SQUARE-INTEGRABLE REPRESENTATIONS OF NILPOTENT GROUPS

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1. Notion of square-integrable representation. Let G be a unimodular locally compact group and Z the center of G. As usual, $\hat{G}$ denotes the set of all equivalence classes $[\pi]$ of irreducible unitary representations π of G, and H_π denotes the representation space of π. So $\hat{Z}$ is the group of unitary characters on Z. If $[\pi]\in\hat{G}$ then $\pi|_Z$ is a multiple of the *central character* $\zeta_\pi\in\hat{Z}$ of $[\pi]$. If $\zeta\in\hat{Z}$ then $L_2(G/Z, \zeta)$ denotes the space of functions on G that is the representation space for $\mathrm{Ind}_{Z\uparrow G}(\zeta)$.

If $[\pi]\in\hat{G}$ and $\zeta_\pi\in\hat{Z}$ one knows that the following conditions are equivalent: (1) There exist nonzero $\phi, \psi\in H_\pi$ such that $\langle\pi(\cdot)\phi, \psi\rangle\in L_2(G/Z, \zeta)$. (2) If $\phi, \psi\in H_\pi$ then $\langle\pi(\cdot)\phi, \psi\rangle\in L_2(G/Z, \zeta)$. (3) $[\pi]$ is a discrete summand of $\mathrm{Ind}_{Z\uparrow G}(\zeta)$. Then we say that $[\pi]$ is *square-integrable*. The usual case is the case where Z is compact, but the more general setting is useful for reductive and for nilpotent groups.

If $[\pi]$ is square-integrable, there is a number $d_\pi>0$ such that

$$\int_{G/Z} \langle\pi(g)\phi_1, \psi_1\rangle \overline{\langle\pi(g)\phi_2, \psi_2\rangle}\, d(gZ)=d_\pi^{-1}\langle\phi_1, \phi_2\rangle \overline{\langle\psi_1, \psi_2\rangle}$$

for all $\phi_i, \psi_i\in H_\pi$. The number d_π is the *formal degree* of $[\pi]$.

If π_1 and π_2 are inequivalent square-integrable representations with the same central character, one has the *orthogonality relations*

$$\int_{G/Z} \langle\pi(g)\phi_1, \psi_1\rangle \overline{\langle\pi(g)\phi_2, \psi_2\rangle}\, d(gZ)=0 \quad \text{for } \phi_i, \psi_i\in H_{\pi_i}.$$

2. The case of a connected, simply connected nilpotent Lie group. Let N be a

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nilpotent group as just described, $\mathfrak{n}$ its Lie algebra and $\mathfrak{z}$ the center of $\mathfrak{n}$. Then $Z=\exp\mathfrak{z}$ is the center of N and $\log:Z\to\mathfrak{z}$ denotes the inverse of $\exp:\mathfrak{z}\to Z$. Let $\mathfrak{n}^*$ denote the (real) dual space of $\mathfrak{n}$; we denote the representation of N on $\mathfrak{n}^*$ by ad^*. If $f\in\mathfrak{n}^*$ then its orbit $O_f=\mathrm{ad}^*(N)(f)$ determines a class $[\pi_f]=[\pi_{O_f}]$ by the Kirillov theory. The central character of $[\pi_f]$ is $\zeta_{\pi_f}(z)=\exp\{2\pi\, if(\log z)\}$. Here note $f|_{\mathfrak{z}}=f'|_{\mathfrak{z}}$ whenever $f'\in O_f$.

Let $\mathfrak{z}^\perp=\{f\in\mathfrak{n}^*:f(\mathfrak{z})=0\}$. Now O_f is contained in the affine hyperplane $H(f|_{\mathfrak{z}})=f+\mathfrak{z}^\perp$ of $\mathfrak{n}^*$.

Finally recall the antisymmetric form b_f on $\mathfrak{n}$ defined by $b_f(x,y)=f[x,y]$. Evidently b_f can be viewed as an antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$.

