THE SPECTRUM OF A REDUCTIVE LIE GROUP

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Harish-Chandra's constructions of various series of representations, and his Plancherel formula, apply (roughly speaking) to those reductive Lie groups G such that the analytic subgroup for the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ has finite center. See Peter Trombi's summary just preceding. Here I want to indicate the extension of that work to a class of reductive groups which includes all semisimple groups and is stable under passage to the reductive part of a cuspidal parabolic subgroup. The extension is definitive for construction of the various series. However, it is provisional for the Plancherel theorem; when the details of Harish-Chandra's work become available his method should extend to give a sharper result with less effort.

1. Relative discrete series

1.1. Notion of relative discrete series. Let G be a unimodular locally compact group and Z a closed normal abelian subgroup. Given a unitary character $\zeta \in \hat{Z}$ we have the representation space

$$L_2(G/Z, \zeta) = \left\{ f : G \to \mathbf{C} : f(gz) = \zeta(z)^{-1} f(g), \forall z \in Z, g \in G \text{ and } \int_{G/Z} |f(g)|^2 \, d(gZ) < \infty \right\}$$

for $l_\zeta = \mathrm{Ind}_{Z \uparrow G}(\zeta)$. Evidently $L_2(G) = \int_{\hat{Z}} L_2(G/Z, \zeta)\, d\zeta$ and G has left regular representation $\int_{\hat{Z}} l_\zeta \, d\zeta$.

$\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G. If $\zeta \in \hat{Z}$ denote $\hat{G}_\zeta = \{[\pi] \in \hat{G} : \zeta \text{ is a summand of } \pi|_Z\}$. A class $[\pi] \in \hat{G}$ is *ζ-discrete* if π is equivalent to a subrepresentation of l_ζ. The ζ-discrete classes form the

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ζ-discrete series $\hat{G}_{\zeta\text{-disc}} \subset \hat{G}_\zeta \subset \hat{G}$. The *relative* (to Z) *discrete series* is $\hat{G}_{\text{disc}} = \bigcup_{\zeta \in \hat{Z}} \hat{G}_{\zeta\text{-disc}}$.

Suppose Z central in G. If $[\pi] \in \hat{G}_\zeta$ the following are equivalent:

(1) There exist nonzero φ, ψ in the representation space H_π such that $\langle \varphi, \pi(\cdot)\, \psi \rangle \in L_2(G/Z, \zeta)$.

(2) If $\varphi, \psi \in H_\pi$ then $\langle \varphi, \pi(\cdot)\, \psi \rangle \in L_2(G/Z, \zeta)$.

(3) $[\pi] \in \hat{G}_{\zeta\text{-disc}}$.

Under those conditions there is a number $d_\pi > 0$ such that

$$\int_{G/Z} \langle \varphi_1, \pi(g)\, \psi_1 \rangle \overline{\langle \varphi_2, \pi(g)\, \psi_2 \rangle}\, d(gZ) = d_\pi^{-1} \langle \varphi_1, \varphi_2 \rangle \overline{\langle \psi_1, \psi_2 \rangle}$$

for all $\varphi_i, \psi_i \in H_\pi$. The number d_π is the *formal degree* of π.

1.2. Exact working hypotheses. From now on, G is reductive Lie group, i.e. its Lie algebra $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1$ with $\mathfrak{c}$ central and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ semisimple. We suppose

(1.2.1) if $g \in G$ then $\operatorname{ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$.

We also suppose that the closed normal abelian subgroup $Z \subset G$ has the following properties:

(1.2.2a) Z centralizes the identity component G^0 and $|G/ZG^0| < \infty$.

(1.2.2b) $Z \cap G^0$ is cocompact in the center Z_{G^0} of G^0.

Two comments. If $|G/G^0| < \infty$ then Z_{G^0} satisfies (1.2.2). And $\hat{G}_{\text{disc}}$ is independent of choice of subgroup $Z \subset G$ that satisfies (1.2.2).

Without comment we use the notation

(1.2.3a) $G^\dagger = \{g \in G : \operatorname{ad}(g) \text{ is an inner automorphism on } G^0\}$.

