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Partially Harmonic Spinors and Representations of Reductive Lie Groups

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Let G be a reductive Lie group subject to some minor technical restrictions. The Plancherel Theorem for G uses several series of unitary representation classes, one series for each conjugacy class of Cartan subgroups of G. Given a Cartan subgroup $H \subset G$, we construct a G-homogeneous family $X \to Y$ of oriented riemannian symmetric spaces, some G-homogeneous bundles $\mathscr{V}_{\mu,\sigma} \to X$, and some Hilbert spaces $H_2^{\pm}(\mathscr{V}_{\mu,\sigma})$ of partially harmonic spinors with values in $\mathscr{V}_{\mu,\sigma}$. Then G acts on $H_2^{\pm}(\mathscr{V}_{\mu,\sigma})$ by a unitary representation $\pi_{\mu,\sigma}^{\pm}$. We then show that these $\pi_{\mu,\sigma}^{\pm}$ realize the series of representation classes of G associated to the conjugacy class of H.

INTRODUCTION

In a recent memoir [13] we extended part of Harish-Chandra's theory of harmonic analysis [5, 6, 7] to a class of reductive Lie groups that includes all semisimple groups, and we worked out geometric realizations for the corresponding unitary representation classes on certain partially holomorphic cohomology spaces. Here we realize those representation classes on Hilbert spaces of square integrable partially harmonic spinors. This extends results of R. Parthasarathy [11] and W. Schmid (unpublished) from the case of discrete series representations of connected linear semisimple Lie groups. See [14] for another special case.

In Section 1 we review the structure, representation theory and Plancherel formula for our class of reductive Lie groups G. There is a series

$$\{[\pi_{\chi,\nu,\sigma}]: \nu \in i\mathfrak{t}^* \text{ consistent with } [\chi] \in Z_M(M^0)^\wedge \text{ and } \sigma \in \mathfrak{a}^*\}$$

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of unitary representation classes associated to any conjugacy class of Cartan subgroups $H \subset G$. Here $H = T \times A$ where T is the anisotropic component, lower case german letters denote Lie algebras, and $Z_Q(R)$ means the Q-centralizer of R. That series is called the H-series. Plancherel measure of G is concentrated on the union of the various H-series. Important facts: (i) $\pi_{\chi,\nu,\sigma}(\text{Casimir}) = \| \nu \|^2 + \| \sigma \|^2 - \| \rho \|^2$ where ρ is half the sum of the positive roots, and (ii) if c is a number then the set of all irreducible unitary representation classes $[\pi] \in \hat{G}$ not of the "relative discrete series," such that $\pi(\text{Casimir}) = c$, has Plancherel measure zero.

In Section 2 we construct the Hilbert spaces of square integrable partially harmonic spinors on which an H-series class $[\pi_{\chi,\nu,\sigma}]$ will be realized. Let $P = MAN$ cuspidal parabolic subgroup of G associated to $H = T \times A$. Consider the G-equivariant fibration

$$p: X = G/UAN \to G/M^{\dagger}AN = Z$$

where $M^{\dagger} = Z_M(M^0)M^0$ and U is the maximal compactly embedded subgroup of $M^{\dagger}$ that contains T. The fibres $Y_z = p^{-1}(z)$ are oriented riemannian symmetric spaces of noncompact type. $\mathscr{S}^{\pm} \to X$ are the G-homogeneous bundles whose every Y_z-restriction are the half spin bundles there. If $[\mu] \in \hat{U}$ and $\sigma \in \mathfrak{a}^*$ we have the G-homogeneous bundle $\mathscr{V}_{\mu,\sigma} \to X$ associated to the representation $uan \to e^{\rho_{\mathfrak{a}}+i\sigma}(a)\, \mu(u)$ of UAN where $\rho_{\mathfrak{a}}$ is half the sum of the positive $\mathfrak{a}$-roots of $\mathfrak{g}$. The Dirac operators of the $\mathscr{V}_{\mu,\sigma}|_{Y_z}$ fit together to form a G-invariant first-order essentially self-adjoint operator $D = D^+ \oplus D^-$ where $D^{\pm}: L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}) \to L_2(\mathscr{S}^{\mp} \otimes \mathscr{V}_{\mu,\sigma})$. Then G has a natural action $\pi^{\pm}_{\mu,\sigma}$ on the space

$$H_2^{\pm}(\mathscr{V}_{\mu,\sigma}) = \{\phi \in L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}) : \tilde{D}^{\pm}\phi = 0\}$$

of $\mathscr{V}_{\mu,\sigma}$-valued square integrable partially harmonic spinors on X, and $\pi^{\pm}_{\mu,\sigma}$ is a unitary representation.

In Section 3 we work out a formula for D^2 on $L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$ in terms of the Casimir operator of M. This formula is of crucial importance in the identification of the $[\pi^{\pm}_{\mu,\sigma}]$.

In Section 4 we more or less identify the $[\pi^{\pm}_{\mu,\sigma}]$. The Plancherel formula recalled in Section 1 is combined with the formula of Section 3 to show that $[\pi^{\pm}_{\mu,\sigma}]$ is a finite sum of H-series classes of the form $[\pi_{\chi,\beta,\sigma}]$ where: (i) $[\mu] = [\chi \otimes \mu^0]$ using $U = Z_M(M^0)U^0$, (ii) μ^0 has highest weight $\nu + \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$, and (iii) $\| \beta \|^2 = \| \nu + \rho_{\mathfrak{t}} \|^2$ where $\rho_{\mathfrak{t}}$ (respectively, $\rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$) is half the sum of the positive (respectively, positive noncompact) $\mathfrak{t}_C$-roots of $\mathfrak{m}_C$. Thus $[\pi^{\pm}_{\mu,\sigma}]$ has well-defined distribution character $\Theta_{\pi^{\pm}_{\mu,\sigma}}$. We then show

$$\Theta_{\pi^{+}_{\mu,\sigma}} - \Theta_{\pi^{-}_{\mu,\sigma}} = (-1)^{p_{\mathfrak{t}}(\nu+\rho_{\mathfrak{t}})}\, \Theta_{\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}}$$

where $\nu + \rho_{\mathfrak{t}}$ is $\mathfrak{m}$-regular and $p_{\mathfrak{t}}(\nu + \rho_{\mathfrak{t}})$ counts the noncompact positive $\mathfrak{t}_C$-roots ϕ of $\mathfrak{m}_C$ with $\langle \nu + \rho_{\mathfrak{t}}, \phi\rangle > 0$. Next we show that, under rather mild conditions, there is a sign $\pm$ such that $H_2^{\mp}(\mathscr{V}_{\mu,\sigma}) = 0$, so then $[\pi^{\pm}_{\mu,\sigma}]$ is the H-series class $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$. Finally we indicate how any H-series class is realized on one of the $H_2^{\pm}(\mathscr{V}_{\mu,\sigma})$.

The reader will note that we make considerable use of ideas from [6, 10, 11, and 13].

1. Unitary Representations of Reductive Lie Groups

We review the representation theory [13, Sects. 2, 3, 4, 5] for a class of reductive Lie groups that contains all the connected semisimple groups and has certain hereditary properties convenient for harmonic analysis. If G is a group in that class, there is a series of unitary representations associated to every conjugacy class of Cartan subgroups. Those series of representations support the Plancherel measure of G. In Section 4 we will realize those representations on Hilbert spaces of square integrable partially harmonic spinors with values in certain G-homogeneous vector bundles.

In Section 1.1 we recall the general notion and basic facts concerning relative discrete series representations. Then we specify our class of reductive Lie groups in Section 1.2, and in Section 1.3 we specify the relative discrete series for the groups in that class. The relative discrete series is building block both for the other series and for the geometric realizations.

In Section 1.4 we go over the structure theory of cuspidal parabolic subgroups associated to a Cartan subgroup H of a reductive Lie group G in our class. The corresponding "H-series" of unitary representation classes of G is described in Section 1.5. Then in Section 1.6 we give the Plancherel Formula for G using the distribution characters of the various H-series of representation classes.

The material of Section 1 is our extension [13, Sects. 2, 3, 4, 5] of some results of Harish-Chandra [5, 6, 7].

1.1. *General Notion of Relative Discrete Series*

Let G be a unimodular locally compact group and Z a closed normal abelian subgroup. $\hat{G}$ denotes the set of all equivalence classes

$[\pi]$ of irreducible unitary representations of G, and similarly $\hat{Z}$ denotes the multiplicative group of all unitary characters ζ on Z. If π is a unitary representation of G, then H_π denotes the representation space, and the *coefficients* of $[\pi]$ are the functions

$$f_{\phi,\psi}: G \to C \text{ by } f_{\phi,\psi}(x) = \langle \phi, \pi(x)\psi \rangle \qquad \text{for} \quad \phi, \psi \in H_\pi . \tag{1.1.1}$$

Every $\zeta \in \hat{Z}$ specifies a Hilbert space

$$L_2(G/Z, \zeta) = \left\{ f: G \to C: f(gz) = \zeta(z)^{-1} f(g) \text{ and } \int_{G/Z} |f(g)|^2 \, d(gZ) < \infty \right\} \tag{1.1.2}$$

where f is measurable and we identify functions that agree a.e. G. $L_2(G/Z, \zeta)$ is representation space for $l_\zeta^G = \mathrm{Ind}_{Z\uparrow G}(\zeta)$. Induction by stages relates the left regular representations l^Z of Z and l^G of G:

$$l^Z = \mathrm{Ind}_{\{1\}\uparrow Z}\{1\} = \int_{\hat{Z}} \zeta \, d\zeta \quad \text{and} \quad l^G = \mathrm{Ind}_{Z\uparrow G}(l^Z) = \int_{\hat{Z}} l_\zeta^G \, d\zeta. \tag{1.1.3a}$$

That gives a direct integral decomposition

$$L_2(G) = \int_{\hat{Z}} L_2(G/Z, \zeta) \, d\zeta \qquad \text{for} \quad l^G = \int_{\hat{Z}} l_\zeta^G \, d\zeta. \tag{1.1.3b}$$

Let $\zeta \in \hat{Z}$ and denote $\hat{G}_\zeta = \{[\pi] \in \hat{G}: \zeta \subset \pi|_Z\}$. Thus l_ζ^G is a direct integral over $\hat{G}_\zeta$. A class $[\pi] \in \hat{G}$ is *ζ-discrete* if π is equivalent to a subrepresentation of l_ζ^G. The ζ-discrete classes form the *ζ-discrete series* $\hat{G}_{\zeta\text{-disc}} \subset \hat{G}_\zeta$ of G. The *relative* (to Z) *discrete series* of G is $\hat{G}_{\text{disc}} = \bigcup_{\zeta\in\hat{Z}} \hat{G}_{\zeta\text{-disc}}$. If Z is compact then $\hat{G}_{\text{disc}}$ is the discrete series of G in the ordinary (subrepresentation of l^G) sense.

If $[\pi] \in \hat{G}_{\text{disc}}$ then its coefficients satisfy $|f_{\phi,\psi}| \in L_2(G/Z)$. In effect, if $\phi, \psi \in H_\pi \subset L_2(G/Z, \zeta)$ and ϕ is projection of $\phi' \in L_1(G/Z, \zeta) \cap L_2(G/Z, \zeta)$, then

$$\begin{aligned} \int_{G/Z} |f_{\phi,\psi}(x)|^2 \, d(xZ) &= \int_{G/Z} \left| \int_{G/Z} \phi'(g) \overline{\psi(x^{-1}g)} \, d(gZ) \right|^2 d(xZ) \\ &= \int_{G/Z} |\psi(x^{-1})|^2 \cdot \left| \int_{G/Z} \phi'(g) \, d(gZ) \right|^2 d(xZ) < \infty. \end{aligned}$$

In general ϕ is approximated by projections from $L_1(G/Z, \zeta) \cap L_2(G/Z, \zeta)$. Conversely, if $[\pi] \in \hat{G}_\zeta$ has a nonzero coefficient $f_{\phi,\psi}$ with $|f_{\phi,\psi}| \in L_2(G/Z)$, then the closed linear span of $l_\zeta^G(G)(f_{\phi,\psi})$ is representation space for a multiple of $[\pi]$, so $[\pi] \in \hat{G}_{\zeta\text{-disc}}$.

Now suppose that Z is central in G. Then the above remark sharpens as follows. If $[\pi] \in \hat{G}_\zeta$ then the following are equivalent:

$$\text{there exist nonzero } \phi, \psi \in H_\pi \text{ such that } f_{\phi,\psi} \in L_2(G/Z, \zeta); \tag{1.1.4a}$$

$$\text{if } \phi, \psi \in H_\pi \text{ then } f_{\phi,\psi} \in L_2(G/Z, \zeta); \text{ and} \tag{1.1.4b}$$

$$[\pi] \in \hat{G}_{\zeta\text{-disc}} . \tag{1.1.4c}$$

Under these conditions there is a number $\deg(\pi) > 0$ called the *formal degree* of $[\pi]$, uch that

$$\langle f_{\phi_1,\psi_1}, f_{\phi_2,\psi_2}\rangle = \deg(\pi)^{-1}\langle\phi_1, \phi_2\rangle\overline{\langle\psi_1, \psi_2\rangle} \quad \text{for} \quad \phi_i, \psi_i \in H_\pi . \tag{1.1.5a}$$

Writing $(f * h)(x) = \int_{G/Z} f(g)\, h(g^{-1}x)\, d(gZ)$ this is equivalent to

$$f_{\phi,\psi} * f_{\alpha,\beta} = \deg(\pi)^{-1}\langle\phi, \beta\rangle f_{\alpha,\psi} \quad \text{for} \quad \phi, \psi, \alpha, \beta \in H_\pi . \tag{1.1.5b}$$

Further, if $[\pi] \neq [\pi']$, both in $\hat{G}_{\zeta\text{-disc}}$, then $\langle f_{\phi,\psi}, f_{\phi',\psi'}\rangle = 0$ for $\phi, \psi \in H_\pi$ and $\phi', \psi' \in H_{\pi'}$. These results are due to R. Godement [2] (see [1, Sect. 14]), Harish–Chandra [3], and M. Rieffel [12] in varying degrees of generality. If $Z = \{1\}$ and G is compact, they reduce to the classical Frobenius–Schur Relations.

1.2. *A Class of Reductive Lie Groups*

From now on, G is a reductive Lie group. Thus its Lie algebra $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1$ where $\mathfrak{c}$ is the center and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. We suppose

$$\text{if } g \in G \text{ then } \mathrm{ad}(g) \text{ is an inner automorphism on } \mathfrak{g}_{\mathbf{C}} . \tag{1.2.1}$$

As usual,

$$G^0 \text{ is the identity component of } G \text{ and } Z_G(G^0) \text{ is its } G\text{-centralizer.}$$

Then (1.2.1) says that $G/Z_G(G^0)$ is a subgroup of the linear algebraic group $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$. Note that $G/Z_G(G^0)$ has finite index in the real form $\{\gamma \in \mathrm{Int}(\mathfrak{g}_{\mathbf{C}}): \gamma(\mathfrak{g}) = \mathfrak{g}\}$.

We further assume that G has a closed normal abelian subgroup Z such that

$$Z \subset Z_G(G^0), \quad |G/ZG^0| < \infty \quad \text{and} \quad Z \cap G^0 \text{ is cocompact in } Z_{G^0} . \tag{1.2.2}$$

Here Z_{G^0} is the center of G^0. If $|G/G^0| < \infty$ then Z_{G^0} satisfies

(1.2.2). In any case, $\hat{G}_{\text{disc}}$ is independent of choice of subgroup $Z \subset G$ that satisfies (1.2.2). Denote

$$G^{\dagger} = \{g \in G\colon \mathrm{ad}(g) \text{ inner automorphism of } G^0\} = Z_G(G^0)G^0. \quad (1.2.3)$$

Then (1.2.2) gives $ZG^0 \subset G^{\dagger}$ and forces $Z_G(G^0)/Z$ to be compact.

From now on, the reductive Lie group G satisfies (1.2.1) and (1.2.2). We recall some general results of Harish-Chandra, slightly extended [13, Sect. 3.2] to G. Choose

$$K/Z\colon \text{maximal compact subgroup of } G/Z.$$

Thus (1.2.2) $Z_G(G^0) \subset K$ and $K/Z_G(G^0)$ is a maximal compact subgroup of $G/Z_G(G^0)$. The basic fact is the existence of an integer $n_G \geqslant 1$ with the following property. If $[\kappa] \in \hat{K}$ and $[\pi] \in \hat{G}$ then

$$\text{the multiplicity } m(\kappa, \pi|_K) \leqslant n_G \dim(\kappa) < \infty. \quad (1.2.4)$$

The first consequence: G is CCR, i.e., if $[\pi] \in \hat{G}$ and $f \in L_1(G)$ then $\pi(f) = \int_G f(g)\, \pi(g)\, dg$ is a compact operator on H_π. In particular G is of type I. The second consequence: if $[\pi] \in \hat{G}$ and $f \in C_c^\infty(G)$ then $\pi(f)$ is of trace class, and

$$\Theta_\pi\colon C_c^\infty(G) \to C \text{ by } \Theta_\pi(f) = \operatorname{trace} \pi(f) \quad (1.2.5)$$

is a Schwartz distribution on G. Θ_π is the *global character* or *distribution character* of $[\pi]$. Classes $[\pi] = [\pi']$ if and only if $\Theta_\pi = \Theta_{\pi'}$.

