Essential Self Adjointness for the Dirac Operator and Its Square

JOSEPH A. WOLF

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§0. Introduction. Some of the natural operators in differential geometry are the Hodge–Laplace–Beltrami operator $\triangle$ on differential forms, the Kodaira–Hodge–Laplace operator $\square$ on differential forms with values in a holomorphic vector bundle, and the Dirac operator D on spinors with values in certain vector bundles. The latter two are quite versatile because one can vary the bundle. In each case the operator is symmetric (formally self adjoint) and elliptic. If the base manifold is compact, then the operator has well defined spectrum (*i.e.*, is essentially self adjoint), which is discrete with finite multiplicities. The study of those spectra is one of the developing chapters in modern geometry.

When the base manifold is noncompact these operators need not be essentially self adjoint. However, they turn out to be essentially self adjoint, *i.e.*, to have well defined spectra, when the riemannian or hermitian metric on the base manifold is complete. Gaffney ([3], [4], [5]) proved $\triangle$ essentially self adjoint under conditions that he showed to be automatic for complete riemannian manifolds. Andreotti and Vesentini [1, §6] obtained estimates that show $\square$ essentially self adjoint over a complete hermitian manifold. Parthasarathy [7] stated that one of the Andreotti–Vesentini estimates (see below) could be proved for D^2 by a similar method. Here we prove that D and D^2 are essentially self adjoint over a complete riemannian spin manifold. This has applications (*cf.* §7–10 below) to the problem of explicit realization of unitary representations of Lie groups, and we work in the generality suitable for construction of unitary representations.

In §1 we recall the Clifford algebra construction for spin groups and spin representations.

In §2 we establish the setting for our Dirac operators. Let Y be an oriented riemannian n-manifold and α: $K \to SO(n)$ a Lie group homomorphism that factors through Spin(n). The setting consists of a principal K-bundle $\mathfrak{F}_K \to Y$ and a connection Γ_K on $\mathfrak{F}_K$, such that α sends $\mathfrak{F}_K$ to the oriented orthonormal frame bundle of Y and sends Γ_K to the riemannian connection. The case α: $K \cong$

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Spin(n) is the classical spin manifold structure. We find criteria for the existence of a "riemannian (K, α)-structure" ($\mathfrak{F}_K$, Γ_K) similar to the standard spin manifold condition.

In §3 we construct the Dirac operators associated to a riemannian (K, α)-structure ($\mathfrak{F}_K$, Γ_K) on Y. Let κ be a finite dimensional unitary representation of K and $\mathcal{V}_\kappa \to Y$ the hermitian vector bundle associated to $\mathfrak{F}_K$. Then our Dirac operator D acts on the $\mathcal{V}_\kappa$-valued spinors on Y. D is a first order elliptic operator. In terms of a local moving orthonormal frame $\{e_i\}$ it has formula $D(u) = \sum e_i \cdot \nabla_{e_i}(u)$ where "$\cdot$" is Clifford product and ∇ is covariant differentiation specified by Γ_K.

In §4 we set up the Hilbert space of square integrable $\mathcal{V}_\kappa$-valued spinors on Y, and we check that D is a symmetric operator there with dense domain consisting of the C_c^∞ spinors. In fact we prove a stronger symmetry result ($\langle Du, v\rangle = \langle u, Dv\rangle$ for u, v locally Lipschitz and one of them compactly supported) that we need in §§5 and 6.

In §§5 and 6 we further assume that the riemannian metric on Y is complete. Local regularization and an examination of the gradient of the distance function, yields essential self adjointness of D in §5. Then in §6 we combine that with an L_2-norm estimate

$$||D(u)||^2 \leqq t\,||D^2(u)||^2 + \frac{1}{t}\,||u||^2 \leqq \infty \quad \text{for} \quad t > 0 \quad \text{and} \quad u \in C^2$$

to prove essential self adjointness of D^2. That estimate is similar to the "Stampacchia inequality" of Andreotti–Vesentini [1].

In §7 we discuss the space of square integrable Dirac spinors with values in $\mathcal{V}_\kappa$ and indicate its role in construction of unitary representations.

In §§8, 9 and 10 we illustrate the realization of unitary representations on spaces of Dirac spinors, for a nilpotent group (the Heisenberg group N_3), a semisimple group (the universal covering group G of $SL(2; R)$) and a non-unimodular solvable group (a parabolic subgroup B of G). As it happens, essential self adjointness of D^2 (thus D) is easy in these cases from classical function theory. That is not the situation in general, although one can sometimes use [9]. At any rate, in each case the representations obtained are irreducible, and they are precisely those of the relative discrete series. This means that for some unitary character on the center of the group, they are subrepresentations of the corresponding component of the left regular representation. In the unimodular cases it means that we obtain precisely those irreducible representations whose coefficients are square integrable over the group modulo its center. That appears to be the general pattern.

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§1. Spin construction. Let $\mathfrak{p}$ be a real n-dimensional vector space with **positive definite** inner product $\langle e, f\rangle$. The **Clifford algebra** over $\mathfrak{p}$ is Cliff ($\mathfrak{p}$) =

$\mathbf{T}(\mathfrak{p})/\mathbf{I}$ where $\mathbf{T}(\mathfrak{p})$ is the tensor algebra and $\mathbf{I}$ is the ideal generated by all $e \otimes e + \langle e, e \rangle 1$, $e \in \mathfrak{p}$. Thus if $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $\mathfrak{p}$, then Cliff $(\mathfrak{p})$ is the real associative algebra generated by $\{e_i\}$ with relations

(1.1) $$e_j^2 = -1 \quad \text{and} \quad e_j \cdot e_k + e_k \cdot e_j = 0, \qquad i \leqq j, k \leqq n, \quad j \neq k.$$

It has basis $\{e_{j_1} \cdot e_{j_2} \cdots e_{j_k} : 1 \leqq j_1 < \cdots < j_k \leqq n \text{ and } 0 \leqq k \leqq n\}$; thus dimension 2^n. Also Cliff $(\mathfrak{p})$ = Cliff$^+$ $(\mathfrak{p})$ + Cliff$^-$ $(\mathfrak{p})$ sum of subspaces spanned by products of even (resp. odd) number of the e_j.

The map $e \to -e$ of $\mathfrak{p}$ extends uniquely to an involutive automorphism $x \to \bar{x}$ of Cliff $(\mathfrak{p})$. If $x = e_{j_1} \cdots e_{j_k}$ as above, then $\bar{x} = (-1)^k e_{j_k} \cdots e_{j_1}$. The **spin group** is the multiplicative group Spin $(\mathfrak{p})$ = Spin (n) of all invertible elements $x \in$ Cliff $(\mathfrak{p})$ such that

(1.2) $$x \in \text{Cliff}^+ (\mathfrak{p}), \quad x \cdot \mathfrak{p} \cdot x^{-1} = \mathfrak{p} \quad \text{and} \quad x \cdot \bar{x} = 1.$$

Spin $(\mathfrak{p})$ has **vector representation** v on $\mathfrak{p}$ given by

(1.3a) $$v(x): \mathfrak{p} \to \mathfrak{p} \quad \text{by} \quad v(x)e = x \cdot e \cdot x^{-1}.$$

That vector representation has image $SO(\mathfrak{p}) = SO(n)$, special orthogonal group, and kernel $\{\pm 1\}$. Thus

(1.3b) v: Spin $(\mathfrak{p}) \to SO(\mathfrak{p})$ is a 2-sheeted covering group for $n \geqq 2$.

If $n > 2$ it is the universal covering group.

Suppose $n = 2m + 1$ with $m \geqq 1$. Then $SO(\mathfrak{p})$ has center $\{I\}$ so Spin$(\mathfrak{p})$ has center $v^{-1}(I) = \{\pm 1\}$.

Suppose $n = 2m$. Fix an orientation on $\mathfrak{p}$ and define

(1.4a) $\epsilon = e_1 \cdots e_{2m}$ where $\{e_1, \cdots, e_{2m}\}$ is an oriented orthonormal basis.

We note that SO $(\mathfrak{p})$ has center $\{\pm I\}$ and we compute

(1.4b) $\epsilon^2 = (-1)^m$, $v^{-1}(-I) = \{\pm \epsilon\}$ and $\epsilon \cdot x = \pm x \cdot \epsilon$ for $x \in$ Cliff$^\pm$ $(\mathfrak{p})$.

In particular if $m > 1$, then Spin $(\mathfrak{p})$ has center $\{\pm 1, \pm \epsilon\}$ which is cyclic Z_4 for m odd, noncyclic $Z_2 \times Z_2$ for m even.

Left multiplication in the complexified Clifford algebra

$$\text{Cliff}\ (\mathfrak{p})_C = \text{Cliff}\ (\mathfrak{p}) \otimes_R C$$

restricts to

(1.5a) $l^\pm$: linear representation of Spin $(\mathfrak{p})$ on Cliff$^\pm$ $(\mathfrak{p})_C$.

To decompose $l^\pm$ we fix an oriented orthonormal basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{p}$ and define

$$a_j = ie_{2j-1} \cdot e_{2j} \quad \text{for} \quad 1 \leqq j \leqq m, \quad n = 2m \quad \text{or} \quad 2m + 1.$$

Then $a_j^2 = 1$ and $a_j \cdot a_k = a_k \cdot a_j$. Each left multiplication $l^\pm(x)$ commutes with every right multiplication $y \to y \cdot a_j$. Thus $\mathrm{Cliff}^{\pm}(\mathfrak{p})_C$ is direct sum of 2^m subspaces

(1.5b) $$E_q^{\pm} = \{y \in \mathrm{Cliff}^{\pm}(\mathfrak{p})_C : y \cdot a_j = q_j y, \quad q_j = \pm 1, \quad 1 \leqq j \leqq m\}$$

on which $l^{\pm}(\mathrm{Spin}(\mathfrak{p}))$ acts irreducibly. That decomposes

(1.5c) $l^{\pm}$ is direct sum of 2^m irreducible representations of Spin $(\mathfrak{p})$.

Suppose $n = 2m + 1$. Then the irreducible summands of $l^{\pm}$ all are equivalent. Thus $\dim_C \mathrm{Cliff}(\mathfrak{p})_C = 2^{2m+1}$ gives us

(1.6) $l^{\pm} = 2^m s$ where s is irreducible and of degree 2^m.

The representation s is the **spin representation** of $\mathrm{Spin}(\mathfrak{p}) = \mathrm{Spin}(2m + 1)$.

Suppose $n = 2m$. Retain (1.4) and (1.5). If $y \in E_q^{\pm}$ and r is the number of q_i equal to -1, then

$$l^{\pm}(\epsilon)y = \epsilon \cdot y = \pm y \cdot \epsilon = \pm y \cdot a_1 \cdots a_m i^{-m} = \pm(-1)^r i^{-m} y.$$

This scalar value on ϵ determines the representation, so

(1.7a) $$l^{\pm} = 2^{m-1}s, \quad s = s^+ \oplus s^-, \quad s^{\pm} \text{ irreducible of degree } 2^{m-1}$$

where the orientation of $\mathfrak{p}$ distinguishes s^+ and s^- by

(1.7b) $$s^{\pm}(\epsilon) = \pm i^{-m} \quad \text{and} \quad s^{\pm}(-1) = -I.$$

The $s^{\pm}$ are the **half spin representations** of $\mathrm{Spin}(\mathfrak{p}) = \mathrm{Spin}(2m)$ and $s = s^+ \oplus s^-$ is the **spin representation.**

Multiplication in $\mathrm{Cliff}(\mathfrak{p})_C$ restricts to two maps

(1.8a) $$\tilde{\mu}: \mathfrak{p}_C \otimes \mathrm{Cliff}^{\pm}(\mathfrak{p})_C \to \mathrm{Cliff}^{\mp}(\mathfrak{p})_C .$$

Left multiplication by an element of $\mathfrak{p}_C$ commutes with right multiplication by any a_j . Thus, in the notation (1.5b),

(1.8b) $\tilde{\mu}: \mathfrak{p}_C \otimes E_q^{\pm} \to E_q^{\mp}$ linear isomorphism.

If we denote

(1.9a) S: representation space of the spin representation s,

then (1.6) and (1.7) show that $\tilde{\mu}$ induces maps

(1.9b) $\mu: \mathfrak{p}_C \otimes S \to S$ linear isomorphism.

In case $n = 2m$ we further denote

(1.10a) $S^{\pm}$: representation space of $s^{\pm}$; so $S = S^+ \oplus S^-$.

Then (1.7b) and the calculation just before (1.7a) show that

(1.10b) $\mu = \mu^+ \oplus \mu^-$ where $\mu^\pm : \mathfrak{p}_C \otimes S^\pm \to S^\mp$ linear isomorphisms.

We will need the multiplication maps μ and $\mu^\pm$ for construction of the Dirac operators.

§2. Differential-geometric preliminaries. We work over an oriented n-dimensional riemannian manifold Y. Thus the bundle $\pi : \mathfrak{F} \to Y$ of oriented orthonormal frames is a principal SO (n)-bundle. Let

(2.1) $\alpha : K \to \mathrm{SO}\ (n)$ Lie group homomorphism that factors through Spin (n).