Then evidently

(1.2.3b) $G^\dagger = Z_G(G^0)\, G^0$ where $Z_G(G^0)$ is the G-centralizer of G^0.

Note $Z \subset Z_G(G^0)$ with $Z_G(G^0)/Z$ compact. So $ZG^0 \subset G^\dagger$.

1.3. Discrete series for connected groups with compact center. The Harish-Chandra analysis of discrete series for connected reductive acceptable groups extends without change to the groups G^0 of § 1.2 for which Z_{G^0} is compact. We state the result.

If G^0 has no compact Cartan subgroup then $(G^0)\hat{}_{\text{disc}}$ is empty.

Let $H^0 \subset G^0$ compact Cartan subgroup. Denote $L = \{\lambda \in i\mathfrak{h}^* : e^\lambda$ is well-defined on $H^0\}$. Choose a positive root system Σ^+ and make the usual definitions:

$$(1.3.1) \quad \varrho = \tfrac{1}{2} \sum_{\varphi \in \Sigma^+} \varphi, \qquad \varpi(\lambda) = \prod_{\varphi \in \Sigma^+} \langle \varphi, \lambda \rangle, \qquad \Delta = \prod_{\varphi \in \Sigma^+} (e^{\varphi/2} - e^{-\varphi/2}).$$

We arrange $\varrho \in L$ by passing to a 2-sheeted "cover" of G if necessary; then Δ is well defined on H^0. Let $L' = \{\lambda \in L : \varpi(\lambda) \neq 0\}$, the regular set in L. If $\lambda \in L'$ then

$$q(\lambda) = |\{\varphi \in \Sigma^+ \text{ compact} : \langle \varphi, \lambda \rangle < 0\}| + |\{\varphi \in \Sigma^+ \text{ noncompact} : \langle \varphi, \lambda \rangle > 0\}|.$$

Suppose $\lambda \in L'$ and $\xi = e^{\lambda - \varrho}|_{Z_{G^0}}$. Then there is a unique class $[\pi_\lambda] = \omega(\lambda) \in (G^0)\hat{}_{\xi\text{-disc}}$ whose distribution character has restriction to the regular elliptic set given by

$$(1.3.2) \qquad \Theta_{\pi_\lambda}|_{H^0 \cap G'} = (-1)^{q(\lambda)} \Delta^{-1} \sum_{W(G^0, H^0)} \det(w)\, e^{w\lambda}.$$

Every class in $(G^0)\hat{}_{\text{disc}}$ is one of these $[\pi_\lambda]$, and $[\pi_\lambda] = [\pi_{\lambda'}]$ precisely when λ' is in the Weyl group orbit $W(G^0, H^0)(\lambda)$. Dual class $[\pi_\lambda^*] = [\pi_{-\lambda}]$. The infinitesimal character of $[\pi_\lambda]$ is χ_λ, so the Casimir element goes to $\|\lambda\|^2 - \|\varrho\|^2$. Finally, for appropriate normalization of Haar measure, $[\pi_\lambda]$ has formal degree $|\varpi(\lambda)|$.

1.4. Relative discrete series for connected groups. In §1.4 we suppose Z central in G. In particular, our considerations apply to $Z \cap G^0$ in G^0.

Let $S = \{s \in C : |s| = 1\}$, the circle group. $1 \in \hat{S}$ is defined by $1(s) = s$. Given $\zeta \in \hat{Z}$ we have the quotient group

$$(1.4.1) \qquad G[\zeta] = \{S \times G\}/\{(\zeta(z)^{-1}, z) : z \in Z\}.$$

It is the Mackey central extension $1 \to S \to G[\zeta] \to G/Z \to 1$ for $\delta\zeta \in Z^2(G/Z; S)$. Anyway, $G[\zeta]$ is a reductive Lie group with Lie algebra $\mathfrak{s} \oplus (\mathfrak{g}/\mathfrak{z})$, with identity component of compact center, and with $|G[\zeta]/G[\zeta]^0| < \infty$. Projection $S \times G \to G[\zeta]$ restricts to a homomorphism

$$(1.4.2) \quad p : G \to G[\zeta] \quad \text{where } f \to f \cdot p \text{ maps } L_2(G[\zeta]/S, 1) \cong L_2(G/Z, \zeta).$$