View the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}_C$ as the associative algebra of all left-invariant differential operators on G. Then the center $\mathfrak{Z}$ of $\mathfrak{G}$ consists of the biinvariant operators; this is equivalent to (1.2.1). $\mathfrak{Z}$ acts on distributions by $(z\Theta)(f) = \Theta({}^t z \cdot f)$ where $\int_G ({}^t z \cdot f)(x)\, h(x)\, dx = \int_G f(x)(z \cdot h)(x)\, dx$. If Θ is an eigendistribution of $\mathfrak{Z}$ ($\dim \mathfrak{Z}(\Theta) \leqslant 1$) then we have an algebra homomorphism $\chi_\Theta\colon \mathfrak{Z} \to C$ defined by $z\Theta = \chi_\Theta(z)\Theta$. If $[\pi] \in \hat{G}$, then its distribution character Θ_π is an $\mathrm{ad}(G)$-invariant eigendistribution of $\mathfrak{Z}$, and the associated homomorphism

$$\chi_\pi\colon \mathfrak{Z} \to C \text{ by } z\Theta_\pi = \chi_\pi(z)\Theta_\pi \quad (1.2.6)$$

is called the *infinitesimal character* of $[\pi]$.

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\mathbf{I}(\mathfrak{h}_C)$ denote the algebra of polynomials on $\mathfrak{h}_C^*$ invariant by the Weyl group $W(\mathfrak{g}_C, \mathfrak{h}_C)$.

Then [4] there is an isomorphism $\gamma: \mathfrak{Z} \to \mathbf{I}(\mathfrak{h}_C)$ such that, if $\lambda \in \mathfrak{h}_C^*$ then

$$\chi_\lambda: \mathfrak{Z} \to C \text{ by } \chi_\lambda(z) = [\gamma(z)](\lambda) \tag{1.2.7}$$

is a homomorphism. Every homomorphism $\mathfrak{Z} \to C$ is one of these χ_λ, and $\chi_\lambda = \chi_{\lambda'}$ precisely when $\lambda' \in W(\mathfrak{g}_C, \mathfrak{h}_C)(\lambda)$. Combining (1.2.6) and (1.2.7), one sees [4] that Θ_π is a locally L_1 function on G which is analytic on the *regular set*

$$G' = \{g \in G: \{\xi \in \mathfrak{g}: \operatorname{ad}(g)\xi = \xi\} \text{ is a Cartan subalgebra of } \mathfrak{g}\}. \tag{1.2.8}$$

G' is a dense open subset of G and $G - G'$ has measure zero. The differential equations (1.2.6) and (1.2.7) also show that only finitely many classes in $\hat{G}$ can have the same infinitesimal character.

1.3. *Relative Discrete Series for Reductive Groups*

G is a reductive Lie group that satisfies (1.2.1) and (1.2.2). We recall the structure of $\hat{G}_{\text{disc}}$ (relative to Z) from [13, Sect. 3].

$\hat{G}_{\text{disc}}$ is nonempty if, and only if, G/Z has a compact Cartan subgroup [13, Theorem 3.5.8].

If G/Z has a compact Cartan subgroup H/Z then [13, Theorem 3.5.9] $\hat{G}_{\text{disc}}$ consists of the classes $[\pi_{\chi,\lambda}]$ constructed as follows. Let

$$L = \{\lambda \in i\mathfrak{h}^*: e^\lambda \text{ is well defined on } H^0\}. \tag{1.3.1}$$

Choose a positive $\mathfrak{h}_C$-root system Σ^+ of $\mathfrak{g}_C$ and define

$$\rho = \tfrac{1}{2} \sum_{\phi \in \Sigma^+} \phi \quad \text{and} \quad \Delta = \prod_{\phi \in \Sigma^+} (e^{\phi/2} - e^{-\phi/2}). \tag{1.3.2}$$

Replacing G by a Z_2-extension if necessary we may assume $\rho \in L$, so Δ is a well-defined function on H^0. Denote

$$\tilde{\omega}(\lambda) = \prod_{\phi \in \Sigma^+} \langle \phi, \lambda \rangle \quad \text{and} \quad L' = \{\lambda \in L: \tilde{\omega}(\lambda) \neq 0\}. \tag{1.3.3}$$

Then $\rho \in L'$. If $\lambda \in L'$, define the sign $(-1)^{q(\lambda)}$ by

$$(-1)^{q(\lambda)} = (-1)^q \cdot \operatorname{sign} \tilde{\omega}(\lambda) \quad \text{where} \quad q = (1/2) \dim G/K. \tag{1.3.4}$$

Step 1. If $\lambda \in L'$ there is a unique class $[\pi_\lambda] \in (G^0)\hat{}_{\text{disc}}$ whose distribution character satisfies

$$\Theta_{\pi_\lambda}|_{H^0 \cap G'} = (-1)^{q(\lambda)} \frac{1}{\Delta} \sum \det(w) e^{w\lambda} \tag{1.3.5}$$

where w runs over the Weyl group $W(G^0, H^0)$. These $[\pi_\lambda]$ exhaust $(G^0)^\wedge_{\text{disc}}$. Classes $[\pi_\lambda] = [\pi_{\lambda'}]$ if and only if $\lambda' \in W(G^0, H^0)(\lambda)$. The class $[\pi_\lambda]$ has dual $[\pi_\lambda{}^*] = [\pi_{-\lambda}]$, has central character $e^{\lambda-\rho}|_{Z_{G^0}}$, has infinitesimal character χ_λ; so $\pi_\lambda(\text{Casimir}) = \| \lambda \|^2 - \| \rho \|^2$. For appropriate normalization of Haar measure, $\deg(\pi_\lambda) = | \tilde{\omega}(\lambda)|$.

Step 2. Let $\lambda \in L'$, $\xi = e^{\lambda-\rho}|_{Z_{G^0}} \in (Z_{G^0})^\wedge$, and $[\chi] \in Z_G(G^0)_\xi$. Then $[\chi \otimes \pi_\lambda]$ is the unique class in $(G^\dagger)^\wedge_{\text{disc}}$ whose distribution character satisfies

$$\Theta_{\chi\otimes\pi_\lambda}(zg) = \operatorname{trace} \chi(z) \cdot \Theta_{\pi_\lambda}(g) \qquad \text{for} \quad z \in Z_G(G^0) \quad \text{and} \quad g \in G^0 \quad (1.3.6)$$

where Θ_{π_λ} is determined by (1.3.5). These $[\chi \otimes \pi_\lambda]$ exhaust $(G^\dagger)^\wedge_{\text{disc}}$. $[\chi \otimes \pi_\lambda]$ has infinitesimal character χ_λ.

Step 3. Let $\lambda \in L'$, $\xi = e^{\lambda-\rho}|_{Z_{G^0}} \in (Z_{G^0})^\wedge$ and $[\chi] \in Z_G(G^0)^\wedge_\xi$. Then

$$[\pi_{\chi,\lambda}] = [\operatorname{Ind}_{G^\dagger\uparrow G}(\chi \otimes \pi_\lambda)] \in \hat{G}_{\text{disc}}, \qquad (1.3.7\text{a})$$

its distribution character $\Theta_{\pi_{\chi,\lambda}}$ is supported in $G^\dagger$, and

$$\Theta_{\pi_{\chi,\lambda}}(zg) = \sum_{xG^\dagger \in G/G^\dagger} \operatorname{trace} \chi(x^{-1}zx)\, \Theta_{\pi_\lambda}(x^{-1}gx) \qquad \text{for} \quad z \in Z_G(G^0) \text{ and } g \in G^0 \quad (1.3.7\text{b})$$

where Θ_{π_λ} is determined by (1.3.5). These $[\pi_{\chi,\lambda}]$ exhaust $\hat{G}_{\text{disc}}$. Classes $[\pi_{\chi,\lambda}] = [\pi_{\chi',\lambda'}]$ precisely when $([\chi'], \lambda') \in W(G, H)([\chi], \lambda)$. The class $[\pi_{\chi,\lambda}]$ has dual $[\pi^*_{\chi,\lambda}] = [\pi_{\chi^*,-\lambda}]$ and has infinitesimal character χ_λ. Thus $\pi_{\chi,\lambda}(\text{Casimir}) = \| \lambda \|^2 - \| \rho \|^2$.

1.4. *Cuspidal Parabolic Subgroups*

Let G be a reductive Lie group that satisfies (1.2.1) and (1.2.2). If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then the associated *Cartan subgroup* is $H = \{g \in G\colon \operatorname{ad}(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}\}$. Note $Z_G(G^0) \subset H$. If G^0 is a linear group, or if H/Z is compact, then $H \cap G^0$ is commutative.

By *Cartan involution* of G we mean an involutive automorphism θ of G whose fixed point set

$$K = \{g \in G\colon \theta(g) = g\} \qquad (1.4.1)$$

satisfies (i) $Z_G(G^0) \subset K$ and (ii) K/Z is a maximal compact subgroup of G/Z. If L/Z is a maximal compact subgroup of G/Z then [13, Lemma 4.1.1] there is a unique Cartan involution of G with fixed point set L. Any two Cartan involutions of G are $\operatorname{ad}(G^0)$-conjugate, and every Cartan subgroup of G is stable under a Cartan involution [13, Lemma 4.1.2].

Fix a Cartan subgroup $H \subset G$ and a Cartan involution θ with $\theta(H) = H$, and let K denote the fixed point set of θ on G. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ where $\mathfrak{t}$ is the 1-eigenspace of $\theta|_{\mathfrak{h}}$ and $\mathfrak{a}$ is the (-1)-eigenspace. That splits

$$H = T \times A \quad \text{where} \quad T = H \cap K \quad \text{and} \quad A = \exp(\mathfrak{a}). \tag{1.4.2}$$

The G-centralizer of A has unique splitting [13, Lemma 4.1.5]

$$Z_G(A) = M \times A \quad \text{with} \quad \theta(M) = M. \tag{1.4.3}$$

The reductive Lie group M inherits (1.2.1) and (1.2.2) from G [13, Proposition 4.1.6]: every $\mathrm{ad}(m)$ is an inner automorphism on $\mathfrak{m}_{\mathbf{C}}$, $Z \subset Z_M(M^0)$, $|M/ZM^0| < \infty$ and $Z \cap M^0$ is cocompact in Z_{M^0}. Further, T is a Cartan subgroup of M and T/Z is compact, so $\hat{M}_{\mathrm{disc}}$ is nonempty.

The $\mathfrak{a}$-*roots* of $\mathfrak{g}$ are the nonzero real linear functionals $\phi \in \mathfrak{a}^*$ such that

$$0 \neq \mathfrak{g}^{\phi} = \{\xi \in \mathfrak{g}\colon [\alpha, \xi] = \phi(\alpha)\xi \text{ for all } \alpha \in \mathfrak{a}\}.$$

Let $\Sigma_{\mathfrak{a}}$ denote the set of all $\mathfrak{a}$-roots of $\mathfrak{g}$. Every $\phi \in \Sigma_{\mathfrak{a}}$ defines a hyperplane $\phi^{\perp} = \{\alpha \in \mathfrak{a}\colon \phi(\alpha) = 0\}$. If $\mathfrak{d}$ is a topological component of $\mathfrak{a} - \bigcup \phi^{\perp}$, then no $\mathfrak{a}$-root changes sign in $\mathfrak{d}$, and $\Sigma_{\mathfrak{a}}^+ = \{\phi \in \Sigma_{\mathfrak{a}}\colon \phi > 0 \text{ on } \mathfrak{d}\}$ is a ***positive*** $\mathfrak{a}$-***root system.*** If $\Sigma_{\mathfrak{a}}^+$ is a positive $\mathfrak{a}$-root system on $\mathfrak{g}$ and $\Sigma_{\mathfrak{t}}^+$ is a positive $\mathfrak{t}_{\mathbf{C}}$-root system on $\mathfrak{m}_{\mathbf{C}}$ then [13, Lemma 4.1.7] there is a unique positive $\mathfrak{h}_{\mathbf{C}}$-root system Σ^+ on $\mathfrak{g}_{\mathbf{C}}$ such that $\Sigma_{\mathfrak{a}}^+ = \{\gamma|_{\mathfrak{a}}\colon \gamma \in \Sigma^+ \text{ and } \gamma|_{\mathfrak{a}} \neq 0\}$ and $\Sigma_{\mathfrak{t}}^+ = \{\gamma|_{\mathfrak{t}}\colon \gamma \in \Sigma^+ \text{ and } \gamma(\mathfrak{a}) = 0\}$.

Fix a positive $\mathfrak{a}$-root system $\Sigma_{\mathfrak{a}}^+$ on $\mathfrak{g}$. Then we have

$$\mathfrak{n} = \sum_{\phi \in \Sigma_{\mathfrak{a}}^+} \mathfrak{g}^{-\phi} \text{ nilpotent subalgebra of } \mathfrak{g}, \tag{1.4.4a}$$

$$N = \exp(\mathfrak{n}) \text{ unipotent analytic subgroup of } G, \text{ and} \tag{1.4.4b}$$

$$P = \{g \in G\colon \mathrm{ad}(g)N = N\} \text{ normalizer of } N \text{ in } G. \tag{1.4.4c}$$

One knows [13, Lemma 4.2.2] that P is a real parabolic subgroup of G with unipotent radical N and reductive part $M \times A$. Thus

$$P = MAN \text{ in the sense of smooth unique factorization.} \tag{1.4.5}$$

One says that P is a *cuspidal parabolic subgroup* of G *associated* to H. The cuspidal parabolic subgroups are characterized by the property [13, Proposition 4.2.3] that their reductive parts have relative discrete series representations.

1.5. *Nondegenerate Series for Reductive Groups*

G is a reductive Lie group that satisfies (1.2.1) and (1.2.2). We fix a Cartan subgroup $H \subset G$ and a Cartan involution θ under which H is stable. Then, as in (1.4.2) and (1.4.3),

$$H = T \times A \qquad \text{and} \qquad Z_G(A) = M \times A.$$

Fix a positive $\mathfrak{a}$-root system $\Sigma_{\mathfrak{a}}^+$ on $\mathfrak{g}$, thus fixing the cuspidal parabolic subgroup $P = MAN$ (1.4.4) of G. We will need

$$\rho_{\mathfrak{a}} = \tfrac{1}{2} \sum_{\phi \in \Sigma_{\mathfrak{a}}^+} (\dim \mathfrak{g}^{\phi})\phi \text{ so } \mathfrak{a} \text{ acts on } \mathfrak{n} \text{ with trace } -2\rho_{\mathfrak{a}}.$$

If $[\eta] \in \hat{M}$ and $\sigma \in \mathfrak{a}^*$ we have $[\eta \otimes e^{i\sigma}] \in \hat{P}$ defined by $(\eta \otimes e^{i\sigma})(man) = e^{i\sigma}(a)\, \eta(m)$. The unitarily induced representation

$$\pi_{\eta,\sigma} = \operatorname{Ind}_{MAN \uparrow G}(\eta \otimes e^{i\sigma})$$

is the representation of G on the Hilbert space of all measurable

$$f\colon G \to (\text{representation space of } \eta)$$

such that

$$f(gman) = \{e^{\rho_{\mathfrak{a}}+i\sigma}(a)\, \eta(m)\}^{-1} f(g) \qquad \text{and} \qquad \int_{K/Z} \| f(k) \|^2\, d(kZ) < \infty.$$

The *H-series* of G consists of the $[\pi_{\eta,\sigma}]$ with $[\eta] \in \hat{M}_{\text{disc}}$. The *nondegenerate series* are the various H-series.

The unitary representation classes $[\pi_{\eta,\sigma}]$, whether of the H-series or not, are pretty well understood [13, Theorem 4.3.8]. Let $\zeta \in \hat{Z}$ such that $[\eta] \in \hat{M}_\zeta$, let χ_ν be the infinitesimal character of $[\eta]$ relative to $\mathfrak{t}$, and let Ψ_η be the distribution character of $[\eta]$. First, $[\pi_{\eta,\sigma}]$ has infinitesimal character $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}$. Second, $[\pi_{\eta,\sigma}]$ is a finite sum from $\hat{G}_\zeta$, so it has distribution character $\Theta_{\pi_{\eta,\sigma}}$ that is a locally summable function analytic on the regular set G'. Third, $\Theta_{\pi_{\eta,\sigma}}$ is supported in the closure of the union of the G-conjugacy classes of Cartan subgroups of MA. And fourth one has a formula there for $\Theta_{\pi_{\eta,\sigma}}$ in terms of Ψ_η and $e^{i\sigma}$. A corollary: $[\pi_{\eta,\sigma}]$ is independent of the choice $P = MAN$ of cuspidal parabolic subgroup associated to $H = T \times A$.

