J. DIFFERENTIAL GEOMETRY
6 (1972) 317-342

# ON THE GEOMETRY AND CLASSIFICATION OF ABSOLUTE PARALLELISMS. I

JOSEPH A. WOLF

## 1. Introduction and summary

### *A. General introduction*

Within the context of riemannian geometry, the euclidean spaces are distinguished as the only complete simply connected manifolds in which parallel translation of tangent vectors is independent of path. When Élie Cartan developed the general notion of affine connection he saw that this absolute sort of parallelism was (at least locally) a matter of vanishing curvature. Then Cartan and Schouten [3] described curvature-free connections on Lie groups, thus exhibiting absolute parallelisms on group manifolds. This generalized Clifford's parallelism on the 3-sphere, which had previously been an isolated phenomenon. Cartan and Schouten [4] also gave a local description of the riemannian manifolds which have an absolute parallelism whose parallel vector fields have constant length and integrate to geodesics; they are the products of euclidean spaces, compact simple groups and 7-spheres. Unfortunately their reduction to the irreducible case may have gaps, and the cause of the parallelism on the 7-sphere was not too clear.

Here we extend the work of Cartan and Schouten to pseudo-riemannian manifolds. This means that the metric form $ds^2$ is of some nondegenerate signature[1] $(p, q)$, but not necessarily of positive definite signature $(n, 0)$. The de Rham decomposition theorem fails for indefinite signatures of metric; in fact, our example (3.7) shows that it fails for bi-invariant pseudo-riemannian metrics on nilpotent Lie groups. Thus we adopt an algebraic curvature condition ("reductive type"; see (5.7) below) which is automatic in the riemannian case and ensures us of a de Rham decomposition. Our main results are proved under that condition.

Let $(M, ds^2)$ be a connected pseudo-riemannian manifold of "reductive type" with an absolute parallelism $\phi$ which satisfies the Cartan-Schouten consistency conditions described above. Our main result (Theorem 9.1) says that $(M, ds^2)$

Received March 8, 1971. Research partially supported by National Science Foundation Grant GP-16651. Due to its length this paper is being published in two parts; the second part will appear in the first issue of the next volume.

[1] Signature $(p, q)$ means $p$ positive squares and $q$ negative squares in dimension $p + q$, as $\sum_1^p (x^i)^2 - \sum_1^q (x^{i+p})^2$ on $R^{p+q}$.

is locally isometric to a globally symmetric pseudo-riemannian manifold

$$(\tilde{M}, d\sigma^2) = (M_{-1}, ds^2_{-1}) \times (M_0, ds^2_0) \times (M_1, ds^2_1) \times \cdots \times (M_t, ds^2_t)$$

with $\phi$ corresponding to a consistent absolute parallelism

$$\tilde{\phi} = \phi_{-1} \times \phi_0 \times \phi_1 \times \cdots \times \phi_t ,$$

where the $(M_i, \phi_i, ds^2_i)$ are given as follows. $M_{-1}$ belongs to a certain class of simply connected nilpotent Lie groups of even dimension $2r$, $1 \neq r \neq 2$, with center of dimension $r$, and $ds^2_{-1}$ is a flat bi-invariant metric of signature $(r, r)$ on $M_{-1}$. $M_0$ is a real vector group and $ds^2_0$ is a (necessarily flat) translation-invariant metric there. Other $M_i$, say for $1 \leq i \leq u$, are simply connected real[2] simple Lie groups with bi-invariant metric $ds^2_i$ derived from a nonzero real multiple of the real[2] Killing form. For $-1 \leq i \leq u$ the parallelism $\phi_i$ on $M_i$ is the one whose parallel vector fields are the left-invariant fields on the group $M_i$. Then, for $u + 1 \leq i \leq t$, $M_i$ is a 7-sphere $SO(8)/SO(7)$, a certain indefinite metric version $SO(4,4)/SO(3,4)$ of the 7-sphere, or the complexification $SO(8, C)/SO(7, C)$ of the 7-sphere; there $ds^2_i$ is the invariant metric induced by a nonzero real multiple of the Killing form of $SO(8), SO(4,4)$ or $SO(8, C)$, and $\phi_i$ is obtained in an explicit manner from the triality automorphism. The $(M_i, ds^2_i), 1 \leq i \leq t$, are the irreducible factors of the de Rham decomposition of $(\tilde{M}, d\sigma^2)$. The flat factor is $(M_{-1}, ds^2_{-1}) \times (M_0, ds^2_0)$. The parallelism $\phi_0$ on $M_0$ is the euclidean parallelism, and is the largest euclidean factor of the parallelism $\phi_{-1} \times \phi_0$ on $M_{-1} \times M_0$. Here $\phi_{-1}$ is noneuclidean (unless $M_{-1}$ is reduced to a point); if $\xi$ is a non-isotropic $\phi_{-1}$-parallel vector field on $M_{-1}$, then there is another such field $\eta$ with $[\xi, \eta] \neq 0$. The $(M_{-1}, \phi_{-1}, ds^2_{-1})$ represent a new phenomenon which starts in signature (3, 3).

If $(M, ds^2)$ is (geodesically) complete, then Theorem 9.1 also provides a pseudo-riemannian covering $\pi: (\tilde{M}, d\sigma^2) \to (M, ds^2)$ such that $\tilde{\phi} = \pi^*\phi$. All such coverings are classified in Theorem 9.7. Thus we classify the complete pseudo-riemannian manifolds with consistent absolute parallelism of reductive type, picking out those which are globally symmetric or compact or riemannian. One interesting consequence (Corollary 9.10) is that the parallelism $\phi$ is consistent with a riemannian metric $dr^2$ on $M$ if, and only if, $(\tilde{M}, d\sigma^2)$ has a compact globally symmetric quotient manifold on which $\tilde{\phi}$ induces an absolute parallelism.

## *B. Summary*

§ 2 is a discussion of absolute parallelism $\phi$ on a differentiable manifold $M$, the flat connection $\Gamma$ associated to $\phi$, and the torsion tensor $T$ of $\Gamma$. Most of

[2] The group may be a complex simple Lie group viewed as a real simple (but not absolutely simple) Lie group. Then we use its Killing form as real Lie group.

the results are classical [3]. Then we characterize the case where $\Gamma$ is complete and $T$ is parallel, as $M = D\backslash G$ where $G$ is a Lie group, $D$ is a discrete subgroup, and $\phi$ is induced from the parallelism of left translation on $G$. That result is due to N. J. Hicks [6]; our proof is more direct.

§ 3 is a discussion of the Cartan-Schouten consistency conditions for an absolute parallelism $\phi$ on a pseudo-riemannian manifold $(M, ds^2)$. We show that the Levi-Cività connection $'\Gamma$ of $ds^2$ is given by $'\Gamma = \Gamma - \frac{1}{2}T$, a fact essentially due to Cartan and Schouten [4]. We also show that, if $\xi, \eta$ and $\zeta$ are $\phi$-parallel vector fields, then $ds^2([\xi, \eta], \zeta) = ds^2(\xi, [\eta, \zeta])$; that forms the basis of most subsequent developments. In the case where $\Gamma$ is complete and $T$ parallel, so $M = D\backslash G$ as described above, the latter fact says that $ds^2$ comes from a bi-invariant metric on the Lie group $G$, and thus from a nondegenerate invariant bilinear form on the Lie algebra $\mathfrak{g}$ of $G$.

§ 4 is a discussion of the curvature of a pseudo-riemannian manifold $(M, ds^2)$ with consistent absolute parallelism $\phi$. We start by direct computation of the sectional curvatures in terms of the torsion tensor $T$ of $\phi$. In the riemannian case it follows immediately that $(M, ds^2)$ has every sectional curvature $\geq 0$. Next we show that every $\phi$-parallel vector field is a Killing field of $(M, ds^2)$; so $(M, ds^2)$ is homogeneous if it is complete and connected. Then we compute the curvature tensor $'R$ of $ds^2$, and show that its coefficients are constant in a $\phi$-parallel frame; the result is that if $\xi$, $\eta$ and $\zeta$ are $\phi$-parallel, then $'R(\xi, \eta)\cdot\zeta = -\frac{1}{4}[[\xi, \eta], \zeta]$ and that is $\phi$-parallel. It follows that $(M, ds^2)$ is locally symmetric. Some of these results were worked out by Cartan and Schouten [4] in the irreducible riemannian case. The arguments in the last half of § 4 follow some ideas developed by J. E. D'Atri and H. K. Nickerson [5] in another context, and the riemannian case of Theorem 9.1 completes the classification they started in [5, § 4]; Corollary 4.15 gives the relation between absolute parallelism and the D'Atri-Nickerson work.

§§ 2, 3 and 4 comprise the general theory. To go farther one needs a de Rham decomposition. The idea is suggested by the facts that, if $\xi, \eta$ and $\zeta$ are $\phi$-parallel vector fields, then

$$'R(\xi, \eta)\cdot\zeta = -\tfrac{1}{4}[[\xi, \eta], \zeta]\ , \qquad \phi\text{-parallel,}$$
$$ds^2([\xi, \eta], \zeta) = ds^2(\xi, [\eta, \zeta])\ .$$

The first of these facts says that the $\phi$-parallel vector fields form a Lie triple system $\mathfrak{p}$ under the composition $[\xi\eta\zeta] = [[\xi, \eta], \zeta]$, and the second suggests some sort of invariant bilinear form on $\mathfrak{p}$.