1.4.3. PROPOSITION. *$\varepsilon[\psi] = [\psi \cdot p]$ defines a bijection $\varepsilon : G[\zeta]\hat{}_1 \to \hat{G}_\zeta$ that carries Plancherel measure to Plancherel measure and maps $G[\zeta]\hat{}_{1\text{-disc}}$ onto $\hat{G}_{\zeta\text{-disc}}$. Distribution characters satisfy $\Theta_{\varepsilon[\psi]} = \Theta_\psi \cdot p$.*

We know $G[\zeta]\hat{}_{1\text{-disc}}$ (for connected G) from §1.3. Apply Proposition 1.4.3. Then $(G^0)\hat{}_{\text{disc}}$ is given as follows:

If $G^0/Z\cap G^0$ has no compact Cartan subgroup then $(G^0)^{\wedge}_{\text{disc}}$ is empty.

Let $H^0/Z\cap G^0$ be a compact Cartan subgroup of $G^0/Z\cap G^0$. Define L, ϱ, Δ, ϖ, L' and q as in §1.3. Replace G by a 2-sheeted cover if necessary, Z by a subgroup of index 2 if necessary, so that e^{ϱ} is well defined on $H^0/Z\cap G^0$. If $\lambda\in L'$ and $\xi=e^{\lambda-\varrho}|_{Z_{G^0}}$, then there is a unique class $[\pi_\lambda]\in(G^0)^{\wedge}_{\text{disc}}$ whose distribution character

$$\Theta_{\pi_\lambda}|_{H^0\cap G'}=(-1)^{q(\lambda)}\,\Delta^{-1}\sum_{W(G^0,H^0)}\det(w)\,e^{w\lambda}. \tag{1.4.4}$$

Every class in $(G^0)^{\wedge}_{\text{disc}}$ is one of those $[\pi_\lambda]$. Classes $[\pi_\lambda]=[\pi_{\lambda'}]$ just when $\lambda'\in W(G^0, H^0)(\lambda)$. $[\pi^*_\lambda]=[\pi_{-\lambda}]$. The infinitesimal character of $[\pi_\lambda]$ is χ_λ and the formal degree $d_{\pi_\lambda}=|\varpi(\lambda)|$.

1.5. Relative discrete series in general. One passes from $(G^0)^{\wedge}_{\text{disc}}$ to $(G^\dagger)^{\wedge}_{\text{disc}}$ by (1.2.3b) and a $\otimes$ construction, then up to $\hat{G}_{\text{disc}}$ by (1.2.1), (1.2.2) and $\text{Ind}_{G^\dagger\uparrow G}$.

Suppose that G/Z has a compact Cartan subgroup H/Z. Let $\lambda\in L'$, $\xi=e^{\lambda-\varrho}|_{Z_{G^0}}$ and $[\chi]\in Z_G(G^0)^{\wedge}_{\xi}$. Note $[\chi\otimes\pi_\lambda]\in(G^\dagger)^{\wedge}_{\zeta\text{-disc}}$ where $\zeta\in\hat{Z}$ is a summand of $\xi|_Z$. Then

$$[\pi_{\chi,\lambda}]=[\text{Ind}_{G^\dagger\uparrow G}(\chi\otimes\pi_\lambda)]\text{ is in }\hat{G}_{\zeta\text{-disc}}. \tag{1.5.1}$$

Further, every element of $\hat{G}_{\zeta\text{-disc}}$ is one of these $[\pi_{\chi,\lambda}]$.

