L_0-orbits. Case 4 now follows from Cases 2 and 3. Q.E.D.

3. Conjugacy of parabolic subgroups

If G is a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and if P is a complex Lie subgroup with Lie algebra $\mathfrak{p}$, then one has equivalent conditions

(i) G/P is compact,

(ii) G/P is a compact simply connected kaehler manifold,

(iii) G/P is a complex projective variety,

(iv) G/P is a closed G-orbit in a projective representation.

Under those circumstances, P is a *parabolic subgroup* of G, $\mathfrak{p}$ is a *parabolic subalgebra* of $\mathfrak{g}$, and one can see that $P=\{g\in G:\mathrm{Ad}(g)\,\mathfrak{p}=\mathfrak{p}\}$.

Let G be a complex Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{s}$ the solvable radical of $\mathfrak{g}$, and $\pi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{s}$ the projection. The *parabolic subalgebras* of $\mathfrak{g}$ are the $\pi^{-1}(\mathfrak{q})$ where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}/\mathfrak{s}$. The *parabolic subgroups* of G are the normalizers $P=\{g\in G:\mathrm{Ad}(g)\,\mathfrak{p}=\mathfrak{p}\}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$; then $\mathfrak{p}$ is the Lie algebra of P.

Let G be a real Lie group and $\mathfrak{g}$ its Lie algebra. Then *parabolic subalgebra* of $\mathfrak{g}$ means a subalgebra $\mathfrak{p}=\mathfrak{g}\cap\mathfrak{q}$ where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}^C$ stable under complex conjugation over $\mathfrak{g}$. The *parabolic subgroups* of G are the normalizers $P=\{g\in G:\mathrm{Ad}(g)\,\mathfrak{p}=\mathfrak{p}\}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and then P has Lie algebra $\mathfrak{p}$.

THEOREM 2. *Let G be a real Lie group, σ an involutive automorphism of G, and $L=G^\sigma$ its fixed point set. Then there are only finitely many L_0-conjugacy classes of parabolic subgroups of G.*

Proof. We will show that $\mathfrak{g}$ has only finitely many L_0-conjugacy classes of parabolic subalgebras. For this we may assume G connected and simply connected. Now G has σ-stable Levi decomposition $G_1\cdot S$ where S is the solvable radical and we may replace G by its semisimple quotient $G/S\cong G_1$. Thus we may assume $\mathfrak{g}$ semisimple.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{p}\cap\sigma(\mathfrak{p})$ is σ-stable and contains a Cartan subalgebra of $\mathfrak{g}$, so it contains a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. There are only finitely many parabolic subalgebras of $\mathfrak{g}$ containing any given Cartan subalgebra, and Theorem 1 says that there are only finitely many possibilities for $\mathfrak{h}$ up to L_0-conjugacy. Thus there are only finitely many possibilities for $\mathfrak{p}$ up to L_0-conjugacy. Q.E.D.

4. Application to real flag manifolds

By *real flag manifold* we mean a homogeneous space $X=G/P$ where G is a real Lie group and P is a parabolic subgroup. Since P is its own normalizer in G, we may view X as the set of all G-conjugates of P under $x\leftrightarrow\{g\in G:g(x)=x\}$. If S is a subgroup of G, then the orbit $S(x)$ corresponds to the S-conjugacy class of $\{g\in G:g(x)=x\}$. Now Theorem 2 gives us

THEOREM 3. *Let G be a real Lie group, P a parabolic subgroup, and $X=G/P$ the corresponding real flag manifold. Let σ be an involutive automorphism of G and $L=G^\sigma$ its fixed point set. Then there are only finitely many*

L_0-orbits on X. In particular, L and L_0 each has both open and closed orbits on X, and each L-orbit is a finite union of L_0-orbits.

A trivial consequence: X has only finitely many topological components. Here note that X is connected if and only if every G-conjugate of P is G_0-conjugate to P.