To define our Dirac operators, we must lift both the bundle $\pi : \mathfrak{F} \to Y$ and its riemannian connection to a principal K-bundle over Y. This will entail some restrictions on K and α.

By **smooth** (K, α)**-structure** on Y we mean a principal K-bundle $\pi_K : \mathfrak{F}_K \to Y$ such that bundle projections are related by a commutative diagram

(2.2)
$$\begin{array}{ccc} F_K & \xrightarrow{\ \bar{\alpha}\ } & F \\ & {\scriptstyle \pi_K} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & Y & \end{array}$$
with $\bar{\alpha}$ given by (2.1) on each π_K-fibre.

By **spin structure** we mean a smooth (Spin (n), v)-structure. Since α factors through Spin (n), every (K, α)-structure specifies a spin structure.

It is easy to enumerate the smooth (K, α)-structures in sheaf language. If A is a Lie group write $\mathbf{A} \to Y$ for the sheaf of germs of C^∞ functions from Y to A. Then the set of all equivalence classes of principal A-bundles over Y is in one to one correspondence with the elements of the cohomology set $H^1(Y; \mathbf{A})$. In effect, the transition functions of a bundle form a cocycle relative to any locally trivializing open cover, and equivalence modifies the cocycle within the same cohomology class. Now denote

(2.3) $\tau \in H^1(Y; \mathbf{SO}(n))$: element corresponding to $\pi : \mathfrak{F} \to Y$

and

(2.4) $\alpha_\# : H^1(Y; \mathbf{K}) \to H^1(Y; \mathbf{SO}(n))$ coefficient map defined by α.

Then the smooth (K, α)-structures on Y are in one to one correspondence with the elements of the (possibly empty) set $\alpha_\#^{-1}(\tau)$.

If α has kernel L that is central in K, we have an exact diagram

(2.5)
$$\begin{array}{ccccccccc} & & & & & & 1 & & \\ & & & & & & \downarrow & & \\ 1 & \to & L & \to & K & \to & K/L & \to & 1 \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{SO}(n) & & \end{array}$$

and that gives an exact cohomology diagram

$$
\begin{array}{ccccccc}
 & & & & \{1\} & & \\
 & & & & \downarrow & & \\
\rightarrow H^1(Y; \mathbf{L}) & \xrightarrow{i} & H^1(Y; \mathbf{K}) & \xrightarrow{a} & H^1(Y; \mathbf{K}/\mathbf{L}) & \xrightarrow{\delta} & H^2(Y; \mathbf{L}). \\
 & & & & j\downarrow & & \\
 & & & & H^1(Y; \mathbf{SO}(n)) & &
\end{array} \tag{2.6}
$$

Thus there is a smooth (K, α)-structure if and only if (i) $\tau = j(\tau')$ and (ii) $\delta(\tau') = 0$; and in that case they are enumerated by $iH^1(Y, \mathbf{L})$. For example, if Y is connected this says that there is a spin structure just when the Stiefel–Whitney class $\delta(\tau) = w_2(Y)$ is zero, and in that case it says that the spin structures are enumerated by $H^1(Y; Z_2)$.

Now fix a smooth (K, α)-structure $\pi_K : \mathfrak{F}_K \rightarrow Y$. We see about lifting the riemannian connection from $\mathfrak{F}$ to $\mathfrak{F}_K$. If Γ_K is a connection on $\pi_K : \mathfrak{F}_K \rightarrow Y$ we recall that $\bar{\alpha}(\Gamma_K)$ is the connection on $\pi: \mathfrak{F} \rightarrow Y$ such that the $\bar{\alpha}(\Gamma_K)$-horizontal space at $\bar{\alpha}(f)$ is the $\bar{\alpha}$-image of the Γ_K-horizontal space at $f \in \mathfrak{F}_K$. In other words, if ω_K is the connection form of Γ_K, then $\alpha \cdot \omega_K$ is the $\bar{\alpha}^*$-image of the connection form of $\bar{\alpha}(\Gamma_K)$. If $\bar{\alpha}(\Gamma_K)$ is the riemannian connection on $\pi: \mathfrak{F} \rightarrow Y$, then we say that $(\mathfrak{F}_K, \Gamma_K)$ is a **riemannian** (K, α)**-structure** on Y.

A smooth spin structure specifies a riemannian spin structure. For over every component of Y, $\bar{\alpha}: \mathfrak{F}_K \rightarrow \mathfrak{F}$ is a (two sheeted) covering, so the riemannian connection lifts from $\mathfrak{F}$ to $\mathfrak{F}_K$. But in general one needs some conditions.

Denote Lie algebras by small German letters, so

(2.7) $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{l}$ is its ideal for $L = \text{kernel}\ (\alpha)$.

2.8. Lemma. *Suppose that* $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$ *direct sum of ideals. Let* Γ *be the riemannian connection on* $\mathfrak{F}$ *and* ω *its connection form. Then the following conditions are equivalent.*

1. $\mathfrak{F}_K$ *has a connection* Γ_K *such that* $\bar{\alpha}(\Gamma_K) = \Gamma$.
2. $\alpha(K)$ *contains the holonomy group of every component of* Y.
3. *The restriction of* ω *to* $\bar{\alpha}(\mathfrak{F}_K)$ *takes values in* $\alpha(\mathfrak{k})$.

Let Γ_K *be one connection on* $\mathfrak{F}_K$ *with* $\bar{\alpha}(\Gamma_K) = \Gamma$, *and let* ω_K *be its connection form. If* Γ_K' *is another connection on* $\mathfrak{F}_K$, *then* $\bar{\alpha}(\Gamma_K') = \Gamma$ *if and only if* Γ_K' *has connection form* $\omega_K + \lambda$ *where* λ *is an* $\mathfrak{l}$*-valued linear differential form on* $\mathfrak{F}_K$ *that annihilates the vertical spaces.*

Proof. Given (1), let ω_K be the connection form of Γ_K, so $\alpha \cdot \omega_K = \bar{\alpha}^*(\omega)$. Then $\bar{\alpha}^*(\omega)$ takes values in $\alpha(\mathfrak{k})$. Now (2) follows because the values of $\omega | \bar{\alpha}(\mathfrak{F}_K)$ are the values of $\bar{\alpha}^*(\omega)$.

Given (2), $\bar{\alpha}(\mathfrak{F}_K)$ is a sub-bundle of $\mathfrak{F}$ stable under the riemannian parallelism, so (3) follows from the Holonomy Theorem.

Assume (3) and denote $\eta = \bar{\alpha}^*(\omega)$. Then η is a linear differential form on $\mathfrak{F}_K$ with values in $\alpha(\mathfrak{k})$. By hypothesis $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$ we may view η as having values in $\mathfrak{m}$. If $\xi \in \mathfrak{k}$ then ξ' denotes the corresponding vertical vector field on $\mathfrak{F}_K$. Now, for every $f \in \mathfrak{F}_K$,

$$\eta(\xi'_f) = 0 \text{ if } \xi \in \mathfrak{l} \qquad \text{and} \qquad \eta(\xi'_f) = \xi \text{ if } \xi \in \mathfrak{m}.$$

Choose a K-invariant riemannian metric on $\mathfrak{F}_K$ such that $\{\xi'_f : \xi \in \mathfrak{l}\} \perp \{\xi'_f : \xi \in \mathfrak{m}\}$ for every $f \in \mathfrak{F}_K$. Define an $\mathfrak{l}$-valued linear differential form β on $\mathfrak{F}_K$ by

$$\beta(\xi'_f) = \xi \text{ if } \xi \in \mathfrak{l} \text{ and } \beta(\{\xi'_f : \xi \in \mathfrak{l}\}^{\perp}) = 0.$$

Then $\omega_K = \eta \oplus \beta$ is a $\mathfrak{k}$-valued linear differential form on $\mathfrak{F}_K$ such that $\omega_K(\xi'_f) = \xi$ for all $\xi \in \mathfrak{k}$, and such that the zero-spaces of ω_K form a K-invariant distribution Γ_K on $\mathfrak{F}_K$. Now Γ_K is a connection and ω_K is its connection form, and $\bar{\alpha}(\Gamma_K) = \Gamma$ because $\alpha \cdot \omega_K = \eta = \bar{\alpha}^*(\omega)$.

We have shown (1), (2) and (3) equivalent.

Let Γ_K and Γ'_K be connections on $\mathfrak{F}_K$, ω_K and ω'_K their connection forms. Then ω_K and ω'_K agree on vertical vectors, sending ξ'_f to $\xi \in \mathfrak{k}$. Further $\bar{\alpha}(\Gamma_K) = \bar{\alpha}(\Gamma'_K)$ precisely when $\alpha \cdot \omega_K = \alpha \cdot \omega'_K$, *i.e.*, when $\omega'_K - \omega_K$ takes values in $\mathfrak{l}$. The last assertion follows. Q.E.D.

Note that the hypothesis that $\mathfrak{l}$ be a direct summand of $\mathfrak{k}$, is automatic if (i) L is discrete or (ii) $\mathfrak{k}$ is reductive. If L is discrete, Lemma 2.8 shows that there is at most one riemannian (K, α)-structure.

We summarize as follows, retaining the notation (2.3) through (2.7).

2.9. Proposition. *Let $\mathfrak{l}$ be a direct summand of $\mathfrak{k}$. Then there is a riemannian (K, α)-structure on Y if and only if*

(2.10a) *$\alpha(K)$ contains the holonomy group of every component of Y*

and

(2.10b) $$\alpha_{\#}^{-1}(\tau) \text{ is not empty.}$$

If L is central in K and if (2.10a) *holds, then $\tau' = j^{-1}(\tau)$ exists and* (2.10b) *is equivalent to*

(2.10c) $$\delta j^{-1}(\tau) = 0.$$

§3. Definition of Dirac operators. Y is an oriented n-dimensional riemannian manifold and $\alpha: K \to SO(n)$ is a Lie group homomorphism that factors through Spin (n). We suppose that Y has a riemannian (K, α)-structure, and we fix one such structure $(\mathfrak{F}_K, \Gamma_K)$.

Viewing α as a unitary representation $K \to U(n)$, we have the vector bundle associated to $\mathfrak{F}_K$:

(3.1) $\mathfrak{Z} \to Y$ complexified tangent bundle.

As α factors through Spin (n) we have

$$\begin{array}{ccc} & & \mathrm{Spin}\,(n) \\ & \nearrow^{\tilde{\alpha}} & \downarrow v \\ K & & \\ & \searrow_{\alpha} & \\ & & \mathrm{SO}(n) \end{array} \qquad \text{commutative diagram.}$$

The vector bundle associated to $\mathfrak{F}_K$ by $s\cdot\tilde{\alpha}$ is

(3.2a) $\mathcal{S} \to Y$ **spin bundle.**

If n is even, then $s = s^+ \oplus s^-$ and so $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ where

(3.2b) $\mathcal{S}^\pm \to Y$ **half spin bundles.**

Now we fix

(3.3a) κ: finite dimensional unitary representation of K

and denote

(3.3b) $\mathcal{V}_\kappa \to Y$ vector bundle associated to $\mathfrak{F}_K$ by κ.

All these bundles carry hermitian metrics because they come from unitary representations of K.

The connection Γ_K specifies covariant differentiation of sections of vector bundles associated to $\mathfrak{F}_K$. Writing $C^\infty(\cdot)$ for C^∞ sections, we denote the covariant differentials by

(3.4a) $\nabla: C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa) \to C^\infty(\mathfrak{T}^* \otimes \mathcal{S} \otimes \mathcal{V}_\kappa)$ in general,

(3.4b) $\nabla^\pm: C^\infty(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa) \to C^\infty(\mathfrak{T}^* \otimes \mathcal{S}^\pm \otimes \mathcal{V}_\kappa)$ for n even.

The hermitian metric on $\mathfrak{T}$, *i.e.*, the riemannian metric of Y, specifies

(3.5a) $h: \mathfrak{T}^* \to \mathfrak{T}$ conjugate-linear bundle isomorphism.

The map (1.9b) on fibres specifies

(3.5b) $\bar{\mu}: \mathfrak{T} \otimes \mathcal{S} \to \mathcal{S}$ bundle isomorphism.

Similarly, if n is even, then $\bar{\mu} = \bar{\mu}^+ \oplus \bar{\mu}^-$ where (1.10b) on fibres gives

(3.5c) $\bar{\mu}^\pm: \mathfrak{T} \otimes \mathcal{S}^\pm \to \mathcal{S}^\mp$ bundle isomorphism.

By definition, the compositions

$$(\bar{\mu} \otimes 1)\cdot(h \otimes 1 \otimes 1)\cdot\nabla \quad \text{in general,}$$

$$(\bar{\mu}^\pm \otimes 1)\cdot(h \otimes 1 \otimes 1)\cdot\nabla^\pm \quad \text{if } n \text{ is even}$$

are the **Dirac operators**

(3.6a) $D: C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa) \to C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$ in general,

(3.6b) $D^\pm: C^\infty(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa) \to C^\infty(\mathcal{S}^\mp \otimes \mathcal{V}_\kappa)$ for n even.