Choose $\{x_1,\dots,x_r\}$ representatives of G modulo $G^\dagger$ with $\text{ad}(x_i)H=H$. Let $w_i\in W(\mathfrak{g}_C,\mathfrak{h}_C)$ be the element specified (using (1.2.1)) by x_i. Then the distribution character $\Theta_{\pi_{\chi,\lambda}}$ has support in $G^\dagger$, where it is given by

$$\Theta_{\pi_{\chi,\lambda}}(xg)=\sum_{1\leq i\leq r}\{\text{trace }\chi(x_i^{-1}xx_i)\}\,\Theta_{\pi_{w_i(\lambda)}}(g) \tag{1.5.2}$$

$$\text{for } x\in Z_G(G^0)\text{ and } g\in(G^0)'.$$

Classes $[\pi_{\chi,\lambda}]=[\pi_{\chi',\lambda'}]$ just when there is an x_i with $[\chi']=[\chi\cdot\text{ad}(x_i)^{-1}]$ and $\lambda'\in W(G^0, H^0)(w_i\lambda)$. Also $[\pi_{\chi,\lambda}]$ has dual $[\pi_{\chi^*,-\lambda}]$, and infinitesimal character χ_λ.

2. The nondegenerate series

2.1. Cuspidal parabolic subgroups. Let K/Z be a maximal compact subgroup of G/Z. In other words, K is the fixed point set of a Cartan involution θ of G. Now choose

$$\{H_1,\dots,H_l\}:\theta\text{-stable Cartan subgroups of }G \tag{2.1.1}$$

such that every Cartan subgroup is conjugate to just one of the H_i. Stability under θ gives splittings

$$\mathfrak{h}_j=\mathfrak{t}_j\oplus\mathfrak{a}_j\quad\text{and}\quad H_j=T_j\times A_j \tag{2.1.2}$$

where $T_j = H_j \cap K$, $\mathfrak{a}_j = \{x \in \mathfrak{h}_j : \theta x = -x\}$ and $A_j = \exp(\mathfrak{a}_j)$.

The $\mathfrak{a}_j$-roots of $\mathfrak{g}$ are the nonzero real linear functionals φ on $\mathfrak{a}_j$ such that

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} : [\alpha, x] = \varphi(\alpha)\, x \text{ for all } \alpha \in \mathfrak{a}_j\} \neq 0.$$

Let $\Sigma_{\mathfrak{a}_j}$ be the $\mathfrak{a}_j$-root system and choose a positive subsystem $\Sigma^+_{\mathfrak{a}_j}$. That specifies

(2.1.3) $$\mathfrak{n}_j = \sum_{\varphi \in \Sigma^+_{\mathfrak{a}_j}} \mathfrak{g}^\varphi \quad \text{and} \quad N_j = \exp_G(\mathfrak{n}_j),$$

and

(2.1.4) $$P_j = \{g \in G : \mathrm{ad}(g)\, N_j = N_j\}.$$

Then P_j is a (real) parabolic subgroup of G with unipotent radical $P^u_j = N_j$. Also $P_j = P^r_j \cdot P^u_j$ (semidirect) $= M_j A_j N_j$ where

(2.1.5) $$P^r_j = \{g \in G : \mathrm{ad}(g)\, \alpha = \alpha \text{ all } \alpha \in \mathfrak{a}_j\} = M_j \times A_j.$$

The P_j are *cuspidal parabolic subgroups* of G. They are characterized by the fact that M_j/Z has a compact Cartan subgroup T_j/Z.

2.1.6. LEMMA. *M_j inherits* (1.2.1) *and* (1.2.2) *from G: Every* $\mathrm{ad}(m)$ *is inner on $\mathfrak{m}_{j\mathbb{C}}$, Z centralizes M^0_j and $|M_j/ZM^0_j| < \infty$, and $Z \cap M^0_j$ is cocompact in the center of M^0_j.*

