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FINITENESS OF ORBIT STRUCTURE
FOR REAL FLAG MANIFOLDS

ABSTRACT. Let $G$ be a reductive real Lie group, $\sigma$ an involutive automorphism of $G$, and $L = G^\sigma$ the fixed point set of $\sigma$. It is shown that $G$ has only finitely many $L$-conjugacy classes of parabolic subgroups, so if $P$ is a parabolic subgroup of $G$ then there are only finitely many $L$-orbits on the real flag manifold $G/P$. This is done by showing that $G$ has only finitely many $L$-conjugacy classes of $\sigma$-stable Cartan subgroups. These results extend known facts for the case where $G$ is a complex group and $L$ is a real form of $G$.

Key words and phrases: flag manifold, reductive Lie group, semisimple Lie group, parabolic subgroup, Cartan subgroup, Cartan subalgebra.

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1. INTRODUCTION

One knows [4] that there are only finitely many conjugacy classes of Cartan subalgebras in a reductive real Lie algebra. If $G$ is a complex reductive Lie group and $L$ is a real form, it follows [8] that there are only finitely many $L$-conjugacy classes of parabolic subgroups of $G$. In particular, if $X = G/P$ is a complex flag manifold of $G$, then [8] there are only finitely many $L$-orbits on $X$. Here we extend these results to the case where $G$ is a reductive real Lie group and $L$ is the fixed point set of an involutive automorphism, and we indicate the scope of applicability of the extension.

2. CONJUGACY OF CARTAN SUBALGEBRAS

If $G$ is a Lie group then $G_0$ denotes its identity component, $g$ denotes its Lie algebra, and $\text{Int}(g)$ denotes the inner automorphism group $\{\text{Ad}(g) : g \in G_0\}$ of $g$.

THEOREM 1. Let $g$ be a reductive real Lie algebra, $\sigma$ an involutive automorphism of $g$, and $I = g^\sigma$ its fixed point set. Let $L_0$ denote the analytic subgroup of $\text{Int}(g)$ for $I$. Then there are only finitely many $L_0$-conjugacy classes of $\sigma$-stable Cartan subalgebras of $g$.

Proof. It suffices to consider the case where $g$ is semisimple and has no proper $\sigma$-stable ideal. We now assume that, and we fix the notation $g = I + m$ where $m = \{x \in g : \sigma(x) = -x\}$.

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Case 1: \( \text{Int}(g) \) is compact. This case follows directly from Kostant's proof [4] that \( g^* = I + \mathbb{R} \text{m} \) has only finitely many conjugacy classes of Cartan subalgebras. Kostant shows that every Cartan subalgebra of \( g^* \) is conjugate to a \( \sigma \)-stable one and that \( g^* \) has only finitely many \( L_0 \)-conjugacy classes of \( \sigma \)-stable Cartan subalgebras \( \mathfrak{h}^* \). Then \( \mathfrak{h}^* \leftrightarrow \mathfrak{h} = (\mathfrak{h}^* \cap I) + i(\mathfrak{h}^* \cap \mathbb{R} \text{m}) \) gives our finiteness assertion.

Case 2: \( \text{Int}(g) \) is complex and \( \sigma \) is conjugate-linear. Then \( I \) is a real form of \( g \), and our assertion is Kostant's result [4] on finiteness of the number of conjugacy classes of Cartan subalgebras of \( I \).