We refer to sections of $\mathcal{S} \otimes \mathcal{V}_\kappa$ and $\mathcal{S}^\pm \otimes \mathcal{V}_\kappa$ as **spinors** on Y **with values in** $\mathcal{V}_\kappa$. Those annihilated by the Dirac operator are **Dirac spinors** on Y with values in $\mathcal{V}_\kappa$.

3.7. Lemma. *The Dirac operators are elliptic.*

3.8. Lemma. *Let $\{e_1, \cdots, e_n\}$ be an orthonormal moving frame on an open set $U \subset Y$. Then the Dirac operators are given on U by*

$$D(u) = \sum_{1 \leq i \leq n} e_i \cdot \nabla_i(u), \qquad \nabla_i = \nabla_{e_i}, \tag{3.9}$$

where "·" denotes Clifford multiplication.

Proof. The orthonormal frame is a trivialization of $\mathfrak{F}$ over U, thus also trivializes $\mathfrak{F}_K$ and the vector bundles over U. Let $\{t_1, \cdots, t_r\}$ be sections of $\mathcal{S} \otimes \mathcal{V}_\kappa$ over U that give a basis of the fibre at every point. In the frames $\{e_i\}$, $\{t_p\}$ the connection form (ω_p^q) is a matrix of 1-forms on U such that $\nabla(t_p) = \sum_q \omega_p^q \otimes t_q$, *i.e.*, $\nabla_i(t_p) = \sum_q \omega_p^q(e_i)t_q$. Let $u = \sum_p f^p t_p$ section of $\mathcal{S} \otimes \mathcal{V}_\kappa$ over U. Then

$$\nabla(u) = \sum_p \{df^p \oplus t_p + f^p \sum_q \omega_p^q \oplus t_q\} \text{ and } \nabla_i(u) = \sum_p \{e_i(f^p)t_p + \sum_q \omega_p^q(e_i)t_q\}.$$

Thus

$$\begin{aligned} D(u) &= \sum_p \{h(df^p)\cdot t_p + f^p \sum_q h(\omega_p^q)\cdot t_q\} \\ &= \sum_{i,p} \{df^p(e_i)e_i\cdot t_p + f^p \sum_q \omega_p^q(e_i)e_i\cdot t_q\} = \sum_i e_i \cdot \nabla_i(u). \end{aligned}$$

That proves Lemma 3.8. If $\theta \in \mathfrak{I}^*$ now the symbol $\sigma(D)$ (θ) is Clifford multiplication by $h(\theta)$, which has square $h(\theta)\cdot h(\theta) = -\langle h(\theta), h(\theta)\rangle = -\,||\theta||^2$. Thus, if $\theta \neq 0$ real $\sigma(D)$ (θ) is bijective. This proves Lemma 3.7. Q.E.D.

§4. **Symmetry of D on square integrable spinors.** Y is an oriented riemannian n-manifold with a fixed riemannian (K, α)-structure $(\mathfrak{F}_K, \Gamma_K)$. We have a finite dimensional unitary representation κ of K, and D is the Dirac operator $C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa) \to C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$ on spinors with values in $\mathcal{V}_\kappa$.

If f is a (Borel-) measurable function on Y, then $\int_Y f(y)\, dy$ denotes its integral against the measure specified by the orientation and riemannian metric of Y.

If $\mathcal{W} \to Y$ is a hermitian vector bundle, and if ϕ and ψ are measurable sections, then the pointwise inner product $\langle \phi, \psi\rangle_y = \langle \phi(y), \psi(y)\rangle$ is a measurable function on Y. The global inner product is

$$\langle \phi, \psi\rangle = \int_Y \langle \phi, \psi\rangle_y\, dy. \tag{4.1a}$$

As usual, the L_2-norm is $||\phi||^2 = \langle \phi, \phi\rangle \leqq \infty$, and

$$L_2(\mathcal{W}) = \{\text{measurable sections } \phi \text{ of } \mathcal{W}\colon ||\phi|| < \infty\} \tag{4.1b}$$

is a Hilbert space with inner product (4.1a). It is important to note that the space $C_c^\infty(\mathcal{W})$ of smooth compactly supported sections, satisfies

(4.1c) $$C_c^\infty(\mathcal{W}) \text{ is dense in } L_2(\mathcal{W}).$$

We now have Hilbert spaces

(4.2) $$L_2(\mathcal{S} \otimes \mathcal{V}_\kappa) \quad \text{and} \quad L_2(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa):$$

square integrable spinors on Y with values in $\mathcal{V}_\kappa$.

The Dirac operators may be viewed as densely defined linear operators on these spaces with domains $C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$ and $C_c^\infty(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa)$. In this §4 we prove that D, with domain $C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$, is a symmetric operator on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$. More precisely we will prove

4.3. Proposition. *Let u and v be sections of $\mathcal{S} \otimes \mathcal{V}_\kappa$ such that*
(i) *u has compact support and satisfies a Lipschitz condition,*
(ii) *v is locally Lipschitz on a neighborhood of support (u).*
Then $\langle D(u), v\rangle = \langle u, D(v)\rangle$.

We set up the procedure for integration by parts used in the proof of Proposition 4.3.

Let ξ be a C^1 tangent vector field on an open set $U \subset Y$. If s and t are locally Lipschitz sections of $\mathcal{S} \otimes \mathcal{V}_\kappa$ over U, we claim

(4.4) $$\xi(\langle s, t\rangle) = \langle \nabla_\xi(s), t\rangle + \langle s, \nabla_\xi(t)\rangle \text{ a.e. in } U.$$

This holds at points of U at which both s and t are differentiable, because $\mathcal{S} \otimes \mathcal{V}_\kappa$ is associated to $\mathfrak{F}_K$ by a unitary representation of K. Since s and t are each differentiable a.e. in U, now (4.4) follows.

Let ξ be a Lipschitz tangent vector field on an open set $U \subset Y$. Its **divergence** is the function div (ξ) on U that is contraction of the covariant differential $\nabla\xi$. In moving orthonormal frame $\{e_1, \cdots, e_n\}$, if ξ is differentiable at y,

(4.5a) $$\operatorname{div}(\xi)(y) = \sum_{1 \leq i \leq n} \langle \nabla_i(\xi), e_i\rangle_y\,, \qquad \nabla_i = \nabla_{e_i}\,.$$

Let Ω be an open set with compact closure $\bar\Omega \subset U$, whose boundary $\partial\Omega = \bar\Omega - \Omega$ is the union of a smooth $(n-1)$-manifold $\partial\Omega'$ and a singular set $\partial\Omega''$ of dimension $\leq n-2$. Let ν denote the outward unit normal on $\partial\Omega'$. Orient $\partial\Omega'$ so that it has volume element $db = \theta^2 \wedge \cdots \wedge \theta^n$ whenever $\{e_i\}$ is an orthonormal moving frame with $\nu = e_1|_{\partial\Omega'}$ such that, in the dual co-frame $\{\theta^i\}$, Y has volume element $dy = \theta^1 \wedge \cdots \wedge \theta^n$. The Lipschitz condition ensures that ξ is differentiable a.e. and div (ξ) is measurable. The **divergence theorem** says

(4.5b) $$\int_\Omega \operatorname{div}(\xi)(y)\, dy = \int_{\partial\Omega'} \langle \xi, \nu\rangle_b\, db.$$

In effect, $\xi = \sum \xi^i e_i$ locally and the Cartan structure equations give

$$d\left(\sum(-1)^{i-1}\xi^i\theta^1 \wedge \cdots \wedge \theta^{i-1} \wedge \theta^{i+1} \wedge \cdots \wedge \theta^n\right) = \operatorname{div}(\xi)\theta^1 \wedge \cdots \wedge \theta^n. \tag{4.5c}$$

Now (4.5b) follows from (4.5c), Stokes' Theorem and partition of unity.

Retain U, ξ and Ω as above. Let f be a Lipschitz function on U. Substituting $f\xi$ for ξ in (4.5a) we get

$$\operatorname{div}(f\xi) = \xi(f) + f \operatorname{div}(\xi) \quad \text{a.e. in } U. \tag{4.6a}$$

If we replace ξ by $f\xi$ in (4.5b), that says

$$\int_\Omega \xi(f)(y)\, dy = \int_{\partial\Omega'} f(b)\langle \xi, \nu\rangle_b\, db - \int_\Omega f(y) \operatorname{div}(\xi)(y)\, dy, \tag{4.6b}$$

which is our formula for integration by parts. In particular, if f has compact support in Ω, then the boundary term drops out and we are left with

$$\int_\Omega \xi(f)(y)\, dy = -\int_\Omega f(y) \operatorname{div}(\xi)(y)\, dy. \tag{4.6c}$$

One more preliminary remark. If ξ is a tangent vector at $y \in Y$, then left Clifford multiplication by ξ has square $-\langle \xi, \xi\rangle_y$ on the fibre $(\mathcal{S} \otimes \mathcal{V}_\kappa)_y$. Polar decomposition thus implies

$$\langle \xi\cdot s, t\rangle_y + \langle s, \xi\cdot t\rangle_y = 0 \quad \text{for} \quad s, t \in (\mathcal{S} \otimes \mathcal{V}_\kappa)_y. \tag{4.7}$$

Proof of Proposition 4.3. It suffices to consider the case where support (u) is contained in an open set that carries a moving orthonormal frame. For then a partition of unity argument gives $u = u_1 + \cdots + u_l$ where each u_k is Lipschitz with compact support in an open set that carries an orthonormal moving frame, so $\langle D(u), v\rangle = \sum \langle D(u_k), v\rangle = \sum \langle u_k, D(v)\rangle = \langle u, D(v)\rangle$.

Now u has compact support in an open set U that carries an orthonormal moving frame $\{e_1, \cdots, e_n\}$. Using (3.9), then (4.7), then (4.4), then (4.6c), we compute

$$\begin{aligned}
\langle D(u), v\rangle &= \sum \int_U \langle e_i \cdot \nabla_i(u), v\rangle_y\, dy \\
&= -\sum \int_U \langle \nabla_i(u), e_i\cdot v\rangle_y\, dy \\
&= -\sum \int_U e_i(\langle u, e_i\cdot v\rangle_y)\, dy + \sum \int_U \langle u, \nabla_i(e_i\cdot v)\rangle_y\, dy \\
&= \sum \int_U \langle u, e_i\cdot v\rangle_y \operatorname{div}(e_i)(y)\, dy + \sum \int_U \langle u, \nabla_i(e_i\cdot v)\rangle_y\, dy \\
&= \int_U \langle u, \sum \{\nabla_i(e_i\cdot v) + \operatorname{div}(e_i)e_i\cdot v\}\rangle_y\, dy.
\end{aligned}$$

Thus the adjoint of D is given on U by the formula

(4.8) $$D^*(v) = \sum_{1 \leq i \leq n} (\nabla_i + \operatorname{div}(e_i))(e_i \cdot v).$$

Using (3.9) and (4.8) one sees that the first order terms cancel in $D^* - D$. In other words,

$$(D^* - D)(fv) = f(D^* - D)(v) \text{ for } f \text{ and } v \text{ differentiable.}$$

Thus $[(D^* - D)(v)](y)$ is determined by $v(y)$. To compute it, we change v and $\{e_i\}$ in a neighborhood of y (but not at y) as follows. We replace v (resp. e_i) by the spinor (resp. vector) obtained by parallel translation of $v(y)$ (resp. $e_i(y)$) out along geodesic rays that start at y. That change made,

$$\nabla e_i\,,\ \nabla v \text{ and } \nabla(e_i \cdot v) \quad \text{all vanish at } y.$$

Thus every term vanishes at y in the expression of $(D^* - D)(v)$ by formulae (3.9) and (4.8), so $(D^* - D)(v)$ vanishes at y. As y was arbitrary now $(D^* - D)(v) = 0$. Thus $\langle D(u), v\rangle = \langle u, D^*(v)\rangle = \langle u, D(v)\rangle$. Q.E.D.

One can add a potential and consider operators $D_{\xi,f}(u) = D(u) + i\xi \cdot u + fu$ where ξ is a real tangent vector field, f is a real function, and ξ and f are locally L_2. Then again $\langle D_{\xi,f}(u), v\rangle = \langle u, D_{\xi,f}(v)\rangle$ if u and v are Lipschitz and one of them is compactly supported, so $D_{\xi,f}$ is a symmetric operator on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ with dense domain $C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$.

§5. Essential self adjointness for the Dirac operator. Y is an oriented riemannian n-manifold with a fixed riemannian (K, α)-structure, and D is the corresponding Dirac operator on $\mathcal{S} \otimes \mathcal{V}_\kappa$. We have just seen that D is symmetric as operator on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ with dense domain $\mathbf{D}(D) = C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$.