2.2. The series for a Cartan subgroup. The relative discrete series of M_j is given as in §1.5. Denote $L_j = \{\nu \in i\mathfrak{t}^*_j : e^\nu \text{ well defined on } T^0_j\}$. Choose a positive $\mathfrak{t}_{j\mathbb{C}}$-root system $\Sigma^+_{\mathfrak{t}_j}$ on $\mathfrak{m}_{j\mathbb{C}}$. Define $\varrho_{\mathfrak{t}_j}$, $\varpi_{\mathfrak{t}_j}(\nu)$ and $\Delta_{\mathfrak{t}_j}$ as in (1.3.1). We may assume $\varrho_{\mathfrak{t}_j} \in L_j$, thus is in its $\mathfrak{m}_j$-regular set $L''_j = \{\nu \in L_j : \varpi_{\mathfrak{t}_j}(\nu) \neq 0\}$. Let $\nu \in L''_j$, $\xi = \exp(\nu - \varrho_{\mathfrak{t}_j})\,|_{\text{center of } M^0_j}$ and $[\chi] \in Z_{M_j}(M^0_j)^\wedge_\xi$. That gives the relative discrete classes $[\eta_\nu]$ of M^0_j, $[\chi \otimes \eta_\nu]$ of $M^\dagger_j = Z_{M_j}(M^0_j)\, M^0_j$, and $[\eta_{\chi,\nu}] = [\mathrm{Ind}_{M^\dagger_j \uparrow M_j}(\chi \otimes \eta_\nu)]$ of M_j.

$(P^r_j)^\wedge_{\mathrm{disc}}$ consists of the $[\eta_{\chi,\nu} \otimes e^{i\sigma}]$, χ and ν as above and $\sigma \in \mathfrak{a}^*_j$. Extend $\eta_{\chi,\nu} \otimes e^{i\sigma}$ to $P_j = M_j A_j N_j$ by $(\eta_{\chi,\nu} \otimes e^{i\sigma})\,(man) = \eta_{\chi,\nu}(m) \cdot e^{i\sigma}(a)$. Then we have the (unitarily) induced representations

(2.2.1) $$\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_{P_j \uparrow G}(\eta_{\chi,\nu} \otimes e^{i\sigma}).$$

By *H_j-series* of G we mean the set of all unitary equivalence classes of representations (2.2.1). The H_j-series depends only on the conjugacy class of H_j. The various H_j-series are the *nondegenerate series*. Two cases are the relative discrete series (H_j/Z compact) and the principal series (P_j minimal parabolic).

2.3. Nondegenerate series characters. Here are the basic facts. Let $[\pi_{\chi,\nu,\sigma}]$ be an H_j-series class. Then $\pi_{\chi,\nu,\sigma}$ is a finite sum of irreducible classes, and is irreducible itself whenever $\langle\sigma,\varphi\rangle\neq 0$ for every $\mathfrak{h}_{jC}$-root φ of $\mathfrak{g}_C$ such that $\varphi|_{\mathfrak{a}_j}\neq 0$. Every irreducible summand of $\pi_{\chi,\nu,\sigma}$ is in $\hat{G}_\zeta$ where $\zeta\in\hat{Z}$ and $[\chi]\in Z_{M_j}(M_j^0)\hat{}_\zeta$. The class $[\pi_{\chi,\nu,\sigma}]$ has infinitesimal character $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}_j$. The distribution character $\Theta_{\pi_{\chi,\nu,\sigma}}$ exists and is a locally integrable function with support in the closure of

$$\bigcup_{g\in G^\dagger}\ \bigcup_{H\subset M_jA_j} gHg^{-1}\subset G^\dagger \qquad (H \text{ is any Cartan subgroup of } M_jA_j). \tag{2.3.1}$$

Finally that character is given on $H_j\cap G'$ by

$$\Theta_{\pi_{\chi,\nu,\sigma}}(ta)=|\Delta_{\mathfrak{t}_j}(t)/\Delta_{\mathfrak{h}_j}(ta)|\sum |N_{M_j}(T_j)(w(t))|^{-1}\Psi_{\eta_{\chi,\nu}}(w(t))\,e^{i\sigma}(w(a)) \tag{2.3.2}$$

where $t\in T_j$ and $a\in A_j$. Here $N_G(H_j)$ is the G-normalizer of H_j, the sum runs over the finite set of all $w(ta)$ in $N_G(H_j)(ta)$, $N_{M_j}(T_j)$ is defined similarly, and $\Psi_{\eta_{\chi,\nu}}$ is the character of $\eta_{\chi,\nu}$.