§ 10 is a summary of the theory of Lie triple systems (LTS) due to N. Jacobson [7] (§ 10A) and W. G. Lister [9] (§ 10B). Then there is a theory of invariant bilinear forms (§ 10C) on LTS which we develop to fit the considerations mentioned above. By "invariant bilinear form" on a LTS $\mathfrak{m}$ we mean a symmetric bilinear form $b$ such that

$$b(z, [yxw]) = b([xyz], w) = b(x, [wzy]) .$$

Then the pair $(\mathfrak{m}, b)$ is said to be of "reductive type" if $b$ is nondegenerate both on $\mathfrak{m}$ and on the center of $\mathfrak{m}$, and if every ideal $\mathfrak{i} \subset \mathfrak{m}$ with $[\mathfrak{i}\mathfrak{m}\mathfrak{i}] = 0$ is central in $\mathfrak{m}$. We show that $(\mathfrak{m}, b)$ is of reductive type if, and only if, $\mathfrak{m}$ is a $b$-orthogonal direct sum $\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t$ where $\mathfrak{m}_0$ is the center and the other $\mathfrak{m}_i$ are simple ideals. If $\mathfrak{m}$ is a Lie algebra, then this says whether $\mathfrak{m}$ is reductive, and even that seems to be new. Evidently this LTS information is just what we need for a de Rham decomposition. It is separated as an appendix in order to preserve continuity of exposition, but it is used in §§ 5, 6, 7 and 8.

§ 5 is a de Rham decomposition theory for pseudo-riemannian symmetric spaces of reductive type. Let $(M, ds^2)$ be a locally symmetric pseudo-riemannian manifold, $x \in M$, $\mathfrak{g}_x$ the Lie algebra of germs of Killing vector fields at $x$, and $\mathfrak{g}_x = \mathfrak{k}_x + \mathfrak{m}_x$ the Cartan decomposition under the local symmetry of $(M, ds^2)$ at $x$. Then $\mathfrak{m}_x$ is the space of germs of infinitesimal transvections of $(M, ds^2)$ at $x$, and it is a LTS under the composition $[uvw] = [[u, v], w]$. $ds^2$ induces a nondegenerate symmetric bilinear form $b_x$ on $\mathfrak{m}_x$, and we show that $b_x$ is an invariant bilinear form on $\mathfrak{m}_x$ in the sense of LTS. If $M$ is connected, then the isomorphism class of $(\mathfrak{m}_x, b_x)$ is independent of choice of $x$ and determines $(M, ds^2)$ up to local isometry; a splitting $(\mathfrak{m}_x, b_x) \cong (\mathfrak{m}'_x, b'_x) \oplus (\mathfrak{m}''_x, b''_x)$ gives an isometric splitting of a neighborhood of $x$. The connection with the curvature tensor $'R$ of $(M, ds^2)$ is that $'R(u_x, v_x)\cdot w_x = -[uvw]_x$ for $u, v, w \in \mathfrak{m}_x$. Now we say that $(M, ds^2)$ is of "reductive type" if the $(\mathfrak{m}_x, b_x)$ are of reductive type, except that we say it with the curvature tensor. The local and global de Rham decompositions follow when $(M, ds^2)$ is of reductive type.

§ 6 translates the results of § 5 to a consistent absolute parallelism $\phi$ on $(M, ds^2)$. The delicate matter is the decomposition of $\phi$ under a product decomposition of $(M, ds^2)$; that is the gap in [4]. Let $x \in M, (\mathfrak{m}_x, b_x)$ be as in the description of § 5, and $\mathfrak{p}$ be the LTS of $\phi$-parallel vector fields. Note $\mathrm{germ}_x(\mathfrak{p}) \subset \mathfrak{g}_x$. If $\sigma_x$ is the symmetry we show that $\xi \mapsto \mathrm{germ}_x(\xi) - \sigma_x\, \mathrm{germ}_x(\xi)$ is a LTS isomorphism of $\mathfrak{p}$ onto $\mathfrak{m}_x$. Then we characterize the submanifolds $N \subset M$ on which $\phi$ induces an absolute parallelism. With that information it is fairly straightforward to prove: if $(M, ds^2)$ is of reductive type, then $\phi$ induces an absolute parallelism on each factor of the de Rham decomposition. Thus questions of classification come down to the cases where $(M, ds^2)$ is either flat or locally irreducible.

§ 7 is the classification of flat pseudo-riemannian manifolds $(M, ds^2)$ with consistent absolute parallelism $\phi$. Let $\mathfrak{p}$ be the LTS of $\phi$-parallel vector fields, and $T$ the torsion tensor of $\phi$. We first show that $T$ is parallel so that $\mathfrak{p}$ is a Lie algebra and $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] = 0$. We then classify all pairs $(\mathfrak{g}, b)$ such that $\mathfrak{g}$ is a Lie algebra with $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$ and $b$ is a nondegenerate invariant bilinear form on $\mathfrak{g}$. From that we obtain local product structure (global in the complete simply connected case) of $(M, ds^2)$ as $(M_{-1}, ds^2_{-1}) \times (M_0, ds^2_0)$, with $\phi$ corresponding to $\phi_{-1} \times \phi_0$, where $\phi_0$ is the euclidean component of the parallelism and $(M_{-1}, \phi_{-1}, ds^2_{-1})$ corresponds to elements of a certain class of metabelian Lie algebras.

§ 8 is the classification of irreducible pseudo-riemannian manifolds $(M, ds^2)$ with consistent absolute parallelism $\phi$. There the LTS of $\phi$ satisfies either $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$ or $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$. If $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$, then we are in the case of a real simple group manifold. If $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$, then $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ is the Lie algebra $\mathfrak{g}$ of all Killing vector fields; thus we have two decompositions

$$[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p} = \mathfrak{g} = \mathfrak{k} + \mathfrak{m} , \qquad \mathfrak{g} \text{ simple,}$$

where the second is Cartan decomposition under local symmetry $\sigma_x$ at a point $x \in M$. Also, the LTS isomorphism $\mathfrak{p} \to \mathfrak{m}$ extends to an automorphism $\varepsilon_x$ of $\mathfrak{g}$. With a bit of technical fuss we show that 1, $\sigma_x$ and $\varepsilon_x$ represent different components of the automorphism group of $\mathfrak{g}$, so $\mathfrak{g}$ is of type $D_4$ and $\varepsilon_x$ is a triality automorphism. It then follows that the complete simply connected symmetric space for $(\mathfrak{m}_x, b_x)$ is the 7-sphere $SO(8)/SO(7)$, a quadric $SO(4,4)/SO(3,4)$, or $SO(8, \mathrm{C})/SO(7, \mathrm{C})$. Conversely we use triality automorphisms to construct all consistent absolute parallelisms on those three spaces.

§ 9 is the synthesis of the results of §§ 6, 7 and 8, described above in the general introduction.

## *C. Notes to the reader*

In general we follow the conventions of [11]. As most computation is done in the flat connection associated to the parallelism, we denote

$$\Gamma, \Gamma^i_{jk}, T, T^i_{jk}, \nabla: \text{ for the flat connection}$$

and

$$'\Gamma, '\Gamma^i_{jk}, 'R, 'R^i_{jkl}, '\nabla: \text{ for the Levi-Cività connection.}$$

This paper is written so that the following segments can be read separately.

§§ 2, 3 and 4 give the general theory of absolute parallelisms on differentiable and pseudo-riemannian manifolds. § 10 is a summary of the general theory of LTS and a theory of reductive LTS. §§ 10 and 5 in that order are a de Rham decomposition theory for pseudo-riemannian symmetric spaces. § 7 with a few glances at §§ 4 and 10 is the theory of absolute parallelisms on flat pseudo-riemannian manifolds. § 8 with a few references to §§ 3, 5, 6 and 10 is the theory of absolute parallelisms on irreducible pseudo-riemannian manifolds.

## 2. Absolute parallelism on differentiable manifolds

By *absolute parallelism* on a differentiable manifold $M$, we mean a rule $\phi$, for translation of tangent vectors between any two points $x, y \in M$, which does not depend on additional choices. More precisely, it is a system of linear isomorphisms

$$\text{(2.1a)} \qquad \phi = \{\phi_{yx}\} , \quad \phi_{yx}: M_x \cong M_y \quad (\text{for all } x, y \in M)$$

of tangent spaces, with consistency condition

$$\text{(2.1b)} \qquad \phi_{zy} \cdot \phi_{yx} = \phi_{zx} , \quad \phi_{xx} = \text{ident.} \quad (\text{for all } x, y, z \in M)$$

and regularity condition

$$\text{(2.1c)} \qquad \text{if} \quad \xi_x \in M_x \quad \text{then} \quad \xi = \{\xi_y\}, \xi_y = \phi_{yx}\xi_x , \quad \text{is smooth.}$$

Let $\phi = \{\phi_{yx}\}$ be an absolute parallelism on $M$. We say that tangent vectors $\xi_x \in M_x$ and $\xi_y \in M_y$ are *parallel* if $\xi_y = \phi_{yx}\xi_x$. This is an equivalence relation on the set of all tangent vectors to $M$ by (2.1b), and the equivalence classes are smooth vector fields on $M$ by (2.1c). We call these equivalence classes the *parallel vector fields* of $(M, \phi)$ or *$\phi$-parallel* vector fields on $M$.