We specialize to the H-series. In analogy to the situation of Section 1.3, $\hat{M}_{\text{disc}}$ consists of the classes $[\eta_{\chi,\nu}]$ constructed as follows. Let

$$L_{\mathfrak{t}} = \{\nu \in i\mathfrak{t}^*\colon e^\nu \text{ is well defined on } T^0\}.$$

Choose a positive $\mathfrak{t}_{\mathbf{C}}$-root system $\Sigma_{\mathfrak{t}}^{+}$ on $\mathfrak{m}_{\mathbf{C}}$ and denote

$$\rho_{\mathfrak{t}} = \tfrac{1}{2} \sum_{\phi \in \Sigma_{\mathfrak{a}}^{+}} \phi \qquad \text{and} \qquad \Delta_{\mathfrak{t}} = \prod_{\phi \in \Sigma_{\mathfrak{a}}^{+}} (e^{\phi/2} - e^{-\phi/2}).$$

We may adjust $\rho_{\mathfrak{t}} \in L_{\mathfrak{t}}$, so $\Delta_{\mathfrak{t}}$ is a well-defined function on T^0. Let

$$\tilde{\omega}_{\mathfrak{t}}(\nu) = \prod_{\phi \in \Sigma_{\mathfrak{t}}^{+}} \langle \phi, \nu \rangle \qquad \text{and} \qquad L_{\mathfrak{t}}'' = \{\nu \in L_{\mathfrak{t}} \colon \tilde{\omega}_{\mathfrak{t}}(\nu) \neq 0\}.$$

Finally $(-1)^{q_{\mathfrak{t}}(\nu)} = (-1)^{q_{\mathfrak{t}}} \operatorname{sign} \tilde{\omega}_{\mathfrak{t}}(\nu)$ where $q_{\mathfrak{t}} = (1/2) \dim(M/K \cap M)$. Now let $\nu \in L_{\mathfrak{t}}''$, $\xi = e^{\nu - \rho_{\mathfrak{t}}}|_{Z_{M^0}}$ and $[\chi] \in Z_M(M^0)\hat{}_{\xi}$. The corresponding class in $\hat{M}_{\text{disc}}$ is

$$[\eta_{\chi,\nu}] = [\operatorname{Ind}_{M^{\dagger}\uparrow M}(\chi \otimes \eta_{\nu})] \tag{1.5.1a}$$

where $M^{\dagger} = Z_M(M^0)M^0$ and $[\eta_\nu]$ is the unique class in $(M^0)\hat{}_{\text{disc}}$ whose distribution character satisfies

$$\Psi_{\eta_\nu}|_{T^0 \cap M''} = (-1)^{q_{\mathfrak{t}}(\nu)} \Delta_{\mathfrak{t}}^{-1} \sum_{W(M^0, T^0)} \det(w) e^{w\nu}. \tag{1.5.1b}$$

The H-series of G consists of the unitary representation classes

$$[\pi_{\chi,\nu,\sigma}] = [\pi_{\eta_{\chi,\nu},\sigma}] = [\operatorname{Ind}_{MAN\uparrow G}(\eta_{\chi,\nu} \otimes e^{i\sigma})]. \tag{1.5.2}$$

An H-series class $[\pi_{\chi,\nu,\sigma}]$ is specified by $\Theta_{\pi_{\chi,\nu,\sigma}}|_{H \cap G'}$ [13, Theorem 4.4.4].

If H and $'H$ are nonconjugate Cartan subgroups of G then [13, Theorem 4.4.6] every H-series class is disjoint from every $'H$-series class.

If $[\pi_{\chi,\nu,\sigma}]$ is an H-series class such that $\nu + i\sigma$ is $\mathfrak{g}$-regular then [13, Corollary 4.5.3] the class $[\pi_{\chi,\nu,\sigma}]$ is irreducible.

1.6. *Plancherel Theorem for Reductive Groups*

Fix a Cartan involution θ of G and a complete system

$$H_j = T_j \times A_j, \qquad 1 \leqslant j \leqslant l, \tag{1.6.1}$$

of θ-stable representatives of the conjugacy classes of Cartan subgroups of G. Let $\zeta \in \hat{Z}$ and denote

$$L_{j,\zeta}'' = \{\nu \in L_{\mathfrak{t}_j}'' \colon e^{\nu - \rho_{\mathfrak{t}_j}} = \zeta \text{ on } Z \cap M_j^0\}. \tag{1.6.2}$$

If $\nu \in L''_{j,\zeta}$ let ξ_ν denote the $Z_{M_j^0}$-restriction of $e^{\nu-\rho_{t_j}}$, and define

$$S(\nu, \zeta) = Z_{M_j}(M_j^0)\hat{}_{\zeta\otimes\xi_\nu} \qquad \text{(finite set)}, \tag{1.6.3a}$$

$$\pi_{j,\zeta,\nu+i\sigma} = \sum_{S(\nu,\zeta)} (\dim \chi)\pi_{\chi,\nu,\sigma} \qquad \text{(finite sum), and} \tag{1.6.3b}$$

$$\Theta_{j,\zeta,\nu+i\sigma} = \sum_{S(\nu,\zeta)} (\dim \chi)\Theta_{\pi_{\chi,\nu,\sigma}} \qquad \text{(finite sum).} \tag{1.6.3c}$$

Then Harish-Chandra's Plancherel Formula [6, 7] goes over to G as follows.

There are unique Borel functions $m_{j,\zeta,\nu}$ on $\mathfrak{a}_j{}^*$, $1 \leqslant j \leqslant l$ and $\nu \in L''_{j,\zeta}$, with the following properties.

(1) The $m_{j,\zeta,\nu}$ are equivariant for the action of the Weyl group $W(G, H_j)$.

(2) If $f \in L_2(G/Z, \zeta)$ is C^∞ with support compact modulo Z, and $[r_x f](g)$ means $f(gx)$ for $x, g \in G$, then

$$f(x) = \sum_{1\leqslant j\leqslant l} \sum_{\nu\in L''_{j,\zeta}} | \tilde{\omega}_{t_j}(\nu)| \int_{\mathfrak{a}_j{}^*} \Theta_{j,\zeta,\nu+i\sigma}(r_x f)\, m_{j,\zeta,\nu}(\sigma)\, d\sigma \tag{1.6.4}$$

where the sums are absolutely convergent.

In particular the various H-series carry the Plancherel measure for each $\hat{G}_\zeta$, $\zeta \in \hat{Z}$.

Let c be a number. In the expansion (1.6.4) of f we may omit those nondiscrete $\Theta_{j,\zeta,\nu+i\sigma}(r_x f)$ for which $\| \nu \|^2 + \| \sigma \|^2 - \| \rho \|^2 = c$. In other words,

$$\{[\pi] \in \hat{G}_\zeta - \hat{G}_{\zeta\text{-disc}}\colon \pi(\text{Casimir}) = c\} \text{ has Plancherel measure zero.} \tag{1.6.5}$$

This applies to M, where it is crucial for the proof of Lemma 4.2.3 below.

2. Square Integrable Partially Harmonic Spinors

We formulate a geometric setting for realization of the various nondegenerate series representations of a reductive Lie group G of the class discussed in Section 1. The setting consists of a fibration

$$p\colon X = G/UAN \to G/M^\dagger AN = Z$$

of G-homogeneous spaces such that every $Y_z = p^{-1}(z)$ is a riemannian symmetric space, G-homogeneous bundles

$$\mathcal{S}^{\pm} \otimes \mathcal{V}_{\mu,\sigma} \to X$$

whose every Y_z-restriction is a bundle of spinors with values in $\mathcal{V}_{\mu,\sigma}|_{Y_z}$, and operators

$$D^{\pm} \text{ densely defined on the space } L_2(\mathcal{S}^{\pm} \otimes \mathcal{V}_{\mu,\sigma}) \text{ of } L_2\text{-sections}$$

such that $D^{\pm}$ is the Dirac operator over every Y_z. Then G acts by a unitary representation $\pi^{\pm}_{\mu,\sigma}$ on the space

$$H_2^{\pm}(\mathcal{V}_{\mu,\sigma}) = \{\phi \in L_2(\mathcal{S}^{\pm} \otimes \mathcal{V}_{\mu,\sigma}) \colon \tilde{D}^{\pm}\phi = 0\}$$

of "square integrable partially harmonic spinors" with values in $\mathcal{V}_{\mu,\sigma}$. Later the $\pi^{\pm}_{\mu,\sigma}$ will be seen to realize the various nondegenerate series representations of G.

In Section 2.1 we recall the Clifford algebra construction for spin groups and spin representations. Then in Section 2.2 we discuss generalized riemannian spin structures and the corresponding notions of spin bundle, Dirac operator and harmonic spinor. These concepts are extended to measurable families of riemannian manifolds in Section 2.3. Finally in Section 2.4 we specialize to the situation of representations of reductive Lie groups.

2.1. *Spin Construction*

We recall the Clifford algebra construction for spin groups and spin representations. See [14, Section 1].

Let E be a real n-dimensional vector space with positive definite inner product $\langle e, e'\rangle$. If $\{e_j\}$ is an orthonormal basis, then the *Clifford algebra* is the real associative algebra $Cl(E)$ with generators and relations

$$e_j \cdot e_j = -1 \text{ and } e_j \cdot e_k + e_k \cdot e_j = 0 \quad \text{for} \quad 1 \leqslant j \leqslant n,\ 1 \leqslant j < k \leqslant n. \tag{2.1.1}$$

$Cl(E)$ has basis $\{e_{j_1} \cdot \cdots \cdot e_{j_k} \colon 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$, hence dimension 2^n. It is Z_2-graded by the subspaces $Cl^{\pm}(E)$ spanned by the $e_{j_1} \cdot \cdots \cdot e_{j_k}$ with $(-1)^k = \pm 1$.

$Cl(E)$ carries an involutive automorphism $x \to \bar{x}$ given by $e_{j_1} \cdot \cdots \cdot e_{j_k} \to (-1)^k e_{j_k} \cdot \cdots \cdot e_{j_1}$. The *spin group* of E is the multiplicative group

$$\mathrm{Spin}(E) = \mathrm{Spin}(n) = \{x \in Cl^{+}(E) \colon x\bar{x} = 1 \text{ and } x \cdot E \cdot \bar{x} = E\}. \tag{2.1.2}$$

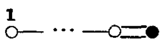
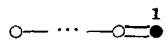
Spin(E) has *vector representation* on E given by

$$v(x)e = x \cdot e \cdot \bar{x}; \qquad \text{so} \quad v\colon \mathrm{Spin}(E) \to SO(E). \tag{2.1.3}$$

This vector representation interprets e_j as reflection in the hyperplane $e_j{}^{\perp}$ of E. It has kernel $\{\pm 1\}$. If $n > 2$ it is the universal covering group $\mathrm{Spin}(n) \to SO(n)$.

Left multiplication is a linear representation l of Spin(E) on the complexified Clifford algebra $Cl(E)_{\mathbf{C}}$.

If $n = 2m + 1$ odd, then $l = 2^{m+1}s$ where s is an irreducible unitary representation of degree 2^m called the *spin representation*. If $m \geqslant 2$, the vector and spin representations have diagram

v:  and s:  (2.1.4)

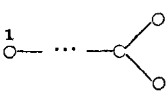
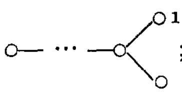
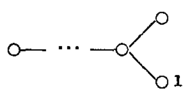
If $n = 2m$ even, we choose an orientation on E and define

$\epsilon = e_1 \cdot \cdots \cdot e_{2m}$ where $\{e_1, \ldots, e_{2m}\}$ is an oriented orthonormal basis. (2.1.5a)

Then Spin(E) has center $\{\pm 1, \pm\epsilon\}$. $l = 2^m s$, $s = s^+ \oplus s^-$, where the $s^{\pm}$ are irreducible unitary representations of degree 2^{m-1} distinguished by the orientation:

$$s^{\pm}(\epsilon) = \pm i^{-m} \qquad \text{and} \qquad s^{\pm}(-1) = -I. \tag{2.1.5b}$$

The $s^{\pm}$ are the *half spin representations* of Spin(E), and $s = s^+ \oplus s^-$ is the *spin representation*. If $m \geqslant 3$, the diagrams are

v:  ; s^+:  ; s^-:  . (2.1.5c)

Let $n = 2m$ even. Choose a representation space $S^+ \subset Cl(E)_{\mathbf{C}}$ for s^+. Then $S^- = e \cdot S^+$ is independent of choice of $0 \neq e \in E$, S^- is representation space for s^-, and $S = S^+ \oplus S^-$ is representation space for s. Thus Clifford multiplication defines linear maps

$$m^{\pm}\colon E_{\mathbf{C}} \otimes S^{\pm} \to S^{\mp} \qquad \text{and} \qquad m = m^+ \oplus m^-\colon E_{\mathbf{C}} \otimes S \to S. \tag{2.1.6}$$

2.2. *Riemannian Spin Structure*

Y is an oriented n-dimensional riemannian manifold. We discuss a generalization of the classical notion of spin structure and the corresponding notions of spin bundle, Dirac operator and harmonic spinor. Compare [14].

Let $\pi: \mathscr{F} \to Y$ denote the oriented orthonormal frame bundle and Γ the riemannian connection on $\mathscr{F}$. This is a principal $SO(n)$-bundle. Now fix a Lie group homomorphism

$$\alpha: U \to SO(n) \text{ that factors } \begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\tilde{\alpha}}{\nearrow} & \big\downarrow v \\ U & \underset{\alpha}{\searrow} & SO(n) \end{array} \tag{2.2.1}$$

By *riemannian* (U, α)*-structure* on Y we mean a principal U-bundle $\pi_U: \mathscr{F}_U \to Y$ with a connection $\Gamma_{\mathscr{F}_U}$, such that

$$\pi_U = \pi \cdot \tilde{\alpha} \text{ with } \tilde{\alpha} \text{ given by (2.2.1) on each } \pi_U\text{-fibre, and} \tag{2.2.2a}$$

$$\tilde{\alpha}(\Gamma_{\mathscr{F}_U}) = \Gamma, \qquad \text{i.e.,} \quad \tilde{\alpha}^*\omega = \alpha \cdot \omega_{\mathscr{F}_U} \text{ on connection forms.} \tag{2.2.2b}$$

Fix a riemannian (U, α)-structure $(\mathscr{F}_U, \Gamma_{\mathscr{F}_U})$ on Y. Given

$$\mu: \text{finite-dimensional unitary representation of } U \tag{2.2.3a}$$

we have

$$\mathscr{V}_\mu \to Y: \text{hermitian vector bundle associated to } \mathscr{F}_U \text{ by } \mu. \tag{2.2.3b}$$

Applying this to the composition of $\tilde{\alpha}: U \to \mathrm{Spin}(n)$ with the spin representation, we get

$$\mathscr{S} = \mathscr{V}_{s \cdot \tilde{\alpha}} \to Y \qquad \textit{spin bundle}, \text{ and} \tag{2.2.4a}$$

$$\mathscr{S} \otimes \mathscr{V}_\mu \to Y \qquad \text{bundle of } \mathscr{V}_\mu\textit{-valued spinors} \tag{2.2.4b}$$

If n is even, the spin bundle is direct sum of its subbundles

$$\mathscr{S}^\pm = \mathscr{V}_{s^\pm \cdot \tilde{\alpha}} \to Y \qquad \textit{half spin bundle}, \tag{2.2.4c}$$

and then $\mathscr{S} \otimes \mathscr{V}_\mu$ is direct sum of its subbundles $\mathscr{S}^\pm \otimes \mathscr{V}_\mu$.

If $\mathscr{W} \to Y$ is an hermitian vector bundle and ϕ, ψ are Borel sections, then the pointwise inner product $\langle \phi, \psi \rangle_y = \langle \phi(y), \psi(y) \rangle$ is a Borel function on Y. Integrating against the volume element we have the global inner product

$$\langle \phi, \psi \rangle = \int_Y \langle \phi, \psi \rangle_y \, dy. \tag{2.2.5a}$$

The L_2-norm of a Borel section is defined by $\|\phi\|^2 = \langle\phi,\phi\rangle$. Throwing out sections that vanish a.e. Y,

$$L_2(\mathscr{W}) = \{\phi \text{ Borel section of } \mathscr{W} : \|\phi\| < \infty\} \tag{2.2.5b}$$

is a Hilbert space with inner product (2.2.5a).