5.1. Theorem. *If the riemannian metric of Y is complete, then D is essentially self adjoint,* i.e., *the closure $\tilde{D}$ of D from $C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$ is the unique self adjoint extension of D.*

Proof. We write L_2 for $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ and C_c^∞ for the domain $\mathbf{D}(D) = C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$ of D. Then the adjoint D^* of D has domain

$$\mathbf{D}(D^*) = \{w \in L_2 : \exists\ w' \in L_2 \quad \text{with} \quad \langle Du, w\rangle = \langle u, w'\rangle, \quad \text{all} \quad u \in C_c^\infty\}$$

and there $D^*(w) = w'$. The closure $\tilde{D}$ of D has domain

$$\mathbf{D}(\tilde{D}) = \{\lim u_n : \{u_n\} \subset C_c^\infty \quad \text{is Cauchy and} \quad \{Du_n\} \quad \text{is Cauchy}\}$$

and there $\tilde{D}(\lim u_n) = \lim D(u_n)$. D^* and $\tilde{D}$ are well defined because D is symmetric with dense domain. $\tilde{D}$ is symmetric, $\tilde{D} = D^{**} \subset D^*$, and $\tilde{D}$ is the minimal closed extension of D. If we prove

(5.2) $$\mathbf{D}(D^*) \subset \mathbf{D}(\tilde{D}),$$

then we will have $\tilde{D} = D^*$, so $\tilde{D}^* = D^{**} = \tilde{D}$; then $\tilde{D}$ will be self adjoint so D will be essentially self adjoint.

$\mathbf{D}(D^*)$ carries the norm $N(w) = \{||w||^2 + ||D^*w||^2\}^{1/2}$. The containment (5.2) is equivalent to

(5.3) $\qquad C_c^\infty$ is dense in $\mathbf{D}(D^*)$ with the norm N.

We will prove Theorem 5.1 by proving (5.3).

Denote $\mathbf{D}_c(D^*) = \{w \in \mathbf{D}(D^*): w \text{ has compact support}\}$. Then $C_c^\infty \subset \mathbf{D}_c(D^*) \subset \mathbf{D}(D^*)$. We are going to prove

(5.4) $\qquad C_c^\infty$ is dense in $\mathbf{D}_c(D^*)$ with the norm N.

Let $w \in \mathbf{D}_c(D^*)$. Choose a locally finite coordinate cover $\{U_i\}$ of Y, closures $\bar{U}_i$ compact, and a C^∞ partition of unity $\{f_i\}$ with support $(f_i) \subset U_i$. We may suppose that $\{1, 2, \cdots, l\}$ are the only indices i such that support (f_i) meets the compact set support (w). Then $w = w_1 + \cdots + w_l$ where $w_i = f_i w$ has compact support in U_i. Fix local trivializations of $\mathcal{S} \otimes \mathcal{V}_\kappa$ over U_i and local coordinates $x_i : U_i \to R^n$. Then w_i goes to a compactly supported distribution on $x_i(U_i)$ with values in the typical fibre $S \otimes V_\kappa$ of $\mathcal{S} \otimes \mathcal{V}_\kappa$. Convolution with a C^∞ approximate identity on R^n gives sequences $\{u_{ik}\}_k$ of C_c^∞ functions $x_i(U_i) \to S \otimes V_\kappa$ which, re-interpreted as C_c^∞ sections of $\mathcal{S} \otimes \mathcal{V}_\kappa$, satisfy $N(w_i - u_{ik}) < 1/k$. Now $u_k = u_{1k} + \cdots + u_{lk} \in C_c^\infty$ satisfy $N(w - u_k) < l/k$, so $\{u_k\} \to w$ in the norm N. That completes the proof of (5.4).

If Y were compact, we would have $\mathbf{D}_c(D^*) = \mathbf{D}(D^*)$, and Theorem 5.1 would be proved. For the general case we now use the methods of Gaffney [4] and Andreotti–Vesentini [1] to obtain the estimate (5.8) below. That estimate will enable us to prove

(5.5) $\qquad \mathbf{D}_c(D^*)$ is dense in $\mathbf{D}(D^*)$ with the norm N.

Combining (5.4) and (5.5) we will have (5.3) and so Theorem 5.1 will be proved.

If u and v are measurable sections $\mathcal{S} \otimes \mathcal{V}_\kappa$ over a Borel set $B \subset Y$, we denote $\langle u, v\rangle_B = \int_B \langle u, v\rangle_y \, dy$ and $||u||_B^2 = \langle u, u\rangle_B$.

Fix $y_0 \in Y$ and let Y_0 be the topological component of Y that contains it. If $y \in Y_0$, then $\rho(y)$ denotes riemannian distance from y_0 to y. Triangle inequality in the metric space Y_0 says $|\rho(y) - \rho(x)| \leqq \text{distance } (y, x)$. Thus ρ is locally Lipschitz, hence differentiable a.e., and at points of differentiability its gradient has length $|\text{grad } (\rho)| \leqq 1$.

If $r > 0$ let $B_r = \{y \in Y_0 : \rho(y) < r\}$. Completeness of Y says that the open set B_r has compact closure $\bar{B}_r$.

Choose a C^∞ function $a: R^1 \to [0, 1]$ such that $a(-\infty, 1] = 1$ and $a[2, \infty) = 0$. Denote $M = \max |a'(t)|$. If $r > 0$ define $b_r : Y_0 \to [0, 1]$ by $b_r(y) = a(\rho(y)/r)$. Then $b_r \equiv 1$ on B_r and support $(b_r) \subset \bar{B}_{2r}$. From the corresponding properties of ρ, b_r is locally Lipschitz, thus a.e. differentiable, and at points of differentiability it has

$$(5.6) \qquad |\text{grad}\,(b_r)|^2 = \left|\frac{1}{r}\,a'(\rho/r)\,\text{grad}\,(\rho)\right|^2 \leqq M^2/r^2.$$

Fix $w \in \mathbf{D}(D^*)$ and consider the sequence $\{w_k\} \subset \mathbf{D}_c(D^*)$ given by $w_k = b_k w$ ($k = 1, 2, \cdots$). Here we are using completeness of Y so that the set $\bar{B}_{2k}$ containing support (b_k) is compact. Using (3.9) to compute locally in an orthonormal moving frame $\{e_i\}$, Proposition 4.3 gives $D^*(w_k) = D^*(b_k w) = \sum e_i \cdot \nabla_i(b_k w) = \sum e_i \cdot (e_i(b_k)w + b_k \nabla_i(w)) = \text{grad}\,(b_k)\cdot w + b_k D^*(w)$ where b_k is differentiable. Thus

$$(5.7) \qquad D^*(w_k) = \text{grad}\,(b_k)\cdot w + b_k D^*(w) \text{ a.e. on } Y_0\,.$$

Using (5.7) and $b_k \equiv 1$ on B_k we see

$$\begin{aligned} ||D^*(w - w_k)||^2_{Y_0} &= ||(1 - b_k)D^*(w) + \text{grad}\,(b_k)\cdot w||^2_{Y_0} \\ &\leqq ||D^*(w)||^2_{Y_0 - B_k} + \frac{M^2}{k^2}\,||w||^2_{Y_0}\,. \end{aligned}$$

That gives us

$$(5.8) \qquad N(w\,|_{Y_0} - w_k)^2 \leqq ||D^*(w)||^2_{Y_0 - B_k} + ||w||^2_{Y_0 - B_k} + \frac{M^2}{k^2}\,||w||^2_{Y_0}\,.$$

Let $\{Y_\beta\}$ be the topological components of Y, so L_2 is discrete direct sum of the $L_2((\mathcal{S} \otimes \mathcal{V}_\kappa)|_{Y_\beta})$. As $\sum ||w||^2_{Y_\beta} = ||w||^2 < \infty$, only countably many $w|_{Y_\beta}$ are nonzero, so there is a (possibly finite) sequence $\{Y_a\}$ of components of Y with $w = \sum w|_{Y_a}$ and $D^*(w) = \sum D^*(w)|_{Y_a}$. If we set $Y_a = Y_0$ in (5.8) we get a sequence $\{v_{al}\}_l \subset \mathbf{D}_c(D^*)$ of elements supported in Y_a such that $N(w|_{Y_a} - v_{al})^2 \leqq 1/2^a l$. Then the $u_l = \sum_{1 \leqq a \leqq l} v_{al}$ form a sequence in $\mathbf{D}_c(D^*)$ such that $N(w - u_l)^2 \leqq \sum_{a > l} ||w||^2_{Y_a} + 1/l$. We conclude $\{u_l\} \to w$ in the norm N. That completes the proof of (5.5). Now Theorem 5.1 is proved. Q.E.D.

§6. The Dirac operator has essentially self adjoint square. Y is a complete oriented riemannian n-manifold with a fixed riemannian (K, α)-structure, and D is the corresponding Dirac operator on $\mathcal{S} \otimes \mathcal{V}_\kappa$. We have just seen that D is essentially self adjoint as operator on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ with dense domain $C^\infty_c(\mathcal{S} \otimes \mathcal{V}_\kappa)$. For applications it is useful to know that D^2 has well defined spectrum.

6.1. Theorem. *If the riemannian metric of Y is complete, then D^2 is essentially self adjoint with domain $C^\infty_c(\mathcal{S} \otimes \mathcal{V}_\kappa)$, and D^2 has closure $\tilde{D}^*\tilde{D} = \tilde{D}^2$.*

Our proof consists of combining essential self adjointness of D with the following estimate (in which D and D^2 act as differential operators).

6.2. Proposition. *If the riemannian metric of Y is complete, $u \in C^2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ and $t > 0$, then*

(6.3a) $$||D(u)||^2 \leqq t\,||D^2(u)||^2 + \frac{1}{t}\,||u||^2 \leqq \infty .$$

In particular (let $t = 1$)

(6.3b) *if $||u|| < \infty$ and $||D^2(u)|| < \infty$, then $||D(u)|| < \infty$*

and (let $t \to \infty$)

(6.3c) *if $||u|| < \infty$ and $D^2(u) = 0$, then $D(u) = 0$.*

Proof of Proposition 6.1. We write L_2, C^k and C_c^k for $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$, $C^k(\mathcal{S} \otimes \mathcal{V}_\kappa)$ and $C_c^k(\mathcal{S} \otimes \mathcal{V}_\kappa)$, and we retain the notation established between (5.5) and (5.6). We are going to use the method of Andreotti–Vesentini [1, §6] to prove:

(6.4) if $r, t > 0$ and $u \in C^2$, then

$$||D(u)||^2_{B_r} \leqq t\,||D^2(u)||^2_{B_{2r}} + \left(\frac{1}{t} + \frac{4M^2}{r^2}\right)||u||^2_{B_{2r}} .$$

In an orthonormal moving frame $\{e_i\}$ we compute $D(b_r^2 v) = \sum e_i \cdot \nabla_i(b_r^2 v) = \sum e_i \cdot (2b_r e_i(b_r)v + b_r^2\,\nabla_i(v)) = 2b_r \operatorname{grad}(b_r)\cdot v + b_r^2 D(v)$ where b_r and v are differentiable. Thus

$$D(b_r^2 v) = 2b_r \operatorname{grad}(b_r)\cdot v + b_r^2 D(v) \text{ a.e. on } Y_0 \text{ for } v \text{ Lipschitz.}$$

As b_r has compact support in $\bar{B}_{2r}$ now, using Proposition 4.3,

$$\begin{aligned} ||b_r D(u)||^2_{B_{2r+\epsilon}} &= \langle b_r^2 D(u), D(u)\rangle_{B_{2r+\epsilon}} = \langle D(b_r^2 D(u)), u\rangle_{B_{2r+\epsilon}} \\ &= \langle 2b_r \operatorname{grad}(b_r)\cdot D(u), u\rangle_{B_{2r+\epsilon}} + \langle b_r^2 D^2(u), u\rangle_{B_{2r+\epsilon}} \end{aligned}$$

for every $\epsilon > 0$. In view of (4.7),

(6.5a) $$||b_r D(u)||^2_{B_{2r}} = \langle D^2(u), b_r^2 u\rangle_{B_{2r}} - \langle b_r D(u), 2 \operatorname{grad}(b_r)\cdot u\rangle_{B_{2r}} .$$

The Schwarz inequality and (5.6) combine to give

(6.5b) $$|\langle b_r D(u), 2 \operatorname{grad}(b_r)\cdot u\rangle_{B_{2r}}| \leqq \tfrac{1}{2}\,||b_r D(u)||^2_{B_{2r}} + \frac{2M^2}{r^2}\,||u||^2_{B_{2r}} .$$

As $b_r \leqq 1$, the Schwarz inequality also gives

(6.5c) $$|\langle D^2(u), b_r^2 u\rangle_{B_{2r}}| \leqq \frac{t}{2}\,||D^2(u)||^2_{B_{2r}} + \frac{1}{2t}\,||u||^2_{B_{2r}} \quad \text{for} \quad t > 0.$$

Calculating with (6.5),

$$\begin{aligned} ||D(u)||^2_{B_r} \leqq ||b_r D(u)||^2_{B_{2r}} &= 2\,||b_r D(u)||^2_{B_{2r}} - ||b_r D(u)||^2_{B_{2r}} \\ &\leqq t\,||D^2(u)||^2_{B_{2r}} + \left(\frac{1}{t} + \frac{4M^2}{r^2}\right)||u||^2_{B_{2r}} . \end{aligned}$$

That proves (6.4). Let $r \to \infty$ and (6.4) becomes $||D(u)||^2_{Y_0} \leqq t\, ||D^2(u)||^2_{Y_0} + (1/t)\, ||u||^2_{Y_0}$. Summing over the topological components of Y we obtain (6.3a). Q.E.D.