The following classical theorem contains the basic facts about absolute parallelisms on differentiable manifolds.

**2.2. Proposition.** *Let $M$ be a connected differentiable manifold. Then there are natural one-one correspondences between*

( i ) *absolute parallelisms $\phi$ on $M$,*

( ii ) *smooth trivializations $X$ of the frame bundle $B \to M$,*

(iii) *smooth connections $\Gamma$ on $B \to M$ with holonomy group reduced to the identity.*[3]

*The correspondences are* (a) $X = \{\xi_1, \cdots, \xi_n\}$, $n = \dim M$, *where the $\xi_i$ are parallel vector fields of $(M, \phi)$ which are linearly independent at some (thus every) point*; (b) *the $\Gamma$-horizontal space at $X(x) \in B$ is the tangent space to $X(M)$ at $X(x)$*; *and* (c) *$\phi$ is the $\Gamma$-parallelism.*

*Proof.* Given $\phi$, let $X' = \{\xi'_1, \cdots, \xi'_n\}$ be a second parallel frame on $M$. Choose $x \in M$ and define $g \in GL(n, R)$ by $X'(x) = X(x) \cdot g$. Then $X' = X \cdot g$ globally on $M$, so $X'$ and $X$ define the same smooth trivialization of $B \to M$.

Given $X$ the $\Gamma$-horizontal space at an arbitrary point $X(x) \cdot g \in B$ ($x \in M$, $g \in GL(n, R)$) is the tangent space there to $X(M) \cdot g$. If $\sigma(t), a \leq t \leq b$, is a sectionally smooth curve in $M$ based at $x$, then its horizontal lift to $X(x) \cdot g$ is $t \mapsto X(\sigma(t)) \cdot g$. Since the lift has endpoint $X(x) \cdot g$, the $\Gamma$-parallel translation around $\sigma$ is trivial. Thus $\Gamma$ has trivial holonomy group.

Finally, notice that $X$ is $\Gamma$-parallel. q.e.d.

---

[3] It follows from the Cartan structure equations that $\Gamma$ is flat (i.e., has zero curvature).

We remark that when $M$ is simply connected, the class (iii) of Proposition 2.2 coincides with the class

(iii)′ *flat (zero curvature) connections $\Gamma$ on $B \to M$.*

**2.3. Example.** Let $G$ be a Lie group. Define $\lambda_{yx}: G_x \to G_y$ to be the tangent space map of the left translation $g \mapsto yx^{-1}g$. This defines an absolute parallelism $\lambda = \{\lambda_{yx}\}$ on $G$. Euclidean parallelism is the case $G = R^n$ of real vector groups, and Clifford parallelism on the 3-sphere is the case $G = SU(2) \cong S^3$. Similarly, $\rho_{yx}: G_x \to G_y$ from $g \mapsto gx^{-1}y$ defines another absolute parallelism $\rho = \{\rho_{yx}\}$ on $G$.

Fix an absolute parallelism $\phi$ on $M$. By the connection *associated* to $\phi$ we mean the connection $\Gamma$ on the frame bundle provided by Proposition 2.2, so that $\phi$ is the parallelism of $\Gamma$. If $\Gamma$ is complete then we say that *$\phi$ is complete.*

Choose a parallel frame $X = \{\xi_1, \cdots, \xi_n\}$ on $M$, and let $\theta = \{\theta^1, \cdots, \theta^n\}$ be the dual coframe. Then the $\nabla_{\xi_j}(\xi_k) = \sum \Gamma^i_{jk}\xi_i$ vanish identically because the $\xi_k$ are $\Gamma$-parallel, so $\Gamma^i_{jk} = 0$ and the connection forms $\omega^i_j = 0$. Now the equations of structure of $\Gamma$ in the frame $X$ are reduced to

$$d\theta^i = \tfrac{1}{2} \sum T^i_{jk}\theta^j \wedge \theta^k \quad \text{and} \quad \omega^i_j = 0 \,, \tag{2.4a}$$

where the torsion tensor of $\Gamma$ is given by

$$T(\xi_j, \xi_k) = \sum T^i_{jk}\xi_i = -[\xi_j, \xi_k] \,. \tag{2.4b}$$

In particular,

(2.4c) $T$ is parallel if, and only if, the $T^i_{jk}$ are constants.

**2.5. Proposition.** *Let $M$ be connected. Then there exist a connected Lie group $G$ and a discrete subgroup $D \subset G$ such that $M$ has the structure of the coset space $D\backslash G = \{Dg: g \in G\}$ with $\phi$ induced from the absolute parallelism of left translation on $G$ if, and only if,* (i) *$\phi$ is complete and* (ii) *$T$ is parallel.*

*Proof.* Let $G$ be a connected Lie group, $D \subset G$ a discrete subgroup, and $\lambda$ the parallelism of left translation on $G$. Then left translation by any $g \in D$ preserves every $\lambda$-parallel vector field, so $\lambda$ induces an absolute parallelism on $D\backslash G$. Choose a basis $X' = \{\xi'_1, \cdots, \xi'_n\}$ for the Lie algebra $\mathfrak{g}$ of left invariant vector fields on $G$, and let $X = \{\xi_1, \cdots, \xi_n\}$ be the global frame induced on $D\backslash G$. Then $[\xi_j, \xi_k] = \sum c^i_{jk}\xi_i$ where the $c^i_{jk}$ are the structure constants of $\mathfrak{g}$, i.e., $[\xi'_j, \xi'_k] = \sum c^i_{jk}\xi'_i$. So the $T^i_{jk} = -c^i_{jk}$ are constant. Thus $T$ is parallel, and completeness on $D\backslash G$ comes from completeness of $\lambda$.

Conversely, suppose $\phi$ complete and $T$ parallel. Then the parallel vector fields of $(M, \phi)$ form a Lie algebra $\mathfrak{g}$. Let $G$ be the simply connected Lie group for $\mathfrak{g}$. If $\xi \in \mathfrak{g}$ and $x \in M$, then the 1-parameter group $\{\exp(t\xi)\}$ acts by: $\{\exp(t\xi) \cdot x\}$ is the integral curve for $\xi$ through $x$. This defines an action of $G$ on $M$ corresponding to the action of $G$ on itself by *right* translation. All $G$-orbits in $M$ are open, so the action is transitive.

Choose $x_0 \in M$ and let $D = \{g \in G: g(x_0) = x_0\}$. $D$ is a discrete subgroup and $M \cong D\backslash G$ with the action corresponding to right translation. The parallel fields of $(M, \phi)$ correspond to the fields on $D\backslash G$ induced by elements of $\mathfrak{g}$, from construction of the action of $G$. Thus $\phi$ is induced by the *left* translation parallelism of $G$.

## 3. Consistent pseudo-Riemannian metrics

Let $M$ be a differentiable manifold, $\phi$ an absolute parallelism on $M$, and $ds^2$ a pseudo-riemannian metric on $M$. We say that $\phi$ and $ds^2$ are *consistent* if $ds^2$ is $\phi$-invariant in the sense

(3.1a) $$ds^2(\phi_{yx}\xi_x, \phi_{yx}\eta_x) = ds^2(\xi_x, \eta_x) \qquad (x, y \in M; \xi_x, \eta_x \in M_x)$$

and, modulo parameterization,

(3.1b) the $ds^2$-geodesics are the $\phi$-geodesics.

Here $\phi$-geodesic means geodesic for the connection associated to $\phi$.

Choose a $\phi$-parallel frame $X = \{\xi_1, \cdots, \xi_n\}$ on $M$, let $\theta = \{\theta^1, \cdots, \theta^n\}$ be the dual coframe, let $\Gamma$ be the connection associated to $\phi$, and recall the structure equations (2.4) of $\Gamma$. Now denote

(3.2a) $$ds^2 = \sum g_{ij}\theta^i\theta^j , \qquad \text{global expression,}$$

and

(3.2b) $'\Gamma$: Levi-Cività connection of $ds^2$ .

**3.3. Lemma.** *$ds^2$ is $\phi$-invariant if, and only if, the functions $g_{ij}$ are constants.*

*Proof.* $\nabla_{\xi_k}(\xi_i) = 0 = \nabla_{\xi_k}(\xi_j)$. If $g_{ij}$ is constant, then

$$\xi_k(g_{ij}) = 0 = ds^2(\nabla_{\xi_k}(\xi_i), \xi_j) + ds^2(\xi_i, \nabla_{\xi_k}(\xi_j)) ,$$

so $ds^2$ is $\phi$-invariant. Conversely, if $ds^2$ is $\phi$-invariant, then

$$\xi_k(g_{ij}) = ds^2(\nabla_{\xi_k}(\xi_i), \xi_j) + ds^2(\xi_i, \nabla_{\xi_k}(\xi_j)) = 0 ,$$

so the $g_{ij}$ are constants.

**3.4. Lemma.** *Let $ds^2$ be $\phi$-invariant. Then the following conditions are equivalent.*

(1) *$\phi$ and $ds^2$ are consistent.*

(2) *The $ds^2$-geodesics are the $\phi$-geodesics with the same affine parameterization.*

(3) *In the $\phi$-parallel frame $X$, the $'\Gamma_{ijk}$ are alternating in every pair of indices.*

(4) $'\Gamma = \Gamma - \frac{1}{2}T$, i.e., *in some* (*hence every*) *moving frame one has* $'\Gamma^i_{jk} = \Gamma^i_{jk} - \frac{1}{2}T^i_{jk}$.