THEOREM 1. *Let $f\in\mathfrak{n}^*$ and ζ the central character of π_f. Then the following conditions are equivalent:*

(1) *π_f is square-integrable.*

(2) *$\mathrm{Ind}_{Z\uparrow N}(\zeta)$ is a primary representation of N.*

(3) *The orbit O_f is the hyperplane $H(f|_{\mathfrak{z}})=f+\mathfrak{z}^\perp$.*

(4) *The form b_f is nondegenerate on $\mathfrak{n}/\mathfrak{z}$.*

The proof is an induction on $\dim N$. If N is abelian the equivalence is routine; now suppose N nonabelian. If $\dim Z>1$, choose a subalgebra $\mathfrak{z}^0\subset\mathfrak{z}$ of codimension 1 contained in kernel(f). Then $\pi=\pi_f$ is the lift of a representation $\pi^0=\pi^0_{f^0}$ of $N^0=N/Z^0$, $Z^0=\exp(\mathfrak{z}^0)$. Theorem 1 applies to N^0, π^0 and f^0 by induction on dimension, and it follows for N, π and f.

Now $\dim Z=1$ and $f(\mathfrak{z})\neq 0$. Choose $x\in\mathfrak{n}-\mathfrak{z}$ with $[x,\mathfrak{n}]\subset\mathfrak{z}$ and $f(x)=0$. Let $\mathfrak{n}_0=\{u\in\mathfrak{n}:[u,x]=0\}$, ideal of codimension 1 in $\mathfrak{n}$, and $N_0=\exp(\mathfrak{n}_0)$. Then $f_0=f|_{\mathfrak{n}_0}$ has $\mathrm{ad}^*(N_0)$-orbit $O_0\subset\mathfrak{n}_0^*$ which in turn gives $[\pi_0]\in\hat{N}_0$. The Kirillov machine gives $\pi=\pi_f$ as $\mathrm{Ind}_{N_0\uparrow N}(\pi_0)$. Choose $y\notin\mathfrak{n}_0$ such that $f(z=[x,y])=1$. Then $f_s=\mathrm{ad}^*(\exp(sy))f|_{\mathfrak{n}_0}$ has $f_s(x)=s$. Let O_s be the orbit of f_s in $\mathfrak{n}_0^*$. From Kirillov, the projection $p:\mathfrak{n}^*\to\mathfrak{n}_0^*$, kernel $\mathfrak{n}_0^\perp$, satisfies $p^{-1}(O_s)=\{f'\in O_f:f'(x)=s\}$ and $p\{f'\in O_f:f'(x)=s\}=O_s$. Let $\mathfrak{z}_0=(x)+\mathfrak{z}$. With some linear algebra and the fact that unipotent orbits are closed, one sees: $O_f=f+\mathfrak{z}^\perp \Leftrightarrow O_s=f_s+\mathfrak{z}_0^\perp$ *for some real* $s\Leftrightarrow O_s=f_s+\mathfrak{z}_0^\perp$ *for all real s; in that case $\mathfrak{z}_0$ is the center of* $\mathfrak{n}_0$. Induction by stages and some direct integral theory then give: *π is a discrete summand of* $\mathrm{Ind}_{Z\uparrow N}(\zeta)$ $\Leftrightarrow O=f+\mathfrak{z}^\perp$; *in that case* $\mathrm{Ind}_{Z\uparrow N}(\zeta)$ *is primary.* Using the results quoted above in §1, now assertions (1), (2) and (3) of Theorem 1 are equivalent. Some linear algebra proves (3) and (4) equivalent.

3. Square-integrability and the Pfaffian. Retain N, Z, etc., as in §2. Make a definite choice of Haar measure $\mu_{N/Z}$ on N/Z. That gives a volume element (alternating form of degree $\dim(\mathfrak{n}/\mathfrak{z})$), say α, on $\mathfrak{n}/\mathfrak{z}$. Let $f\in\mathfrak{n}^*$ and recall that the alternating bilinear form b_f on $\mathfrak{n}/\mathfrak{z}$ has Pfaffian (relative to α) defined as follows: If

$\dim \mathfrak{n}/\mathfrak{z}$ is odd, then $Pf(b_f)=0$. If $\dim \mathfrak{n}/\mathfrak{z}=2m$ even, then the mth exterior power $b_f^m = Pf(b_f)\,\alpha$. Evidently $P(f)=Pf(b_f)$ is a real polynomial function on $\mathfrak{n}^*$, and $P(f)\neq 0$ precisely when b_f is nondegenerate on $\mathfrak{n}/\mathfrak{z}$. Using Theorem 1, now $P(\cdot)$ is constant on $\mathrm{ad}^*(N)$-orbits and $P(f)$ depends only on $f|_{\mathfrak{z}}$. Thus we view P as a polynomial on $\mathfrak{z}^*$.