In the complex case (G is a complex Lie group, P is a complex parabolic subgroup and L is a real form of G) there is just one closed L_0-orbit on every topological component of $X=G/P$ [8, Theorem 3.3] Here is an example in the real case where there are two closed orbits. Let $G=SL(2;R)$ and

$$P=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a, b\in R \quad \text{with} \quad a\neq 0\right\}$$

so that $X=G/P$ is the circle $R\cup\{\infty\}$ bounding the upper half plane, the action being

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}: \quad x\mapsto \frac{ax+b}{cx+d}.$$

Let σ be conjugation by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix};$$

then $L=G^\sigma$ is the Cartan subgroup

$$\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}: \quad 0\neq a\in R\right\}$$

with L_0 given by $a>0$. As

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}: \quad x\mapsto a^2x,$$

L and L_0 have the same orbits on X, which are

two closed orbits $\{0\}$ and $\{\infty\}$, and
two open orbits $\{x\in R: x>0\}$ and $\{x\in R: x<0\}$.

This is a variation on the Šilov boundary example in 5 below.

5. Application to symmetric R-spaces

A *symmetric R-space* is a compact Riemannian symmetric space $X=K/V$ which admits a connected effective Lie transformation group G such that (i) $K\subset G$ and (ii) if Y is a G-stable Riemannian factor of X then G acts on Y as a noncompact group. Then [5] G is semisimple, K is a maximal com-

pact subgroup of G, and, up to finite covering, X is a real flag manifold G/P. It often happens that G contains an interesting noncompact form L of K.

EXAMPLE: *Grassmann Manifolds.* Let F denote the field of real, complex or quaternion numbers. If $n>0$ then $U(n;F)$ denotes its unitary group, which is the orthogonal group $O(n)$, the complex unitary group $U(n)$ or the unitary symplectic group $\mathrm{Sp}(n)$. The grassmannian of k-planes in F^{k+l} is the compact Riemannian symmetric space

$$X = K/V = U(k+l;F)/U(k;F) \times U(l;F).$$

Under the action of the general linear group $G=\mathrm{GL}(k+l;F)$ it is a real flag G/P where

$$P = \left\{ g \in G : g \text{ has matrix } \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \right\}.$$

Let L be the F-unitary group of any of the nondegenerate F-Hermitian forms $h(x,y)=(x_1\bar{y}_1+\cdots+x_p\bar{y}_p)-(x_{p+1}\bar{y}_{p+1}+\cdots+x_{p+q}\bar{y}_{p+q})$ where $p+q=k+l$. Then Witt's Theorem (over F) gives us the L-orbit structure of X.

EXAMPLE: *Hermitian Symmetric Spaces.* Here $X=K/V$ is an Hermitian symmetric space of compact type and $G=K^{\mathbf{C}}$. The dual bounded symmetric domain $D=K^*/V$ where K^* is a certain noncompact real form of G, and the Borel embedding realizes D as a certain open K^*-orbit on X. One knows the K^*-orbit structure of X ([7–9]). Let L be an arbitrary real form of G.

EXAMPLE: *Šilov Boundaries of Tube Domains.* Let $X=K/V$ be an hermitian symmetric space of compact type and $D=K^*/V$ the dual bounded symmetric domain. Suppose that D is holomorphically equivalent to a tube domain. Then its Bergman-Šilov boundary is a symmetric R-space $S=V/E=K^*/\mathrm{W}$; see [3]. Let L be an isotropy subgroup of K^* on an open K^*-orbit in X. If one expresses the open orbit as $K^*(c_\Sigma^2 x_0)$ where x_0 is the base point and c_Σ is a partial Cayley transform then $L=K^* \cap \mathrm{Ad}(c_\Sigma^2)\, V^{\mathbf{C}}$, a possibly-noncompact form of V.

These examples all fit into the following pattern, and the result allows us to apply Theorem 3 to symmetric R-spaces. Here note that any reductive subalgebra of $\mathfrak{g}$ is invariant under a Cartan involution.