Theorem 5.1 and Proposition 6.2 combine as follows.

6.6. Lemma. *If u is in the domain of $(D^2 + 1)^*$, and if $(D^2 + 1)^*(u) = 0$, then $u = 0$ in $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$.*

Proof. As D^2 is elliptic, $u \in C^\infty$ and $(D^2 + 1)(u) = 0$ as differential equation. Now $u \in L_2$ implies $D^2(u) \in L_2$ and (6.3b) forces $D(u) \in L_2$. Define $v = (D - i)(u)$. Then $v \in C^\infty \cap L_2$ and $(D + i)(v) = (D + i)(D - i)(u) = (D^2 + 1)(u) = 0$. Proposition 4.3 gives $v \in \mathbf{D}(D^*)$, so $D^*(v) = D(v) = -iv$. Now Theorem 5.1 says $v = 0$. Thus $D(u) = iu$. Again, Proposition 4.3 gives $D^*(u) = iu$ and Theorem 5.1 says $u = 0$. Q.E.D.

Proof of Theorem 6.1. Define $A = D^2 + 1$ with domain $\mathbf{D}(A) = C_c^\infty$. Lemma 6.6 says that the range $\mathbf{R}(A) = A(C_c^\infty)$ is dense in L_2. Further $A \geqq 1$ on C_c^∞ so the closure $\tilde{A} \geqq 1$ on $\mathbf{D}(\tilde{A})$. Now $B = \tilde{A}^{-1}$ is a well defined closed bounded symmetric operator with dense domain $\mathbf{D}(B) = \mathbf{R}(\tilde{A}) \supset \mathbf{R}(A)$. Thus B is self adjoint, so $\tilde{A} = B^{-1}$ is self adjoint, and D^2 has self adjoint closure $\tilde{A} - 1$.

We have just proved D^2 essentially self adjoint with closure $\tilde{A} - 1$. Now $\tilde{A} - 1$ is the unique self adjoint extension of D^2. As $\tilde{D}^*\tilde{D} = \tilde{D}^2$ is a self adjoint extension we must have $\tilde{A} - 1 = \tilde{D}^*\tilde{D}$. Q.E.D.

§7. Square integrable Dirac spinors. Y is an oriented riemannian n-manifold with a fixed riemannian (K, α)-structure $(\mathfrak{F}_K, \Gamma_K)$. Given a finite dimensional unitary representation κ of K we have the hermitian vector bundle $\mathcal{V}_\kappa \to Y$ associated to $\mathfrak{F}_K$, the bundle $\mathcal{S} \otimes \mathcal{V}_\kappa$ of $\mathcal{V}_\kappa$-valued spinors, and the Dirac operator D on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ with domain $C_c^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa)$. The space of **square integrable** $\mathcal{V}_\kappa$-valued **Dirac spinors** on Y is

(7.1a) $H_2(\mathcal{V}_\kappa) = \{u \in L_2(\mathcal{S} \otimes \mathcal{V}_\kappa) : D(u) = 0 \text{ as differential equation}\}.$

In other words, using ellipticity of D,

(7.1b) $H_2(\mathcal{V}_\kappa) = \{u \in L_2(\mathcal{S} \otimes \mathcal{V}_\kappa) : D^*(u) = 0\} \subset C^\infty(\mathcal{S} \otimes \mathcal{V}_\kappa).$

Further, if n is even, then we have

(7.1c) $H_2(\mathcal{V}_\kappa) = H_2^+(\mathcal{V}_\kappa) \oplus H_2^-(\mathcal{V}_\kappa)$ where $H_2^\pm(\mathcal{V}_\kappa) = H_2(\mathcal{V}_\kappa) \cap L_2(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa).$

If D (resp. D^2) is essentially self adjoint, then its closure is its adjoint, and every eigenspace of the latter is closed. Thus, from the results of §§5 and 6, we have

7.2. Proposition. *Suppose that the riemannian metric of Y is complete. Then $H_2(\mathcal{V}_\kappa)$ is a closed subspace of $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$, given by the elliptic equation*

$$H_2(\mathcal{V}_\kappa) = \{u \in L_2(\mathcal{S} \otimes \mathcal{V}_\kappa) : D^2(u) = 0\}.$$

Suppose further that n is even. Then $H_2(\mathcal{V}_\kappa)$ is orthogonal direct sum of its closed subspaces $H_\kappa^\pm(\mathcal{V}_\kappa)$, and they are given by elliptic equations:

$$H_2^\pm(\mathcal{V}_\kappa) = \{u \in L_2(\mathcal{S}^\pm \otimes \mathcal{V}_\kappa) : D^\mp D^\pm(u) = 0\}.$$

Let G be a Lie group and $\Phi: G \times Y \to Y$ a differentiable action by orientation-preserving isometries. Then Φ has a natural lift to the oriented orthonormal frame bundle,

$$\begin{array}{ccc} G \times \mathfrak{F} & \xrightarrow{\Psi} & \mathfrak{F} \\ \Big\downarrow{\scriptstyle 1 \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times Y & \xrightarrow{\Phi} & Y \end{array} \quad \text{commuting diagram of group actions,}$$

and the lifted action Ψ preserves the riemannian connection Γ of $\mathfrak{F}$. The action may or may not lift further; for example it lifts to the principal Spin (n)-bundles precisely when G acts trivially on $H^1(Y; Z_2)$. In any case, by **lift of Φ to $(\mathfrak{F}_K, \Gamma_K)$** we mean a differentiable group action $\Psi_K : G \times \mathfrak{F}_K \to \mathfrak{F}_K$ that covers Ψ, *i.e.*,

$$(7.3) \qquad \begin{array}{ccc} G \times \mathfrak{F}_K & \xrightarrow{\Psi_K} & \mathfrak{F}_K \\ \Big\downarrow{\scriptstyle 1 \times \tilde{\alpha}} & & \Big\downarrow{\scriptstyle \tilde{\alpha}} \\ G \times \mathfrak{F} & \xrightarrow{\Psi} & \mathfrak{F} \\ \Big\downarrow{\scriptstyle 1 \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times Y & \xrightarrow{\Phi} & Y \end{array} \quad \begin{array}{l} \text{commuting diagram} \\ \text{of differentiable} \\ \text{group actions,} \end{array}$$

such that each $\Psi_K(g, \cdot)$ preserves the connection Γ_K of $\mathfrak{F}_K$. We say that Φ lifts to $(\mathfrak{F}_K, \Gamma_K)$ if it has such a lift.

Proposition 7.2 and the definitions give us

7.4. Proposition. *Let $\Phi: G \times Y \to Y$ be a differentiable group action by orientation preserving isometries and $\Psi_K : G \times \mathfrak{F}_K \to \mathfrak{F}_K$ a lift of Φ to $(\mathfrak{F}_K, \Gamma_K)$. Then Ψ_K specifies a unitary representation $\tilde{\rho}_\kappa$ of G on $L_2(\mathcal{S} \otimes \mathcal{V}_\kappa)$ such that every $\tilde{\rho}_\kappa(g)$ commutes with D.*

Suppose that the riemannian metric of Y is complete. Then $\tilde{\rho}_\kappa$ restricts to a unitary representation ρ_κ of G on $H_2(\mathcal{V}_\kappa)$. If n is even, then $\rho_\kappa = \rho_\kappa^+ \oplus \rho_\kappa^-$ where $\rho_\kappa^\pm$ is the representation of G on $H_2^\pm(\mathcal{V}_\kappa)$.

We examine a case where ρ_κ are related to induced representations. Suppose that

(7.5a) G is a Lie group and K is a closed subgroup, and

(7.5b) $Y = G/K$ has G-invariant structure of oriented riemannian n-manifold.

Let α denote the (isotropy) representation of K on the tangent space to Y at $y_0 = 1 \cdot K \in G/K = Y$. Then $\alpha: K \to \mathrm{SO}\ (n)$ Lie group homomorphism. We further assume

(7.6a) $\alpha: K \to SO(n)$ factors through Spin (n),

(7.6b) $\alpha(K)$ contains the holonomy group of Y at y_0, and

(7.6c) the Lie algebra $\mathfrak{l}$ of L = kernel (α) is a direct summand of $\mathfrak{k}$.

$G \to G/K$ is our principal K-bundle $\mathfrak{F}_K \to Y$. Choose an oriented orthonormal basis f_0 of the tangent space Y_{y_0}. Then $\bar{\alpha}: G \to \mathfrak{F}$ is given by $\bar{\alpha}(g) = g(f_0)$. Choose a decomposition $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$. According to Lemma 2.8 that specifies a connection Γ_K on $G \to G/K$ such that $\bar{\alpha}(\Gamma_K)$ is the riemannian connection. Now (G, Γ_K) is a riemannian (K, α)-structure on Y. The action of G on Y is lifted by the action of G on itself through left translation. Since homogeneous riemannian metrics are complete, now the unitary representation κ of K gives us a unitary representation ρ_κ of G.

§8. Example: Representations of the Heisenberg group. We apply the construction of representations $\rho_\kappa^\pm$ and obtain all infinite dimensional irreducible unitary representations of the Heisenberg group.

The 3-dimensional Heisenberg group N_3 is the connected simply connected Lie group, whose Lie algebra $\mathfrak{n}_3$ has basis $\{\xi, \eta, \zeta\}$, ζ central and $[\xi, \eta] = \zeta$. The central Z of N_3 is the analytic subgroup for $\mathfrak{z} = \zeta R$. The coset space N_3/Z is identified with the standard euclidean plane R^2 by $\exp(x\xi + y\eta)Z \leftrightarrow (x, y)$, and the action of N_3 is

$$\exp(a\xi + b\eta + c\zeta)\exp(x\xi + y\eta)Z = \exp((x + a)\xi + (y + b)\eta)Z.$$

Thus N_3 acts on R^2 by orientation-preserving isometries for the euclidean metric $dx^2 + dy^2$, and $\{\xi, \eta\}$ gives the global orthonormal frame $\{\partial/\partial x, \partial/\partial y\}$.

The half spin bundles over $R^2 = N_3/Z$ are the trivial complex line bundles $\mathcal{S}^\pm \to R^2$ with respective fibers $S^+ = (1 + i\xi\cdot\eta)C$ and $S^- = (i\xi + \eta)C$. Here "$\cdot$" is Clifford multiplication.

The irreducible unitary representations of Z are the characters $\chi_\lambda : \exp(t\zeta) \to e^{i\lambda t}$, λ real. As Z acts trivially on R^2 the corresponding bundles $\mathcal{V}_\lambda = \mathcal{V}_{\chi_\lambda} \to R^2$ are trivial complex line bundles.

A section u of $\mathcal{S}^+ \otimes \mathcal{V}_\lambda$ now corresponds to a function $U: R^2 \to C$ by the rule $u(x, y) = (1 + i\xi_{(x,y)}\cdot\eta_{(x,y)})U(x, y)$. Similarly a section v of $\mathcal{S}^- \otimes \mathcal{V}_\lambda$ corresponds to a function $V: R^2 \to C$ under $v(x, y) = (i\xi_{(x,y)} + \eta_{(x,y)})V(x, y)$.

8.1. Lemma. *Let $u \in C^1(\mathcal{S}^+ \otimes \mathcal{V}_\lambda)$ and $U: R^2 \to C$ the corresponding function. Then $D^+(u) = 0$ if and only if*

(8.2a) $U(x, y) = e^{-\lambda(x^2+y^2)/4}E(x + iy)$ *with E holomorphic.*

Let $v \in C^1(\mathcal{S}^- \otimes \mathcal{V}_\lambda)$ and $V: R^2 \to C$ the corresponding function. Then $D^-(v) = 0$ if and only if

(8.2b) $V(x, y) = e^{\lambda(x^2+y^2)/4}F(x + iy)$ *with F antiholomorphic.*

Proof. We compute $D^+(u) = \xi\cdot\nabla_\xi(u) + \eta\cdot\nabla_\eta(u)$. For that view $u: N_3 \to (1 + i\xi\cdot\eta)C$ function such that $u(gz) = \chi_\lambda(z)^{-1}u(g)$, all $g \in N_3$ and $z \in Z$, and note from the Campbell–Hausdorff formula that

$$\exp(x\xi + y\eta + w\zeta)\exp(a\xi + b\eta) = \exp\left((x + a)\xi + (y + b)\eta + \left(w + \frac{xb - ya}{2}\right)\zeta\right).$$

Now

$$\begin{aligned}\nabla_\xi(u)(\exp(x\xi + y\eta + w\zeta)) &= \frac{d}{dt}\bigg|_{t=0} u\left(\exp\left((x + t)\xi + y\eta + \left(w - \frac{ty}{2}\right)\zeta\right)\right)\\ &= \frac{d}{dt}\bigg|_{t=0} \{e^{i\lambda ty/2}u(\exp((x + t)\xi + y\eta + w\zeta))\}.\end{aligned}$$

Bringing u back down to R^2, this says

$$\nabla_\xi(u)(x, y) = (1 + i\xi\cdot\eta)_{(x,y)}\left\{\frac{i\lambda y}{2}U(x, y) + \frac{\partial U}{\partial x}(x, y)\right\}.$$

Similarly

$$\nabla_\eta(u)(x, y) = (1 + i\xi\cdot\eta)_{(x,y)}\left\{-\frac{i\lambda x}{2}U(x, y) + \frac{\partial U}{\partial y}(x, y)\right\}.$$

Notice $\xi\cdot(1 + i\xi\cdot\eta) = \xi - i\eta$ and $\eta\cdot(1 + i\xi\cdot\eta) = \eta + i\xi = i(\xi - i\eta)$. Now

$$\begin{aligned}\text{(8.3a)}\quad D^+(u)(x, y) &= \{\xi\cdot\nabla_\xi(u) + \eta\cdot\nabla_\eta(u)\}(x, y)\\ &= (\xi - i\eta)_{(x,y)}\left\{\frac{\lambda(x + iy)}{2}U(x, y) + \left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right](x, y)\right\}.\end{aligned}$$

Defining $E: C \to C$ by $U(x, y) = e^{-\lambda(x^2+y^2)/4}E(x + iy)$, we have

$$\begin{aligned}\text{(8.3b)}\quad &\left[\frac{\partial E}{\partial x} + i\frac{\partial E}{\partial y}\right](x + iy)\\ &= e^{\lambda(x^2+y^2)/4}\left\{\frac{\lambda(x + iy)}{2}U(x, y) + \left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right](x, y)\right\}.\end{aligned}$$

Comparing (8.3a) and (8.3b), we see that $D^+(u) = 0$ precisely when $\partial E/\partial x + i\,\partial E/\partial y = 0$, *i.e.*, when E is holomorphic.