(5) *If* $\xi, \eta$ *and* $\zeta$ *are* $\phi$*-parallel vector fields, then* $ds^2([\xi,\eta],\zeta) + ds^2(\eta,[\xi,\zeta]) = 0$.

*Proof.* $ds^2 = \sum g_{ij}\theta^i\theta^j$ and Lemma 3.3 say that the nonsingular symmetric matrix $(g_{ij})$ is constant.

Assume (1) and let $\sigma$ be a $\phi$-geodesic in affine parameter. Thus $\sigma'(t) = \sum a^i\xi_{i\sigma(t)}$, $a^i$ constant, and $\sigma(t) = \sigma(t(s))$ is a $ds^2$-geodesic with affine parameter $s$. As the $a^i$ and the $g_{ij}$ are constant,

$$ds^2(\sigma'(t), \sigma'(t)) = \sum g_{ij}a^ia^j \text{ constant.}$$

Thus $t$ is a $ds^2$-affine parameter for $\sigma$, proving (2).

Assume (2). If $\sigma$ is a geodesic, then $\sigma' = \sum a^i\xi_i$ with $a^i$ constant, so

$$0 = da^i/dt + \sum {'\Gamma^i_{jk}}a^ja^k = \sum {'\Gamma^i_{jk}}a^ja^k .$$

Thus the $'\Gamma^i_{jk} + {'\Gamma^i_{kj}} = 0$. Now the $'\Gamma_{ijk} = \sum g_{mk}{'\Gamma^m_{ij}}$ satisfy $'\Gamma_{ijk} + {'\Gamma_{jik}} = 0$. On the other hand,

$$\begin{aligned} 0 = \xi_i(g_{jk}) &= ds^2({'\nabla_{\xi_i}}(\xi_j), \xi_k) + ds^2(\xi_j, {'\nabla_{\xi_i}}(\xi_k)) \\ &= ds^2(\textstyle\sum {'\Gamma^m_{ij}}\xi_m, \xi_k) + ds^2(\xi_j, \sum {'\Gamma^m_{ik}}\xi_m) = {'\Gamma_{ijk}} + {'\Gamma_{ikj}} . \end{aligned}$$

Thus $'\Gamma_{ijk}$ is alternating in $(i,j)$ and in $(j,k)$, hence also in $(i,k)$, so (3) is proved.

Assume (3) and consider the tensor field of type $(2,1)$ given by $S(\xi,\eta) = \nabla_\xi(\eta) - {'\nabla_\xi}(\eta)$. Then $S(\xi_j,\xi_k) = -\sum {'\Gamma^i_{jk}}\xi_i$, so $S$ is alternating. As $'\Gamma$ has torsion $'T = 0$ now

$$\begin{aligned} 2S(\xi_j,\xi_k) &= S(\xi_j,\xi_k) - S(\xi_k,\xi_j) \\ &= \{\nabla_{\xi_j}(\xi_k) - \nabla_{\xi_k}(\xi_j)\} - \{{'\nabla_{\xi_j}}(\xi_k) - {'\nabla_{\xi_k}}(\xi_j)\} \\ &= {'\nabla_{\xi_k}}(\xi_j) - {'\nabla_{\xi_j}}(\xi_k) = [\xi_k,\xi_j] = T(\xi_j,\xi_k) . \end{aligned}$$

Thus $S = \frac{1}{2}T$, i.e., $'\Gamma - \Gamma = -\frac{1}{2}T$, proving (4).

Assume (4), so $'\nabla_{\xi_j}(\xi_k) = \nabla_{\xi_j}(\xi_k) - \frac{1}{2}T(\xi_j,\xi_k) = \frac{1}{2}[\xi_j,\xi_k]$. Then

$$\begin{aligned} 0 = \xi_i(g_{jk}) &= ds^2({'\nabla_{\xi_i}}(\xi_j), \xi_k) + ds^2(\xi_j, {'\nabla_{\xi_i}}(\xi_k)) \\ &= \tfrac{1}{2}\{ds^2([\xi_i,\xi_j],\xi_k) + ds^2(\xi_j,[\xi_i,\xi_k])\} , \end{aligned}$$

proving (5).

Assume (5), so the $T_{ijk} = \sum g_{km}T^m_{ij}$ satisfy $T_{ijk} + T_{ikj} = 0$. Thus the $T_{ijk}$ are alternating in every pair of indices. Let $\Delta$ be the connection with components $-\frac{1}{2}T^i_{jk}$, $D$ its covariant differentiation, and $S$ its torsion tensor. Then

$$\xi_i(g_{jk}) = 0 = -\tfrac{1}{2}T_{ijk} - \tfrac{1}{2}T_{ikj} = ds^2(D_{\xi_i}(\xi_j), \xi_k) + ds^2(\xi_j, D_{\xi_i}(\xi_k))$$

shows $ds^2$ to be invariant by $D$, and

$$S(\xi_j, \xi_k) = D_{\xi_j}(\xi_k) - D_{\xi_k}(\xi_j) - [\xi_j, \xi_k]$$
$$= -\tfrac{1}{2} \sum T^i_{jk}\xi_i + \tfrac{1}{2} \sum T^i_{kj}\xi_i + T(\xi_j, \xi_k) = 0$$

shows $\Delta$ to be torsion free. Thus $\Delta = {}'\Gamma$ by uniqueness of the Levi-Cività connection for $ds^2$. Now ${}'\Gamma^i_{jk} + {}'\Gamma^i_{kj} = 0$ so, for any constants $(a^1, \cdots, a^n)$,

$$\frac{da^i}{dt} + \sum {}'\Gamma^i_{jk} a^j a^k = 0 .$$

That proves (2), thus implies (1). q.e.d.

Let $\mathfrak{g}$ be a Lie algebra. By *invariant bilinear form* on $\mathfrak{g}$ one means a symmetric bilinear form $b$ on $\mathfrak{g}$ such that

(3.5a) $$b([\xi, \eta], \zeta) = b(\xi, [\eta, \zeta]) \quad \text{for all} \quad \xi, \eta, \zeta \in \mathfrak{g} .$$

If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, then $G$ acts on the space of symmetric bilinear forms on $\mathfrak{g}$ by the symmetric square of the dual of the adjoint representation, and (3.5a) is equivalent to $G$-invariance of $b$, i.e., to

(3.5b) $$b(\mathrm{ad}(g)\xi, \mathrm{ad}(g)\eta) = b(\xi, \eta) \quad \text{for all} \quad \xi, \eta \in \mathfrak{g} \quad \text{and} \quad g \in G .$$

The main example of an invariant bilinear form on $\mathfrak{g}$ is the *trace form*

$$b_\pi(\xi, \eta) = \text{trace } \pi(\xi) \cdot \pi(\eta) ,$$

where $\pi$ is a linear representation of $\mathfrak{g}$. The Killing form is the trace form of the adjoint representation.

A Lie algebra $\mathfrak{g}$ is called *reductive* if it has a faithful completely reducible representation. It is standard that the following conditions are equivalent.

(3.6a) $\mathfrak{g}$ is reductive.

(3.6b) The adjoint representation of $\mathfrak{g}$ is completely reducible.

(3.6c) $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple.

(3.6d) $\mathfrak{g}$ has a nondegenerate trace form.

It turns out that some non-reductive Lie algebras have non-degenerate invariant bilinear forms. In particular such a form cannot be a linear combination of trace forms. Here is an example. Let $\mathfrak{g}$ be the nilpotent algebra with basis $\{z_1, z_2, z_3, w_1, w_2, w_3\}$ where

(3.7a) $$\{z_1, z_2, z_3\} \text{ spans the center of } \mathfrak{g} \text{ and}$$
$$[w_1, w_2] = z_3 , \quad [w_2, w_3] = z_1 , \quad [w_3, w_1] = z_2 .$$

Now define

$$
\begin{aligned}
& b(w_i, w_j) = 0 = b(z_i, z_j) \quad \text{for all} \quad i, j; \\
(3.7\text{b}) \qquad & b(w_i, z_j) = 0 = b(z_j, w_i) \quad \text{for} \quad i \neq j; \\
& b(w_i, z_i) = 1 = b(z_i, w_i) \quad \text{for all} \quad i\,.
\end{aligned}
$$

Then $b$ is a nondegenerate invariant bilinear form on $\mathfrak{g}$ with signature $(3, 3)$.

Proposition 2.5, Lemmas 3.3 and 3.4, and definition (3.5) have the following immediate consequence.

**3.8. Theorem.** *The triples $(M, \phi, ds^2)$ such that $M$ is a connected manifold, $\phi$ is a complete absolute parallelism on $M$ with parallel torsion tensor, and $ds^2$ is a pseudo-riemannian metric consistent with $\phi$, are precisely the triples $(D\backslash G, \lambda, d\sigma^2)$ where*

(i) *$D$ is a discrete subgroup in a simply connected Lie group $G$ whose Lie algebra has a nondegenerate invariant bilinear form,*

(ii) *$\lambda$ is induced by the absolute parallelism of left translation on $G$, and*

(iii) *$d\sigma^2$ is induced from a bi-invariant pseudo-riemannian metric on $G$,* i.e., *from a metric defined by a nondegenerate invariant bilinear form on the Lie algebra.*

In the positive and negative definite cases, invariant bilinear forms are trace forms, so Theorem 3.8 simplifies as follows.