THEOREM 4. *Let $\mathfrak{g}$ be a semisimple Lie algebra, θ a Cartan involution, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ the eigenspace decomposition under θ, so that $\mathfrak{k}=\mathfrak{g}^\theta$ is the maximal compactly embedded subalgebra. Suppose that $\mathfrak{g}$ has no proper θ-stable ideal,*

i.e. *that* $(\mathfrak{g}, \mathfrak{k})$ *is an irreducible Riemannian symmetric pair of noncompact type. Let* $\mathfrak{l}$ *be a* θ*-stable subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{l}^C \cong \mathfrak{k}^C$.

(1) *There is a complex linear automorphism* α *of* $\mathfrak{g}^C$ *that preserves the compact form* $\mathfrak{k}+i\mathfrak{s}$ *and sends* $\mathfrak{l}^C$ to $\mathfrak{k}^C$.

(2) $\mathfrak{g}$ *is stable under the involutive automorphism* $\sigma=\alpha^{-1}\theta\alpha$, *and* $\mathfrak{l}=\mathfrak{g}^\sigma$.

Remark. I suspect that Theorem 4 is true for $\mathfrak{g}$ reductive and $(\mathfrak{g}, \mathfrak{k})$ assumed irreducible. But this does not affect the application to symmetric R-spaces.

Proof. We first check that (1) implies (2). Since $\alpha(\mathfrak{l}^C)=\mathfrak{k}^C$, the fixed point set $(\mathfrak{g}^C)^\sigma=\alpha^{-1}\cdot(\mathfrak{g}^C)^\theta=\alpha^{-1}(\mathfrak{k}^C)=\mathfrak{l}^C$ which is stable under θ. Thus θ and σ commute. Let $\mathfrak{m}^C$ denote the (-1)-eigenspace of σ, so $\mathfrak{g}^C=\mathfrak{l}^C+\mathfrak{m}^C$ under σ. Then

$$\begin{aligned}\mathfrak{g}^C = (\mathfrak{l}^C \cap \mathfrak{k}) + (\mathfrak{l}^C \cap i\mathfrak{s}) + (\mathfrak{m}^C \cap \mathfrak{k}) + (\mathfrak{m}^C \cap i\mathfrak{s}) + \\ + (\mathfrak{l}^C \cap i\mathfrak{k}) + (\mathfrak{l}^C \cap \mathfrak{s}) + (\mathfrak{m}^C \cap i\mathfrak{k}) + (\mathfrak{m}^C \cap i\mathfrak{s}).\end{aligned}$$

Each of these eight summands is σ-stable because σ preserves $\mathfrak{k}^C$, $\mathfrak{s}^C$ and $\mathfrak{k}+i\mathfrak{s}$. That gives σ-stability of $\mathfrak{g}=(\mathfrak{l}^C\cap\mathfrak{k})+(\mathfrak{l}^C\cap\mathfrak{s})+(\mathfrak{m}^C\cap\mathfrak{k})+(\mathfrak{m}^C\cap\mathfrak{s})$, and now $\mathfrak{g}^\sigma=\mathfrak{g}\cap(\mathfrak{g}^C)^\sigma=\mathfrak{g}\cap\mathfrak{l}^C=\mathfrak{l}$.

We prove (1). Let $\mathfrak{g}^*=\mathfrak{k}+i\mathfrak{s}$ and $\mathfrak{l}^*=(\mathfrak{l}\cap\mathfrak{k})+i(\mathfrak{l}\cap\mathfrak{s})$.

If $\mathfrak{g}$ is not absolutely simple, then $\mathfrak{g}^*\cong\mathfrak{k}\oplus\mathfrak{k}$ with $\mathfrak{k}$ embedded diagonally and θ interchanging the two summands by $\theta(x, y)=(y, x)$. Let $v_j:\mathfrak{g}^*\to\mathfrak{k}_j$ denote projection to the jth summand. $\mathfrak{l}^*$ is in neither summand because $\theta\mathfrak{l}^*=\mathfrak{l}^*$, so $v_j:\mathfrak{l}^*\cong\mathfrak{k}_j$. That gives $\beta\in\mathrm{Aut}(\mathfrak{k})$ with $\mathfrak{l}^*=\{(x, \beta x): x\in\mathfrak{k}\}$. Notice that $\beta^2=1$ because $\mathfrak{l}^*=\theta\mathfrak{l}^*=\{(\beta y, y): y\in\mathfrak{k}\}$. Now $\alpha=1\times\beta$ is an automorphism of $\mathfrak{g}^*$ that interchanges $\mathfrak{l}^*$ and the diagonally embedded $\mathfrak{k}$.