Case 3: \( \text{Int}(g) \) is complex and \( \sigma \) is complex-linear. This is the nontrivial case. Fix a Cartan involution \( \theta \) of \( g \) that commutes with \( \sigma \). Its fixed point set \( g^\theta \) is a \( \sigma \)-stable compact real form of \( g \), so

\[
g^\theta = I^\theta + m^\theta \quad \text{where} \quad I^\theta = I \cap g^\theta \quad \text{and} \quad m^\theta = m \cap g^\theta.
\]

Let \( L_0^\theta \) denote the analytic subgroup of \( L_0 \) for \( I^\theta \). Case 1 tells us that \( g^\theta \) has only finitely many \( L_0^\theta \)-conjugacy classes of \( \sigma \)-stable Cartan subalgebras. Since \( g^\theta \) is a real form of \( g \), this says that \( g \) has only finitely many \( L_0^\sigma \)-conjugacy classes of \( \theta \)-stable \( \sigma \)-stable Cartan subalgebras. \( L_0^\sigma \)-conjugacy implies \( L_0 \)-conjugacy. Now we need only show that every \( L_0 \)-conjugacy class of \( \sigma \)-stable Cartan subalgebras of \( g \) contains a \( \theta \)-stable algebra.

Let \( \mathcal{H} \) be an \( L_0 \)-conjugacy class of \( \sigma \)-stable Cartan subalgebras of \( g \). We first consider the situation

if \( \mathfrak{h} \in \mathcal{H} \) then \( \mathfrak{h} \subset m \) i.e. \( \mathfrak{h} \cap I = 0 \).
σ-stable ideal, so now \((g^\theta, l^\theta)\) is an irreducible hermitian symmetric pair, and \(m = m_+ + m_-\) direct sum of \(l\)-invariant \(l\)-irreducible subspaces that are interchanged by \(\theta\) and also by \(\theta'\). Thus we have a complex number \(\alpha\) with

\[
\zeta_{|m_+} = \alpha \quad \text{and} \quad \zeta_{|m_-} = \bar{\alpha}.
\]

Let \(x_{\pm} \in m_{\pm}\) and compute

\[
x_{+} + x_- = \theta^2 (x_{+} + x_-) = (\theta' \zeta)^2 (x_{+} + x_-) = \theta' \zeta (\theta' \bar{\alpha} x_- + \theta' \alpha x_+) = \theta' (\alpha \theta' \bar{\alpha} x_-).
\]

Thus \(\alpha \bar{\alpha} = 1\) and \(\zeta = \text{Ad} (z)\) where \(z\) is central in \(L_0^g\). Now \(\theta m^\theta = \theta' \text{Ad} (z) m^\theta = \theta' m^\theta = m^\theta\), so

\[
m^\theta = (m^\theta \cap m^\theta) + (m^\theta \cap im^\theta).
\]

The second summand vanishes because \(g'\) is compact. We conclude \(g' = g^\theta\).

In summary: if \(\mathcal{H}\) is an \(L_0\)-conjugacy class of \(\sigma\)-stable Cartan subalgebras of \(g\), and if \(h \in \mathcal{H}\) implies \(h \subset m\), we have shown that \(\mathcal{H}\) contains a \(\theta\)-stable algebra.

Now let \(\mathcal{H}\) be any \(L_0\)-conjugacy class of \(\sigma\)-stable Cartan subalgebras of \(g\). Let \(h \in \mathcal{H}\) and split \(h = (h \cap l) + (h \cap m)\). Passing to the derived algebra of the \(g\)-centralizer of \(h \cap l\), we reduce to the case just considered, obtaining a \(\theta\)-stable \(L_0\)-conjugate of \(h\). This completes the argument for Case 3.

**Case 4:** the general case. Let \(n = \dim g\), \(r = \text{rank } g\) and \(N = \left(\frac{n}{r}\right) - 1\). Use Plücker coordinates to view the Grassmannian of \(r\)-dimensional subspaces of \(g^C\) as a subvariety of the complex projective space \(P^N(C)\). Let \(L_0^C\) denote the analytic subgroup of \(\text{Int}(g^C)\) with Lie algebra \(l^C\). Now every \(L_0^C\)-conjugacy class of Cartan subalgebras of \(g^C\) is an \(L_0^C\)-orbit on \(P^N(C)\) under the rational representation \(\lambda'(|\text{Ad}_{\text{Int}(g^C)}|_{L_0^C})\). If \(\mathcal{O}\) is such an orbit, then [6, Lemma 1.1] \(\mathcal{O} \cap P^N(R)\) is a finite union of \(L_0\)-orbits. Case 4 now follows from Cases 2 and 3. Q.E.D.