**3.9. Corollary.** *In the riemannian case, the triples $(M, \phi, ds^2)$ of Theorem* 3.8 *are precisely the triples $(D\backslash G, \lambda, d\sigma^2)$ where*

(i) *$G = V \times G'$ where $V$ is a real vector group and $G'$ is a compact simply connected semisimple group, and $D$ is a discrete subgroup of $G$,*

(ii) *$\lambda$ is induced by the absolute parallelism of left translation on $G$, and*

(iii) *$d\sigma^2$ is induced from a bi-invariant riemannian metric on $G$.*

**Remarks.** Given $(M, \phi)$, Lemma 3.4 (4) shows that $\phi$ determines the Levi-Cività connection of every consistent metric $ds^2$. One cannot expect more because every translation-invariant metric on $R^n$ is consistent with the euclidean parallelism. Also, the converse is false: $(M, ds^2)$ does not determine $\phi$ modulo isometries of $(M, ds^2)$. For, if $G$ is the 6-dimensional nonabelian nilpotent group for the algebra (3.7a), and $d\sigma^2$ is the bi-invariant metric of signature (3.3) specified by (3.7b), then we will see in (4.7) that $(G, d\sigma^2)$ is flat, hence isometric to the euclidean space of signature $(3, 3)$. The left translation parallelism $\lambda$ on $G$ is not the euclidean one because $[w_i, w_j] \neq 0$ for $i \neq j$. This matter will be explored systematically in § 7.

## 4. Curvature and symmetry of consistent metrics

Let $\phi$ be an absolute parallelism on $M$, and $ds^2$ a consistent pseudo-riemannian metric. We work in a $\phi$-parallel frame $X = \{\xi_1, \cdots, \xi_n\}$ and the dual co-frame $\theta = \{\theta^1, \cdots, \theta^n\}$. Lemma 3.3 says $ds^2 = \sum g_{ij}\theta^i\theta^j$ with the $g_{ij}$ constant. Lemma 3.4 says that the Levi-Cività connection $'\Gamma$ for $ds^2$ has "components"

$$(4.1) \qquad '\Gamma^u_{vw} = -\tfrac{1}{2}T^u_{vw} ,$$

where $T$ is the torsion tensor (2.4) of the connection $\Gamma$ associated to $\phi$. Now denote

$$(4.2a) \qquad 'R\colon \text{ curvature tensor of } ds^2 .$$

Its components in the frame $X$ are given by the

$$(4.2b) \qquad 'R(\xi_k, \xi_l)\xi_j = ([' \nabla_{\xi_k}, '\nabla_{\xi_l}] - '\nabla_{[\xi_k, \xi_l]})\xi_j = \sum {'R}^i_{jkl}\xi_i$$

or, using the metric,

$$(4.2c) \qquad 'R_{ijkl} = \sum {'R}^m_{jkl} g_{mi} = ds^2('R(\xi_k, \xi_l)\xi_j, \xi_i) .$$

**4.3. Theorem.** *Let $x \in M$ and $\alpha, \beta \in M_x$ such that $\alpha$ and $\beta$ span a plane on which $ds^2$ is nondegenerate, i.e., such that*

$$(4.4a) \qquad \|\alpha \wedge \beta\|^2 = ds^2(\alpha, \alpha)ds^2(\beta, \beta) - ds^2(\alpha, \beta)^2$$

*is nonzero. Then the sectional curvature of $(M, ds^2)$ on the plane spanned by $\alpha$ and $\beta$ is*

$$(4.4b) \qquad K_{\alpha\beta} = \tfrac{1}{4}ds^2(T(\alpha, \beta), T(\alpha, \beta))/\|\alpha \wedge \beta\|^2 .$$

*Proof.* Let $\alpha = a^i\xi_{ix}$ and $\beta = b^j\xi_{jx}$ (we use the summation convention). Then

$$\begin{aligned} 4ds^2('R(\alpha, \beta)\alpha, \beta) &= 4ds^2('R(\xi_i, \xi_j)\xi_k, \xi_l)a^ib^ja^kb^l \\ &= 4ds^2('\nabla_{\xi_i}'\nabla_{\xi_j}\xi_k - '\nabla_{\xi_j}'\nabla_{\xi_i}\xi_k - '\nabla_{[\xi_i, \xi_j]}\xi_k, \xi_l)a^ib^ja^kb^l \\ &= 2ds^2(-'\nabla_{\xi_i}T^m_{jk}\xi_m + '\nabla_{\xi_j}T^m_{ik}\xi_m + 2T^m_{ij}'\nabla_{\xi_m}\xi_k, \xi_l)a^ib^ja^kb^l \end{aligned}$$

(interchange $i, k$ to see that the second term vanishes)

$$\begin{aligned} &= ds^2(-2\xi_i(T^m_{jk})\xi_m + T^m_{jk}T^r_{im}\xi_r - 2T^m_{ij}T^r_{mk}\xi_r, \xi_l)a^ib^ja^kb^l \\ &= \{-2\xi_i(T_{jkl}) + T^m_{jk}T_{iml} - 2T^m_{ij}T_{mkl}\}a^ib^ja^kb^l \end{aligned}$$

(interchange $j, l$ to see that the first term vanishes)

$$\begin{aligned} &= T^m_{jk}T_{iml}a^ib^ja^kb^l - 2T^m_{ij}T_{mkl}a^ib^ja^kb^l \\ &= T^m_{ji}T_{kml}a^ib^ja^kb^l - 2T^m_{ij}T_{klm}a^ib^ja^kb^l \\ &= -T^m_{ij}T_{klm}a^ib^ja^kb^l \\ &= -ds^2(a^ib^jT^m_{ij}\xi_m, a^kb^lT^r_{kl}\xi_r) \\ &= -ds^2(T(\alpha, \beta), T(\alpha, \beta)) . \end{aligned}$$

Now recall $K_{\alpha\beta} = -ds^2('R(\alpha,\beta)\alpha,\beta)/\|\alpha\wedge\beta\|^2$.

**4.5. Corollary.** *If $ds^2$ is positive definite, then $(M, ds^2)$ has every sectional curvature $K_{\alpha\beta}\geq 0$.*

Let $\{\phi_k(t)\}$ be the local one-parameter group of local diffeomorphisms whose orbits are the integral curves of $\xi_k$. Suppose $x\in M$, $x_t = \phi_k(t)\cdot x$, and

$$\alpha_t = a^i(t)\xi_{ix_t}\,, \qquad \beta_t = b^j(t)\xi_{jx_t}$$

are vector fields along $\{x_t\}$. Then the Lie derivatives

$$L_{\xi_k}(\alpha) = \xi_k(a^i)\xi_i - a^iT(\xi_k,\xi_i)\,, \qquad L_{\xi_k}(\beta) = \xi_k(b^j)\xi_j - b^jT(\xi_k,\xi_j)\,.$$

In particular, $\{\alpha_t\}$ and $\{\beta_t\}$ are $\phi_k$-invariant if, and only if,

$$\xi_k(a^i) = T^i_{km}a^m \quad\text{and}\quad \xi_k(b^j) = T^j_{kr}b^r\,.$$

In that case,

$$\begin{aligned}\frac{d}{dt}ds^2(\alpha_t,\beta_t) &= ds^2(T^i_{km}a^m\xi_i, b^j\xi_j) + ds^2(a^i\xi_i, T^j_{kr}b^r\xi_j)\\ &= T_{kmj}a^mb^j + T_{kri}a^ib^r = (T_{kij}+T_{kji})a^ib^j = 0\,.\end{aligned}$$

Hence:

**4.6. Proposition.** *Every $\phi$-parallel vector fields is a Killing vector field of $(M, ds^2)$. In particular, if $(M,\phi)$ is complete, and $M$ is connected, then $(M, ds^2)$ is homogeneous.*

We go on to show $(M, ds^2)$ to be locally symmetric. Consider the tensor $S$ of type $(1,3)$ given in the frame $X$ by

$$S(\xi_k,\xi_l)\cdot\xi_j = S^m_{jkl}\xi_m = -\tfrac{1}{4}[[\xi_k,\xi_l],\xi_j]\,.$$

Lowering an index,

$$S_{ijkl} = ds^2(S(\xi_k,\xi_l)\cdot\xi_j,\xi_i) = -\tfrac{1}{4}ds^2([[\xi_k,\xi_l],\xi_j],\xi_i)\,.$$

Evidently we have the identities

$$S_{ijkl} + S_{iljk} + S_{iklj} = 0 = S_{ijkl} + S_{ijlk}\,.$$

Also, as $[\xi_k,\xi_l]$ is a Killing vector field by Proposition 4.6, and as $g_{ji}$ is constant,

$$ds^2([[\xi_k,\xi_l],\xi_j],\xi_i) + ds^2(\xi_j,[[\xi_k,\xi_l],\xi_i]) = 0;$$

so we also have the identity

$$S_{ijkl} + S_{jikl} = 0\,.$$

If $(a^i)$ and $(b^j)$ are $n$-tuples, then Theorem 4.3 ensures

$$S_{ijkl}a^ib^ja^kb^l = {}'R_{ijkl}a^ib^ja^kb^l ,$$

which implies $S_{ijkl} = {}'R_{ijkl}$ in view of the identities satisfied by $S$. Thus we have proved

(4.7) $\quad {}'R(\xi_k, \xi_l)\cdot\xi_j = -\frac{1}{4}[[\xi_k, \xi_l], \xi_j]$ and is a Killing vector field.

