JOSEPH A. WOLF

# LOCAL AND GLOBAL EQUIVALENCE FOR FLAT AFFINE MANIFOLDS WITH PARALLEL GEOMETRIC STRUCTURES*

## 0. Introduction

Euclidean space forms are usually studied under considerations of affine equivalence ([2], [3], [4], [8], [10]). In dimensions 2 and 3, one easily contrives *ad hoc* methods for refining the affine classification to an isometric classification [10, pp. 77–79 and pp. 123–124]. In this regard also see [5]. Here we give a general method for the isometric classification of complete flat riemannian manifolds in a fixed affine equivalence class. We then extend the method to a study of the similar question for complete flat affinely connected manifolds with parallel torsion tensor and an arbitrary family of parallel tensor fields. The latter could be a riemannian or pseudo-riemannian structure, a kaehler structure, a product structure, an absolute parallelism, or some combination.

## 1. Euclidean space forms

Fix a connected $n$-dimensional complete flat riemannian manifold $M$. It has universal riemannian covering $p:\mathbf{E}^n\to M$ where $\mathbf{E}^n$ is euclidean space, and this identifies $M$ with $D\backslash\mathbf{E}^n$ where $D$ is a properly discontinuous group of rigid motions acting freely on $\mathbf{E}^n$.

Identify $\mathbf{E}^n$ with real number space $\mathbf{R}^n$ by choice of an origin 0 and an orthonormal frame at 0. As usual, $\mathbf{GL}(n, \mathbf{R})$ denotes the group of all invertible linear transformations of $\mathbf{R}^n$, and $\mathbf{O}(n)$ is the orthogonal group. The *affine group* of $\mathbf{R}^n$ is the group $\mathbf{A}(n)=\mathbf{R}^n\cdot\mathbf{GL}(n, \mathbf{R})$ consisting of all

$$(t, g): x\to t+g(x),\ x\in\mathbf{R}^n;\quad \text{here}\quad t\in\mathbf{R}^n\quad\text{and}\quad g\in\mathbf{GL}(n, \mathbf{R}).$$

The *euclidean group* is the subgroup $\mathbf{E}(n)=\mathbf{R}^n\cdot\mathbf{O}(n)$; it consists of all the rigid motions.

Recall $M=D\backslash\mathbf{E}^n$ and note $D\subset\mathbf{E}(n)\subset\mathbf{A}(n)$. We denote normalizer by $N_{\mathbf{A}(n)}(D)=\{\alpha\in\mathbf{A}(n):\alpha D\alpha^{-1}=D\}$. For example, let $\mathbf{Z}^n$ denote the integer lattice in $\mathbf{R}^n$ and let $\mathbf{GL}(n, \mathbf{Z})$ denote the group of integral matrices of determinant $\pm 1$; then $N_{\mathbf{A}(n)}(\mathbf{Z}^n)=\mathbf{R}^n\cdot\mathbf{GL}(n, \mathbf{Z})$.

* Research partially supported by N.S.F. Grant GP-16651.

*Geometriae Dedicata* 2 (1973) 127–132. 

THEOREM 1. *The subset* $\mathbf{E}(n)\backslash\{\gamma\in\mathbf{A}(n)\colon \gamma D\gamma^{-1}\subset\mathbf{E}(n)\}/N_{\mathbf{A}(n)}(D)$ *of the double coset space* $\mathbf{E}(n)\backslash\mathbf{A}(n)/N_{\mathbf{A}(n)}(D)$, *is in bijective correspondence with the set of all isometry classes of riemannian manifolds that are affinely equivalent to* $M=D\backslash\mathbf{E}^n$. *The double coset* $\mathbf{E}(n)\cdot\gamma\cdot N_{\mathbf{A}(n)}(D)$ *corresponds to the isometry class of* $(\gamma D\gamma^{-1})\backslash\mathbf{E}^n$.

Every flat riemannian $n$-torus is affinely equivalent to $\mathbf{Z}^n\backslash\mathbf{E}^n$. If $\gamma=(t, g)\in\in\mathbf{A}(n)$ then $\gamma\mathbf{Z}^n\gamma^{-1}=g(\mathbf{Z}^n)\subset\mathbf{E}(n)$. Thus the special case $D=\mathbf{Z}^n$ of Theorem 1 is as follows; this is equivalent to [10, Lemma 3.5.11].