We now have the Hilbert space

$$L_2(\mathscr{S} \otimes \mathscr{V}_\mu): \textit{square integrable } \mathscr{V}_\mu\textit{-valued spinors} \text{ on } Y \tag{2.2.6}$$

If n is even, it is orthogonal direct sum of its subspaces $L_2(\mathscr{S}^\pm \otimes \mathscr{V}_\mu)$.

Now assume $n = 2m$ even. Let $\mathscr{T} \to Y$ denote the complexified tangent bundle. We denote the covariant differential on C^∞ sections of $\mathscr{S}^\pm \otimes \mathscr{V}_\mu$, for the connection associated to $\Gamma_{\mathscr{F}_U}$, by

$$\nabla^\pm: C^\infty(\mathscr{S}^\pm \otimes \mathscr{V}_\mu) \to C^\infty(\mathscr{T}^* \otimes \mathscr{S}^\pm \otimes \mathscr{V}_\mu) \quad \text{and} \quad \nabla = \nabla^+ \oplus \nabla^-.$$

The riemannian metric of Y gives a bundle isomorphism

$$h: \mathscr{T}^* \to \mathscr{T} \quad \text{by} \quad \langle h(f), e\rangle = f(e).$$

The multiplication maps (2.1.6) define bundle maps

$$m^\pm: \mathscr{T} \otimes \mathscr{S}^\pm \to \mathscr{S}^\mp \quad \text{and} \quad m: \mathscr{T} \otimes \mathscr{S} \to \mathscr{S}.$$

The *Dirac operators*

$$D^\pm: C^\infty(\mathscr{S}^\pm \otimes \mathscr{V}_\mu) \to C^\infty(\mathscr{S}^\mp \otimes \mathscr{V}_\mu) \quad \text{and} \quad D = D^+ \oplus D^- \tag{2.2.7a}$$

are defined to be the compositions

$$D^\pm = (m^\pm \otimes 1) \circ (h \otimes 1 \otimes 1) \circ \nabla^\pm \quad \text{and} \quad D = (m \otimes 1) \circ (h \otimes 1 \otimes 1) \circ \nabla. \tag{2.2.7b}$$

Let $\{e_1, ..., e_n\}$ be a moving orthonormal frame on an open set $\mathbf{0} \subset Y$. Then [14, Sect. 3] D is given on $\mathbf{0}$ by

$$D\phi = \sum_{1 \leqslant j \leqslant n} e_j \cdot \nabla_{e_j}(\phi) \text{ where } \cdot \text{ is Clifford product.} \tag{2.2.8}$$

It follows that D is elliptic. Viewing D as an operator on $L_2(\mathscr{S} \otimes \mathscr{V}_\mu)$ with dense domain $C_c^\infty(\mathscr{S} \otimes \mathscr{V}_\mu)$, the compactly supported C^∞ sections, it also follows [14, Sect. 4] that D is symmetric.

Now suppose that the riemannian metric of Y is complete. Then [14, Sects. 5 and 6] D and D^2 are essentially self-adjoint on $L_2(\mathscr{S} \otimes \mathscr{V}_\mu)$. This means that each has closure (denoted $\tilde{}$) equal to its adjoint, which thus is the unique self-adjoint extension, so each has well-defined spectral decomposition. The space of *square integrable $\mathscr{V}_\mu$-valued harmonic spinors* is their common kernel

$$H_2(\mathscr{V}_\mu) = \{\phi \in L_2(\mathscr{S} \otimes \mathscr{V}_\mu): \tilde{D}\phi = 0\}. \tag{2.2.9a}$$

It is closed in $L_2(\mathscr{S} \otimes \mathscr{V}_\mu)$, is contained in $C^\infty(\mathscr{S} \otimes \mathscr{V}_\mu)$, and is orthogonal direct sum of its subspaces

$$H_2^{\pm}(\mathscr{V}_\mu) = \{\phi \in L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_\mu): \tilde{D}^{\pm}\phi = 0\}. \tag{2.2.9b}$$

2.3. *Measurable Families of Riemannian Spin Manifolds*

We consider a measurable family of oriented riemannian n-manifolds and extend the notions of Section 2.2. Thus we discuss partial spin structures and the corresponding notions of partial spin bundle, partial Dirac operator and partially harmonic spinor.

By *measurable family of differentiable manifolds* we mean

$$p: X \to Z \text{ locally trivial Borel fibre space} \tag{2.3.1a}$$

where X and Z are analytic Borel spaces, together with

$$\text{the structure of } C^\infty\ n\text{-manifold on each fibre } Y_z = p^{-1}(z), \tag{2.3.1b}$$

with the compatibility condition that

$$X \text{ induces the intrinsic Borel structure on each manifold } Y_z. \tag{2.3.1c}$$

In practice, $p: X \to Z$ will be a C^∞ fibre bundle.

Fix a measurable family $p: X \to Z$ of differentiable manifolds. By *partially C^∞ fibre bundle* over X we mean a locally trivial Borel fibre space $\mathscr{W} \to X$ such that all the $\mathscr{W}|_{Y_z} \to Y_z$ are C^∞ fibre bundles with the same structure group. Then *partially C^∞ section* means a Borel section that is C^∞ over each Y_z. The case of the product bundle $X \times \mathbf{C} \to X$ gives us the notion of *partially C^∞ function*: Borel function $f: X \to \mathbf{C}$ such that each $f|_{Y_z}$ is C^∞.

Let us denote

$$\mathscr{T}_z \to Y_z\text{: complexified tangent bundle.}$$

We define the *partial tangent bundle* to be

$$\mathscr{T} = \bigcup_Z \mathscr{T}_z \to X \qquad \text{by} \quad \mathscr{T}|_{Y_z} = \mathscr{T}_z.$$

If f is a partially C^∞ function on X, each differential $d(f|_{Y_z})\colon \mathscr{T}_z \to \mathrm{C}$ is C^∞. The *differential of f* is

$$df\colon \mathscr{T} \to \mathrm{C} \qquad \text{defined by} \quad (df)|_{\mathscr{T}_z} = d(f|_{\mathscr{T}_z}).$$

We give $\mathscr{T}$ the smallest Borel structure such that, if f is a partially C^∞ function on X then df is a Borel function on $\mathscr{T}$. Combining local triviality of $X \to Z$ with local triviality of the $\mathscr{T}_z \to Y_z$, one sees that $\mathscr{T} \to X$ is a locally trivial Borel fibre space. Thus the partial tangent bundle is a partially C^∞ fibre bundle.

Let $\mathfrak{S}$ be a geometric structure, e.g., riemannian. By ***measurable family of*** $\mathfrak{S}$***-manifolds*** we mean a measurable family $p\colon X \to Z$ of differentiable manifolds, and a collection of partially C^∞ fibre bundles and partially C^∞ sections over X whose restrictions to each Y_z define an $\mathfrak{S}$-structure there. For example, a measurable family of riemannian manifolds comes from the partial tangent bundle $\mathscr{T} \to X$ by an appropriate choice of partially C^∞ section of $\mathscr{T}^* \otimes \mathscr{T}^*$; the section will be a Borel assignment of hermitian metrics on the $\mathscr{T}_z \to Y_z$.

We now fix a measurable family $p\colon X \to Z$ of oriented n-dimensional riemannian manifolds and a Lie group homomorphism

$$\alpha\colon U \to SO(n) \text{ that factors } U \xrightarrow{\tilde{\alpha}} \mathrm{Spin}(n) \xrightarrow{v} SO(n). \quad (2.3.2)$$

The union of the oriented orthonormal frame bundles of the $Y_z = p^{-1}(z)$ is the partially C^∞ fibre bundle

$$\pi\colon \mathscr{F} \to X \textit{ partial oriented orthonormal frame bundle}$$

with Borel structure defined by the maps

$$df\colon \mathscr{F} \to \mathrm{C}^n \text{ for all partially } C^\infty \text{ functions } f \text{ on } X.$$

It is a principal $SO(n)$ bundle in the category of Borel spaces. Now we define ***partial riemannian*** (U, α)***-structure*** on X to mean a partially C^∞ principal U-bundle $\pi_U\colon \mathscr{F}_U \to X$ with a measurable assignment $\Gamma_{\mathscr{F}_U}$ of connections on the $\mathscr{F}_U|_{Y_z} \to Y_z$ such that

$$\pi_U = \pi \cdot \tilde{\alpha} \text{ with } \tilde{\alpha} \text{ given by (2.3.2) on each } \pi_U\text{-fibre and} \quad (2.3.3a)$$

$$\tilde{\alpha} \text{ sends the connection on } \mathscr{F}_U|_{Y_z} \text{ to the riemannian connection.} \quad (2.3.3b)$$

Fix a partial riemannian (U, α)-structure $(\mathscr{F}_U, \Gamma_{\mathscr{F}_U})$ on X. If μ is a finite-dimensional unitary representation of U and V_μ is the representation space, then we have

$$\mathscr{V}_\mu = \mathscr{F}_U \times_U V_\mu \to X \text{ associated partially } C^\infty \text{ vector bundle.} \quad (2.3.4a)$$

In particular we have

$$\mathscr{S} = \mathscr{V}_{s\cdot\tilde{\alpha}} \to X \textit{ partial spin bundle}, \tag{2.3.4b}$$

and thus

$$\mathscr{S} \otimes \mathscr{V}_\mu \to X \text{ bundle of } \mathscr{V}_\mu\textit{-valued partial spinors}. \tag{2.3.4c}$$

If n is even, then $\mathscr{S}$ is direct sum of its subbundles $\mathscr{S}^\pm = \mathscr{V}_{s^\pm\cdot\tilde{\alpha}} \to X$, so $\mathscr{S} \otimes \mathscr{V}_\mu$ is direct sum of its subbundles $\mathscr{S}^\pm \otimes \mathscr{V}_\mu$. Associated to unitary representations of U, all these bundles carry natural, partially C^∞, hermitian metrics.

Fix a positive σ-finite Borel measure dz on Z. If $\mathscr{W} \to X$ is a partially C^∞ hermitian vector bundle and ϕ, ψ are Borel sections, then the global inner product is

$$\langle \phi, \psi \rangle = \int_Z \langle \phi, \psi \rangle_{Y_z}\, dz = \int_Z \left\{ \int_{Y_z} \langle \phi, \psi \rangle_y\, dy \right\} dz. \tag{2.3.5a}$$

The L_2-norm of a Borel section is defined by $\| \phi \|^2 = \langle \phi, \phi \rangle$. Throwing out sections ϕ such that $\langle \phi, \phi \rangle_{Y_z} = 0$ a.e. (Z, dz),

$$L_2(\mathscr{W}) = \{\phi \text{ Borel section of } \mathscr{W}: \| \phi \| < \infty\} \tag{2.3.5b}$$

is a Hilbert space with inner product (2.3.5a). Given $z \in Z$ denote $\mathscr{W}_z = \mathscr{W}|_{Y_z} \to Y_z$ and define the Hilbert space $L_2(\mathscr{W}_z)$ as in (2.2.5). Then $z \to L_2(\mathscr{W}_z)$ is a measurable assignment of Hilbert spaces on Z and

$$L_2(\mathscr{W}) = \int_Z L_2(\mathscr{W}_z)\, dz \qquad \text{(direct integral)}. \tag{2.3.5c}$$

This uses analyticity of the Borel structures on X and Z. We now have the Hilbert space

$$L_2(\mathscr{S} \otimes \mathscr{V}_\mu)\text{: } \textit{square integrable } \mathscr{V}_\mu\textit{-valued partial spinors} \text{ on } X. \tag{2.3.6}$$

If n is even, it is orthogonal direct sum of its subspaces $L_2(\mathscr{S}^\pm \otimes \mathscr{V}_\mu)$.

Now assume $n = 2m$ even. Let $C^\infty(\cdot)$ denote the set of all partially C^∞ sections. The covariant differential associated to $\Gamma_{\mathscr{F}_U}$ is denoted

$$\nabla^\pm: C^\infty(\mathscr{S}^\pm \otimes \mathscr{V}_\mu) \to C^\infty(\mathscr{T}^* \otimes \mathscr{S}^\pm \otimes \mathscr{V}_\mu) \qquad \text{and} \qquad \nabla = \nabla^+ \oplus \nabla^-.$$

Let $h: \mathscr{T}^* \to \mathscr{T}$ be the isomorphism from the riemannian metrics of the Y_z. The multiplication maps (2.1.6) define bundle maps

$$m^\pm: \mathscr{T} \otimes \mathscr{S}^\pm \to \mathscr{S}^\mp \qquad \text{and} \qquad m: \mathscr{T} \otimes \mathscr{S} \to \mathscr{S}.$$

The *partial Dirac operators*

$$D^{\pm}\colon C^{\infty}(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu}) \to C^{\infty}(\mathscr{S}^{\mp} \otimes \mathscr{V}_{\mu}) \quad \text{and} \quad D = D^{+} \oplus D^{-} \tag{2.3.7a}$$

are defined to be the compositions

$$D^{\pm} = (m^{\pm} \otimes 1) \circ (h \otimes 1 \otimes 1) \circ \nabla^{\pm} \quad \text{and} \quad D = (m \otimes 1) \circ (h \otimes 1 \otimes 1) \circ \nabla. \tag{2.3.7b}$$

Let $\mathbf{0} \subset X$ be a Borel set with each $\mathbf{0} \cap Y_z$ open in Y_z and $\{e_1, \ldots, e_n\}$ a partially C^{∞} section of $\mathscr{F}$ over $\mathbf{0}$. We apply (2.2.8) to each $\mathbf{0} \cap Y_z$ to see that D is given on $\mathbf{0}$ by

$$D\phi = \sum_{1 \leqslant j \leqslant n} e_j \cdot \nabla_{e_j}(\phi) \text{ where } \cdot \text{ is Clifford product.} \tag{2.3.8}$$

It follows that D is transverse elliptic for $X \to Z$ in the sense that its symbol $\sigma(D)(\xi)$ is nonsingular whenever ξ is a nonzero element of the real partial cotangent bundle.

Now suppose that the riemannian metric of Y_z is complete a.e. (Z, dz). Then almost every $D|_{Y_z}$ and $D^2|_{Y_z}$ is essentially self-adjoint on $L_2(\mathscr{S} \otimes \mathscr{V}_{\mu}|_{Y_z})$ with domain $C_c^{\infty}(\mathscr{S} \otimes \mathscr{V}_{\mu}|_{Y_z})$, and their common kernel

$$H_2(\mathscr{V}_{\mu}|_{Y_z}) = \{\phi_z \in L_2(\mathscr{S} \otimes \mathscr{V}_{\mu}|_{Y_z})\colon (D|_{Y_z})^{\sim}\phi_z = 0\} \tag{2.3.9a}$$

is closed in $L_2(\mathscr{S} \otimes \mathscr{V}_{\mu}|_{Y_z})$, consists of C^{∞} sections of $\mathscr{S} \otimes \mathscr{V}_{\mu}|_{Y_z}$, and is orthogonal direct sum of its subspaces

$$H_2^{\pm}(\mathscr{V}_{\mu}|_{Y_z}) = \{\phi_z \in L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu}|_{Y_z}) = (D^{\pm}|_{Y_z})^{\sim}\phi_z = 0\}. \tag{2.3.9b}$$

The spaces of *square integrable $\mathscr{V}_{\mu}$-valued partially harmonic spinors* on X are the

$$H_2^{\pm}(\mathscr{V}_{\mu}) = \int_Z H_2^{\pm}(\mathscr{V}_{\mu}|_{Y_z})\, dz \quad \text{and} \quad H_2(\mathscr{V}_{\mu}) = \int_Z H_2(\mathscr{V}_{\mu}|_{Y_z})\, dz. \tag{2.3.10a}$$

Evidently these Hilbert spaces satisfy $H_2(\mathscr{V}_{\mu}) = H_2^{+}(\mathscr{V}_{\mu}) \oplus H_2^{-}(\mathscr{V}_{\mu})$, and

$$\begin{aligned} H\,(\mathscr{V}_{\mu}) &= \{\phi \in L_2(\mathscr{S} \otimes \mathscr{V}_{\mu})\colon \phi|_{Y_z} \in H_2(\mathscr{V}_{\mu}|_{Y_z}) \text{ a.e. } (Z, dz)\} \\ &= \{\phi \in L_2(\mathscr{S} \otimes \mathscr{V}_{\mu})\colon \tilde{D}\phi|_{Y_z} = 0 \text{ a.e. } (Z, dz)\} \\ &= \{\phi \in L_2(\mathscr{S} \otimes \mathscr{V}_{\mu})\colon \tilde{D}\phi = 0 \text{ in } L_2(\mathscr{S} \otimes \mathscr{V}_{\mu})\}. \end{aligned} \tag{2.3.10b}$$