### 3. Conjugacy of parabolic subgroups

If \(G\) is a connected complex semisimple Lie group with Lie algebra \(g\), and if \(P\) is a complex Lie subgroup with Lie algebra \(p\), then one has equivalent conditions

(i) \(G/P\) is compact,

(ii) \(G/P\) is a compact simply connected kaehler manifold,
(iii) $G/P$ is a complex projective variety.
(iv) $G/P$ is a closed $G$-orbit in a projective representation.

Under those circumstances, $P$ is a parabolic subgroup of $G$, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and one can see that $P = \{ g \in G : \text{Ad}(g) \mathfrak{p} = \mathfrak{p} \}$.

Let $G$ be a complex Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{s}$ the solvable radical of $\mathfrak{g}$, and $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{s}$ the projection. The parabolic subalgebras of $\mathfrak{g}$ are the $\pi^{-1}(\mathfrak{q})$ where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}/\mathfrak{s}$. The parabolic subgroups of $G$ are the normalizers $P = \{ g \in G : \text{Ad}(g) \mathfrak{p} = \mathfrak{p} \}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$; then $\mathfrak{p}$ is the Lie algebra of $P$.

Let $G$ be a real Lie group and $\mathfrak{g}$ its Lie algebra. Then parabolic subalgebra of $\mathfrak{g}$ means a subalgebra $\mathfrak{p} = \mathfrak{g} \cap \mathfrak{q}$ where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g^C}$ stable under complex conjugation over $\mathfrak{g}$. The parabolic subgroups of $G$ are the normalizers $P = \{ g \in G : \text{Ad}(g) \mathfrak{p} = \mathfrak{p} \}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and then $P$ has Lie algebra $\mathfrak{p}$.

**THEOREM 2.** Let $G$ be a real Lie group, $\sigma$ an involutive automorphism of $G$, and $L = G^\sigma$ its fixed point set. Then there are only finitely many $L_0$-conjugacy classes of parabolic subgroups of $G$.

**Proof.** We will show that $\mathfrak{g}$ has only finitely many $L_0$-conjugacy classes of parabolic subalgebras. For this we may assume $G$ connected and simply connected. Now $G$ has $\sigma$-stable Levi decomposition $G = S \cdot \mathfrak{s}$ where $\mathfrak{s}$ is the solvable radical and we may replace $G$ by its semisimple quotient $G/S \cong G_1$. Thus we may assume $\mathfrak{g}$ semisimple.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{p} \cap \sigma(\mathfrak{p})$ is $\sigma$-stable and contains a Cartan subalgebra of $\mathfrak{g}$, so it contains a $\sigma$-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. There are only finitely many parabolic subalgebras of $\mathfrak{g}$ containing any given Cartan subalgebra, and Theorem 1 says that there are only finitely many possibilities for $\mathfrak{h}$ up to $L_0$-conjugacy. Thus there are only finitely many possibilities for $\mathfrak{p}$ up to $L_0$-conjugacy. Q.E.D.

### 4. Application to Real Flag Manifolds

By real flag manifold we mean a homogeneous space $X = G/P$ where $G$ is a real Lie group and $P$ is a parabolic subgroup. Since $P$ is its own normalizer in $G$, we may view $X$ as the set of all $G$-conjugates of $P$ under $x \leftrightarrow \{ g \in G : g(x) = x \}$. If $S$ is a subgroup of $G$, then the orbit $S(x)$ corresponds to the $S$-conjugacy class of $\{ g \in G : g(x) = x \}$. Now Theorem 2 gives us

**THEOREM 3.** Let $G$ be a real Lie group, $P$ a parabolic subgroup, and $X = G/P$ the corresponding real flag manifold. Let $\sigma$ be an involutive automorphism of $G$ and $L = G^\sigma$ its fixed point set. Then there are only finitely many
$L_0$-orbits on $X$. In particular, $L$ and $L_0$ each has both open and closed orbits on $X$, and each $L$-orbit is a finite union of $L_0$-orbits.

