GEOMETRIC REALIZATIONS OF REPRESENTATIONS OF REDUCTIVE LIE GROUPS

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1. General idea. Let G be a reductive Lie group, $H = T_H \times A_H$ a Cartan subgroup, and $P_H = MAN$ a cuspidal parabolic subgroup associated to H. We find complex manifolds X on which G acts, and certain orbits $Y_H = G(x_H) \subset X$, such that P_H is the G-stabilizer of the maximal complex analytic piece $S_{[x_H]}$ of Y_H that passes through x_H. This is done so that the isotropy subgroup of G at x_H is UAN with $T \subset U \subset M$, and a certain quotient U/Z is compact. If $[\mu] \in \hat{U}$ and $e^{i\sigma} \in \hat{A}$ then $[\mu \otimes e^{i\sigma}] \in (UAN)^\wedge$ defines a G-homogeneous Hermitian vector bundle $\mathscr{V}_{\mu,\sigma} \to Y_H$ that is holomorphic over the complex analytic pieces. Then G acts on the space $H_2^{0,q}(\mathscr{V}_{\mu,\sigma})$ of L_2 partially harmonic $(0, q)$-forms with values in $\mathscr{V}_{\mu,\sigma}$, by a unitary representation $\pi^q_{\mu,\sigma}$. Roughly speaking we realize every H-series representation of G by the $\pi^q_{\mu,\sigma}$. The relative discrete series, which is an interesting special case, plays a key role.

2. The flag manifold orbits. We work under the following fixed hypotheses. G is a reductive Lie group, i.e., its Lie algebra $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1$ with $\mathfrak{c}$ central and $\mathfrak{g}_1$ semisimple. Further

(2.1) if $g \in G$ then $\mathrm{ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$.

Finally, G has a closed normal abelian subgroup Z such that

(2.2a) Z centralizes the identity component G^0 and $|G/ZG^0| < \infty$,

(2.2b) $Z \cap G^0$ is cocompact in the center Z_{G^0} of G^0.

Let $\bar{G} = G^0/Z_{G^0}$ and $\bar{G}_C$ its complexification. If $\bar{P}$ is a parabolic subgroup of $\bar{G}_C$ then by (2.1), G acts on the complex flag manifold $X = \bar{G}_C/\bar{P}$ by: $g(\bar{x}\bar{P})$ is the point at which $\bar{G}_C$ has isotropy group $\mathrm{ad}(g)\,\mathrm{ad}(\bar{x})\,\bar{P}$. This action is holomorphic.

Fix a Cartan subgroup $H \subset G$. One can construct pairs (X, x_H) such that

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$x_H \in X$ complex flag, with the following properties: The G-normalizer $N_{[x_H]}$ of the holomorphic arc component (maximal complex analytic piece) $S_{[x_H]}$ of $G(x_H)$ through x_H has the same Lie algebra as a cuspidal parabolic subgroup $P_H = MAN$ associated to H. Further $S_{[x_H]}$ has an $N_{[x_H]}$-invariant positive Radon measure. Finally G has isotropy group UAN at x_H with $T \subset U \subset M$ and U/Z compact. For example, one could take $\bar{P}$ to be a Borel subgroup of $\bar{G}_C$.

We remark that G permutes the holomorphic arc components of $G(x_H)$, and that the component through gx_H (which is $S_{[gx_H]} = gS_{[x_H]}$) has G-normalizer $\mathrm{ad}(g)\, N_{[x_H]}$.

3. Partially harmonic L_2 forms. Let $[\mu] \in \hat{U}$ and $\sigma \in \mathfrak{a}^*$. Denote $\varrho(\alpha) = \frac{1}{2}\mathrm{trace}_{\mathfrak{n}}(\mathrm{ad}\,\alpha)$. Define a representation of UAN on the space V_μ of $[\mu]$ by $\gamma_{\mu,\sigma}(uan) = e^{i\sigma+\varrho}(a)\,\mu(u)$. Then we have

$$(3.1) \qquad p : \mathscr{V}_{\mu,\sigma} \to G/UAN = G(x_H) \text{ associated complex vector bundle.}$$

There is a unique assignment of complex structures to the pieces $p^{-1} S_{[gx_H]}$, stable under G, such that $\mathscr{V}_{\mu,\sigma}|_{S_{[gx_H]}}$ is a holomorphic vector bundle.