2.4. *Families of Riemannian Symmetric Spaces*

Let G be a reductive Lie group of the class considered in Section 1. Fix

$$\theta\text{: Cartan involution of } G, \tag{2.4.1a}$$

$$K\text{: fixed point set of } \theta, \tag{2.4.1b}$$

$$H = T \times A\text{: } \theta\text{-stable Cartan subgroup of } G, \text{ and} \tag{2.4.1c}$$

$$P = MAN\text{: associated cuspidal parabolic subgroup of } G. \tag{2.4.1d}$$

Recall our notation that, if L is a topological group then L^0 is its identity component, $Z_L(L^0)$ is the L-centralizer of L^0, and $L^\dagger = \{x \in L\text{: ad}(x) \text{ is an inner automorphism on } L^0\}$; so $L^\dagger = Z_L(L^0)L^0$. Now define

$$U = K \cap M^\dagger, \quad X = G/UAN \quad \text{and} \quad Z = G/M^\dagger AN = K/U. \tag{2.4.2}$$

Then p: $X \to Z$, given by $gUAN \to gM^\dagger AN$, is a real analytic fibre bundle. The fibres

$$Y_{kU} = kM^\dagger/U \text{ riemannian symmetric space of noncompact type} \tag{2.4.3}$$

with metric derived from the Killing form of $\text{ad}(k)\mathfrak{m}$. Since $M^\dagger$ acts on $M^\dagger/U$ as a connected group, the linear isotropy (real tangent space) representation maps U into $SO(n)$ where $n = \dim M^\dagger/U$. Replacing G by a Z_2-extension if necessary, that linear isotropy representation

$$\alpha\text{: } U \to SO(n) \text{ factors } U \xrightarrow{\tilde{\alpha}} \text{Spin}(n) \xrightarrow{v} SO(n). \tag{2.4.4}$$

Choose an orientation on Y_{1U}. If $k \in K$, give Y_{kU} the orientation such that k: $Y_{1U} \to Y_{kU}$ is orientation-preserving. Now we have the partial oriented orthonormal frame bundle π: $\mathscr{F} \to X$. Choose $F \in \pi^{-1}(1 \cdot UAN)$ oriented orthonormal frame on Y_{1U} at $1 \cdot U$. Since AN is normal in UAN we have

$$\mathscr{F}_U = G/AN \to G/UAN = X \text{ principal } U\text{-bundle}, \tag{2.4.5a}$$

$$\tilde{\alpha}\text{: } \mathscr{F}_U \to \mathscr{F} \text{ bundle map given by } \tilde{\alpha}(gAN) = g(F). \tag{2.4.5b}$$

Let Γ_{kU} denote the riemannian connection on Y_{kU} and ω_{kU} the restriction of its connection form to $\tilde{\alpha}\pi_U^{-1}(Y_{kU}) = kM^\dagger(F)$. Then ω_{kU} takes values in $\mathfrak{u} \cap [\mathfrak{m}, \mathfrak{m}]$. $\mathfrak{m}$ has center $\mathfrak{c}_m$ that is the Lie algebra of $Z_M(M^0)$, and $\mathfrak{u} = \mathfrak{c}_m \oplus (\mathfrak{u} \cap [\mathfrak{m}, \mathfrak{m}])$, so we may view ω_{kU} as taking values in $\mathfrak{u}$. Now let

$$\Gamma_{\mathscr{F}_U} = \{'\Gamma_{kU}\} \text{ where } '\Gamma_{kU} \text{ has connection form } \tilde{\alpha}^*\omega_{kU}. \quad (2.4.5c)$$

Then $(\mathscr{F}_U, \Gamma_{\mathscr{F}_U})$ is a partial riemannian (U, α)-structure for $p: X \to Z$.

There is a unique K-invariant probability measure on $Z = K/U$. We denote it $dz = d(kU)$.

If $\Sigma_{\mathfrak{t}}^+$ is a system of positive $\mathfrak{t}_{\mathbb{C}}$-roots on $\mathfrak{m}_{\mathbb{C}}$ then $n = \dim M^{\dagger}/U$ is twice the number of noncompact roots in $\Sigma_{\mathfrak{t}}^+$. Thus n is even.

Let μ be a finite-dimensional unitary representation of U. As in Section 2.3, we have the Hilbert spaces

$$L_2(\mathscr{S} \otimes \mathscr{V}_\mu) = L_2(\mathscr{S}^+ \otimes \mathscr{V}_\mu) \oplus L_2(\mathscr{S}^- \otimes \mathscr{V}_\mu)$$

of square integrable $\mathscr{V}_\mu$-valued partial spinors, and the Dirac operators $D = D^+ \oplus D^-$. The Y_{kU} are complete riemannian manifolds, so we also have Hilbert spaces $H_2^{\pm}(\mathscr{V}_\mu)$ of square integrable $\mathscr{V}_\mu$-valued partially harmonic spinors on X. Now we want to realize these Hilbert spaces as unitary G-modules.

Let $\Sigma_{\mathfrak{a}}^+$ denote the positive $\mathfrak{a}$-root system on $\mathfrak{g}$ such that $\mathfrak{n}$ is the sum of the negative $\mathfrak{a}$-root spaces. As usual, $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\phi \in \Sigma_{\mathfrak{a}}^+} (\dim \mathfrak{g}^{\phi})\phi$, so $\mathfrak{a}$ acts on $\mathfrak{n}$ with trace $-2\rho_{\mathfrak{a}}$. Let

$$[\mu] \in \hat{U} \quad \text{and} \quad \sigma \in \mathfrak{a}^*, \quad \text{i.e.} \quad [\mu \otimes e^{i\sigma}] \in (U \times A)\hat{}. \quad (2.4.6a)$$

Now UAN acts on the representation space V_μ of μ by

$$\gamma_{\mu,\sigma}(uan) = e^{\rho_{\mathfrak{a}} + i\sigma}(a)\, \mu(u). \quad (2.4.6b)$$

Similarly UAN acts on the representation space $S^{\pm} \otimes V_\mu$ of $(s^{\pm} \cdot \tilde{\alpha}) \otimes \mu$ by

$$[(s^{\pm} \cdot \tilde{\alpha}) \otimes \gamma_{\mu,\sigma}](uan) = e^{\rho_{\mathfrak{a}} + i\sigma}(a)\{s^{\pm}(\tilde{\alpha}(u)) \otimes \mu(u)\}. \quad (2.4.6c)$$

Now consider the associated G-homogeneous complex vector bundles,

$$\mathscr{V}_{\mu,\sigma} \to G/UAN = X \text{ associated to } \gamma_{\mu,\sigma} \text{ and} \quad (2.4.7a)$$

$$\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma} \to G/UAN = X \text{ associated to } (s^{\pm} \cdot \tilde{\alpha}) \otimes \gamma_{\mu,\sigma}. \quad (2.4.7b)$$

Each has K-invariant hermitian metric that is $\mathrm{ad}(k)M^{\dagger}$-homogeneous over Y_{kU} for all $k \in K$. Thus

$$\mathscr{V}_\mu = \mathscr{F}_U \times_U V_\mu = (G/AN) \times_U V_\mu = G \times_{UAN} V_\mu = \mathscr{V}_{\mu,\sigma}$$

and

$$\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu} = (G/AN) \times_U (S^{\pm} \otimes V_{\mu}) = G \times_{UAN} S^{\pm} \otimes V_{\mu} = \mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}$$

are K-equivariant hermitian bundle isomorphisms. Now (2.4.7) are realization of the bundles of Section 2.3 as G-homogeneous vector bundles. $L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$ is the Hilbert space of Borel functions $f\colon G \to S^{\pm} \otimes V_{\mu}$ such that

$$f(guan) = [(s^{\pm} \cdot \tilde{\alpha}) \otimes \gamma_{\mu,\sigma}](uan)^{-1} f(g) \text{ a.e. } G \text{ and} \tag{2.4.8a}$$

$$\|f\|^2 = \int_{K/U} \left\{ \int_{M^{\dagger}/U} \|f(km)\|^2 \, d(mU) \right\} d(kU) < \infty \tag{2.4.8b}$$

with inner product

$$\langle f, f' \rangle = \int_{K/U} \left\{ \int_{M^{\dagger}/U} \langle f(km), f'(km) \rangle \, d(mU) \right\} d(kU) \tag{2.4.8c}$$

The partial Dirac operators of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma} \to X$ act on $L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$ with dense domain consisting of all partially C^{∞} sections f such that $f|_{Y_{kU}}$ is compactly supported a.e. K. As the riemannian manifolds Y_{kU} are complete, $D = D^{+} \oplus D^{-}$ and its square are essentially self-adjoint, and the kernels

$$H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma}) = \{f \in L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})\colon \tilde{D}^{\pm} f = 0\}$$

are closed subspaces consisting of partially C^{∞} sections.

LEMMA 2.4.9. *The natural action of G on sections of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma} \to X$ preserves L_2-norm; it defines*

$$\tilde{\pi}^{\pm}_{\mu,\sigma}\colon \textit{unitary representation of } G \textit{ on } L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}). \tag{2.4.10a}$$

If $g \in G$ then $\tilde{\pi}^{\pm}_{\mu,\sigma}(g)$ commutes with $D^{\pm}$ on $L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$; thus $\tilde{\pi}^{\pm}_{\mu,\sigma}$ restricts to

$$\pi^{\pm}_{\mu,\sigma}\colon \textit{unitary representation of } G \textit{ on } H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma}). \tag{2.4.10b}$$

Proof. Define $'\gamma_{\mu,\sigma}(uan) = e^{i\sigma}(a)\,\mu(u)$ unitary, so $\gamma_{\mu,\sigma} = e^{\rho_{\mathfrak{a}}} \cdot {}'\gamma_{\mu,\sigma}$. Note that uan acts on the real tangent space $\mathfrak{g}/(\mathfrak{u} + \mathfrak{a} + \mathfrak{n})$ of X with determinant $e^{2\rho_{\mathfrak{a}}}(a)$. By definition now, the action of G on L_2 sections of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}$ is the unitarily induced representation

$$\tilde{\pi}^{\pm}_{\mu,\sigma} = \operatorname{Ind}_{UAN\uparrow G}((s^{\pm} \cdot \tilde{\alpha}) \otimes {}'\gamma_{\mu,\sigma}). \tag{2.4.11}$$

The ingredients of the definition of $D^{\pm}$ all are K-invariant, so each $\tilde{\pi}^{\pm}_{\mu,\sigma}(k)$ commutes with $D^{\pm}$.

Let $man \in M^{\dagger}AN$. Then n acts trivially both on Y_{1U} and the fibres of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}|_{Y_{1U}}$, a acts trivially on Y_{1U} and acts as multiplication by the scalar $e^{\rho_{\mathfrak{a}}}(a)$ on the fibres of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}|_{Y_{1U}}$, and m preserves all the ingredients of the definition on the Dirac operator of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma}|_{Y_{1U}}$. Now man commutes with $D^{\pm}$ over Y_{1U}. Conjugating by an arbitrary element of K, we see: if $x \in \{g \in G: gY_{kU} = Y_{kU}\} = \text{ad}(k)(M^{\dagger}AN)$, then x commutes with $D^{\pm}$ over Y_{kU}.

Let $g \in G$. If $k \in K$ we have $k' \in K$ such that $k'gY_{kU} = Y_{kU}$. Now $k'g$ commutes with $D^{\pm}$ over Y_{kU}, i.e.,

$$k'g \cdot (D^{\pm}|_{Y_{kU}}) = (D^{\pm}|_{Y_{kU}}) \cdot k'g.$$

But k' commutes with $D^{\pm}$, so

$$(D^{\pm}|_{Y_{kU}}) \cdot k' = k' \cdot (D^{\pm}|_{Y_{gkU}}).$$

Combining these, and noting that $k \in K$ was arbitrary,

$$g \cdot (D^{\pm}|_{Y_{kU}}) = (D^{\pm}|_{Y_{gkU}}) \cdot g \qquad \text{for all } k \in K.$$

We conclude that $\tilde{\pi}^{\pm}_{\mu,\sigma}(g)$ commutes with $D^{\pm}$. Q.E.D.

The point of this paper is the realization of the H-series of unitary representation classes of G by the representations $\pi^{\pm}_{\mu,\sigma}$ of G on the $H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma})$.

3. Formula for D^2

Let $\mathscr{V}_{\mu,\sigma} \to X = G/UAN$ be one of the bundles of Section 2.4. We obtain a formula for D^2 on the spaces $C^{\infty}(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$ of partially C^{∞} $\mathscr{V}_{\mu,\sigma}$-valued spinors on X. The formula involves a certain G-invariant operator Ξ derived from the Casimir element Ω_M of $\mathfrak{M}$ and the highest weight of $\mu|_{U^0}$. As consequence of the formula, the spaces $H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma})$ of $\mathscr{V}_{\mu,\sigma}$-valued partially harmonic spinors, are eigenspaces of D^2 on the $L_2(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$. That fact later plays a key role in the identification of the representations $\pi^{\pm}_{\mu,\sigma}$ of G on $H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma})$.

3.1. *Statement of Formula*

G is a reductive Lie group of the class discussed in Section 1. As in Section 2.4 we fix the data

$$\theta\text{: Cartan involution of } G, \tag{3.1.1a}$$

$$K\text{: fixed point set of } \theta, \tag{3.1.1b}$$

$$H = T \times A\text{: } \theta\text{-stable Cartan subgroup of } G, \text{ and} \tag{3.1.1c}$$

$$P = MAN\text{: associated cuspidal parabolic subgroup of } G. \tag{3.1.1d}$$

In order to discuss Casimir operators and length of roots consistently for several subgroups of G, we fix

$$\langle\, ,\, \rangle\text{: nondegenerate ad}(G)\text{-invariant symmetric form on } \mathfrak{g} \tag{3.1.2a}$$

such that

$$\langle\, ,\, \rangle \text{ is negative definite on } \mathfrak{k}. \tag{3.1.2b}$$

In other words, split $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where $\mathfrak{c}$ is the center and $\mathfrak{g}_i$ are the simple ideals; then $\langle\, ,\, \rangle$ is direct sum of an ad(G)-invariant negative definite form on $\mathfrak{c}$ with positive multiples of the Killing forms of the $\mathfrak{g}_i$. We extend $\langle\, ,\, \rangle$ to $\mathfrak{g}_C$ by linearity, to $\mathfrak{g}_C{}^*$ by duality, and write $\| x \|^2$ for $\langle x, x \rangle$. Note that $\langle\, ,\, \rangle$ is positive definite on $i\mathfrak{t}^* + \mathfrak{a}^*$.