If $h\in\mathfrak{z}^*$ with $P(h)\neq 0$, denote $\phi(h)=[\pi_f]$ for any $f\in h+\mathfrak{z}^\perp$. Note that $\phi(h)$ is the only class in $\hat{N}$ with central character $\zeta(\exp z)=\exp\{2\pi i h(z)\}$. We often write π_h for $\phi(h)$.

THEOREM 2. *ϕ is a bijection of $\{h\in\mathfrak{z}^*:P(h)\neq 0\}$ onto $\{[\pi]\in\hat{N}:[\pi]$ is square-integrable$\}$. Further, it is a homeomorphism from the natural topology to the Fell (hull-kernel) topology of $\hat{N}$.*

As polynomial function on $\mathfrak{z}^*$, P is in the symmetric algebra $S(\mathfrak{z})$. Let $\mathfrak{Z}$ be the center of the universal enveloping algebra $\mathscr{U}$ of $\mathfrak{n}$. Then $\mathfrak{z}\subset\mathfrak{Z}$ gives $S(\mathfrak{z})\subset\mathfrak{Z}$, so also $P\in\mathfrak{Z}$.

THEOREM 3. *N has square-integrable representations if, and only if, $S(\mathfrak{z})=\mathfrak{Z}$*

4. Formal degree and Plancherel measure. Formal degree and Pfaffian depend in the same way on the normalization of Haar measure on N/Z, so the following is intrinsic.

THEOREM 4. *In the notation of Theorem 2, the square-integrable class $\phi(h)$ has formal degree $|P(h)|$.*

The *infinitesimal character* of a class $[\pi]\in\hat{N}$ is the homomorphism $\chi_\pi:\mathfrak{Z}\to C$ given by $\chi_\pi(z)\,v=d\pi(v)$ on C^∞ vectors. It is related to the central character ζ_π by $\zeta_\pi(\exp z)=\exp\{\chi_\pi(z)\}$ for $z\in\mathfrak{z}\subset\mathfrak{Z}$.

If $[\pi]\in\hat{N}$ is not square-integrable, we understand its formal degree $d_\pi=0$. This is consistent with Theorem 4.

THEOREM 5. *If $[\pi]\in\hat{N}$, then $d_\pi=|\chi_\pi(P)|$.*

The fact $d_\pi^2=|\chi_\pi(P)|^2=\chi_\pi(P\bar{P})$ is in accord with the Weyl formula for the degree of a representation in terms of its highest weight. The analogy may go quite far.

Fix Haar measures μ_N on N and μ_Z on Z consistent with our choice $\mu_{N/Z}$ for N/Z: $d_{\mu_N}(n)=d_{\mu_{N/Z}}(nZ)\,d_{\mu_Z}(z)$. These choices fix Lebesgue measures dn on $\mathfrak{n}$, dz on $\mathfrak{z}$ and $d\dot{n}$ on $\mathfrak{n}/\mathfrak{z}$ with $dn=d\dot{n}\,dz$. In turn Lebesgue measures dn^* on $\mathfrak{n}^*$, dz^* on $\mathfrak{z}^*$ and dv on $(\mathfrak{n}/\mathfrak{z})^*$ are fixed by the condition that Fourier transforms be isometries.

THEOREM 6. *If N has square-integrable representations, then Plancherel measure is concentrated in $\{[\pi]\in\hat{N}:[\pi]$ is square-integrable$\}$. Then the map ϕ of Theorem 2 pulls Plancherel measure back to $c|P(h)|\,dz^*(h)$ where $c=k!\,2^k$ and $k=\frac{1}{2}\dim\mathfrak{n}/\mathfrak{z}$.*

Theorem 4 is proved using the same inductive procedure as that sketched for Theorem 1. Theorem 5 is a corollary. In Theorem 6, the concentration of Plancherel measure comes from induction by stages and the fact that the zeroes of P have Lebesgue measure zero in $\mathfrak{z}^*$. The formula for Plancherel measure then comes by expressing the distribution character Θ_π of a square-integrable class $\pi=\phi(h)$ on a C_c^∞ function f by

$$\Theta_\pi(f)=c^{-1}\int_{O_h}(f\cdot\exp)\hat{\ }(y)\,d_{\mu_h}(y)$$

where $\hat{\ }$ is Fourier transform on $\mathfrak{n}^*$ and μ_h is the symplectic measure on $O_h=h+\mathfrak{z}^\perp$.