If $\mathfrak{g}$ is absolutely simple and rank $\mathfrak{k}=$rank $\mathfrak{g}$, then $\mathfrak{g}^*$ is simple and its subalgebra $\mathfrak{k}$ is a maximal subalgebra of maximal rank. The Borel-de Siebenthal classification [1] shows that any two subalgebras of $\mathfrak{g}^*$ isomorphic to $\mathfrak{k}$ must be $\mathrm{Aut}(\mathfrak{g}^*)$-conjugate.

Now suppose $\mathfrak{g}$ absolutely simple with rank $\mathfrak{k}<$rank $\mathfrak{g}$. If $\mathfrak{g}^*$ is of classical type then the inclusion $\mathfrak{k}\subsetneq\mathfrak{g}^*$ is one of

$$\mathfrak{so}(n)\subsetneq\mathfrak{su}(n),\ \mathfrak{sp}(n)\subsetneq\mathfrak{su}(2n),\ \mathfrak{so}(p)\oplus\mathfrak{so}(q)\subsetneq\mathfrak{so}(p+q).$$

Looking at degrees of representations we see that $\mathfrak{k}\subsetneq\mathfrak{g}^*$ and $\mathfrak{l}^*\subsetneq\mathfrak{g}^*$ are equivalent representations. If η is the equivalence we take $\alpha=\eta\mid_{\mathfrak{g}^*}$. If $\mathfrak{g}^*$ is of exceptional type then $\mathfrak{k}\subsetneq\mathfrak{g}^*$ is one of

$$\mathfrak{sp}(4)\subsetneq\mathfrak{e}_6 \quad\text{or}\quad \mathfrak{f}_4\subsetneq\mathfrak{e}_6$$

and Dynkin's results [2] show that any two subalgebras of $\mathfrak{g}^*$ isomorphic to $\mathfrak{k}$ must be $\mathrm{Aut}(\mathfrak{g}^*)$-conjugate. Q.E.D.

BIBLIOGRAPHY

1. Borel, A. and de Siebenthal, J.: 'Les sous-groupes fermés de rang maximum des groupes de Lie clos', *Comm. Math. Helv.* **23** (1949), 200–221.
2. Dynkin, E. B.: 'Semisimple Subalgebras of Semisimple Lie Algebras', *Math. Sb.* **30** (72), 349–462; also in *Am. Math. Soc. Transl. Ser. 2*, **6**, 111–244.
3. Korányi, A. and Wolf, J. A.: 'Realization of Hermitian Symmetric Spaces as Generalized Half-Planes', *Ann. Math.* **81** (1965), 265-288.
4. Kostant, B.: 'On the Conjugacy of Real Cartan Subalgebras', *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 967–970.
5. Nagano, T.: 'Transformation Groups on Compact Symmetric Spaces', *Trans. Am. Math. Soc.* **118** (1965), 428–453.
6. Richardson, R. W.: 'Principal Orbit Types for Real-Analytic Transformation Groups', *Am. J. Math.* **95** (1973), 193–203.
7. Takeuchi, M.: 'On Orbits in a Compact Hermitian Symmetric Space', *Am. J. Math.* **90** (1968), 657–680.
8. Wolf, J. A.: 'The Action of a Real Semisimple Group on a Complex Flag Manifold, I: Orbit Structure and Holomorphic Arc Components', *Bull. Am. Math. Soc.* **75** (1969), 1121–1237.
9. Wolf, J. A.: 'Fine Structure of Hermitian Symmetric Spaces', in *Symmetric Spaces: Short Courses Presented at Washington University* (ed. by Boothby and Weiss), Marcell Dekker, New York, 1972.

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