A trivial consequence: $X$ has only finitely many topological components. Here note that $X$ is connected if and only if every $G$-conjugate of $P$ is $G_0$-conjugate to $P$.

In the complex case ($G$ is a complex Lie group, $P$ is a complex parabolic subgroup and $L$ is a real form of $G$) there is just one closed $L_0$-orbit on every topological component of $X=G/P$ [8, Theorem 3.3] Here is an example in the real case where there are two closed orbits. Let $G=SL(2;\mathbb{R})$ and

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R} \text{ with } a \neq 0 \right\}$$

so that $X=G/P$ is the circle $\mathbb{R}\cup\{\infty\}$ bounding the upper half plane, the action being

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x \mapsto \frac{ax + b}{cx + d}.$$ 

Let $\sigma$ be conjugation by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

then $L=G^\sigma$ is the Cartan subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : 0 \neq a \in \mathbb{R} \right\}$$

with $L_0$ given by $a>0$. As

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : x \mapsto a^2x,$$

$L$ and $L_0$ have the same orbits on $X$, which are

- two closed orbits $\{0\}$ and $\{\infty\}$, and
- two open orbits $\{x \in \mathbb{R} : x > 0\}$ and $\{x \in \mathbb{R} : x < 0\}$.

This is a variation on the Šilov boundary example in 5 below.

5. APPLICATION TO SYMMETRIC R-SPACES

A symmetric $R$-space is a compact Riemannian symmetric space $X=K/V$ which admits a connected effective Lie transformation group $G$ such that

(i) $K \subset G$ and

(ii) if $Y$ is a $G$-stable Riemannian factor of $X$ then $G$ acts on $Y$ as a noncompact group. Then [5] $G$ is semisimple, $K$ is a maximal com-
compact subgroup of $G$, and, up to finite covering, $X$ is a real flag manifold $G/P$.
It often happens that $G$ contains an interesting noncompact form $L$ of $K$.

**EXAMPLE:** Grassmann Manifolds. Let $F$ denote the field of real, complex
or quaternion numbers. If $n>0$ then $U(n; F)$ denotes its unitary group,
which is the orthogonal group $O(n)$, the complex unitary group $U(n)$ or
the unitary symplectic group $Sp(n)$. The grassmannian of $k$-planes in $F^{k+l}$
is the compact Riemannian symmetric space

$$X = K/V = U(k + l; F)/U(k; F) \times U(l; F).$$

Under the action of the general linear group $G = GL(k+l; F)$ it is a real
flag $G/P$ where

$$P = \left\{ g \in G : g \text{ has matrix } \begin{pmatrix} * & \ast \\ 0 & * \end{pmatrix} \right\}.$$  

Let $L$ be the $F$-unitary group of any of the nondegenerate $F$-Hermitian forms
$h(x, y) = (x_1 \bar{y}_1 + \cdots + x_p \bar{y}_p) - (x_{p+1} \bar{y}_{p+1} + \cdots + x_{p+q} \bar{y}_{p+q})$ where $p+q = k+l$.
Then Witt's Theorem (over $F$) gives us the $L$-orbit structure of $X$.

**EXAMPLE:** Hermitian Symmetric Spaces. Here $X = K/V$ is an Hermitian
symmetric space of compact type and $G = K^c$. The dual bounded symmetric
domain $D = K^*/V$ where $K^*$ is a certain noncompact real form of $G$, and the
Borel embedding realizes $D$ as a certain open $K^*$-orbit on $X$. One knows the
$K^*$-orbit structure of $X$ ([7–9]). Let $L$ be an arbitrary real form of $G$.