COROLLARY. *The double coset space* $\mathbf{O}(n)\backslash\mathbf{GL}(n,\mathbf{R})/\mathbf{GL}(n,\mathbf{Z})$ *is in bijective correspondence with the set of all isometry classes of flat riemannian n-tori. The double coset* $\mathbf{O}(n)\cdot g\cdot\mathbf{GL}(n,\mathbf{Z})$ *corresponds to the class of* $g(\mathbf{Z}^n)\backslash\mathbf{E}^n$.

*Proof.* Let $\kappa\in\mathbf{E}(n)$, $\gamma\in\mathbf{A}(n)$ with $\gamma D\gamma^{-1}\subset\mathbf{E}(n)$, and $\delta\in N_{\mathbf{A}(n)}(D)$. Then the manifolds $(\gamma D\gamma^{-1})\backslash\mathbf{E}^n$ and $(\kappa\gamma\delta D\delta^{-1}\gamma^{-1}\kappa^{-1})\backslash\mathbf{E}^n$ inherit complete flat riemannian structure from $\mathbf{E}^n$, and $\kappa$ induces an isometry of the first onto the second. Thus the correspondence of Theorem 1 is well defined.

Let $M_1$ be a riemannian manifold and $f\colon M\to M_1$ and affine equivalence (connection preserving diffeomorphism). Then $M_1$ is connected, flat and complete, so $\mathbf{E}^n$ is its universal riemannian covering manifold. Let $D_1$ be the group of deck (covering) transformations. Then $D_1\subset\mathbf{E}(n)$ because the covering is riemannian. Now $f_1$ lifts to an affine transformation $\gamma_1\in\mathbf{A}(n)$ such that $\gamma_1 D\gamma_1^{-1}=D_1\subset\mathbf{E}(n)$, so $\mathbf{E}(n)\cdot\gamma_1\cdot N_{\mathbf{A}(n)}(D)$ corresponds to the isometry class of $M_1$. Our correspondence is proved surjective.

Let $M_2$ be another riemannian manifold, $f_2\colon M\to M_2$ an affine equivalence, $D_2\subset\mathbf{E}(n)$ the group of deck transformations of the universal riemannian covering $\mathbf{E}^n\to M_2$, and $\gamma_2\in\mathbf{A}(n)$ a lift of $f_2$. Now suppose that there is an isometry $k\colon M_1\to M_2$. Then $k$ lifts to a rigid motion $\kappa\in\mathbf{E}(n)$ with $\kappa D_1\kappa^{-1}=D_2$. Thus $\gamma_2 D\gamma_2^{-1}=D_2=\kappa\gamma_1 D\gamma_1^{-1}\kappa^{-1}$, so $\gamma_2\in\kappa\gamma_1\cdot N_{\mathbf{A}(n)}(D)$. Now $\mathbf{E}(n)\cdot\gamma_2\cdot\cdot N_{\mathbf{A}(n)}(D)=\mathbf{E}(n)\cdot\gamma_1\cdot N_{\mathbf{A}(n)}(D)$, so our correspondence is bijective.

Q.E.D.

## 2. Flat affine manifolds

Let $G$ be a Lie group. We view its Lie algebra $\mathfrak{g}$ as the set of all tangent vector fields on $G$ invariant by all left translations $t\colon x\to tx$ $(t, x\in G)$. There is a unique affine connection $\Gamma_G$ of $G$ such that the fields $\xi\in\mathfrak{g}$ all are parallel. $\Gamma_G$ is complete, is flat (curvature tensor zero,) and has parallel torsion tensor given by $T(\xi,\eta)=-[\xi,\eta]$. A tensor field $S$ on $G$ is $\Gamma_G$-parallel if and only if it is invariant by every left translation. We refer to $\Gamma_G$ as the *left translation connection* on $G$. For example, $\Gamma_{\mathbf{R}^n}$ is the euclidean connection.

Let $\mathbf{Aut}(G)$ denote the Lie group of all continuous automorphisms of $G$. The semidirect product $\mathbf{A}(G)=G\cdot\mathbf{Aut}(G)$ is the manifold $G\times\mathbf{Aut}(G)$ with the group law $(t, g)(u, h)=(t\cdot g(u), gh)$. $\mathbf{A}(G)$ acts on $G$ by $(t, g): x\to t\cdot g(x)$. $\mathbf{A}(G)$ is the *affine group* of $G$ in the sense that it consists of all connection-preserving diffeomorphisms of $(G, \Gamma_G)$. For example, $\mathbf{Aut}(\mathbf{R}^n)=\mathbf{GL}(n, \mathbf{R})$ and $\mathbf{A}(\mathbf{R}^n)=\mathbf{A}(n)$. Compare [1] and [9].