We need the derivations of the tensor algebra given by

(4.8a) $\quad A_\xi = L_\xi - {}'\nabla_\xi$ where $L$ is Lie derivative.

As ${}'\Gamma$ is torsion free they satisfy

(4.8b) $\quad {}'\nabla_\eta(\xi) = -A_\xi(\eta)$ , for all vector fields $\xi, \eta$.

In particular, $\xi$ is a Killing vector field if and only if

(4.8c) $\quad ds^2(A_\xi\eta, \zeta) + ds^2(\eta, A_\xi\zeta) = 0$ , for all fields $\eta, \zeta$.

**4.9. Lemma** (*D'Atri-Nickerson*). *Let $\xi$ and $\eta$ be Killing vector fields of $(M, ds^2)$. Then $ds^2(\xi, \eta)$ is constant if and only if $A_\xi(\eta) + A_\eta(\xi) = 0$.*

*Proof.* For any vector field $\zeta$ we have $\zeta\cdot ds^2(\xi, \eta) = ds^2({}'\nabla_\zeta\xi, \eta) + ds^2(\xi, {}'\nabla_\zeta\eta) = -ds^2(A_\xi\zeta, \eta) - ds^2(\xi, A_\eta\zeta) = ds^2(\zeta, A_\xi\eta + A_\eta\xi)$.

**4.10. Lemma.** *The ${}'R_{ijkl}$ are constants.*

*Proof.* Let $\eta = {}'R(\xi_k, \xi_l)\xi_j$, so that ${}'R_{ijkl} = ds^2(\eta, \xi_i)$ and $\eta$ is a Killing vector field by (4.7). Then

$$\xi_q\cdot{}'R_{ijkl} = ds^2({}'\nabla_{\xi_q}\eta, \xi_i) + ds^2(\eta, {}'\nabla_{\xi_q}\xi_i) .$$

Now $2{}'\nabla_{\xi_q}(\eta) = [\xi_q, \eta] = L_{\xi_q}(\eta)$ shows $A_{\xi_q}(\eta) = {}'\nabla_{\xi_q}(\eta)$, and ${}'\nabla_{\xi_q}(\xi_i) = -A_{\xi_i}(\xi_q) = A_{\xi_q}(\xi_i)$ by Lemma 4.9. Thus

$$\xi_q\cdot{}'R_{ijkl} = ds^2(A_{\xi_q}\eta, \xi_i) + ds^2(\eta, A_{\xi_q}(\xi_i)) = 0 . \qquad \text{q.e.d.}$$

As the $g_{ij}$ are constants, Lemma 4.10 says that the ${}'R^i_{jkl}$ are constants. In view of (4.7) we have

**4.11. Proposition.** *If $\xi, \eta$ and $\zeta$ are $\phi$-parallel vector fields on $M$, then $[[\xi, \eta], \zeta]$ and ${}'R(\xi, \eta)\zeta = -\frac{1}{4}[[\xi, \eta], \zeta]$ are $\phi$-parallel vector fields on $M$.*

Let $x_0 \in M$, and $\{x_t\}$ be a geodesic arc through $x_0$. We may assume $X$ to be chosen so that $\{x_t\}$ is an integral curve to $\xi_q$, i.e., $x_t = \phi_q(t)\cdot x$ where $\phi_q$ is the local one-parameter group local isometries for the Killing vector field $\xi_q$. Now let $\alpha_0, \beta_0 \in M_{x_0}$. We extend them to two pairs $\{\alpha_t\}, \{\beta_t\}$ and $\{'\alpha_t\}, \{'\beta_t\}$ of vector fields along $\{x_t\}$ by the conditions that $\alpha_t = a^i(t)\xi_{ix_t}$ and $\beta_t = b^j(t)\xi_{jx_t}$ be $\phi_q$-invariant, and $'\alpha_t = {}'a^i(t)\xi_{ix_t}$ and $'\beta_t = {}'b^j(t)\xi_{jx_t}$ be $ds^2$-parallel. In the discussion following Corollary 4.5 we saw that the $\phi_q$-invariance means that $\xi_q(a^i) = T^i_{qm}a^m$ and $\xi_q(b^j) = T^j_{qr}b^r$. Similarly, $'\Gamma^i_{jk} = -\frac{1}{2}T^i_{jk}$ says that $ds^2$-parallel means that $\xi_q('a^i) = \frac{1}{2}T^i_{qm}{}'a^m$ and $\xi_q('b^j) = \frac{1}{2}T^j_{qr}{}'b^r$. Thus

$$\xi_q(a^i) = 2\xi_q('a^i) \quad \text{and} \quad \xi_q(b^j) = 2\xi_q('b^j) \text{ at } x_0 \ . \tag{4.12}$$

As the $\phi_q(t)$ are local isometries of $(M, ds^2)$ we have

$$\frac{d}{dt}\{'R_{ijkl}a^ib^ja^kb^l\} = 0 \ .$$

With (4.12) and Lemma 4.10 this says that

$$\left.\frac{d}{dt}\right|_{t=0} {}'R_{ijkl}{}'a^{i}{}'b^{j}{}'a^{k}{}'b^l = 0 \ ,$$

and thus that derivative vanishes for all $t$. Now $\|'\alpha_t \wedge {}'\beta_t\|^2$ is constant. Thus, if $ds^2$ is nondegenerate on the plane spanned by $\alpha_0$ and $\beta_0$, then

$$\frac{d}{dt}(K_{'\alpha_t{}'\beta_t}) = 0 \ , \tag{4.13}$$

which proves sectional curvature of $(M, ds^2)$ to be invariant under $ds^2$-parallel translation. We summarize as follows.

**4.14. Theorem.** *Let $M$ be a differentiable manifold, $\phi$ an absolute parallelism on $M$, and $ds^2$ a pseudo-riemannian metric consistent with $\phi$.*

1. *$(M, ds^2)$ is locally symmetric.*
2. *If $\xi$ is a $\phi$-parallel vector field on $M$, then $\xi$ is a Killing vector field of $(M, ds^2)$.*
3. *If $\xi$ and $\eta$ are $\phi$-parallel vector fields on $M$, then $ds^2(\xi, \eta)$ is constant.*
4. *If $\xi, \eta$ and $\zeta$ are $\phi$-parallel vector fields on $M$, and $'R$ is the curvature tensor of $(M, ds^2)$, then $'R(\xi, \eta)\zeta$ and $[[\xi, \eta], \zeta] = -4'R(\xi, \eta)\zeta$ are $\phi$-parallel vector fields on $M$.*

We note that Theorem 4.14 can be turned around.

**4.15. Corollary.** *Let $(M, ds^2)$ be a connected pseudo-riemannian manifold. Then $M$ has an absolute parallelism $\phi$ consistent with $ds^2$ if, and only if, $M$ carries a global frame $X = \{\xi_1, \cdots, \xi_n\}$ such that*

(i) *each $\xi_i$ is a Killing vector field of $(M, ds^2)$, and*

(ii) *the $ds^2(\xi_i, \xi_j)$ are constants.*

*Then the $\xi_i$ are $\phi$-parallel, and $ds^2$ is locally symmetric.*

*Proof.* Given $\phi$ and $ds^2$ consistent, the assertions are contained in Theorem 4.14.

Given $X$, define $\phi$ by the condition that the $\xi_i$ be $\phi$-parallel. Then (ii) is $\phi$-invariance (3.1a) of $ds^2$, so we must check (3.1b) that the $ds^2$- and $\phi$-geodesics agree, i.e., that $'\nabla_\xi(\xi) = 0$ for all $\phi$-parallel $\xi$. For that, note (4.8) that $A_\xi(\xi) = 0$ by Lemma 4.9, so $-'\nabla_\xi(\xi) = [\xi, \xi] - {}'\nabla_\xi(\xi) = A_\xi(\xi) = 0$.

## 5. Decomposition of pseudo-riemannian symmetric spaces

Let $(M, ds^2)$ be a locally symmetric pseudo-riemannian manifold. Choose $x \in M$, let $\mathfrak{g}_x$ be the Lie algebra of germs of Killing vector fields at $x$, and let $s_x$ be the local symmetry at $x$. Then $s_x$ induces an involutive automorphism $\sigma_x$ of $\mathfrak{g}_x$, and we denote

$$\mathfrak{k}_x\colon\ +1 \text{ eigenspace of } \sigma_x\,, \qquad \mathfrak{m}_x\colon\ -1 \text{ eigenspace of } \sigma_x\,.$$

As is standard, $\mathfrak{g}_x = \mathfrak{k}_x + \mathfrak{m}_x$ and

$$[\mathfrak{k}_x, \mathfrak{k}_x] \subset \mathfrak{k}_x\,, \quad [\mathfrak{k}_x, \mathfrak{m}_x] \subset \mathfrak{m}_x\,, \quad [\mathfrak{m}_x, \mathfrak{m}_x] \subset \mathfrak{k}_x\,.$$

Thus $\mathfrak{m}_x$ is a *Lie triple system* (LTS) under the composition $\mathfrak{m}_x \times \mathfrak{m}_x \times \mathfrak{m}_x \to \mathfrak{m}_x$ given by

$$(5.1) \qquad (u, v, w) \mapsto [uvw] = [[u, v], w] \qquad \text{(definition)}.$$

We say that $\mathfrak{m}_x$ with the LTS structure (5.1) is the *Lie triple system of* $(M, ds^2)$ *at* $x$.