**EXAMPLE:** Šilov Boundaries of Tube Domains. Let $X = K/V$ be an hermitian
symmetric space of compact type and $D = K^*/V$ the dual bounded symmetric
domain. Suppose that $D$ is holomorphically equivalent to a tube domain.
Then its Bergman-Šilov boundary is a symmetric $R$-space $S = V/E = K^*/W$;
see [3]. Let $L$ be an isotropy subgroup of $K^*$ on an open $K^*$-orbit in $X$. If
one expresses the open orbit as $K^* (c_x x_0)$ where $x_0$ is the base point and $c_x$
is a partial Cayley transform then $L = K^* \cap \text{Ad}(c_x^2) V^c$, a possibly-noncom-
pact form of $V$.

These examples all fit into the following pattern, and the result allows us
to apply Theorem 3 to symmetric $R$-spaces. Here note that any reductive
subalgebra of $\mathfrak{g}$ is invariant under a Cartan involution.

**THEOREM 4.** Let $\mathfrak{g}$ be a semisimple Lie algebra, $\theta$ a Cartan involution, and
$\mathfrak{g} = \mathfrak{t} + \mathfrak{s}$ the eigenspace decomposition under $\theta$, so that $\mathfrak{t} = \mathfrak{g}^\theta$ is the maximal
compactly embedded subalgebra. Suppose that $\mathfrak{g}$ has no proper $\theta$-stable ideal,
i.e. that \((g, \mathfrak{f})\) is an irreducible Riemannian symmetric pair of noncompact type. Let \(\mathfrak{l}\) be a \(\theta\)-stable subalgebra of \(g\) such that \(\mathfrak{l}^C \cong \mathfrak{f}^C\).

1. There is a complex linear automorphism \(\alpha\) of \(g^C\) that preserves the compact form \(\mathfrak{f} + i\mathfrak{s}\) and sends \(\mathfrak{l}^C\) to \(\mathfrak{f}^C\).

2. \(g\) is stable under the involutive automorphism \(\sigma = \alpha^{-1}\theta\alpha\), and \(\mathfrak{l} = g^\sigma\).

Remark. I suspect that Theorem 4 is true for \(g\) reductive and \((g, \mathfrak{f})\) assumed irreducible. But this does not affect the application to symmetric \(R\)-spaces.

Proof. We first check that (1) implies (2). Since \(\alpha(\mathfrak{l}^C) = \mathfrak{f}^C\), the fixed point set \((\mathfrak{g}^C)^\sigma = \alpha^{-1} \cdot (\mathfrak{g}^C)^\theta = \alpha^{-1}(\mathfrak{f}^C) = \mathfrak{l}^C\) which is stable under \(\theta\). Thus \(\theta\) and \(\sigma\) commute. Let \(\mathfrak{m}^C\) denote the \((-1)\)-eigenspace of \(\sigma\), so \(\mathfrak{g}^C = \mathfrak{l}^C + \mathfrak{m}^C\) under \(\sigma\).

Then
\[
\mathfrak{g}^C = (\mathfrak{l}^C \cap \mathfrak{f}) + (\mathfrak{l}^C \cap i\mathfrak{s}) + (\mathfrak{m}^C \cap \mathfrak{f}) + (\mathfrak{m}^C \cap i\mathfrak{s}) + (\mathfrak{m}^C \cap i\mathfrak{f}) + (\mathfrak{m}^C \cap i\mathfrak{s}) + (\mathfrak{m}^C \cap \mathfrak{s}) + (\mathfrak{m}^C \cap \mathfrak{s}^+). 
\]

Each of these eight summands is \(\sigma\)-stable because \(\sigma\) preserves \(\mathfrak{f}^C, \mathfrak{s}^C\) and \(\mathfrak{f} + i\mathfrak{s}\). That gives \(\sigma\)-stability of \(\mathfrak{g} = (\mathfrak{l}^C \cap \mathfrak{f}) + (\mathfrak{l}^C \cap i\mathfrak{s}) + (\mathfrak{m}^C \cap \mathfrak{f}) + (\mathfrak{m}^C \cap i\mathfrak{s})\), and now \(\mathfrak{g}^\sigma = \mathfrak{g} \cap (\mathfrak{g}^C)^\sigma = \mathfrak{g} \cap \mathfrak{l}^C = \mathfrak{l}\).