Let $\mathscr{T} \to G(x_H)$ be the complex G-homogeneous bundle such that each $\mathscr{T}|_{S_{[gx_H]}}$ is the holomorphic tangent bundle there. By *partially smooth* (p, q)*-form* with values in $\mathscr{V}_{\mu,\sigma}$ we mean a measurable section of $\mathscr{V}_{\mu,\sigma} \otimes \Lambda^p \mathscr{T}^* \otimes \Lambda^q \overline{\mathscr{T}}^*$ that is C^∞ over each holomorphic arc component. Let $A^{p,q}(\mathscr{V}_{\mu,\sigma})$ denote the space of all such forms. The Dolbeault operator of X specifies operators $\bar{\partial}: A^{p,q}(\mathscr{V}_{\mu,\sigma}) \to A^{p,q+1}(\mathscr{V}_{\mu,\sigma})$. Using K-invariant metrics, where K is the fixed point set of a Cartan involution that stabilizes H, we get Hodge-Kodaira maps

$$A^{p,q}(\mathscr{V}_{\mu,\sigma}) \xrightarrow{\#} A^{n-p,n-q}(\mathscr{V}^*_{\mu,\sigma}) \xrightarrow{\tilde{\#}} A^{p,q}(\mathscr{V}_{\mu,\sigma})$$

where $n = \dim_C S_{[x_H]}$. That specifies a pre-Hilbert space

$$A_2^{p,q}(\mathscr{V}_{\mu,\sigma}) = \left\{ \omega \in A^{p,q}(\mathscr{V}_{\mu,\sigma}) : \int_{K/Z} \left(\int_{S[kx_H]} \omega \bar{\wedge} \# \omega \right) d(kZ) < \infty \right\}.$$

$L_2^{p,q}(\mathscr{V}_{\mu,\sigma})$ is the Hilbert space completion. The partial Hodge-Kodaira-Laplace operator,

$$(3.2) \qquad \square = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \qquad \bar{\partial}^* = -\tilde{\#}\bar{\partial}\#,$$

is essentially selfadjoint from domain $\{\omega \in A^{p,q}(\mathscr{V}_{\mu,\sigma}) : \mathrm{supp}(\omega) \text{ compact}\}$. Its kernel

$$(3.3) \qquad H_2^{p,q}(\mathscr{V}_{\mu,\sigma}) = \{\omega \in L_2^{p,q}(\mathscr{V}_{\mu,\sigma}) : \square^*(\omega) = 0\}$$

is the space of *square-integrable partially harmonic (p, q)-forms* with values in $\mathscr{V}_{\mu,\sigma}$. G acts there by a unitary representation $\pi^{p,q}_{\mu,\sigma}$. We write $\pi^q_{\mu,\sigma}$ for $\pi^{0,q}_{\mu,\sigma}$.

4. Main theorem. Following the notation used in the preceding article, † denotes the elements that give rise to inner automorphisms: $G^\dagger = Z_G(G^0)\,G^0$, $M^\dagger = Z_M(M^0)\,M^0$ and $U^\dagger = Z_U(U^0)\,U^0$. Let $L''_{\mathfrak{t}}$ denote $\{\nu \in i\mathfrak{t}^* : e^\nu$ defined on T^0 and $\mathfrak{m}$-regular$\}$. We are interested in the classes of the $\mu_{\chi,\nu} = \mathrm{Ind}_{U^\dagger \uparrow U}(\chi \otimes \mu_\nu)$ where $\mu_\nu \in \hat{U}^0$ has highest weight ν and $\chi \in Z_U(U^0)^\wedge_\xi$, $\xi = e^\nu|_{\mathrm{center}(U^0)}$. Note that $\mu_{\chi,\nu}$ is irreducible if $\nu \in L''_{\mathfrak{t}}$.

4.1. THEOREM. *Let $[\mu_{\chi,\nu}] \in \hat{U}$ as above where $\nu + \varrho_{\mathfrak{t}} \in L''_{\mathfrak{t}}$. Let $\sigma \in \mathfrak{a}^*$ and $\pi^q_{\chi,\nu,\sigma}$ be the representation of G on $H^{0,q}_2(\mathscr{V}_{\mu_{\chi,\nu},\sigma})$.*

1. *The irreducible subrepresentations of $\pi^q_{\chi,\nu,\sigma}$ are just its constituents equivalent to irreducible subrepresentations of H-series representations of G. Let $\Theta^H_{\chi,\nu,\sigma,q}$ denote the sum of their distribution characters. Then, in the notation of the preceding article,*

$$\sum_{q \geq 0} (-1)^q\, \Theta^H_{\chi,\nu,\sigma,q} = (-1)^{n + q_H(\nu+\varrho_{\mathfrak{t}})}\, \Theta_{\pi_{\chi,\nu+\varrho_{\mathfrak{t}},\sigma}}. \tag{4.2}$$