Let $D$ be a subgroup of $\mathbf{A}(G)$. Suppose that $D$ acts freely and properly discontinuously on $G$, i.e. that $G\to D\diagdown G$ is a covering space. Then $D\diagdown G$ inherits a connection $\Gamma_{D\setminus G}$ from $\Gamma_G$. Evidently $(D\diagdown G, \Gamma_{D\setminus G})$ is a complete affinely connected manifold with vanishing curvature and parallel torsion.

Let $(M, \Gamma)$ be any connected, complete flat affinely connected manifold with parallel torsion tensor. Using the Lie algebra structure obtained from the negative of the torsion tensor, one obtains a simply connected Lie group $G$, a subgroup $D\subset\mathbf{A}(G)$ that acts freely and properly discontinuously, and an affine equivalence $(M, \Gamma)\to D\diagdown(G, \Gamma_G)=(D\diagdown G, \Gamma_{D\setminus G})$. This follows directly from the Cartan-Ambrose-Hicks theorem ([6]; or see [10, §1.9]). A slightly complicated derivation is given in [7]. The method of [10, §1.9] also gives us

THEOREM 2. *Let $p_i:(G_i, \Gamma_{G_i})\to D_i\diagdown(G_i, \Gamma_{G_i})=(M_i, \Gamma_i)$ be universal affine covering spaces where the $G_i$ are simply connected Lie groups.*

(1) *Some open set in $(M_1, \Gamma_1)$ is affinely equivalent to some open set in $(M_2, \Gamma_2)$ if, and only if, $G_1\cong G_2$.*

(2) *Let $x_i\in G_i$, $m_i=p_i(x_i)$, and $\phi: G_1\cong G_2$. If $U$ is a small connected neighborhood of* 1 *in $G_1$ then $f(p_1(x_1\cdot z))=p_2(x_2\cdot\phi(z))$, $z\in U$, is an affine equivalence of a neighborhood of $m_1$ to a neighborhood of $m_2$. Every affine equivalence $(U_1, m_1)\to(U_2, m_2)$ of connected neighborhoods is obtained this way. Finally $f$ extends to an affine equivalence $(M_1, \Gamma_1)\to(M_2, \Gamma_2)$ if and only if, viewing $x_i=(x_i, 1)\in\mathbf{A}(G_i)$, $x_2^{-1}D_2x_2=\phi x_1^{-1}D_1x_1\phi^{-1}$.*

## 3. Parallel geometric structure

We are going to study triples $(M, \Gamma, P)$ where

(3.1) $M$ is a connected manifold,

(3.2) $\Gamma$ is a flat complete connection with parallel torsion, and

(3.3) $P$ is a family $\{\pi_i\}_{i\in I}$ of $\Gamma$-parallel tensor fields on $M$.

Complete connected flat riemannian manifold is the case where $P$ consists of a riemannian metric and $\Gamma$ is Levi-Cività connection.

Let $(M', \Gamma', P')$ and $(M'', \Gamma'', P'')$ satisfy the conditions (3.1) through (3.3) with the same index set $I$. By *local equivalence* we mean an affine equivalence

$$f:(U', \Gamma'|_{U'}) \to (U'', \Gamma''|_{U''})$$

of open sets such that $f(\pi_i' \mid_{U'}) = \pi_i'' \mid_{U''}$ for every $i \in I$. By *equivalence* or *global equivalence* we mean (of course) that, in addition, $U' = M'$ and $U'' = M'$.

Now fix a triple $(M, \Gamma, P)$ that satisfies (3.1.) through (3.3). Express $(M, \Gamma) = D \backslash (G, \Gamma_G)$ as in §2, where $G$ is simply connected Lie group and $D \subset \mathbf{A}(G)$. For $i \in I$ let $\tilde{\pi}_i$ denote the $\Gamma_G$-parallel (i.e., left-invariant) tensor field on $G$ that projects to $\pi_i$; denote $P_G = \{\tilde{\pi}_i : i \in I\}$. Now

$$(M, \Gamma, P) = D \backslash (G, \Gamma_G, P_G)$$

with $D$ contained in

$$\mathbf{A}(G:P) = \{\gamma \in \mathbf{A}(G) : \gamma(\tilde{\pi}_i) = \tilde{\pi}_i \quad \text{for all} \quad i \in I\}.$$

In the flat riemannian case, $\mathbf{A}(G:P)$ is $\mathbf{E}(n)$.