We prove (1). Let \(\mathfrak{g}^* = \mathfrak{f} + i\mathfrak{s}\) and \(\mathfrak{l}^* = (\mathfrak{l} \cap \mathfrak{f}) + i(\mathfrak{l} \cap \mathfrak{s})\).

If \(g\) is not absolutely simple, then \(\mathfrak{g}^* \cong \mathfrak{f} \oplus \mathfrak{f}\) with \(\mathfrak{f}\) embedded diagonally and \(\theta\) interchanging the two summands by \(\theta(x, y) = (y, x)\). Let \(\nu_j : \mathfrak{g}^* \to \mathfrak{f}_j\) denote projection to the \(j\)th summand. \(\mathfrak{l}^*\) is in neither summand because \(\theta \mathfrak{l}^* = \mathfrak{l}^*\), so \(\nu_j \mathfrak{l}^* \cong \mathfrak{f}_j\). That gives \(\beta \in \text{Aut}(\mathfrak{f})\) with \(\mathfrak{l}^* = \{(x, \beta x) : x \in \mathfrak{f}\}\). Notice that \(\beta^2 = 1\) because \(\mathfrak{l}^* = \theta \mathfrak{l}^* = \{(\beta y, y) : y \in \mathfrak{f}\}\). Now \(\alpha = 1 \times \beta\) is an automorphism of \(\mathfrak{g}^*\) that interchanges \(\mathfrak{l}^*\) and the diagonally embedded \(\mathfrak{f}\).

If \(g\) is absolutely simple and rank \(\mathfrak{f} = \text{rank } g\), then \(\mathfrak{g}^*\) is simple and its subalgebra \(\mathfrak{f}\) is a maximal subalgebra of maximal rank. The Borel-de Sieben-thal classification [1] shows that any two subalgebras of \(\mathfrak{g}^*\) isomorphic to \(\mathfrak{f}\) must be \(\text{Aut}(\mathfrak{g}^*)\)-conjugate.

Now suppose \(g\) absolutely simple with rank \(\mathfrak{f} < \text{rank } g\). If \(\mathfrak{g}^*\) is of classical type then the inclusion \(\mathfrak{f} \subset \mathfrak{g}^*\) is one of
\[
\mathfrak{so}(n) \subset \mathfrak{su}(n), \mathfrak{sp}(n) \subset \mathfrak{su}(2n), \mathfrak{so}(p) \oplus \mathfrak{so}(q) \subset \mathfrak{so}(p + q).
\]

Looking at degrees of representations we see that \(\mathfrak{f} \subset \mathfrak{g}^*\) and \(\mathfrak{l}^* \subset \mathfrak{g}^*\) are equivalent representations. If \(\eta\) is the equivalence we take \(\alpha = \eta \mid_{\mathfrak{g}^*}\). If \(\mathfrak{g}^*\) is of exceptional type then \(\mathfrak{f} \subset \mathfrak{g}^*\) is one of
\[
\mathfrak{sp}(4) \subset \mathfrak{e}_6 \quad \text{or} \quad \mathfrak{f}_4 \subset \mathfrak{e}_6 
\]
and Dynkin's results [2] show that any two subalgebras of \(\mathfrak{g}^*\) isomorphic to \(\mathfrak{f}\) must be \(\text{Aut}(\mathfrak{g}^*)\)-conjugate. Q.E.D.
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