Computation of covariant derivatives of v is the same; we get

$$\nabla_\xi(v)(x, y) = (i\xi + \eta)_{(x,y)}\left\{\frac{i\lambda y}{2}V(x, y) + \frac{\partial V}{\partial x}(x, y)\right\}$$

and

$$\nabla_\eta(v)(x, y) = (i\xi + \eta)_{(x,y)}\left\{-\frac{i\lambda x}{2}V(x, y) + \frac{\partial V}{\partial y}(x, y)\right\}.$$

Notice $\xi\cdot(i\xi+\eta) = -i+\xi\cdot\eta$ and $\eta\cdot(i\xi+\eta) = -i\xi\cdot\eta - 1 = -i(-i+\xi\cdot\eta)$, so

$$\text{(8.4a)}\qquad D^{-}(v)(x, y) = (-i+\xi\cdot\eta)_{(x,y)}\left\{-\frac{\lambda(x-iy)}{2}V(x, y) + \left[\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right](x, y)\right\}.$$

Defining $F\colon C \to C$ by $V(x, y) = e^{\lambda(x^2+y^2)/4}F(x+iy)$, we have

$$\text{(8.4b)}\qquad \left[\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right](x+iy) = e^{-\lambda(x^2+y^2)/4}\left\{-\frac{\lambda(x-iy)}{2}V(x, y) + \left[\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right](x, y)\right\}.$$

Comparing (8.4a) and (8.4b), we see that $D^{-}(v) = 0$ precisely when $\partial F/\partial x - i\,\partial F/\partial y = 0$, *i.e.*, when F is antiholomorphic. Q.E.D.

We carry the action of N_3 on sections u of $\mathcal{S}^{+}\otimes\mathcal{V}_\lambda$ (resp. v of $\mathcal{S}^{-}\otimes\mathcal{V}_\lambda$) over to the action on functions E (resp. F) specified by

$$\text{(8.5a)}\qquad u(x, y) = (1+i\xi\cdot\eta)_{(x,y)}e^{-\lambda(x^2+y^2)/4}E(x+iy)$$

and

$$\text{(8.5b)}\qquad v(x, y) = (i\xi+\eta)_{(x,y)}e^{\lambda(x^2+y^2)/4}F(x+iy).$$

For that, denote

$$\text{(8.6a)}\qquad \Lambda_\lambda^{\pm}(g)\text{: action of } g \in N_3 \text{ on sections of } \mathcal{S}^{\pm}\otimes\mathcal{V}_\lambda\,.$$

Using (8.5) we now define actions $\Phi_\lambda^{\pm}$ by

$$\text{(8.6b)}\qquad [\Lambda_\lambda^{+}(g)u](x, y) = (1+i\xi\cdot\eta)_{(x,y)}e^{-\lambda(x^2+y^2)/4}[\Phi_\lambda^{+}(g)E](x+iy).$$

and

$$\text{(8.6c)}\qquad [\Lambda_\lambda^{-}(g)v](x, y) = (i\xi+\eta)_{(x,y)}e^{\lambda(x^2+y^2)/4}[\Phi_\lambda^{-}(g)F](x+iy).$$

Viewing $u\colon N_3 \to (1+i\xi\cdot\eta)C$ function such that $u(gz) = \chi_\lambda(z)^{-1}u(g)$ for all $g \in N_3$ and $z \in Z$, we calculate

$$\begin{aligned}
&[\Lambda_\lambda^{+}(\exp(x_0\xi + y_0\eta + w_0\xi)u](\exp(x\xi + y\eta + w\zeta))\\
&\qquad = u(\exp(x_0\xi + y_0\eta + w_0\zeta)^{-1}\exp(x\xi + y\eta + w\zeta))\\
&\qquad = u\left(\exp\left((x-x_0)\xi + (y-y_0)\eta + \left(w - w_0 + \frac{xy_0 - yx_0}{2}\right)\zeta\right)\right)\\
&\qquad = e^{i\lambda(w_0-(xy_0-yx_0)/2)}u(\exp((x-x_0)\xi + (y-y_0)\eta + w\zeta)).
\end{aligned}$$

Down on R^2 this says that the corresponding action L_λ^{+} on $U(x, y) = e^{\lambda(x^2+y^2)/4}E(x+iy)$ is

$$[L_\lambda^{+}(\exp(x_0\xi + y_0\eta + w_0\zeta)U](x, y) = e^{i\lambda(w_0-(xy_0-yx_0)/2)}U(x-x_0\,,\,y-y_0).$$

We conclude

$$[\Phi_\lambda^+(\exp(x_0\xi + y_0\eta + w_0\zeta)E](x+iy) = e^f E((x+iy) - (x_0+iy_0))$$

where

$$\begin{aligned} f &= \frac{\lambda}{4}(x^2+y^2) + i\lambda w_0 - \frac{i\lambda}{2}(xy_0 - yx_0) - \frac{\lambda}{4}((x-x_0)^2 + (y-y_0)^2) \\ &= i\lambda w_0 - \frac{\lambda}{4}(x_0^2+y_0^2) + \frac{\lambda}{2}(x+iy)(x_0-iy_0). \end{aligned}$$

In summary,

$$\begin{aligned} (8.7) \qquad & [\Phi_\lambda^+(\exp(x_0\xi + y_0\eta + w_0\zeta)E](z) \\ & \qquad = e^{i\lambda w_0 - \lambda|z_0|^2/4 + \lambda z\bar{z}_0/2} E(z - z_0), \qquad z_0 = x_0 + iy_0 . \end{aligned}$$

A similar calculation gives

$$\begin{aligned} (8.8) \qquad & [\Phi_\lambda^-(\exp(x_0\xi + y_0\eta + w_0\zeta)F](z) \\ & \qquad = e^{i\lambda w_0 + \lambda|z_0|^2/4 - \lambda\bar{z}z_0/2} F(z - z_0), \qquad z_0 = x_0 + iy_0 . \end{aligned}$$

In view of Lemma 8.1 we define Hilbert spaces

$$(8.9a) \qquad \mathbf{H}_\lambda^+ = \left\{E : C \to C \text{ holomorphic: } \iint e^{-\lambda|z|^2/2}\,|E(z)|^2\,dx\,dy < \infty\right\}$$

with inner product

$$(8.9b) \qquad \langle E, E'\rangle = \iint e^{-\lambda|z|^2/2} E(z)\overline{E'(z)}\,dx\,dy.$$

The formula (8.7) defines a unitary representation ϕ_λ^+ of N_3 on $\mathbf{H}_\lambda^+$, which is the classical Fock representation. Here note $\mathbf{H}_\lambda^+ = \{0\}$ for $\lambda \leqq 0$; if $\lambda > 0$ then the linear combinations of functions $e^{cz}P(z)$, c constant and P polynomial, form a dense subspace of $\mathbf{H}_\lambda^+$.

Similarly we define Hilbert spaces

$$(8.10a) \qquad \mathbf{H}_\lambda^- = \left\{F : C \to C \text{ antiholomorphic: } \iint e^{\lambda|z|^2/2}\,|F(z)|^2\,dx\,dy < \infty\right\}$$

with inner product

$$(8.10b) \qquad \langle F, F'\rangle = \iint e^{\lambda|z|^2/2} F(z)\overline{F'(z)}\,dx\,dy.$$

The formula (8.8) defines a unitary representation ϕ_λ^- of N_3 on $\mathbf{H}_\lambda^-$, which is the dual (contragredient) of $\phi_{-\lambda}^+$. Here note $\mathbf{H}_\lambda^- = \{0\}$ for $\lambda \geqq 0$; if $\lambda < 0$, then the linear combinations of functions $e^{c\bar{z}}P(\bar{z})$, c constant and P polynomial, form a dense subspace of $\mathbf{H}_\lambda^-$.

The Fock representations and their duals give us a collection

$$\{\phi_\lambda^+ : \lambda > 0\} \cup \{\phi_\lambda^- : \lambda < 0\} \tag{8.11}$$

of nontrivial unitary representations of N_3. A glance at the dense subspaces $\{\text{span } e^{cz}P(z)\}$ and $\{\text{span } e^{c\bar{z}}P(\bar{z})\}$ shows that these representations are irreducible. Further they are mutually inequivalent because they have distinct central characters, $\phi_\lambda^\pm|_Z$ being a multiple of χ_λ. Finally, if p is any irreducible unitary representation of N_3, then either p is infinite dimensional and equivalent to one of the $\phi_\lambda^\pm$, $\pm\lambda > 0$, or p is 1-dimensional and of the form

$$p_{a,b}(\exp(x\xi + y\eta + w\zeta)) = e^{i(ax+by)}. \tag{8.12}$$

We summarize as follows.

8.13. Theorem. *Let $\rho_\lambda^\pm$ denote the unitary representation of N_3 on the space $H_2^\pm(\mathcal{V}_\lambda)$ of square integrable $\mathcal{V}_\lambda$-valued Dirac spinors on $R^2 = N_3/Z$.*

1. *Suppose $\lambda > 0$. Then $H_2^-(\mathcal{V}_\lambda) = \{0\}$, and ρ_λ^+ is equivalent to the Fock representation ϕ_λ^+ of N_3 on $\mathbf{H}_\lambda^+$.*
2. *If $\lambda = 0$, then $H_2^+(\mathcal{V}_\lambda) = \{0\} = H_2^-(\mathcal{V}_\lambda)$.*
3. *Suppose $\lambda < 0$. Then $H_2^+(\mathcal{V}_\lambda) = 0$, and ρ_λ^- is equivalent to the representation ϕ_λ^- of N_3 on $\mathbf{H}_\lambda^-$, dual to $\phi_{-\lambda}^+$.*
4. *$\{\rho_\lambda^+ : \lambda > 0\} \cup \{\rho_\lambda^- : \lambda < 0\}$ all are irreducible, nontrivial and mutually inequivalent, and they represent all the equivalence classes of irreducible unitary representations of N_3 except for those of the unitary characters* (8.12).

In a later paper [13] I hope to extend this method to the class of all connected nilpotent Lie groups N with a unitary representation whose coefficients are square integrable over $N/(\text{center})$. That class includes the generalized Heisenberg groups N_{2m+1}. The Plancherel measure for a group in that class is in fact carried by the "square integrable" representations [6].

§9. Example: Representations of the universal cover of SL(2, R). We apply the construction of representations $\rho_\kappa^\pm$ and obtain all relative discrete series (*cf.* [11]) representations of the simply connected covering group of $SL(2, R)$.

$SL(2, R)$ is the multiplicative group of 2×2 real matrices of determinant 1. It acts on the upper half plane $\mathbf{h} = \{z = x + iy \in C : y = \text{Im}(z) > 0\}$ by linear fractional transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto (az + b)/(cz + d)$. The action is transitive, orientation-preserving, and isometric for the Poincaré metric $y^{-2}(dx^2 + dy^2)$. The isotropy subgroup at i is $\bar{K} = \{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) : \theta \text{ real}\}$. The Lie algebra $\mathfrak{sl}(2, R)$ consists of the 2×2 real matrices of trace 0. The subalgebra $\mathfrak{k}$ for $\bar{K}$ is spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The global orthonormal frame $\{e_1, e_2\} = \{y(\partial/\partial x), y(\partial/\partial y)\}$ on $\mathbf{h}$ is represented, at the point i, by $\frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $-\frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Denote the universal cover by $q: G \to SL(2, R)$, identify the Lie algebra $\mathfrak{g}$ with $\mathfrak{sl}(2, R)$ under q, and define $K = q^{-1}(\bar{K})$. Then $k: \theta \mapsto \exp_G \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$ is an isomorphism of R^1 onto K, and q has kernel $k(2\pi Z)$ where Z denotes the integers.