THEOREM 3. *The subset* $\mathbf{A}(G:P) \backslash \{\gamma \in \mathbf{A}(G) : \gamma D \gamma^{-1} \subset \mathbf{A}(G:P)\} / N_{\mathbf{A}(G)}(D)$ *of the double coset space* $\mathbf{A}(G:P) \backslash \mathbf{A}(G) / N_{\mathbf{A}(G)}(D)$, *is in bijective correspondence with the set of all equivalence classes of triples that* (i) *satisfy* (3.1) *through* (3.3), (ii) *are affinely equivalent to* $(M, \Gamma)$, *and* (iii) *are locally equivalent to* $(M, \Gamma, P)$. *The double coset* $\mathbf{A}(G:P) \cdot \gamma \cdot N_{\mathbf{A}(G)}(D)$ *corresponds to the equivalence class of* $(\gamma D \gamma^{-1}) \backslash (G, \Gamma_G, P_G)$.

*Proof.* We imitate the proof of Theorem 1. Let $\kappa \in \mathbf{A}(G:P)$, $\gamma \in \mathbf{A}(G)$ with $\gamma D \gamma^{-1} \subset \mathbf{A}(G:P)$, and $\delta \in N_{\mathbf{A}(G)}(D)$. Then $(\gamma D \gamma^{-1}) \backslash (G, \Gamma_G, P_G)$ and $(\kappa \gamma \delta D \delta^{-1} \gamma^{-1} \kappa^{-1}) \backslash (G, \Gamma_G, P_G)$ have the required structure and are equivalent by $\kappa$. Thus the correspondence is well defined.

Let $(M', \Gamma', P')$ satisfy (3.1) through (3.3) and be locally equivalent to $(M, \Gamma, P)$. As $(M', \Gamma')$ is complete, connected and locally affine equivalent to $(M, \Gamma)$, we have $(M', \Gamma') = D^* \backslash (G, \Gamma_G)$ for appropriate $D^* \subset \mathbf{A}(G)$. Define $P^* = \{\tilde{\pi}_i' : i \in I\}$ where $\tilde{\pi}_i'$ is the lift of $\pi_i'$ from $M'$ to $G$. According to Theorem 2, the local equivalence of $(M', \Gamma', P')$ with $(M, \Gamma, P)$ has lift $F \in \mathbf{A}(G)$ such that each $F(\tilde{\pi}_i') = \tilde{\pi}_i$. Now $F$ induces an equivalence of $(M', \Gamma', P')$ with $(FD^*F^{-1}) \backslash (G, \Gamma_G, P_G)$. Thus we have expressed $(M', \Gamma', P') = D' \backslash (G, \Gamma_G, P_G)$ where, necessarily, $D' \subset \mathbf{A}(G:P)$. Now let $f' : (M, \Gamma) \to (M', \Gamma')$ affine equivalence. Then $f'$ has lift $\gamma' \in \mathbf{A}(G)$ such that $\gamma' D \gamma'^{-1} = D' \subset \mathbf{A}(G:P)$. Now $\mathbf{A}(G:P) \cdot \gamma' \cdot N_{\mathbf{A}(G)}(D)$ corresponds to the equivalence class of $(M', \Gamma', P')$. Our correspondence is proved surjective.

Let $(M'', \Gamma'', P'')$ be another triple that satisfies (3.1) through (3.3), is affinely equivalent to $(M, \Gamma)$, and is locally equivalent to $(M, \Gamma, P)$. As above, $(M'', \Gamma'', P'') = D'' \backslash (G, \Gamma_G, P_G)$, and we have $\gamma'' \in \mathbf{A}(G)$, such that $\gamma'' D \gamma''^{-1} = D'' \subset \mathbf{A}(G:P)$. Suppose that we have an equivalence

$$k: (M', \Gamma', P') \to (M'', \Gamma'', P'').$$

Then $k$ lifts to an element $\kappa \in \mathbf{A}(G:P)$ such that $\kappa D' \kappa^{-1} = D''$. Now $\gamma'' D \gamma''^{-1} = = D'' = \kappa \gamma' D \gamma'^{-1} \kappa^{-1}$, so $\gamma'' \in \kappa \gamma' \cdot N_{\mathbf{A}(G)}(D)$. Now $\mathbf{A}(G:P) \cdot \gamma'' \cdot N_{\mathbf{A}(G)}(D) = = \mathbf{A}(G:P) \cdot \gamma' \cdot N_{\mathbf{A}(G)}(D)$, so our correspondence is bijective.