2. *There is a constant $b_H \geq 0$ dependent only on $[\mathfrak{m}, \mathfrak{m}]$ such that if $|\langle \nu + \varrho_{\mathfrak{t}}, \psi \rangle| > b_H$ for all $\psi \in \Sigma^+_{\mathfrak{t}}$, and if $q \neq q_H(\nu + \varrho_{\mathfrak{t}})$, then $H^{0,q}_2(\mathscr{V}_{\mu_{\chi,\nu},\sigma}) = 0$.*
3. *If q_0 is an integer such that $q \neq q_0$ implies $H^{0,q}_2(\mathscr{V}_{\mu_{\chi,\nu},\sigma}) = 0$, then $[\pi^{q_0}_{\chi,\nu,\sigma}]$ is the H-series class $[\pi_{\chi,\nu+\varrho_{\mathfrak{t}},\sigma}]$.*

The rest of this article is a brief sketch of the idea of proof of Theorem 4.1.

5. Reduction to discrete series. Let $\eta^q_{\chi,\nu}$ denote the (unitary) representation of M on $H^{0,q}_2(\mathscr{V}_{\mu_{\chi,\nu}})$ where $\mathscr{V}_{\mu_{\chi,\nu}} = \mathscr{V}_{\mu_{\chi,\nu},\sigma}\,|_{M(x_H)}$. One can prove

$$\pi^q_{\chi,\nu,\sigma} = \mathrm{Ind}_{P_H \uparrow G}(\eta^q_{\chi,\nu} \otimes e^{i\sigma}). \tag{5.1}$$

The Plancherel theorem (3.1.3) of the preceding article combines with (5.1) to prove the assertion on the irreducible constituents of $\pi^q_{\chi,\nu,\sigma}$ in Theorem 4.1. If one knows the corresponding discrete series result for the $\eta^q_{\chi,\nu}$, then Theorem 4.1 follows by standard H-series considerations.

6. Idea of proof for discrete series. Considerations are reduced to the case where H/Z is compact. Thus $G(x_H)$ is an open submanifold of X with a G-invariant Hermitian metric, and $\pi^q_{\chi,\nu,\sigma}$ is properly written $\pi^q_{\chi,\nu}$.

One checks that $\pi^q_{\chi,\nu}$ is induced from the corresponding representation of $G^\dagger$.

This reduces Theorem 4.1 from G to $G^{\dagger}$. There $\pi^q_{\chi,\nu}=\chi\otimes\pi^q_\nu$ where π^q_ν is the corresponding representation of G^0. In summary we may assume G connected and examine its action π^q_ν on $H_2^{0,q}(\mathscr{V}_{\mu_\nu})$.

We may assume $x_H=1\cdot\bar{P}\in\bar{G}_C/\bar{P}=X$. Root orderings give a Borel subgroup $\bar{B}\subset\bar{P}$ of $\bar{G}_C$. Let $y_H=1\cdot\bar{B}\in\bar{G}_C/\bar{B}=Y$. The holomorphic fibration $Y\to X$ gives a proper holomorphic fibration $G(y_H)\to G(x_H)$. Let $\mathscr{L}_\nu\to G(y_H)$ be the holomorphic line bundle for $e^\nu\in\hat{H}$. An L_2-version of the Leray spectral sequence, using the Borel-Weil theorem extended to U/H, gives $H_2^{0,q}(\mathscr{L}_\nu)\cong H_2^{0,q}(\mathscr{V}_{\mu_\nu})$ unitary equivalence. These reduce Theorem 4.1 further to the case $X=\bar{G}_C/\bar{B}$ and $U=H$.

In the case to which we are reduced, cohomology consisting of the elements of $H_2^{0,q}(\mathscr{L}_\nu)$ can be compared with Lie algebra cohomology. The alternating sum formula (4.2) can then be extracted.

The vanishing theorem (part 2 of Theorem 4.1) is a Lie algebra computation of Griffiths and Schmid.

In the case considered (after our reductions) in part 3 of Theorem 4.1, the alternating sum formula shows that $\pi^{q_0}_\nu$ has relative discrete series component $\pi_{\nu+\varrho}$. A consequence of the Plancherel Theorem (see Corollary 3.6.1 of the preceding article) eliminates other constituents from the direct integral expression of $\pi^{q_0}_\nu$. Thus $\pi^{q_0}_\nu=\pi_{\nu+\varrho}$.

7. Remark on harmonic spinors. One can also follow these considerations with L_2 spinors killed by Dirac operators. The vanishing theorem (Parthasarathy) is better, but the result is not so geometric.

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