Let L be a reductive subgroup of G, e.g., $K \cap M$, U, K or G. Then $\mathfrak{L}$ denotes the universal enveloping algebra of $\mathfrak{l}_C$, and Ω_L denotes the Casimir element of $\mathfrak{L}$ relative to the $\mathfrak{l}_C$-restriction of $\langle\, ,\, \rangle$. Thus

$$\text{if } \{x_i\} \text{ is an orthogonal basis of } \mathfrak{l} \text{ then } \Omega_L = \textstyle\sum \| x_i \|^{-2} x_i{}^2. \tag{3.1.3}$$

As in Section 2.4 our basic spaces are

$$p\colon X = G/UAN \to G/M^{\dagger}AN = K/U = Z \quad \text{where } U = K \cap M^{\dagger}. \tag{3.1.4}$$

If $k \in K$ then $Y_{kU} = p^{-1}(kU)$ carries the riemannian symmetric space structure whose metric is induced by the ad(k)$\mathfrak{m}$-restriction of $\langle\, ,\, \rangle$. Now choose $[\mu] \in \hat{U}$ and $\sigma \in \mathfrak{a}^*$, and consider the bundles $\mathscr{V}_{\mu,\sigma} \to X$ and $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma} \to X$. In each case we define an operator Ξ on the space of partially C^∞ sections by

$$(\Xi f) |_{Y_{kU}} = \Omega_{\mathrm{ad}(k)M}(f |_{Y_{kU}}) \qquad \text{for all } k \in K. \tag{3.1.5a}$$

Ξ is well defined because, if $k, k' \in K$ with $kU = k'U$, then $kM = k'M$, so $\mathrm{ad}(k)M = \mathrm{ad}(k')M$ and $\Omega_{\mathrm{ad}(k)M} = \Omega_{\mathrm{ad}(k')M}$. The action of Ω_M on C^∞ sections of the Y_{1U}-restrictions of the bundles, commutes with the action of $M^\dagger A$ because Ω_M is central in $\mathfrak{M}\mathfrak{A}$, commutes with the action of N because the latter is trivial. As $G = KM^\dagger AN$ it follows from (3.1.5a) that

$$\Xi \text{ is a } G\text{-invariant operator.} \tag{3.1.5b}$$

We need a better description of $[\mu] \in \hat{U}$. Observe $M^\dagger = Z_M(M^0)M^0$ with $Z_M(M^0) \subset (T = H \cap K) \subset (U = M^\dagger \cap K)$. Now

$$U = Z_M(M^0)U^0, \quad M^\dagger = UM^0 \quad \text{and} \quad U \cap M^0 = U^0. \tag{3.1.6a}$$

In particular

$$[\mu] = [\chi \otimes \mu^0] \quad \text{where} \quad [\chi] \in Z_M(M^0)^\wedge \quad \text{and} \quad [\mu^0] \in \hat{U}^0. \tag{3.1.6b}$$

Now choose

$$\Sigma_{\mathfrak{t}}^+\text{: positive } \mathfrak{t}_{\mathbf{C}}\text{-root system on } \mathfrak{m}_{\mathbf{C}}. \tag{3.1.7a}$$

It is disjoint union of its subsets

$$\Sigma_{\mathfrak{t},\mathfrak{u}}^+ = \{\phi \in \Sigma_{\mathfrak{t}}^+ : \mathfrak{m}_{\mathbf{C}}^\phi \subset \mathfrak{k}_{\mathbf{C}}\} \quad \text{and} \quad \Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+ = \{\phi \in \Sigma_{\mathfrak{t}}^+ : \mathfrak{m}_{\mathbf{C}}^\phi \not\subset \mathfrak{k}_{\mathbf{C}}\}. \tag{3.1.7b}$$

Thus we have $\rho_{\mathfrak{t}} = \rho_{\mathfrak{t},\mathfrak{u}} + \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ where

$$\rho_{\mathfrak{t}} = \tfrac{1}{2} \sum_{\Sigma_{\mathfrak{t}}^+} \phi, \qquad \rho_{\mathfrak{t},\mathfrak{u}} = \tfrac{1}{2} \sum_{\Sigma_{\mathfrak{t},\mathfrak{u}}^+} \phi, \qquad \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}} = \tfrac{1}{2} \sum_{\Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+} \phi. \tag{3.1.7c}$$

The half sum of any positive $\mathfrak{h}_{\mathbf{C}}$-root system on $\mathfrak{g}_{\mathbf{C}}$ exponentiates to a character on H [13, Lemma 4.3.6], and it follows [4] that $\rho_{\mathfrak{t}}$ exponentiates to a character on T. As $\rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ is a weight of $s^+ \cdot \tilde{\alpha}$, now

$$\rho_{\mathfrak{t},\mathfrak{u}} \text{ and } \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}} \text{ exponentiate to characters on } T. \tag{3.1.7d}$$

We now come to the formula for D^2. The case where G is a connected linear semisimple group and H is compact, is due to R. Parthasarathy [11, Proposition 3.2].

THEOREM 3.1.8. *Let $[\mu] \in \hat{U}$, say $[\mu] = [\chi \otimes \mu^0]$ as in (3.1.6) where $[\mu^0] \in \hat{U}^0$ has highest weight $\nu + \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ relative to $\Sigma_{\mathfrak{t},\mathfrak{m}}^+$. Let $\sigma \in \mathfrak{a}^*$. If f is a partially C^∞ section of $\mathscr{S}^\pm \otimes \mathscr{V}_{\mu,\sigma} \to X$, then*

$$D^2(f) = -\Xi(f) + \{\|\nu + \rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{t}}\|^2\} f \tag{3.1.9}$$

View D^2 and Ξ as essentially self adjoint operators on $L_2(\mathscr{S}^\pm \otimes \mathscr{V}_{\mu,\sigma})$. Then Theorem 3.1.8 gives us

COROLLARY 3.1.10. *The space $H_2^\pm(\mathscr{V}_{\mu,\sigma})$ of square integrable $\mathscr{V}_{\mu,\sigma}$-valued partially harmonic spinors on X, is the $\{\|\nu + \rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{t}}\|^2\}$-eigenspace of $\tilde{\Xi}$ on $L_2(\mathscr{S}^\pm \otimes \mathscr{V}_{\mu,\sigma})$.*

The remainder of Section 3 consists of the proof of Theorem 3.1.8.

3.2. *Proof of Formula*

We first prove (3.1.9) over Y_{1U}, following the general lines of [11, Section 3]. For this purpose we denote

$$Y = Y_{1U} = J/L \text{ where } J = M^\dagger A \text{ and } L = UA; \tag{3.2.1a}$$

$$\mathscr{W} \to Y \text{ the } J\text{-homogeneous bundle for } ((s \cdot \tilde{\alpha}) \otimes \mu) \otimes e^{\rho_{\mathfrak{a}}+i\sigma}; \tag{3.2.1b}$$

$$D_Y \text{ the Dirac operator on } C^\infty \text{ sections of } \mathscr{W} \to Y. \tag{3.2.1c}$$

The space $C^\infty(\mathscr{W})$ of C^∞ sections of $\mathscr{W} \to Y$ consists of the C^∞ functions $h\colon J \to S \otimes V_\mu$ such that

$$h(jua) = \{e^{\rho_{\mathfrak{a}}+i\sigma}(a)[s(\tilde{\alpha}(u)) \otimes \mu(u)]\}^{-1}(h(j)) \qquad \text{for} \quad j \in J, \quad u \in U, \quad a \in A.$$

In other words, writing $\{\ \}^L$ for the L-fixed elements under (right translation) $\otimes$ $(s \cdot \tilde{\alpha}) \otimes (\mu \otimes e^{\rho_{\mathfrak{a}}+i\sigma})$, we view

$$C^\infty(\mathscr{W}) = \{C^\infty(J) \otimes S \otimes V_\mu\}^L. \tag{3.2.2a}$$

If $x \in \mathfrak{j}$, write $l(x)$ for the left translation action of x on $C^\infty(J)$. If $e \in \mathfrak{e} = \{x \in \mathfrak{m}\colon \theta(x) = -x\}$, write $cl(e)$ for left Clifford multiplication on the subspace $S \subset Cl(\mathfrak{e})_{\mathbf{C}}$. Choose an orthonormal basis $\{e_1, \ldots, e_{2m}\}$, $m = |\Sigma^+_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}|$, of $\mathfrak{e}$. Now (2.2.8) goes over to the formulation (3.2.2a) as

$$D_Y = \sum_{1 \leqslant i \leqslant 2m} l(e_i) \otimes cl(e_i) \otimes 1. \tag{3.2.2b}$$

We will prove (3.1.9) over Y by squaring (3.2.2b). For that, we need

$$(s \cdot \tilde{\alpha})(\Omega_U) \text{ is scalar multiplication by } \|\rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{t},\mathfrak{u}}\|^2, \text{ and} \tag{3.2.3a}$$

$$\text{if } x \in \mathfrak{u} \text{ then } (s \cdot \tilde{\alpha})(x) = \tfrac{1}{4} \sum_{1 \leqslant i,j \leqslant 2m} \langle [x, e_i], e_j \rangle\, cl(e_i)\, cl(e_j). \tag{3.2.3b}$$

We check (3.2.3b). If $i \neq j$ then $e_i \cdot e_j \in Cl(\mathfrak{e})$ satisfies $(e_i \cdot e_j)^2 = -1$, so $\exp(te_i \cdot e_j) = \cos(t) + \sin(t)\, e_i \cdot e_j$. That gives us

$$\exp(te_i \cdot e_j) \cdot e_k \cdot \exp(-te_i \cdot e_j) = \begin{cases} e_k, & \text{if } i \neq k \neq j, \\ \cos(2t)e_i + \sin(2t)e_j, & \text{if } k = i \neq j, \\ \cos(2t)e_j - \sin(2t)e_i, & \text{if } k = j \neq i. \end{cases}$$

First, we conclude that $e_i \cdot e_j$ is in the Lie algebra $\mathfrak{spin}(\mathfrak{e})$ of Spin($\mathfrak{e}$), thus by dimension that $\{e_i \cdot e_j: 1 \leqslant i < j \leqslant 2m\}$ is a basis of $\mathfrak{spin}(\mathfrak{e})$. Second, we conclude that the differential of the vector representation v: Spin($\mathfrak{e}$) $\to SO(\mathfrak{e})$ satisfies

$$v(e_i \cdot e_j): e_k \to 0 \text{ if } i \neq k \neq j, \qquad 2e_j \text{ if } k = i, \qquad -2e_i \text{ if } k = j.$$

If $x \in \mathfrak{u}$ then we express $(s \cdot \tilde{\alpha})(x) = \sum_{i<j} p_{ij} cl(e_i) \cdot cl(e_j)$. Since $v(s(\tilde{\alpha}(x))) = \mathrm{ad}(x)|_{\mathfrak{e}}$, the above calculation says

$$[x, e_k] = \sum_{i<j} p_{ij} v(e_i \cdot e_j) e_k = 2 \sum_{j>k} p_{kj} e_j - 2 \sum_{i<k} p_{ik} e_i .$$

Thus $p_{ij} = (1/2)\langle [x, e_i], e_j \rangle$ for $i < j$, and (3.2.3b) follows.

We check (3.2.3a). Enumerate $\Sigma^+_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}} = \{\phi_1, ..., \phi_m\}$. Then $s^\pm \cdot \tilde{\alpha}$ has weight system $\{\frac{1}{2}(\epsilon_1\phi_1 + \cdots + \epsilon_m\phi_m): \epsilon_i = 1 \text{ or } -1 \text{ and } \prod \epsilon_i = \pm 1\}$, the multiplicity of a weight being the number of ways it can be so expressed. Now

$$\text{trace } s^+(\tilde{\alpha}(\exp x)) - \text{trace } s^-(\tilde{\alpha}(\exp x)) = \prod_{1 \leqslant i \leqslant m} (e^{\phi_i(x)/2} - e^{-\phi_i(x)/2}), \qquad x \in \mathfrak{t}. \tag{3.2.4}$$

Denote Weyl groups and a subset by

$$W_{\mathfrak{m}_C}: \mathfrak{m}_C \text{ for } \mathfrak{t}_C\,; \quad W_{\mathfrak{m}} = W_{\mathfrak{u}}: M^0 \text{ for } T^0; \quad W_{\mathfrak{m}/\mathfrak{u}} = \{w \in W_{\mathfrak{m}_C}: \Sigma^+_{\mathfrak{t},\mathfrak{u}} \subset w\Sigma^+_{\mathfrak{t}}\}.$$

Then $W_{\mathfrak{m}/\mathfrak{u}}$ is a set of representatives for $W_{\mathfrak{m}} \backslash W_{\mathfrak{m}_C}$, and if $w \in W_{\mathfrak{m},\mathfrak{u}}$ then $w\rho_{\mathfrak{t}} - \rho_{\mathfrak{t},\mathfrak{u}}$ is $\Sigma^+_{\mathfrak{t},\mathfrak{u}}$-dominant. Write $\mu_\nu{}^0$ for the irreducible representation of U^0 with highest weight ν. Using (3.2.4) and the Weyl character formula, we calculate on $T^0 = \exp(\mathfrak{t})$:

$$\begin{aligned}
&\text{trace } s^+ \cdot \tilde{\alpha} - \text{trace } s^- \cdot \tilde{\alpha} \\
&= \prod_{\phi \in \Sigma_{\mathfrak{t}}^+} (e^{\phi/2} - e^{-\phi/2}) \cdot \prod_{\psi \in \Sigma^+_{\mathfrak{t},\mathfrak{u}}} (e^{\psi/2} - e^{-\psi/2})^{-1} \\
&= \left(\sum_{w \in W_{\mathfrak{m}_C}} \det(w)\, e^{w\rho_{\mathfrak{t}}} \right) \Big/ \left(\sum_{v \in W_{\mathfrak{m}}} \det(v)\, e^{v\rho_{\mathfrak{t},\mathfrak{u}}} \right) \\
&= \sum_{w \in W_{\mathfrak{m}/\mathfrak{u}}} \det(w) \left\{ \left(\sum_{v \in W_{\mathfrak{m}}} \det(v)\, e^{vw\rho_{\mathfrak{t}}} \right) \Big/ \left(\sum_{v \in W_{\mathfrak{m}}} \det(v)\, e^{v\rho_{\mathfrak{t},\mathfrak{u}}} \right) \right\} \\
&= \sum_{w \in W_{\mathfrak{m}/\mathfrak{u}}} \det(w) \text{ trace } \mu^0_{w\rho_{\mathfrak{t}} - \rho_{\mathfrak{t},\mathfrak{u}}} .
\end{aligned}$$

There is no cancellation because $s^+ \cdot \tilde{\alpha}$ and $s^- \cdot \tilde{\alpha}$ have no weight in common: every weight of $s^+ \cdot \tilde{\alpha}$ (respectively, $s^- \cdot \tilde{\alpha}$) is an $\rho_{\mathfrak{t},\mathfrak{u}/\mathfrak{m}} - (\phi_{i_1} + \cdots + \phi_{i_r})$, $\phi_{i_j} \in \Sigma^+_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ and r even (respectively, r odd).

So

$$s^{\pm} \cdot \tilde{\alpha} \,|_{U^0} = \sum_{w \in W_{\mathfrak{m}/\mathfrak{u}}, \det(w) = \pm 1} \mu^0_{w\rho_{\mathfrak{t}} - \rho_{\mathfrak{t},\mathfrak{u}}}. \tag{3.2.5}$$

As $\mu^0_{w\rho_{\mathfrak{t}}-\rho_{\mathfrak{t},\mathfrak{u}}}(\Omega_U) = \| w\rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t},\mathfrak{u}} \|^2 = \| \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t},\mathfrak{u}} \|^2$, (3.2.3a) is proved. We now have established (3.2.3).

We calculate $D_Y{}^2$ by combining (3.2.2) and (3.2.3):

$$\begin{aligned} D_Y{}^2 &= \sum_{1 \leqslant i \leqslant m} l(e_i)^2 \otimes cl(e_i)^2 \otimes 1 + \sum_{i \neq j} l(e_i)\, l(e_j) \otimes cl(e_i)\, cl(e_j) \otimes 1 \\ &= -\sum_{1 \leqslant i \leqslant m} l(e_i)^2 \otimes 1 \otimes 1 + \tfrac{1}{2} \sum_{i \neq j} l[e_i, e_j] \otimes cl(e_i)\, cl(e_j) \otimes 1. \end{aligned}$$

Let $\{ix_1, ..., ix_p\}$ be an orthonormal basis of $i\mathfrak{u}$. Then $[e_i, e_j] = -\sum \langle [e_i, e_j], x_k \rangle x_k$ so

$$\begin{aligned} &\tfrac{1}{2} \sum_{i \neq j} l[e_i, e_j] \otimes cl(e_i)\, cl(e_j) \otimes 1 \\ &= -\tfrac{1}{2} \sum_k \sum_{i \neq j} l(x_k) \otimes \langle [e_i, e_j], x_k \rangle\, cl(e_i)\, cl(e_j) \otimes 1 \\ &= -2 \sum_k l(x_k) \otimes (s \cdot \tilde{\alpha})(x_k) \otimes 1 \\ &= -\sum_k (l \otimes s \cdot \tilde{\alpha})(x_k)^2 \otimes 1 + \sum_k l(x_k)^2 \otimes 1 \otimes 1 + \sum_k 1 \otimes (s \cdot \tilde{\alpha})(x_k)^2 \otimes 1 \\ &= \sum_k 1 \otimes 1 \otimes (\mu \otimes e^{i\sigma})(x_k)^2 + \sum_k l(x_k)^2 \otimes 1 \otimes 1 + \sum_k 1 \otimes (s \cdot \tilde{\alpha})(x_k)^2 \otimes 1 \\ &= 1 \otimes 1 \otimes \mu(\Omega_U) - l(\Omega_U) \otimes 1 \otimes 1 - 1 \otimes (s \cdot \tilde{\alpha})(\Omega_U) \otimes 1 \\ &= \{(\| \nu + \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t},\mathfrak{u}} \|^2) - l(\Omega_U) - (\| \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t},\mathfrak{u}} \|^2)\} \otimes 1 \otimes 1. \end{aligned}$$

Thus,

$$\begin{aligned} D_Y{}^2 &= \Big\{ -\sum_{1 \leqslant i \leqslant m} l(e_i)^2 - l(\Omega_U) + (\| \nu + \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t}} \|^2) \Big\} \otimes 1 \otimes 1 \\ &= \{ -l(\Omega_M) + (\| \nu + \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t}} \|^2) \} \otimes 1 \otimes 1. \end{aligned}$$

In summary, following the lines of [11, Sect. 3] we have proved (3.1.9) over the fibre Y_{1U} of $p: X \to Z$.

Consider the operator $B = D^2 + \Xi$ on the space $C^\infty(\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma})$ of partially C^∞ sections of $\mathscr{S}^{\pm} \otimes \mathscr{V}_{\mu,\sigma} \to X$. B commutes with the action of every $g \in G$, thus with the action of every $k \in K$. Since $B = \| \nu + \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t}} \|^2$ over Y_{1U}, now $B = \| \nu + \rho_{\mathfrak{t}} \|^2 - \| \rho_{\mathfrak{t}} \|^2$ over every Y_{kU}. That proves (3.1.9). Q.E.D.