3. Plancherel measure

3.1. Statement of result. Fix $\zeta\in\hat{Z}$. Define

$$L_{j,\zeta}=\{\nu\in L_j : e^\nu\in(T_j^0)\hat{}_\zeta\} \quad\text{and}\quad L''_{j,\zeta}=L_{j,\zeta}\cap L''_j. \tag{3.1.1}$$

Given $\nu\in L''_{j,\zeta}$ and $\sigma\in\mathfrak{a}_j^*$, the corresponding H-series classes that transform by ζ are the $[\pi_{\chi,\nu,\sigma}]$ with $[\chi]\in Z_{M_j}(M_j^0)\hat{}_\zeta$. They give us discrete sums

$$\pi_{j,\zeta,\lambda}=\sum(\dim\chi)\,\pi_{\chi,\nu,\sigma} \quad\text{and}\quad \Theta_{j,\zeta,\lambda}=\Theta_{\pi_{j,\zeta,\lambda}}=\sum(\dim\chi)\,\Theta_{\pi_{\chi,\nu,\sigma}} \tag{3.1.2}$$

where $\lambda=\nu+i\sigma$ and $[\chi]$ runs over the appropriate subset of $Z_{M_j}(M_j^0)\hat{}_\zeta$. Here is our extension of a weak form of Harish-Chandra's Plancherel formula.

3.1.3. THEOREM. *There are unique measurable functions $m_{j,\zeta,\nu}$ on $\mathfrak{a}_j^*, \nu\in L''_{j,\zeta}$, with these properties.*

1. *$m_{j,\zeta,\nu}$ is invariant by the Weyl group $W(G,A_j)$.*
2. *If $f\in L_2(G/Z,\zeta)$ is C^∞ with support compact modulo Z, then*

$$\sum_{1\leq j\leq l}\ \sum_{\nu\in L''_{j,\zeta}} |\varpi_{\mathfrak{t}_j}(\nu)| \int_{\mathfrak{a}_j^*} |\Theta_{j,\zeta,\nu+i\sigma}(f)\,m_{j,\zeta,\nu}(\sigma)|\,d\sigma<\infty \tag{3.1.4a}$$

and

(3.1.4b) $$f(1)=\sum_{1\leq j\leq l}\sum_{\nu\in L''_{j,\zeta}}|\varpi_{t_j}(\nu)|\int_{\mathfrak{a}_j^*}\Theta_{j,\zeta,\nu+i\sigma}(f)\,m_{j,\zeta,\nu}(\sigma)\,d\sigma.$$

3.2. Reduction from G to ZG^0. Denote $G^1=ZG^0$, $M_j^1=M_j\cap G^1$, etc. Define $\pi^1_{j,\zeta,\lambda}$ as in (3.1.2) on G^1. Then $\pi_{j,\zeta,\lambda}=\operatorname{Ind}_{G^1\uparrow G}(\pi^1_{j,\zeta,\lambda})$. If Theorem 3.1.3 holds for G^1 with functions $m^1_{j,\zeta,\nu}$ now it holds for G with functions $m_{j,\zeta,\nu}=|G/G^1|^{-1}m^1_{j,\zeta,\nu}$.

3.3. Reduction from ZG^0 to $(ZG^0)[\zeta]$. For simplicity now let $G=ZG^0$. Enlarge Z to that $|Z_G/Z|<\infty$. The considerations of §1.4 apply. Define $b:L_j[\zeta]_1\to L_{j,\zeta}$ by $e^\nu\cdot q=\zeta\otimes e^{b(\nu)}$. If $\lambda=\nu+i\sigma$ let $b(\lambda)=b(\nu)+i\sigma$. In Proposition 1.4.3 we have $[\pi_{j,1,\lambda}\cdot p]=[\pi_{j,\zeta,b(\lambda)}]$ and $\Theta_{j,1,\lambda}\cdot p=\Theta_{j,\zeta,b(\lambda)}$. If Theorem 3.1.3 holds for $(G[\zeta], S, 1)$ with functions $m_{j,1,\nu}$ then it holds for (G, Z, ζ) with functions $m_{j,\zeta,b(\nu)}=m_{j,1,\nu}\cdot p_*$.