G acts on $\mathbf{h}$ by $g(z) = q(g)(z)$; the isotropy subgroup at i is K; that identifies G/K with $\mathbf{h}$ under $gK \leftrightarrow g(i)$.

The irreducible unitary representations of K are the characters $\chi_\lambda(k(\theta)) = e^{2i\lambda\theta}$, λ real. Let $\mathcal{V}_\lambda \to \mathbf{h}$ denote the complex line bundle associated to $G \to G/K = \mathbf{h}$ by χ_λ. $k(\theta)$ rotates the real tangent space $\mathbf{h}_i$ by 2θ, so its image in Spin $(\mathbf{h}_i)$ is $-(\cos\theta\, e_1 + \sin\theta\, e_2)\cdot e_1 = \cos\theta + \sin\theta\, e_1\cdot e_2$. Thus the K-lifts of the half spin representations satisfy $\mathfrak{s}^+(k(\theta))(1 + ie_1\cdot e_2) = (\cos\theta + \sin\theta\, e_1\cdot e_2)\cdot(1 + ie_1\cdot e_2) = e^{-i\theta}(1 + ie_1\cdot e_2)$ and (similarly) $\mathfrak{s}^-(k(\theta))(ie_1 + e_2) = e^{i\theta}(ie_1 + e_2)$. Now $\mathfrak{s}^\pm = \chi_{\mp 1/2}$.

In the moving frame $\{e_1, e_2\} = \{y(\partial/\partial x), y(\partial/\partial y)\}$, $\mathbf{h}$ has Christoffel symbols $\Gamma^2_{11} = 1 = -\Gamma^1_{12}$ and all others zero. Thus $\nabla_1(e_1) = e_2$, $\nabla_1(e_2) = -e_1$, $\nabla_2(e_j) = 0$ ($\nabla_k = \nabla_{e_k}$). In other words the $\{e_1, e_2\}^*$-image of the connection form of the oriented orthonormal frame bundle is given by

$$\omega(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad\text{and}\quad \omega(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the connection form on $\mathcal{V}_\lambda \to \mathbf{h}$, in the frame $\{e_1, e_2\}$, is

$$\omega_\lambda(e_1) = (\chi_\lambda)_*\omega(e_1) = i\lambda \quad\text{and}\quad \omega_\lambda(e_2) = 0.$$

9.1. Lemma. *Let* $u(z) = (1 + ie_1\cdot e_2)U(z) = (1 + ie_1\cdot e_2)y^{-\lambda+1/2}E(z)$ *section of* $\mathfrak{S}^+ \otimes \mathcal{V}_\lambda \to \mathbf{h}$. *Then*

$$\text{(9.2a)}\qquad [D^+(u)](z) = (e_1 - ie_2)\left\{i(\lambda - \tfrac{1}{2})U(z) + y\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right](z)\right\}.$$

In particular $D^+(u) = 0$ *if and only if* E *is holomorphic.*

Let $v(z) = (ie_1 + e_2)V(z) = (ie_1 + e_2)y^{\lambda+1/2}F(z)$ *section of* $\mathfrak{S}^-\otimes \mathcal{V}_\lambda \to \mathbf{h}$. *Then*

$$\text{(9.2b)}\qquad [D^-(v)](z) = -i(1 + ie_1\cdot e_2)\left\{i(\lambda + \tfrac{1}{2})V(z) + y\left[\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right](z)\right\}.$$

In particular $D^-(v) = 0$ *if and only if* F *is antiholomorphic.*

Proof. Note $\mathfrak{S}^\pm \otimes \mathcal{V}_\lambda = \mathcal{V}_{\lambda\mp 1/2}$, so it has connection form $\omega_{\lambda\mp 1/2}(e_1) = i(\lambda \mp \tfrac{1}{2})$ and $\omega_{\lambda\mp1/2}(e_2) = 0$. Compute

$$\nabla_1(u) = (1 + ie_1\cdot e_2)\left\{i(\lambda - \tfrac{1}{2})U + y\frac{\partial U}{\partial x}\right\} \quad\text{and}\quad \nabla_2(u) = (1 + ie_1\cdot e_2)y\frac{\partial U}{\partial y}.$$

Using $D^+(u) = e_1\cdot\nabla_1(u) + e_2\cdot\nabla_2(u)$ and $e_2\cdot(1 + ie_1\cdot e_2) = i(e_1 - ie_2) = ie_1\cdot(1 + ie_1\cdot e_2)$, the assertions on $D^+(u)$ follow. Similarly

$$\nabla_1(v) = (ie_1 + e_2)\left\{i(\lambda + \tfrac{1}{2})V + y\frac{\partial V}{\partial x}\right\} \quad\text{and}\quad \nabla_2(v) = (ie_1 + e_2)y\frac{\partial V}{\partial y},$$

and the assertions on $D^-(v) = e_1\cdot\nabla_1(v) + e_2\cdot\nabla_2(v)$ now follow using $e_2\cdot(ie_1 + e_2) = -i(e_1\cdot e_2 - i) = -ie_1\cdot(ie_1 + e_2)$. Q.E.D.

To compute the action of G we need some algebra. Define

(9.3a) $$b : \mathbf{h} \to G \quad \text{by} \quad b(x+iy) = \exp_G \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \exp_G \begin{pmatrix} 1/2 \log (y) & 0 \\ 0 & -1/2 \log (y) \end{pmatrix}.$$

Then $z = b(z)(i)$, so b is a section to $G \to G/K = \mathbf{h}$, and we have a diffeomorphism

(9.3b) $$\mathbf{h} \times R \to G \quad \text{by} \quad (z, \theta) \mapsto b(z)k(\theta).$$

Note that the unique factorization

(9.3c) $$g = \exp_G \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot \exp_G \begin{pmatrix} 1/2 \log (y) & 0 \\ 0 & -1/2 \log (y) \end{pmatrix} k(\theta), \qquad x + iy = g(i),$$

is the lift $G = NAK$ of the Iwasawa decomposition $SL(2, R) = q(N)q(A)q(K)$.

Now define a function

(9.4a) $$\phi: G \times \mathbf{h} \to R \quad \text{by} \quad gb(z) = b(gz)k(-\phi(g, z)).$$

Then the function

(9.4b) $$\Phi: G \times \mathbf{h} \to C \text{ by } \Phi(g, z) = \{\mathrm{Im}\,(z)/\mathrm{Im}\,(g(z))\}^{1/2} e^{i\phi(g,z)}$$

has the property that

(9.4c) $$\Phi(g, z) = cz + d \quad \text{where} \quad q(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The point, however, is that we can take arbitrary real powers

(9.4d) $$\Phi(g, z)^{\nu} = \{\mathrm{Im}\,(z)/\mathrm{Im}\,(g(z))\}^{\nu/2} e^{i\nu\phi(g,z)} \text{ holomorphic in } z.$$

Now we compute the action of G. Denote

(9.5) $$\Lambda_\lambda^{\pm}(g)\text{: action of } g \in G \text{ on sections of } \mathcal{S}^{\pm} \otimes \mathcal{V}_\lambda .$$

If $u(z) = (1 + ie_1 \cdot e_2)y^{-\lambda+1/2}E(z)$, then we denote

(9.6a) $$[\Lambda_\lambda^{+}(g)u](z) = (1 + ie_1 \cdot e_2)y^{-\lambda+1/2}[\Psi_\lambda^{+}(g)E](z).$$

Identify u with the function $\tilde{u}: G \to C$ given by

$$b(z)k(\theta) \to \chi_{\lambda-1/2}(k(\theta))^{-1} y^{-\lambda+1/2} E(z).$$

Then we compute

$$\begin{aligned}[\Psi_\lambda^{+}(g)E](z) &= y^{\lambda-1/2}\tilde{u}(g^{-1}b(z)) \\ &= y^{\lambda-1/2}\tilde{u}(b(g^{-1}z)k(-\phi(g^{-1}, z))) \\ &= y^{\lambda-1/2} e^{i(2\lambda-1)\phi(g^{-1},z)}\ \mathrm{Im}\,(g^{-1}(z))^{-\lambda+1/2} E(g^{-1}z).\end{aligned}$$

That says

(9.6b) $$[\Psi_\lambda^+(g)E](z) = \Phi(g^{-1}, z)^{2\lambda-1}E(g^{-1}z).$$

If $v(z) = (ie_1 + e_2)y^{\lambda+1/2}F(z)$ section of $\mathcal{S}^- \otimes \mathcal{V}_\lambda$, we denote

(9.7a) $$[\Lambda_\lambda^-(g)v](z) = (ie_1 + e_2)y^{\lambda+1/2}[\Psi_\lambda^-(g)F](z).$$

Calculating as above,

$$\begin{aligned}[\Psi_\lambda^-(g)F](z) &= y^{-\lambda-1/2}\tilde{v}(g^{-1}b(z)) \\ &= y^{-\lambda-1/2}\tilde{v}(b(g^{-1}z)k(-\phi(g^{-1}, z))) \\ &= y^{-\lambda-1/2}e^{i(2\lambda+1)\phi(g^{-1},z)} \operatorname{Im}(g^{-1}(z))^{\lambda+1/2}F(g^{-1}z).\end{aligned}$$

That says

(9.7b) $$[\Psi_\lambda^-(g)F](z) = \bar{\Phi}(g^{-1}, z)^{-2\lambda-1}F(g^{-1}z)$$

where $\bar{\Phi}$ is the complex conjugate function of Φ.

In view of Lemma 9.1 we define Hilbert spaces

(9.8a) $$\mathbf{H}_\lambda^+ = \left\{E : \mathbf{h} \to C \quad \text{holomorphic:} \quad \int_{\mathbf{h}} y^{1-2\lambda}\, |E(z)|^2\, d\mu(z) < \infty\right\}$$

where μ is the G-invariant measure

$$d\mu(z) = y^{-2}\, dx\, dy.$$

The inner product on $\mathbf{H}_\lambda^+$ is

(9.8b) $$\langle E, E'\rangle = \int_{\mathbf{h}} y^{1-2\lambda}E(z)\overline{E'(z)}\, d\mu(z) = \int_0^\infty dy \int_{-\infty}^\infty E(z)\overline{E'(z)}y^{-2\lambda-1}\, dx.$$

Formula (9.6b) gives a unitary representation ψ_λ^+ of G on $\mathbf{H}_\lambda^+$. It is straightforward to check that $\mathbf{H}_\lambda^+ = \{0\}$ for $\lambda \geqq 0$. If $\lambda < 0$ then, fixing a branch of $\log(z + i)$ on $\mathbf{h}$, we see that the

$$E_{\lambda,m}(z) = (z + i)^{2\lambda-1}\left(\frac{z - i}{z + i}\right)^m, \qquad m = 0, 1, \cdots,$$

form a complete orthogonal system in $\mathbf{H}_\lambda^+$ consisting of K-eigenvectors. Irreducibility of Ψ_λ^+ follows immediately from (9.6b).

Similarly we define

(9.9a) $$\mathbf{H}_\lambda^- = \left\{F : \mathbf{h} \to C \quad \text{antiholomorphic:} \quad \int_{\mathbf{h}} y^{2\lambda+1}\, |F(z)|^2\, d\mu(z) < \infty\right\},$$

Hilbert space with inner product

(9.9b) $$\langle F, F'\rangle = \int_{\mathbf{h}} y^{2\lambda+1}F(z)\overline{F'(z)}\, d\mu(z) = \int_0^\infty dy \int_{-\infty}^\infty F(z)\overline{F'(z)}y^{2\lambda-1}\, dx.$$

Formula (9.7b) gives a unitary representation ψ_λ^- of G on $\mathbf{H}_\lambda^-$. Note that $F \to \bar{F}$ is an isometry of $\mathbf{H}_\lambda^-$ onto $\mathbf{H}_{-\lambda}^+$ that carries ψ_λ^- to $\psi_{-\lambda}^+$. Now $\mathbf{H}_\lambda^- = \{0\}$ for $\lambda \leqq 0$; if $\lambda > 0$, then the $\bar{E}_{-\lambda,m}$ form a complete orthogonal system in $\mathbf{H}_\lambda^-$ consisting of K-eigenvectors; and the ψ_λ^- are irreducible.

In Pukánszky's notation [8], $\psi_\lambda^\pm$ is equivalent to the representation $D_l^\pm$ where $l = \mp\lambda + \frac{1}{2}$. Thus, if 2λ is an integer, then $\psi_\lambda^\pm$ factors through Bargmann's representation [2] $D_l^\pm$ of $SL(2, R)$, again with $l = \mp\lambda + \frac{1}{2}$.