4. Identification of the Representations

Let $\mathscr{V}_{\mu,\sigma} \to X = G/UAN$ be one of the bundles of Section 2.4 and $\pi^{\pm}_{\mu,\sigma}$ the unitary representation of G on $H_2{}^{\pm}(\mathscr{V}_{\mu,\sigma})$. We prove that $[\pi^{\pm}_{\mu,\sigma}]$ always is a finite sum of H-series representation classes. In particular $\pi^{\pm}_{\mu,\sigma}$ has a distribution character. We calculate the difference $\Theta_{\pi^+_{\mu,\sigma}} - \Theta_{\pi^-_{\mu,\sigma}}$ of those characters and see that it is $\pm\Theta_{\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}}$ where $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$ is a certain H-series class given in terms of μ and σ. Finally, we show for a certain choice $\pm$ of sign, that $H_2{}^{\mp}(\mathscr{V}_{\mu,\sigma}) = 0$ and $[\pi^{\pm}_{\mu,\sigma}] = [\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$. Afterwards we note that $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$ could, with appropriate choice of μ and σ, be any H-series class.

4.1. *Formulation of Main Theorem*

G is a reductive Lie group of the class discussed in Section 1. As in Sections 2.4 and 3.1 we fix a Cartan involution θ of G, a θ-stable Cartan subgroup $H = T \times A$ of G, and an associated cuspidal parabolic subgroup $P = MAN$ of G. Then we have

$$p: X = G/UAN \to G/M^{\dagger}AN = K/U = Z,$$

where K is the fixed point set of θ and $U = K \cap M^{\dagger}$. The fibres $Y_{kU} = p^{-1}(kU)$ are riemannian symmetric spaces of the $\mathrm{ad}(k)M^{\dagger}$ as indicated in Section 3.1.

Our choice of P specifies the positive $\mathfrak{a}$-root system $\Sigma_{\mathfrak{a}}{}^+$ on $\mathfrak{g}$ such that $\mathfrak{n}$ is the sum of the negative $\mathfrak{a}$-root spaces. We also choose a positive $\mathfrak{t}_C$-root system $\Sigma_{\mathfrak{t}}{}^+$ on $\mathfrak{m}_C$ and decompose it into the subset $\Sigma^+_{\mathfrak{t},\mathfrak{u}}$ of compact roots and the subset $\Sigma^+_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ of noncompact roots, and define $\rho_{\mathfrak{t}}$, $\rho_{\mathfrak{t},\mathfrak{u}}$ and $\rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ to be the respective half-sums of the elements of $\Sigma_{\mathfrak{t}}{}^+$, $\Sigma^+_{\mathfrak{t},\mathfrak{u}}$ and $\Sigma^+_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$. Σ^+ is the positive $\mathfrak{h}_C$-root system on $\mathfrak{g}_C$ that induces $\Sigma_{\mathfrak{a}}{}^+$ and $\Sigma_{\mathfrak{t}}{}^+$. We assume G replaced by a Z_2-extension if necessary [13, Section 4.3] so that $\rho = \frac{1}{2}\sum_{\gamma\in\Sigma^+}\gamma$ exponentiates to a character on H; so then $\rho_{\mathfrak{t}}$, $\rho_{\mathfrak{t},\mathfrak{u}}$ and $\rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$ exponentiate to characters on T. In particular they are contained in the lattice

$$L_{\mathfrak{t}} = \{\nu \in i\mathfrak{t}^*: e^{\nu} \text{ is well defined on } T^0\}. \tag{4.1.1a}$$

Further $\rho_{\mathfrak{t}}$ is contained in the $\mathfrak{m}$-regular set

$$L_{\mathfrak{t}}'' = \{\nu \in L_{\mathfrak{t}}: \tilde{\omega}_{\mathfrak{t}}(\nu) \neq 0\} \qquad \text{where} \quad \tilde{\omega}_{\mathfrak{t}}(\nu) = \prod_{\phi\in\Sigma_{\mathfrak{t}}{}^+} \langle \nu, \phi\rangle. \tag{4.1.1b}$$

Finally, define a function from $i\mathfrak{t}^*$ to the nonnegative integers by

$$p_\mathfrak{t}(\nu) = |\{\phi \in \Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+ : \langle \nu, \phi \rangle > 0\}|. \tag{4.1.1c}$$

For example, $p_\mathfrak{t}(\rho_\mathfrak{t}) = |\Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+|$.

Recall $U = Z_M(M^0)U^0$, so every $[\mu] \in \hat{U}$ decomposes

$$[\mu] = [\chi \otimes \mu^0] \qquad \text{where} \quad [\chi] \in Z_M(M^0)^\wedge \text{ and } [\mu^0] \in \hat{U}^0. \tag{4.1.2a}$$

Further $[\mu^0]$ is characterized by its highest weight. That highest weight can be expressed in the form $\nu + \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}$:

$$[\mu^0] = [\mu_\nu{}^0] \qquad \text{where} \quad \nu + \rho_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}} \text{ is its highest weight for } \Sigma_{\mathfrak{t},\mathfrak{u}}^+. \tag{4.1.2b}$$

We can now state our main result.

THEOREM 4.1.1. *Let $[\mu] \in \hat{U}$, say $[\mu] = [\chi \otimes \mu_\nu{}^0]$ as in* (4.1.2), *and suppose $\nu + \rho_\mathfrak{t} \in L_\mathfrak{t}''$. Let $\sigma \in \mathfrak{a}^*$ and recall the unitary representation $\pi_{\mu,\sigma}^\pm$ of G on the space $H_2{}^\pm(\mathscr{V}_{\mu,\sigma})$ of $\mathscr{V}_{\mu,\sigma}$-valued square integrable partially harmonic spinors on X.*

(1) *$[\pi_{\mu,\sigma}^\pm]$ is a finite sum of H-series classes $[\pi_{\chi,\beta,\sigma}]$ where $\beta \in L_\mathfrak{t}''$ with $\|\beta\|^2 = \|\nu + \rho_\mathfrak{t}\|^2$.*

(2) *The $[\pi_{\mu,\sigma}^\pm]$ have well defined distribution characters $\Theta_{\pi_{\mu,\sigma}^\pm}$, and*

$$\Theta_{\pi_{\mu,\sigma}^+} - \Theta_{\pi_{\mu,\sigma}^-} = (-1)^{p_\mathfrak{t}(\nu+\rho_\mathfrak{t})}\, \Theta_{\pi_{\chi,\nu+\rho_\mathfrak{t},\sigma}}. \tag{4.1.2}$$

(3) *There is a unique Weyl group element $w \in W(\mathfrak{m}_\mathbf{C}, \mathfrak{t}_\mathbf{C})$ such that $w\Sigma_{\mathfrak{t},\mathfrak{u}}^+ \subset \Sigma_\mathfrak{t}^+$ and $\langle w(\nu + \rho_\mathfrak{t}), \phi \rangle > 0$ for all $\phi \in \Sigma_\mathfrak{t}^+$. Fix the sign $\pm$ so that* $\det(w) = \pm 1$. *If*

$$\langle \nu + \rho_\mathfrak{t} - w^{-1}\rho_\mathfrak{t}, \phi \rangle \neq 0 \qquad \textit{for all } \phi \in \Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+ \tag{4.1.3}$$

then $H_2{}^\mp(\mathscr{V}_{\mu,\sigma}) = 0$ and $[\pi_{\mu,\sigma}^\pm]$ is the H-series class $[\pi_{\chi,\nu+\rho_\mathfrak{t},\sigma}]$.

If G is a connected linear semisimple group, and if the series under consideration is the discrete series (i.e., H is compact), then the result is due to R. Parthasarathy [11]. W. Schmid independently obtained some results in that case but did not write them up. Our proof combines ideas of Parthasarathy [11] and of Narasimhan–Okamoto [10] with certain induced representation techniques [13, Section 8.2]. The proof is distributed through the remainder of Section 4.

4.2. *Analysis Along the Fibres*

Note that K meets every component of M and choose a system $\{1 = k_1, k_2, \ldots, k_r\} \subset K$ of representatives of M modulo $M^{\dagger}$. Define

$$Y = \bigcup_{1 \leqslant i \leqslant r} Y_{k_i U} = M/U \qquad \text{and} \qquad \mathscr{W}_\mu = \mathscr{V}_{\mu,\sigma}|_Y. \tag{4.2.1}$$

Then we have unitary representations of M, independent of σ:

$$\tilde{\eta}_\mu^{\pm} \text{ on } L_2(\mathscr{S}^{\pm} \otimes \mathscr{W}_\mu) \qquad \text{and} \qquad \eta_\mu^{\pm} \text{ on } H_2^{\pm}(\mathscr{W}_\mu). \tag{4.2.2}$$

Recall the normal abelian subgroup $Z \subset Z_G(G^0)$ of G from (1.2.2), observe $Z \subset U$, and let $\zeta \in \hat{Z}$ such that $[\mu] \in \hat{U}_\zeta$. Cut Z to a subgroup of index 2 if necessary so that it is annihilated by the $s^{\pm} \cdot \tilde{\alpha}$. Then $\tilde{\eta}_\mu^{\pm}(mz) = \zeta(z)^{-1}\, \tilde{\eta}_\mu^{\pm}(m)$ for $m \in M$ and $z \in Z$.

LEMMA 4.2.3. *Let ${}^0\eta_\mu^{\pm}$ denote the sum of the irreducible subrepresentations of $\tilde{\eta}_\mu^{\pm}$. Then $\eta_\mu^{\pm}$ is a subrepresentation of ${}^0\eta_\mu^{\pm}$, and $[{}^0\eta_\mu^{\pm}]$ is a finite sum of ζ-discrete classes in $\hat{M}$.*

Proof. $\tilde{\eta}_\mu^{\pm}$ is contained in the left representation of M on $L_2(M/Z, \zeta) \otimes S^{\pm} \otimes V_\mu$, so its irreducible subrepresentations are just its ζ-discrete subrepresentations. Corollary 3.1.10 and the Plancherel Theorem for $\hat{M}_\zeta$ force $[\eta_\mu^{\pm}]$ to be a discrete sum of classes from $\hat{M}_{\zeta\text{-disc}}$, so $\eta_\mu^{\pm}$ is a subrepresentation of ${}^0\eta_\mu^{\pm}$.

Since ZM^0 has finite index in M, the restriction $({}^0\eta_\mu^{\pm})|_{ZM^0} = \sum n_i \eta_i$ discrete sum with $[\eta_i] \in (ZM^0)^{\wedge}_{\text{disc}}$. Since μ is finite-dimensional the restriction $\mu|_{ZU^0} = \sum m_j \mu_j$ finite sum with $m_j < \infty$ and $[\mu_j] \in (ZU^0)^{\wedge}$. Our extension [13, Theorem 2.5.1] of Kunze's Frobenius Reciprocity Theorem for square integrable representations [9] says that the multiplicities

$$m(\eta_i, \operatorname{Ind}_{ZU^0 \uparrow ZM^0}(\mu_j)) = m(\mu_j, \eta_i|_{ZU^0}).$$

Write m_{ij} for that multiplicity. Every $m_{ij} < \infty$; given j, only finitely many $m_{ij} > 0$. Note $n_i = \sum_j m(\eta_i, \operatorname{Ind}_{ZU^0 \uparrow ZM^0}(\mu_j)) = \sum_j m_{ij}$. Now $\sum_i n_i = \sum_j (\sum_i m_{ij}) < \infty$, so $({}^0\eta_\mu^{\pm})|_{ZM^0}$ is a finite sum from $(ZM^0)^{\wedge}_{\text{disc}}$. As ZM^0 has finite index in M, it follows that ${}^0\eta_\mu^{\pm}$ is a finite sum from $\hat{M}_{\text{disc}}$. Q.E.D.

If ϕ is a rapidly decreasing C^∞ function on M, we decompose $\phi = \int_{\hat{Z}} \phi_\xi \, d\xi$ in the sense

$$\phi(m) = \int_{\hat{Z}} \phi_\xi(m)\, d\xi \qquad \text{where} \quad \phi_\xi(m) = \int_Z \phi(mz)\, \xi(z)\, dz. \tag{4.2.4a}$$

Then $\phi_\xi \in L_2(M/Z, \xi)$ is smooth and rapidly decreasing on M/Z, and its projection to the ξ-discrete part of $L_2(M/Z, \xi)$ is the C^∞ function ${}^0\phi_\xi$ given by

$$ {}^0\phi_\xi(m) = \sum_{[\eta]\in \hat{M}_{\xi\text{-disc}}} \deg(\eta)\, \Psi_\eta(r_m\phi_\xi) \tag{4.2.4b}$$

where $[\eta]$ has formal degree $\deg(\eta)$ and distribution character Ψ_η, and $(r_m\phi_\xi)(x) = \phi_\xi(xm)$.

Denote orthogonal projection of $L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu)$ to the subspace ${}^0L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu)$ for ${}^0\eta_\mu{}^\pm$ by

$$E\colon L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu) \to {}^0L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu). \tag{4.2.5}$$

The analysis of ${}^0\eta_\mu{}^\pm$ is based on

PROPOSITION 4.2.6. *Let ϕ be a rapidly decreasing C^∞ function on M such that ϕ_ζ is U-finite. Define*

$$K_\phi{}^\pm(x, y) = \int_{U/Z} {}^0\phi_\zeta(xuy^{-1})[(s^\pm \cdot \tilde{\alpha}) \otimes \mu](u)\, d(uZ). \tag{4.2.7a}$$

Then $K_\phi{}^\pm\colon M \times M \to$ (linear transformations of $S^\pm \otimes V_\mu$) is well defined and C^∞. If $f \in L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu)$ then

$$[\tilde{\eta}_\mu{}^\pm(\phi) \cdot Ef](x) = \int_{M/Z} K_\phi{}^\pm(x, y) f(y)\, d(yZ). \tag{4.2.7b}$$

Further $\tilde{\eta}_\mu{}^\pm(\phi) \cdot E$ is an operator of finite rank on $L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu)$, and

$$\operatorname{trace}({}^0\eta_\mu{}^\pm)(\phi) = \int_{M/Z} \operatorname{trace} K_\phi{}^\pm(x, x)\, d(xZ) \tag{4.2.7c}$$

Proof. Since ${}^0\phi_\zeta$ is rapidly decreasing on M/Z we can differentiate (4.2.7a) under the integral sign; so $K_\phi{}^\pm$ is C^∞. Let $f \in L_2(\mathscr{S}^\pm \otimes \mathscr{W}_\mu)$. Calculating with convolutions over M/Z,

$$\begin{aligned}
&\int_{M/Z} K_\phi{}^\pm(x, y) f(y)\, d(yZ) \\
&\quad = \int_{M/Z} \left\{\int_{U/Z} [(s^\pm \cdot \tilde{\alpha}) \otimes \mu](u) \cdot {}^0\phi_\zeta(xuy^{-1}) f(y)\, d(uZ)\right\} d(yZ) \\
&\quad = \int_{U/Z} [(s^\pm \cdot \tilde{\alpha}) \otimes \mu](u) \cdot ({}^0\phi_\zeta * f)(xu)\, d(uZ) \\
&\quad = ({}^0\phi_\zeta * f)(x) = ({}^0\phi_\zeta * Ef)(x) = (\phi_\zeta * Ef)(x) = [\tilde{\eta}_\mu{}^\pm(\phi) \cdot Ef](x).
\end{aligned}$$

That proves (4.2.7b).

Since ϕ_ζ is U-finite, there is a finite subset $B \subset \hat{U}_\zeta$ such that $\tilde{\eta}_\mu^\pm(\phi) \cdot E$ has range contained in the sum of the β-primary subspaces of ${}^0L_2(\mathcal{S}^\pm \otimes \mathcal{W}_\mu)$, $\beta \in B$. Lemma 4.2.3 ensures that that sum of β-primary subspaces is finite dimensional. Thus $\tilde{\eta}_\mu^\pm(\phi) \cdot E$ has finite rank.

Choose finite subsets $\{v_i\}, \{w_j\}$ of the sum of the β-primary subspaces of ${}^0L_2(\mathcal{S}^\pm \otimes \mathcal{W}_\mu)$, $\beta \in B$ as above, consisting of C^∞ sections such that

$$(\tilde{\eta}_\mu^\pm(\phi) \cdot E)(f) = \sum_{i,j} \langle w_j, f\rangle v_i \qquad \text{for } f \in L_2(\mathcal{S}^\pm \otimes \mathcal{W}_\mu).$$

Then trace $K_\phi^\pm(x, x) = \sum_{i,j} \langle w_j(x), v_i(x)\rangle$ and we compute

$$\int_{M/Z} \text{trace } K_\phi^\pm(x, x)\, d(xZ) = \int_{M/U} \text{trace } K_\phi^\pm(x, x)\, d(xU)$$
$$= \sum_{i,j} \langle w_j, v_i\rangle = \text{trace } \tilde{\eta}_\mu^\pm(\phi) \cdot E = \text{trace}({}^0\eta_\mu^\pm)(\phi).$$

Q.E.D.