We may identify $\mathfrak{m}_x$ with the tangent space $M_x$ under $v \mapsto v_x$. Then [11, Theorem 8.4.1] the curvature tensor of $M$ is given at $x$ by

$$(5.2) \qquad 'R(u_x, v_x)w_x = -[uvw]_x \qquad \text{for } u, v, w \in \mathfrak{m}_x\,.$$

Also, this identification carries $ds^2_x$ over to the real bilinear form

$$(5.3) \qquad b_x(u, v) = ds^2(u_x, v_x) \qquad \text{for } u, v \in \mathfrak{m}_x\,.$$

The main fact on $b_x$ is the following lemma.

**5.4. Lemma.** *Viewing* $\mathfrak{m}_x$ *as a* LTS, $b_x$ *is a nondegenerate invariant bilinear form on* $\mathfrak{m}_x$. *In other words* (10.11), *if* $t, u, v, w \in \mathfrak{m}_x$ *then*

$$b_x(v, [utw]) = b_x([tuv], w) = b_x(t, [wvu])\,,$$

*and* $b_x$ *is nondegenerate as a symmetric bilinear form.*

*Proof.* The nondegeneracy of $b_x$ on $\mathfrak{m}_x$ follows from nondegeneracy of $ds^2_x$ on $M_x$.

Let $t, u, v, w \in \mathfrak{m}_x$ and denote $r = [t, u] \in \mathfrak{k}_x$. Then $r$ preserves $ds^2_x$, i.e.,

$$ds^2([r, v]_x, w_x) + ds^2(v_x, [r, w]_x) = 0\,,$$

which implies

$$b_x([tuv], w) = -b_x(v, [tuw]) = b_x(v, [utw])\,.$$

Let $\{y_1, \cdots, y_m\}$ be a basis of $\mathfrak{m}_x$, and extend it to a moving frame on a neighborhood of $x$. In that frame the curvature tensor $'R$ has components whose values at $x$ are given by

$$'R_{ijkl} = ds^2('R(y_k, y_l)y_j, y_i) = -b_x([y_k y_l y_j], y_i) .$$

If $t = y_k, u = y_l, v = y_j$ and $w = y_i$, then

$$b_x([tuv], w) = -'R_{ijkl} = -'R_{klij} = b_x([wvu], t) = b_x(t, [wvu]) .$$

It follows that $b_x([tuv], w) = b_x(t, [wvu])$ in general. q.e.d.

Lemma 5.4 allows us to apply the theory of LTS with nondegenerate invariant bilinear forms to $(\mathfrak{m}_x, b_x)$. That theory is worked out in § 10 and forms an appendix to this note. From now on we use the facts on LTS, as collected in § 10, without much comment.

Let $(\mathfrak{m}_i, b_i)$ be LTS with invariant bilinear forms. By *isomorphism* of $(\mathfrak{m}_1, b_1)$ to $(\mathfrak{m}_2, b_2)$ we mean a LTS isomorphism $f: \mathfrak{m}_1 \to \mathfrak{m}_2$ such that $b_2(u, v) = b_1(f^{-1}u, f^{-1}v)$.

**5.5. Lemma.** *If $(M, ds^2)$ is a connected locally symmetric pseudo-riemannian manifold and $x, z \in M$, then $(\mathfrak{m}_x, b_x)$ is isomorphic to $(\mathfrak{m}_z, b_z)$.*

(For there is an isometry of a neighborhood of $x$ onto a neighborhood of $z$, which sends $x$ to $z$.)

We collect some standard facts in LTS formulation.

**5.6. Theorem.** *If $(\mathfrak{m}, b)$ is a real LTS with nondegenerate invariant bilinear form, then there is a unique (up to global isometry) simply connected globally symmetric pseudo-riemannian manifold $(M, ds^2)$ such that $(\mathfrak{m}, b) \cong (\mathfrak{m}_x, b_x)$ for $x \in M$. If $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $b = b_1 \oplus b_2$, then $(M, ds^2) = (M_1, ds_1^2) \times (M_2, ds_2^2)$ where the factors correspond to $(\mathfrak{m}_1, b_1)$ and $(\mathfrak{m}_2, b_2)$.*

*Let $(M, ds^2)$ and $(N, d\sigma^2)$ be locally symmetric. Let $x \in M$ and $z \in N$. Let $(\mathfrak{m}_x, b_x)$ and $(\mathfrak{n}_z, b_z)$ be the associated LTS and invariant bilinear forms. If $f: (\mathfrak{m}_x, b_x) \cong (\mathfrak{n}_z, b_z)$ is an isomorphism, then*

$$\exp_x(u_x) \mapsto \exp_z(f(u)_z) , \qquad u \in \mathfrak{m}_x ,$$

*gives an isometry, from a local coordinate neighborhood of $x$ to one of $z$, which carries $x$ to $z$. In particular, if $M$ is connected, then $(M, ds^2)$ is locally isometric to the simply connected globally symmetric space corresponding to $(\mathfrak{m}_x, b_x)$.*

*Proof.* For the first assertion let $\mathfrak{g}$ be the standard Lie enveloping algebra of $\mathfrak{m}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ in $\mathfrak{g}$, vector space direct sum; so $\mathfrak{g}$ has an automorphism $\sigma$ which is $+1$ on $\mathfrak{k}$ and $-1$ on $\mathfrak{m}$. Let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$, and $K$ the analytic subgroup for $\mathfrak{k}$. Now $\sigma$ extends to $G$; here $K$ is the identity component of its fixed point set, so $K$ is closed in $G$. Thus $M = G/K$ is a simply connected manifold whose tangent space at $x = 1 \cdot K$ is represented by $\mathfrak{m}$. As $b([k, u], v) + b(u, [k, v]) = 0$ for all $k \in \mathfrak{k}$ and $u, v \in \mathfrak{m}$, and as $b$ is nondegenerate, now $b$ defines a $G$-invariant pseudo-riemannian metric $ds^2$ on $M$ such that $(\mathfrak{m}, b) = (\mathfrak{m}_x, b_x)$. As $b(\sigma u, \sigma v) = b(u, v)$ for $u, v \in \mathfrak{m}$, $\sigma$ is an isometry of $(M, ds^2)$, so $(M, ds^2)$ is globally symmetric.

If $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $b = b_1 \oplus b_2$, then the corresponding $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ because we use the standard Lie enveloping algebra, and the product statement follows.

The second assertion is an application [11, Theorem 1.7.20] of the polar coordinate form of Cartan's structure equations, and gives the uniqueness in the first assertion. q.e.d.

We say that the locally symmetric space $(M, ds^2)$ is *of reductive type at* $x$ if

$$\begin{aligned} &ds_x^2 \text{ is nondegenerate on the subspace} \\ &Z_x = \{\xi \in M_x : {}'R(\xi, M_x) = 0\} \text{ of } M_x \ , \end{aligned} \tag{5.7a}$$

and

$$\begin{aligned} &\text{if } S \subset M_x \text{ subspace such that } {}'R(S, M_x)\cdot M_x \subset S \\ &\text{and } \quad {}'R(S, M_x)\cdot S = 0 \ , \quad \text{then} \quad S \subset Z_x \ . \end{aligned} \tag{5.7b}$$

We say that $(M, ds^2)$ is *of reductive type* if it is of reductive type at every $x \in M$.

We translate the definition (5.7) to the LTS of $(M, ds^2)$ at $x$. Using (5.2) we see that

$$Z_x = \{u_x \in M_x : u \in \mathfrak{m}_x \quad \text{and} \quad [u\mathfrak{m}_x\mathfrak{m}_x] = 0\} \ .$$

In other words (10.8),

$$Z_x = \{u_x : u \in \mathfrak{z}_x\} \text{ where } \mathfrak{z}_x \text{ is the center of } \mathfrak{m}_x.$$

Thus (5.7a) says that $b_x$ is nondegenerate on the center of $\mathfrak{m}_x$. Similarly, (5.7b) says that if $\mathfrak{i}$ is an ideal in $\mathfrak{m}_x$ such that $[\mathfrak{i}\mathfrak{m}_x\mathfrak{i}] = 0$ then $\mathfrak{i}$ is central in $\mathfrak{m}_x$. Thus (10.15) and Lemma 5.4 say:

$$\begin{aligned} &(M, ds^2) \text{ is of reductive type at } x \text{ if, and} \\ &\text{only if, } (\mathfrak{m}_x, b_x) \text{ is of reductive type.} \end{aligned} \tag{5.8}$$

**5.9. Theorem.** *Let $(M, ds^2)$ be a connected pseudo-riemannian manifold of reductive type. Then there exist simply connected globally symmetric pseudo-riemannian manifolds $(M_i, ds_i^2)$, $0 \leq i \leq t$, unique up to global isometry and permutation of $\{1, 2, \cdots, t\}$, with the following properties.*

(1) *$(M_0, ds_0^2)$ is flat (curvature $\equiv 0$).*

(2) *If $i > 0$, then $(M_i, ds_i^2)$ is irreducible (in the strongest sense: the infinitesimal holonomy group at each point is real-irreducible in the tangent space).*

(3) *If $x \in M$, then a neighborhood of $x$ is isometric to an open set in $(M_0, ds_0^2) \times (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2)$.*

(4) *If* $(M, ds^2)$ *is complete, then there is a pseudo-riemannian covering*

$$(M_0, ds_0^2) \times (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2) \to (M, ds^2);$$

*if* $(M, ds^2)$ *is complete and simply connected, then it is isometric to* $(M_0, ds_0^2) \times \cdots \times (M_t, ds_t^2)$.