3.4. The function E. As seen above, the proof of Theorem 3.1.3 reduces to the case where G is connected, Z_G is a finite extension of the circle group S, and $\zeta=1\in\hat{S}$. We may also assume $K=Z_K^0\times[K,K]$. Then one can construct a class function $E:G\to S$, analytic on the regular set, with the following properties. $E(g)=E(g_{ss})$ where g_{ss} is the semisimple part. If $s\in S$ and $g\in G$ then $E(sg)=sE(g)$. Each $E|_{H_j}\in\hat{H}_j$ with A_j in its kernel. And $E|_K\in\hat{K}$. In effect S is a direct factor of K, $E|_K$ is projection of K to S, and then E is specified by the other properties.

3.5. Reduction from $(ZG^0)[\zeta]$ to $G^0/Z\cap G^0$. We take $(G,Z,\zeta)=(G,S,1)$ as in §3.4. Denote $L_{j,n}=\{\nu\in L_j: e^\nu(s)=s^n \text{ for } s\in S\}$ and $L''_{j,n}=L_{j,n}\cap L''_j$. Define $\varepsilon_j\in L_{j,1}$ by $\exp(\varepsilon_j)=E|_{T_j^0}$, so $L_{j,0}=\{\nu-\varepsilon_j:\nu\in L_{j,1}\}$. Arguing from Harish-Chandra's Plancherel formula, and from the explicit form of G' and the $\Theta_{j,\zeta,\lambda}$, one can prove

3.5.1. PROPOSITION. *Let $f\in L_2(G/S,1)$ be continuous at 1, C^∞ on G', and bounded by a rapidly decreasing function. Let $B_{j,\nu}\subset\mathfrak{a}_j^*$ be sets of Lebesgue measure zero. Suppose $\Theta_{j,1,\lambda}(f)=0$ whenever $1\leq j\leq l$, $\lambda=\nu+i\sigma$ with $\nu\in L''_{j,1}$, and $\sigma\in\mathfrak{a}_j^*-B_j$. Then $\Theta_{j,0,\lambda}(Ef)=0$ whenever $1\leq j\leq l$ and $\lambda\in L''_{j,0}+i\mathfrak{a}_j^*$.*

Proposition 3.5.1 and Harish-Chandra's formula on G/S give absolutely continuous Borel measures $\mu_{j,1,\nu}$ on $\mathfrak{a}_j^*$ such that

$$f(1)=\sum_{1\leq j\leq l}\sum_{\nu\in L''_{j,1}}|\varpi(\nu)|\int_{\mathfrak{a}_j^*}\Theta_{j,1,\nu+i\sigma}(f)\,d\mu_{j,1,\nu}(\sigma)$$

in $C_c^\infty(G)\cap L_2(G/S,1)$. Theorem 3.1.3 follows for $(ZG^0)[\zeta]$, then finally for (G,Z,ζ).

3.6. Two consequences. For realization of nondegenerate series representations on partially holomorphic cohomology spaces one needs

3.6.1. COROLLARY. *Suppose that* $\hat{G}_{\text{disc}}$ *is not empty. If* $[\pi] \in \hat{G}$ *let* T_π *be the distribution* $f \mapsto \text{trace} \int_K f(k)\, \pi(k)\, dk$ *on* K. *If* $\zeta \in \hat{Z}$ *then* $\{[\pi] \in \hat{G}_\zeta - \hat{G}_{\zeta\text{-disc}} : T_\pi|_{K \cap G'} \neq 0\}$ *has Plancherel measure zero in* $\hat{G}_\zeta$.

For realization of nondegenerate series representations on spaces of partially-harmonic-spinors one needs

3.6.2. COROLLARY. *Let* $\Omega \in \mathfrak{G}$ *be the Casimir element. If* c *is a number and* $\zeta \in \hat{Z}$ *then* $\{[\pi] \in \hat{G}_\zeta - \hat{G}_{\zeta\text{-disc}} : \chi_\pi(\Omega) = c\}$ *has Plancherel measure zero in* $\hat{G}_\zeta$.

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