4.3. *Difference Formula Along the Fibres*

Retain the notation of Sections 4.1 and 4.2. It follows from Lemma 4.2.3 that $[\eta_\mu^\pm]$ has distribution character $\Psi_{\eta_\mu^\pm}$ that is a locally integrable function on M, analytic on the regular set M'' and determined by its restriction to $T \cap M''$. Write $\Psi_{\eta_{\chi,\beta}}$, $[\chi] \in Z_M(M^0)^\wedge$ and $\beta \in L_{\mathfrak{t}}''$, for the distribution character of $[\eta_{\chi,\beta}] \in \hat{M}_{\text{disc}}$. We are going to prove

PROPOSITION 4.3.1. $\Psi_{\eta_\mu^+} - \Psi_{\eta_\mu^-} = (-1)^{p_{\mathfrak{t}}(\nu+\rho_{\mathfrak{t}})}\, \Psi_{\eta_{\chi,\nu+\rho_{\mathfrak{t}}}}$.

The proof is a calculation with (4.2.7) and some character formulas. In order to simplify the calculation we first prove

LEMMA 4.3.2. *In proving Proposition* 4.3.1 *we may assume that* M *is connected.*

Proof. Let $\xi_\mu^\pm$ denote the representation of $M^\dagger$ on $H_2^\pm(\mathcal{V}_{\mu,\sigma}|_{Y_{1U}})$. Then (4.2.1) says $\eta_\mu^\pm = \text{Ind}_{M^\dagger\uparrow M}(\xi_\mu^\pm)$. By construction we have $\eta_{\chi,\nu+\rho_{\mathfrak{t}}} = \text{Ind}_{M^\dagger\uparrow M}(\chi \otimes \eta_{\nu+\rho_{\mathfrak{t}}})$. If $[\xi] \in \hat{M}^\dagger$ and Ψ_ξ is its distribution character, and if $[\eta] = \text{Ind}_{M^\dagger\uparrow M}(\xi)$, then $[\eta]$ has distribution character Ψ_η supported in $M^\dagger$ and given there by

$$\Psi_\eta(x) = \sum_{1 \leqslant i \leqslant r} \Psi_\xi(k_i x k_i^{-1}).$$

In proving Proposition 4.3.1 we thus may assume $M = M^\dagger$.

Suppose $M = M^{\dagger}$. Let $\eta^{\pm}_{\mu^0}$ denote the representation of M^0 on $H_2{}^{\pm}(\mathscr{V}_{\mu^0})$ where $\mathscr{V}_{\mu^0} \to Y_{1U}$ is the homogeneous bundle for $[\mu^0] \in \hat{U}^0$. The representation spaces satisfy $V_\mu = V_\chi \otimes V_{\mu^0}$, so $\mathscr{V}_{\mu,\sigma}|_{Y_{1U}} = V_\chi \otimes \mathscr{V}_{\mu^0}$ and it follows that $\eta_\mu{}^{\pm} = \chi \otimes \eta^{\pm}_{\mu^0}$. If $z \in Z_M(M^0)$ and $m \in M^0$ now

$$\Psi_{\eta_\mu{}^{\pm}}(zm) = \operatorname{trace} \chi(z)\, \Psi_{\eta^{\pm}_{\mu^0}}(m) \qquad \text{and} \qquad \Psi_{\eta_{\chi,\nu+\rho_{\mathfrak{t}}}}(zm) = \operatorname{trace} \chi(z)\, \Psi_{\eta_{\nu+\rho_{\mathfrak{t}}}}(m).$$

In proving Proposition 4.3.1 we may thus assume $M = M^0$. Q.E.D.

Proof of Proposition. Let ϕ be a C^∞ function on M that satisfies the conditions of Proposition 4.2.6. Define

$$J = \operatorname{trace}({}^0\eta_\mu{}^+)(\phi) - \operatorname{trace}({}^0\eta_\mu{}^-)(\phi). \tag{4.3.3}$$

To calculate J we need

$$\Delta_{\mathfrak{t}} = \prod_{\Sigma_{\mathfrak{t}}^+} (e^{\alpha/2} - e^{-\alpha/2}), \qquad \Delta_{\mathfrak{t},\mathfrak{u}} = \prod_{\Sigma_{\mathfrak{t},\mathfrak{u}}^+} (e^{\alpha/2} - e^{-\alpha/2}),$$
$$\Delta_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}} = \prod_{\Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

Evidently, if $t \in T$ then

$$|\Delta_{\mathfrak{t},\mathfrak{u}}(t)|^2 \Delta_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}(t) = (-1)^p \Delta_{\mathfrak{t},\mathfrak{u}}(t)\, \Delta_{\mathfrak{t}}(t) \text{ where } p = |\Sigma_{\mathfrak{t},\mathfrak{u}}^+|. \tag{4.3.4}$$

Now assume M connected, as is allowed by Lemma 4.3.2. Then U is connected, so the Weyl Character Formula for $\psi_\mu(u) = \operatorname{trace} \mu(u)$ is

$$\psi_\mu(t) = \Delta_{\mathfrak{t},\mathfrak{u}}(t)^{-1} \sum_{w \in W_U} \det(w)\, e^{w(\nu+\rho_{\mathfrak{t}})}(t) \qquad \text{for } U\text{-regular } t \in T.$$

The formula for the distribution character $\Psi_{\eta_{\nu+\rho_{\mathfrak{t}}}}$ of the discrete class $[\eta_{\nu+\rho_{\mathfrak{t}}}] \in \hat{M}$ now tells us

$$\Delta_{\mathfrak{t},\mathfrak{u}}(t)\, \psi_\mu(t) = (-1)^{q+p(\nu+\rho_{\mathfrak{t}})}\, \Delta_{\mathfrak{t}}(t)\, \Psi_{\eta_{\nu+\rho_{\mathfrak{t}}}}(t) \qquad \text{for } \ t \in T \cap M''. \tag{4.3.5}$$

where $q = |\Sigma_{\mathfrak{t}}^+|$.

We are ready to evaluate J. Using (4.2.7), then (3.2.4) and the Weyl Integration Formula on U, then (4.3.4) and (4.3.5), we calculate

$$\begin{aligned}
J &= \int_{M/Z} \left\{ \int_{U/Z} [\operatorname{trace} s^+(\tilde{\alpha}(u)) - \operatorname{trace} s^-(\tilde{\alpha}(u))]\, \psi_\mu(u) \cdot {}^0\phi_\zeta(xux^{-1})\, d(uZ) \right\} d(xZ) \\
&= \int_{M/Z} \left\{ \frac{1}{|W_U|} \int_{U/Z} d(uZ) \int_{T/Z} |\Delta_{\mathfrak{t},\mathfrak{u}}(t)|^2 \Delta_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}(t)\, \psi_\mu(t) \right. \\
&\qquad \left. \cdot\, {}^0\phi_\zeta(xutu^{-1}x^{-1})\, d(tZ) \right\} d(xZ) \\
&= (-1)^{p+q+p(\nu+\rho_{\mathfrak{t}})} \frac{1}{|W_U|} \\
&\qquad \cdot \int_{M/Z} \left\{ \int_{U/Z} d(uZ) \int_{T/Z} \Delta_{\mathfrak{t}}(t)^2\, \Psi_{\eta_{\nu+\rho_{\mathfrak{t}}}}\, {}^0\phi_\zeta(xutu^{-1}x^{-1})\, d(tZ) \right\} d(xZ).
\end{aligned}$$

Denote

$$F({}^0\phi_\zeta, t) = \Delta_t(t) \int_{M/Z} {}^0\phi_\zeta(xtx^{-1})\, d(xZ) \qquad \text{and} \qquad \Phi_{\nu+\rho_t} = \Delta_t \Psi_{\eta_{\nu+\rho_t}}.$$

Now the integration on U/Z drops out and

$$J = (-1)^{p+q+p(\nu+\rho_t)} \frac{1}{|W_U|} \int_{T/Z} F({}^0\phi_\zeta, t)\, \Phi_{\nu+\rho_t}(t)\, d(tZ).$$

Since ${}^0\phi_\zeta$ is finite under the center of the enveloping algebra of $\mathfrak{m}_C$, [5, Lemma 79] adapted to $L_2(M/Z, \zeta)$ implies

$$\Psi_{\eta_{\nu+\rho_t}}({}^0\phi_\zeta) = (-1)^{p+q} \frac{1}{|W_M|} \int_{T/Z} F({}^0\phi_\zeta, t)\, \Phi_{\nu+\rho_t}(t)\, d(tZ).$$

Thus, using $W_U = W_M$,

$$J = (-1)^{p(\nu+\rho_t)}\, \Psi_{\eta_{\nu+\rho_t}}({}^0\phi_\zeta) = (-1)^{p(\nu+\rho_t)}\, \Psi_{\eta_{\nu+\rho_t}}(\phi).$$

In summary we have shown

$$\Psi_{({}^0\eta_\mu{}^+)} - \Psi_{({}^0\eta_\mu{}^-)} = (-1)^{p(\nu+\rho_t)}\, \Psi_{\eta_{\chi,\nu+\rho_t}}. \tag{4.3.6}$$

Lemma 4.2.3 gives us an M-equivariant exact sequence

$$0 \longrightarrow H_2{}^+(\mathscr{W}_\mu) \longrightarrow {}^0L_2(\mathscr{S}^+ \otimes \mathscr{W}_\mu) \xrightarrow{D} {}^0L_2(\mathscr{S}^- \otimes \mathscr{W}_\mu) \longrightarrow H_2{}^-(\mathscr{W}_\mu) \longrightarrow 0.$$

It follows that

$$\Psi_{({}^0\eta_\mu{}^+)} - \Psi_{({}^0\eta_\mu{}^-)} = \Psi_{\eta_\mu{}^+} - \Psi_{\eta_\mu{}^-}.$$

In view of (4.3.6), this completes the proof of Proposition 4.3.1.
Q.E.D.

4.4. *Vanishing Theorem Along the Fibres*

Retain the notation of Sections 4.1 and 4.2. We are going to prove

PROPOSITION 4.4.1. *There is a unique element w of the complex Weyl group $W(\mathfrak{m}_C, \mathfrak{t}_C)$ such that*

$$w\Sigma_{\mathfrak{t},\mathfrak{u}}^+ \subset \Sigma_{\mathfrak{t}}{}^+ \quad \textit{and} \quad \langle w(\nu + \rho_t), \phi\rangle > 0 \quad \textit{for all } \phi \in \Sigma_{\mathfrak{t}}{}^+. \tag{4.4.2}$$

Define $\pm$ by $\det(w) = \pm 1$. *If $\langle \nu + \rho_t - w^{-1}\rho_t, \phi\rangle \neq 0$ for all $\phi \in \Sigma_{\mathfrak{t},\mathfrak{m}/\mathfrak{u}}^+$, then $H_2{}^{\mp}(\mathscr{W}_\mu) = 0$ and $[\eta_\mu{}^{\pm}] = [\eta_{\chi,\nu+\rho_t}] \in \hat{M}_{\zeta\text{-disc}}$.*

Proof. Since $\nu + \rho_{t,m/u}$ is the highest weight of a representation of U^0 we have $\langle \nu + \rho_{t,m/u}, \phi \rangle \geqslant 0$ for all $\phi \in \Sigma_{t,u}^+$. On the other hand $\langle \rho_{t,u}, \phi \rangle > 0$ for all $\phi \in \Sigma_{t,u}^+$. Now

$$\nu \in \{\nu'' \in L_t : \langle \nu'' + \rho_t, \phi \rangle > 0 \text{ for all } \phi \in \Sigma_{t,u}^+\}.$$

Let $W^1 = \{v \in W(\mathfrak{m}_C, \mathfrak{t}_C): v^{-1}\Sigma_{t,u}^+ \subset \Sigma_t^+\}$. Then [8, Lemma 6.4] the map $(\nu', v) \to v(\nu' + \rho_t) - \rho_t$ bijects

$$\{\nu' \in L_t : \langle \nu' + \rho_t, \phi \rangle > 0 \text{ for all } \phi \in \Sigma_t^+\} \times W^1$$

onto the set just mentioned as containing ν. Now there are unique elements $\nu' \in L_t$ and $v \in W^1$ such that

$$\langle \nu' + \rho_t, \phi \rangle > 0 \qquad \text{for all } \phi \in \Sigma_t^+ \qquad \text{and} \qquad v(\nu' + \rho_t) = \nu + \rho_t .$$

For the first assertion we set $w = v^{-1}$. Note that $\det(w) = \det(v)$ here, so $\det(v) = \pm 1$ with $\pm$ as in the statement of the Proposition. Note also that $v(\nu') = \nu + \rho_t - w^{-1}\rho_t$.

Suppose $\langle \nu + \rho_t - w^{-1}\rho_t, \phi \rangle \neq 0$ for all $\phi \in \Sigma_{t,m/u}^+$. Then $\langle v(\nu'), \phi \rangle \neq 0$ for all $\phi \in \Sigma_{t,m/u}^+$. In view of Theorem 3.1.8, the proof of R. Parthasarathy's vanishing theorem [11, Theorem 2] is valid for $H_2^{\mp}(\mathscr{V}_{\mu,\sigma}|_{Y_{1U}})$; so that space vanishes. Now

$$H_2^{\mp}(\mathscr{W}_\mu) = \sum_{1 \leqslant i \leqslant r} \eta_\mu^{\mp}(k_i) \cdot H_2^{\mp}(\mathscr{V}_{\mu,\sigma}|_{Y_{1U}}) = 0.$$

Finally, Proposition 4.3.1 forces $[\eta_\mu^{\pm}] = [\eta_{\chi,\nu+\rho_t}] \in \hat{M}_{\zeta\text{-disc}}$. Q.E.D.

4.5. *Proof of Main Theorem*

Retain the notation of Sections 4.1 and 4.2. The argument of Lemma 2.4.9 implies

$$\pi_{\mu,\sigma}^{\pm} = \mathrm{Ind}_{MAN \uparrow G}(\eta_\mu^{\pm} \otimes e^{i\sigma}). \tag{4.5.1}$$

$[\eta_\mu^{\pm}]$ is a finite sum of classes $[\eta_{\alpha,\beta}] \in \hat{M}_{\zeta\text{-disc}}$ by Lemma 4.2.3. Each $[\alpha] = [\chi] \in Z_M(M^0)^\wedge$ by the considerations of Lemma 4.3.2, and Corollary 3.1.10 implies that each $\| \beta \|^2 - \| \rho_t \|^2 = \| \nu + \rho_t \|^2 - \| \rho_t \|^2$. Now $[\eta_\mu^{\pm}]$ is a finite sum of classes $[\eta_{\chi,\beta}] \in \hat{M}_{\mathrm{disc}}$ with $\| \beta \|^2 = \| \nu + \rho_t \|^2$, and (4.5.1) says that $[\pi_{\mu,\sigma}^{\pm}]$ is the corresponding sum of H-series classes $[\pi_{\chi,\beta,\sigma}]$ of G. We have proved the first assertion of Theorem 4.1.1. The second and third assertions follow from Propositions 4.3.1 and 4.4.1 using (4.5.1) and the formula [13, Theorem 4.3.8] for induced characters. Q.E.D.

4.6. *Application to Geometric Realization*

Let $[\pi]$ be an element of the H-series of unitary representation classes of G. Express $[\pi] = [\pi_{\chi,\nu+\rho_t,\sigma}]$ where $[\chi] \in Z_M(M^0)^\wedge$, $\nu + \rho_t \in L_t''$ and $\sigma \in \mathfrak{a}^*$. We may assume $\langle \nu + \rho_t, \phi \rangle > 0$ for all $\phi \in \Sigma_{t,u}^+$, and then $\langle \nu + \rho_{t,m/u}, \phi \rangle \geqslant 0$ for all $\phi \in \Sigma_{t,u}^+$. Now let $[\mu^0] \in \hat{U}^0$ be the class with highest weight $\nu + \rho_{t,m/u}$ and define $[\mu] = [\chi \otimes \mu^0] \in \hat{U}$. Theorem 4.1.1 says that $[\pi]$ is a subrepresentation of either $[\pi_{\mu,\sigma}^+]$ or $[\pi_{\mu,\sigma}^-]$. If the mild condition (4.1.3) holds, it further identifies $[\pi]$ as one of the $[\pi_{\mu,\sigma}^\pm]$.

In summary: Theorem 4.1.1 gives an implicit realization of every H-series unitary representation class of G, and it gives an explicit realization of "most" of those classes.

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