5. Two examples. First consider the Heisenberg group N_k, simply connected group for the $(2k+1)$-dimensional Lie algebra $\mathfrak{n}_k$:

$$\{x_1,\ldots,x_k;\quad y_1,\ldots,y_k;\quad z:[x_i,y_i]=z,\quad \text{all others zero}\}.$$

Choose the measure on N_k/Z so that $\mathfrak{n}_k/\mathfrak{z}$ has volume α with $\alpha(x_1,\ldots,y_k)=1$. Then $P(tz^*)=t^k$. So the infinite-dimensional classes in $\hat{N}_k$ all are square-integrable, π_{tz^*} having formal degree $|t|^k$, and the Plancherel formula on N_k is

$$f(1)=c\int_{-\infty}^{\infty}\Theta_{\pi_{tz^*}}(f)\,|t|^k\,dt.$$

This is classical.

For the second example, let (a_{ij}) be a $k\times k$ matrix that gives a $(2k+2)$-dimensional Lie algebra $\mathfrak{n}$ with basis $\{x_1,\ldots,x_k;y_1,\ldots,y_k;z;w\}$ and $[x_i,y_j]=\delta_{ij}z+a_{ij}w$, all others zero. Then $\mathfrak{z}=(z)+(w)$ and we describe $f\in\mathfrak{z}^*$ by $(u,v)=(f(z),f(w))$. Now $P(u,v)=\det(u-va_{ij})$, which is v^k times the value at u/v of the characteristic polynomial of (a_{ij}). In particular P can be any homogeneous polynomial of degree k in two variables.

Suppose $k=1$ and $(a_{ij})=-\lambda$ irrational; then $P(az^*+bw^*)=a+\lambda b$. Let $\Delta=\{\exp(nz+mw):m,n \text{ integers}\}$ and $\bar{N}=N/\Delta$. Then $\bar{N}$ has compact center and $\bar{N}\hat{\ }$ consists of $\{[\pi]\in\hat{N}:\pi \text{ kills } \Delta\}$. Thus the square integrable classes in $\bar{N}\hat{\ }$ are the $\pi_{nz^*+mw^*}$, m and n integers. The formal degree of $\pi_{nz^*+mw^*}$ is $n+\lambda m$, which has 0 as limit point when (n,m) varies over $\boldsymbol{Z}^2-\{0\}$.

6. Square-integrable subrepresentations for $L_2(N/\Gamma)$. Let Γ be a discrete uniform subgroup of N. N acts on $L_2(N/\Gamma)$ by $U = \mathrm{Ind}_{\Gamma\uparrow N}(1_\Gamma)$, and $U = \sum_{\hat{N}} m_\pi \cdot \pi$ discrete sum.

Normalize $\mu_{N/Z}$ by: $N/Z\Gamma$ has volume 1. Then the Pfaffian polynomial P is determined up to sign. $\Gamma \cap Z$ is a lattice in Z and $\log(\Gamma \cap Z)$ a lattice L in $\mathfrak{z}$. Let $L^* \subset \mathfrak{z}^*$ be the lattice dual to $L \subset \mathfrak{z}$.

THEOREM 7. *Let $h \in \mathfrak{z}^*$ with $P(h) \neq 0$. Then the square-integrable representation π_h occurs in U precisely when $h \in L^*$. In that case the multiplicity $m_{\pi_h} = |P(h)|$, formal degree of π_h.*

Γ specifies a rational form $\mathfrak{n}_Q$ of $\mathfrak{n}$. A subalgebra $\mathfrak{h} \subset \mathfrak{n}$ and the group $H = \exp(\mathfrak{h})$ are rational just when $H/H \cap \Gamma$ is compact. One proceeds inductively as in Theorem 1, keeping everything rational. The final induction from N_0 to N is managed by a multiplicity formula of C. C. Moore (Ann. of Math. (2) **82** (1965), 153).

7. Extension to solvable groups. Do not assume G unimodular. The classes in $\hat{G}$ that are subrepresentations of $L_2(G)$ correspond (using results of Auslander, Kostant and Pukanszky) to the simply connected open orbits in $\mathfrak{n}^*$. If $f \in \mathfrak{n}^*$, we expect the associated representations to be square-integrable (in the appropriate sense) just when G_f/Z is compact.

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