Q.E.D.

The flat connection $\Gamma$ on $M = D \backslash G$ is derived from an absolute parallelism precisely when $D$ acts by pure translations ([11, Proposition 2.5]; or see [6]). In that case $D \subset G \subset \mathbf{A}(G)$, we compute

$$\begin{aligned}(t, g)\,(d, 1)\,(t, g)^{-1} &= (t \cdot g\,(d), g)\,(g^{-1}\,(t^{-1}), g^{-1}) = \\ &= (t \cdot g\,(d) \cdot t^{-1}, 1)\end{aligned}$$

and see $(t, g) D (t, g)^{-1} = t \cdot g(D) \cdot t^{-1} \subset G \subset \mathbf{A}(G:P)$. Now define $\mathbf{Aut}(G:P) = \{g \in \mathbf{Aut}(G): g(\tilde{\pi}_i) = \tilde{\pi}_i \text{ for every } i \in I\}$, so that $\mathbf{A}(G:P) = G \cdot \mathbf{Aut}(G:P)$ semi-direct product. Then Theorem 3 specializes as follows.

COROLLARY. *Suppose that $\Gamma$ is the connection of an absolute parallelism on $M$. Then the double coset space* $\mathbf{Aut}(G:P) \backslash \mathbf{Aut}(G) / N_{\mathbf{Aut}(G)}(D)$ *is in bijective correspondence with the set of all equivalence classes of triples that* (i) *satisfy* (3.1) *through* (3.3), (ii) *are affinely equivalent to* $(M, \Gamma)$, *and* (iii) *are locally equivalent to* $(M, \Gamma, P)$. *The double coset* $\mathbf{Aut}(G:P) \cdot g \cdot N_{\mathbf{Aut}(G)}(D)$ *corresponds to the equivalence class of* $g(D) \backslash (G, \Gamma_G, P_G)$.

In case $G = \mathbf{R}^n$, Theorem 3 and its Corollary specialize to Theorem 1 and its Corollary.

## BIBLIOGRAPHY

[1] L. Auslander, 'A Fixed Point Theorem for Nilpotent Lie Groups', *Proc. Amer. Math. Soc.* **9** (1958), 822–823.
[2] L. Bieberbach, 'Über die Bewegungsgruppen der euklidischen Raume I', *Math. Annalen* **70** (1911), 297–336; II, *Ibid.* **72** (1912), 400–412.
[3] J. J. Burckhardt, *Die Bewegungsgruppen der Kristallographie*, second edition, Birkhäuser Verlag, Basel, 1966.
[4] W. Hantzsche und H. Wendt, 'Driedimensionale euklidische Raumformen', *Math. Annalen* **110** (1935), 593–611.
[5] H. G. Helfenstein, 'Local Isometries of Flat Tori', *Pacific J. Math.* **32** (1970), 113–117.
[6] N. J. Hicks, 'A Theorem on Affine Connexions', *Ill. J. Math.* **3** (1959), 242–254.
[7] F. W. Kamber and Ph. Tondeur, 'Flat Manifolds With Parallel Torsion', *J. Differential Geometry* **4** (1968), 385–389.
[8] W. Nowacki, 'Die euklidischen, driedimensionalen geschlossenen und offenen Raumformen', *Comment. Math. Helv.* **7** (1934), 81–93.

[9] J. A. Wolf, 'The Affine Group of a Lie Group', *Proc. Amer. Math. Soc.* **14** (1963), 352–353.
[10] J. A. Wolf, *Spaces of Constant Curvature*, second edition, J. A. Wolf, Berkeley, 1972.
[11] J. A. Wolf, 'On the Geometry and Classification of Absolute Parallelisms I', *J. Differential Geometry* **6** (1972), 317–342.

*Author's address:*

Joseph A. Wolf,
Department of Mathematics,
University of California,
*Berkeley*, Calif. 94720
U.S.A.

(Received December 3, 1972)