The representations just constructed, form a collection

$$\{\psi_\lambda^+ : \lambda < 0\} \cup \{\psi_\lambda^- : \lambda > 0\} \tag{9.10a}$$

of irreducible representations of G, which contains just one element from each class in the relative discrete series [11]. View λ as the linear form $\left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix}\right) \to 2i\lambda t$ that exponentiates to χ_λ on the Cartan subgroup K of G. Then 1 is the (unique, noncompact) positive root. In my notation [11], which is consistent with Harish-Chandra's notation,

$$\text{if } \pm\lambda < 0, \text{ then } \psi_\lambda^\pm \text{ represents the class } [\pi_\lambda]. \tag{9.10b}$$

Thus the distribution character of $\psi_\lambda^\pm$ is the locally integrable class function specified on the regular elliptic set by $k(\theta) \to \pm(e^{2i\theta} - e^{-2i\theta})^{-1}\chi_\lambda(k(\theta))$. Or see [8, p. 132].

We summarize as follows.

9.11. Theorem. *Let $\rho_\lambda^\pm$ denote the representation of G on the space $H_2^\pm(\mathcal{V}_\lambda)$ of $\mathcal{V}_\lambda$-valued square integrable Dirac spinors on* $\mathbf{h} = G/K$.

1. *Suppose $\lambda < 0$. Then $H_2^-(\mathcal{V}_\lambda) = \{0\}$ and ρ_λ^+ is equivalent to the holomorphic relative discrete series representation ψ_λ^+ of G on $\mathbf{H}_\lambda^+$.*
2. *If $\lambda = 0$, then $H_2^+(\mathcal{V}_\lambda) = \{0\} = H_2^-(\mathcal{V}_\lambda)$.*
3. *Suppose $\lambda > 0$. Then $H_2^+(\mathcal{V}_\lambda) = \{0\}$ and ρ_λ^- is equivalent to the antiholomorphic relative discrete series representation ψ_λ^- of G on $\mathbf{H}_\lambda^-$.*
4. *$\{\rho_\lambda^- : \lambda < 0\} \cup \{\rho_\lambda^+ : \lambda > 0\}$ all are irreducible, nontrivial and mutually inequivalent, and they form a system of representatives of the relative discrete series of G.*

In [12] I will extend this method to the nondegenerate series representations of reductive Lie groups. The case of discrete series representations of linear semisimple groups is independently due to Schmid (unpublished) and Parthasarathy [7].

§10. Example: Representations of a parabolic subgroup. We retain the notation of §9 and examine the action of parabolic subgroups of G on the spaces $H_2^\pm(\mathcal{V}_\lambda)$ of square integrable Dirac spinors.

Recall the parametrization (9.3) of G. Denote

$$N = \left\{\exp_G \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in R\right\} \quad \text{and} \quad A = \left\{\exp_G \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in R\right\}, \tag{10.1a}$$

so that b: $\mathbf{h} \to G$ has image NA. Further denote

(10.1b) $$Z = \{k(\theta): \theta/\pi \text{ is an integer}\} = (\text{center of } G).$$

Then the proper parabolic subgroups of G are just the conjugates of

(10.1c) $$B = NAZ = (NA) \times Z = \{g \in G: gNg^{-1} = N\}.$$

We will analyse the representations $\rho_\lambda^\pm|_B$ of B on $H_2^\pm(\mathcal{V}_\lambda)$.

The unitary characters on Z are the

(10.2a) $$\zeta_\lambda = \chi_{\lambda \mp 1/2}|_Z : k(\pi l) \mapsto e^{2\pi i (\lambda - 1/2) l}, \qquad \lambda \text{ specified modulo } 1.$$

As $\mathcal{S}^\pm \otimes \mathcal{V}_\lambda = \mathcal{V}_{\lambda \mp 1/2}$, ζ_λ is the action of Z on the fibres of $\mathcal{S}^\pm \otimes \mathcal{V}_\lambda$. Since Z is central now it acts on sections of $\mathcal{S}^\pm \otimes \mathcal{V}_\lambda$ by ζ_λ, so $\rho_\lambda^\pm|_Z$ is a multiple of ζ_λ:

(10.2b) $$\rho_\lambda^\pm \text{ has central character } \zeta_\lambda .$$

Since the (infinite cyclic) group Z is type I, now

(10.2c) $$\rho_\lambda^\pm|_B = (\rho_\lambda^\pm|_{NA}) \otimes \zeta_\lambda .$$

To calculate $\rho_\lambda^\pm|_{NA}$ we formulate the product in $NA = b(\mathbf{h})$ in terms of $\mathbf{h}$. Denote

(10.3a) $$n(x) = \exp_G \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$a(y) = \exp_G \begin{pmatrix} 1/2 \log (y) & 0 \\ 0 & -1/2 \log (y) \end{pmatrix} \text{ so } b(x + iy) = n(x)a(y).$$

Then $b(x_0 + iy_0)^{-1}b(x + iy) = a(y_0)^{-1}n(x_0)^{-1}n(x)a(y) = a(y_0)^{-1}n(x - x_0)a(y) = n(y_0^{-1}(x - x_0))a(y_0)^{-1}a(y)$, so

(10.3b) $$b(x_0 + iy_0)^{-1}b(x + iy) = b(y_0^{-1}(x - x_0) + iy^{-1}_0 y).$$

In particular the function Φ of (9.4) satisfies

(10.3c) $$\Phi(b(z_0)^{-1}, z) = y_0^{1/2} \quad \text{where} \quad z_0 = x_0 + iy_0 .$$

Combining (10.3) with the results of §9 we arrive at

10.4. Lemma. *The restriction $\rho_\lambda^+|_{NA}$ is equivalent to the representation of NA on $\mathbf{H}_\lambda^+$ given by*

(10.5a) $$[\beta_\lambda^+(b(z_0))E](z) = y_0^{\lambda - 1/2}E(y_0^{-1}(z - x_0)), \qquad z_0 = x_0 + iy_0 .$$

The restriction $\rho_\lambda^-|_{NA}$ is equivalent to the representation of NA on $\mathbf{H}_\lambda^-$ given by

(10.5b) $$[\beta_\lambda^-(b(z_0))F](z) = y_0^{-\lambda - 1/2}F(y_0^{-1}(z - x_0)), \qquad z_0 = x_0 + iy_0 .$$

In particular,

(10.5c) $$[\beta_\lambda^+(n(x_0))E](z) = E(z - x_0) \text{ and } [\beta_\lambda^-(n(x_0))F](z) = F(z - x_0).$$

In order to identify the $\rho_\lambda^\pm|_{NA}$ we recall the structure of

(10.6a) $$l_{NA} : \text{left regular representation of } NA.$$

Evidently it is the unitarily induced representation

$$l_{NA} = \text{Ind}_{\{1\}\uparrow NA}(1) = \text{Ind}_{N\uparrow NA}\,(\text{Ind}_{\{1\}\uparrow N}(1))$$

where 1 denotes the trivial representation of degree 1. Now denote representations of N and NA by

(10.6b) $$\sigma_\xi(n(x)) = e^{i\xi x} \quad \text{and} \quad \tau_\xi = \text{Ind}_{N\uparrow NA}(\sigma_\xi).$$

Then

$$l_{NA} = \text{Ind}_{N\uparrow NA} \int_{-\infty}^{\infty} \sigma_\xi \, d\xi = \int_{-\infty}^{\infty} \tau_\xi \, d\xi.$$

Note $a(y)n(x)a(y)^{-1} = n(yx)$, so $a(y)$ sends τ_ξ to $\tau_{y\xi}$. Thus τ_ξ is irreducible for $\xi \neq 0$, and $\tau_\xi \simeq \tau_\eta$ just when either $\xi = 0 = \eta$ or $\xi\eta > 0$. In summary

(10.6c) $$l_{NA} = \infty\tau_1 \oplus \infty\tau_{-1}.$$

The $\tau_{\pm 1}$ are distinguished by the fact, evident from

$$\tau_1|_N = \int_0^{\infty} \sigma_\xi \, d\xi \quad \text{and} \quad \tau_{-1}|_N = \int_{-\infty}^{0} \sigma_\xi \, d\xi,$$

that

(10.7a) $$-i\tau_1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ has spectrum } (0, \infty)$$

and

(10.7b) $$-i\tau_{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ has spectrum } (-\infty, 0).$$

10.8. Lemma. *If $\lambda < 0$, then $\rho_\lambda^+|_{NA}$ is equivalent to τ_{-1}. If $\lambda > 0$, then $\rho_\lambda^-|_{NA}$ is equivalent to τ_1.*

Proof. By construction, each $\rho_\lambda^\pm|_{NA}$ is a subrepresentation of l_{NA}. Irreducibility follows from Lemma 10.4. Thus each $\rho_\lambda^\pm|_{NA}$, $\pm\lambda < 0$, is equivalent to τ_1 or τ_{-1}. We will use (10.7) to distinguish the two possibilities.

Let $\lambda < 0$ and let $E \in \mathbf{H}_\lambda^+$ be a finite linear combination from the complete orthogonal set $E_{\lambda,m}(z) = (z + i)^{2\lambda-1}((z - i)/(z + i))^m$, $m = 0, 1, \cdots$. Then the Fourier transform

(10.9a) $$\hat{E}_y(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} E(x + iy)e^{-ix\xi} \, dx$$

converges and we may move $\partial/\partial y$ under the integral. As E is holomorphic, $\partial E/\partial y = i\, \partial E/\partial x$, so

$$\frac{\partial}{\partial y}(\hat{E}_y(\xi)) = i(2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{\partial E(x+iy)}{\partial x}e^{-ix\xi}\,dx$$
$$= -i(2\pi)^{-1/2}\int_{-\infty}^{\infty}E(x+iy)\frac{\partial e^{-ix\xi}}{\partial x}\,dx$$
$$= -\xi\hat{E}_y(\xi).$$

That says

(10.9b) $$e^{\xi y}\hat{E}_y(\xi) = \hat{E}_0(\xi) \quad \text{independent of } y.$$

Now compute

$$\|E\|^2 = \int_0^{\infty} y^{-(2\lambda+1)}\,dy\int_{-\infty}^{\infty}|E(x+iy)|^2\,dx$$
$$= \int_0^{\infty} y^{-(2\lambda+1)}\,dy\int_{-\infty}^{\infty}|\hat{E}_y(\xi)|^2\,d\xi$$
$$= \int_{-\infty}^{\infty}|\hat{E}_0(\xi)|^2\,d\xi\int_0^{\infty}e^{-2\xi y}y^{-(2\lambda+1)}\,dy.$$

Fubini's Theorem now says that $\int_0^\infty e^{-2\xi y}y^{-(2\lambda+1)}\,dy$ converges a.e. on the set $\hat{E}_0(\xi) \neq 0$. That integral never converges for $\xi < 0$. We conclude

(10.9c) $$\hat{E}_y(\xi) = 0 \quad \text{a.e. in the interval } \xi < 0.$$

We use (10.5c) to compute $[-i\beta_\lambda^+(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix})E](x+iy) = -idE(x+iy-t)/dt|_{t=0} = i(\partial E/\partial x)(x+iy)$. In other words $[-i\beta_\lambda^+(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix})E]_y^\wedge(\xi) = -\xi\hat{E}_y(\xi)$. From (10.9) we see that the self adjoint operator $-i\beta_\lambda^+(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix})$ has spectrum in $(-\infty, 0]$. Lemma 10.4 now says that $-i\rho_\lambda^+(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix})$ has spectrum in $(-\infty, 0]$. Thus (10.7a) implies $\rho_\lambda^+|_{NA}$ not equivalent to τ_1. We conclude $\rho_\lambda^+|_{NA} \simeq \tau_{-1}$.

If $\lambda > 0$, then a similar argument shows $\rho_\lambda^-|_{NA} \simeq \tau_1$. The difference is that we work on an antiholomorphic function F, so $\partial F/\partial y = -i\,\partial F/\partial x$ and (10.9b) is replaced by $e^{-iy}\hat{F}_y(\xi) = \hat{F}_0(\xi)$ independent of y, so that $\hat{F}_y(\xi) = 0$ a.e. in the interval $\xi > 0$. Q.E.D.

Lemma 10.8 combines with (10.2) as follows

10.10. Theorem. *Let $\rho_\lambda^\pm$ denote the representation of G on the space $H_2^\pm(\mathcal{V}_\lambda)$ of $\mathcal{V}_\lambda$-valued square integrable Dirac spinors on* h $= G/K$.

1. *If* $\lambda < 0$, i.e., *if* $H_2^+(\mathcal{V}_\lambda) \neq \{0\}$, *then $\rho_\lambda^+|_B$ is equivalent to $\tau_{-1} \otimes \zeta_\lambda$.*
2. *If* $\lambda > 0$, i.e., *if* $H_2^-(\mathcal{V}_\lambda) \neq \{0\}$, *then $\rho_\lambda^-|_B$ is equivalent to $\tau_1 \otimes \zeta_\lambda$.*
3. $\{\rho_\lambda^+|_B : 1 \leqq \lambda < 0\} \cup \{\rho_\lambda^-|_B : 0 < \lambda \leqq 1\}$ *all are irreducible, nontrivial and mutually inequivalent, and they form a system of representatives of the relative discrete series of B.*

Taking (9.10) and Theorem 9.11 into account, Theorem 10.10 is contained in M. Vergne's systematic analysis [10] of restrictions of holomorphic discrete series representations.

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University of California, Berkeley
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