*Conversely, if a locally symmetric space is locally isometric to the product of a flat and some irreducible symmetric spaces, then it is of reductive type.*

**Remark.** If $(M, ds^2)$ is riemannian, i.e., if $ds^2$ is positive definite, then it is automatic that $(M, ds^2)$ be of reductive type, and the result of Theorem 5.9 is standard.

*Proof.* Let $x \in M$. Then $(\mathfrak{m}_x, b_x)$ is of reductive type. Theorem 10.16 says $\mathfrak{m}_x = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t$, $b_x$-orthogonal direct sum, where $\mathfrak{m}_0$ is the center and the other $\mathfrak{m}_i$ are the simple ideals of the derived system $\mathfrak{m}_x^{(1)} = [\mathfrak{m}_x\mathfrak{m}_x\mathfrak{m}_x]$. Let $(M_i, ds_i^2)$ be the simply connected globally symmetric space associated to $(\mathfrak{m}_i, b_i)$, $b_i = b_x|_{\mathfrak{m}_i \times \mathfrak{m}_i}$, by Theorem 5.6, and $(\tilde{M}, d\sigma^2)$ be the one associated to $(\mathfrak{m}_x, b_x)$. Theorem 5.6 gives a global isometry

$$(\tilde{M}, d\sigma^2) = (M_0, ds_0^2) \times (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2) .$$

As $\mathfrak{m}_0$ is central in $\mathfrak{m}_x$ we have $[\mathfrak{m}_0\mathfrak{m}_0\mathfrak{m}_0] = 0$. Now (5.2) shows that the curvature tensor of $(M_0, ds_0^2)$ vanishes identically, proving (1).

Let $i > 0$. Then $\mathfrak{m}_i$ is simple, i.e., it has no proper subspace $\mathfrak{i}$ such that $[\mathfrak{i}\mathfrak{m}_i\mathfrak{m}_i] \subset \mathfrak{i}$. If $x_i \in M_i$, then (5.2) says that the tangent space $M_{i,x_i}$ has no proper subspace $S$ such that the curvature tensor satisfies $'R(S, M_{i,x_i}) \cdot M_{i,x_i} \subset S$. This implies irreducibility of the infinitesimal holonomy group, and hence (2) is proved.

Theorem 5.6 says that $(M, ds^2)$ is locally isometric to $(\tilde{M}, d\sigma^2)$, which is isometric to the product of the $(M_i, ds_i^2)$. Hence (3) is proved, and (4) follows.

The converse follows from Theorem 5.6 and the converse statement of Theorem 10.16.

## 6. Decomposition of absolute parallelism

Let $M$ be a smooth manifold with an absolute parallelism $\phi$. Suppose that $ds^2$ is a pseudo-riemannian metric on $M$ consistent with $\phi$.

Theorem 4.14 says that $(M, ds^2)$ is locally symmetric. If $x \in M$, then we have the Lie algebra $\mathfrak{g}_x$ of germs of Killing vector fields at $x$, the involutive automorphism $\sigma_x$ of $\mathfrak{g}_x$ induced by the local symmetry, the $\sigma_x$-eigenspace decomposition $\mathfrak{g}_x = \mathfrak{k}_x + \mathfrak{m}_x$, the LTS $\mathfrak{m}_x$ of $(M, ds^2)$ at $x$, and the nondegenerate invariant bilinear form $b_x$ on $\mathfrak{m}_x$. We also have

(6.1a) $\qquad\mathfrak{p}$: space of all $\phi$-parallel vector fields on $M$.

Theorem 4.14 tells us that

(6.1b) $\xi \mapsto$ (germ of $\xi$ at $x$) injects $\mathfrak{p}$ into $\mathfrak{g}_x$,

and that

(6.1c) $\mathfrak{p}$ is a LTS under $[\xi\eta\zeta] = [[\xi, \eta], \zeta]$.

**6.2. Lemma.** *Let $x \in M$, and consider the evaluations*

$$\pi\colon \mathfrak{p} \to M_x \text{ by } \pi(\xi) = \xi_x \ , \qquad \mu\colon \mathfrak{m}_x \to M_x \text{ by } \mu(u) = u_x \ .$$

*Define*

$$f_x\colon \mathfrak{p} \to \mathfrak{m}_x \text{ by } f_x(\xi) = 2\mu^{-1}\pi(\xi) \ .$$

*In other words,*

$$f_x(\xi) = (\text{germ of } \xi \text{ at } x) - \sigma_x(\text{germ of } \xi \text{ at } x) \ .$$

*Then $f_x$ is a* LTS *isomorphism of $\mathfrak{p}$ onto $\mathfrak{m}$.*

*Proof.* The curvature tensor $'R$ of $(M, ds^2)$ satisfies

$$'R(\mu(u), \mu(v)) \cdot \mu(w) = -\mu[uvw] \quad \text{for} \quad u, v, w \in \mathfrak{m}_x$$

by (5.2), and also satisfies

$$'R(\pi(\xi), \pi(\eta)) \cdot \pi(\zeta) = -\tfrac{1}{4}\pi[\xi\eta\zeta] \quad \text{for} \quad \xi, \eta, \zeta \in \mathfrak{p}$$

by (4.7). Thus

$$\begin{aligned} f_x[\xi\eta\zeta] &= 2\mu^{-1}\pi[\xi, \eta, \zeta] = -8\mu^{-1}\{'R(\pi\xi, \pi\eta) \cdot \pi\zeta\} \\ &= -2\mu^{-1}\{'R(\mu f_x(\xi), \mu f_x(\eta)) \cdot \mu f_x(\zeta)\} = [f_x(\xi) f_x(\eta) f_x(\zeta)] \ . \quad \text{q.e.d.} \end{aligned}$$

Lemma 6.2 is one of the basic ingredients in our decomposition of $\phi$ under a product decomposition of $(M, ds^2)$. Here is the other basic ingredient.

**6.3. Proposition.** *Let $(N, d\sigma^2)$ be a pseudo-riemannian submanifold of $(M, ds^2)$. Then the following conditions are equivalent.*

(1) *$\phi$ induces an absolute parallelism on $N$,* i.e., *the tangent spaces satisfy $\phi_{yx}N_x = N_y$ for all $x, y \in N$.*

(2) *There is a subsystem $\mathfrak{q} \subset \mathfrak{p}$ such that $N_x = \{\xi_x : \xi \in \mathfrak{q}\}$ for every point $x \in N$.*

(3) *$(N, d\sigma^2)$ is totally geodesic in $(M, ds^2)$. Further, if $x \in N$, then $\mathfrak{q}_x = \{\xi \in \mathfrak{p} : \xi_x \in N_x\}$ satisfies $[\xi, \eta]_x \in N_x$ for all $\xi, \eta \in \mathfrak{q}_x$.*

*Proof.* If $x \in N$, define $\mathfrak{q}_x = \{\xi \in \mathfrak{p} : \xi_x \in N_x\}$.

Assume (1). If $x, y \in N$, then $\{\xi_y : \xi \in \mathfrak{q}_y\} = N_y = \phi_{yx}N_x = \phi_{yx}\{\xi_x : \xi \in \mathfrak{q}_x\} = \{\xi_y : \xi \in \mathfrak{q}_x\}$, which says $\mathfrak{q}_y = \mathfrak{q}_x$. Thus we have a linear subspace $\mathfrak{q} \subset \mathfrak{p}$ such that each $N_x = \{\xi_x : \xi \in \mathfrak{q}\}$. As $N$ is a submanifold of $M$, now $x \in N$ and $\xi, \eta \in \mathfrak{q}$ imply $[\xi, \eta]_x \in N_x$; then $\zeta \in \mathfrak{q}$ further implies $[\xi\eta\zeta]_x \in N_x$ so $[\xi\eta\zeta] \in \mathfrak{q}$. Thus $\mathfrak{q}$ is a subsystem of $\mathfrak{p}$ and (2) is proved.

Assume (2). If $x \in N$, then the geodesics of $(M, ds^2)$ tangent to $N$ at $x$ are the integral curves through $x$ of the elements of $\mathfrak{q}$. Those are contained in $N$ locally. Thus $(N, d\sigma^2)$ is totally geodesic in $(M, ds^2)$. By assumption (2), also every $\mathfrak{q}_x = \mathfrak{q}$; as in the proof that (1) implies (2), this forces $[\xi, \eta]_x \in N_x$ for all $\xi, \eta \in \mathfrak{q} = \mathfrak{q}_x$. Thus (2) implies (3).

Assume (3). Let $x, y \in N$, and $\gamma$ be a smooth curve in $N$ from $x$ to $y$. As $(N, d\sigma^2)$ is totally geodesic in $(M, ds^2)$, $ds^2$-parallel displacement along $\gamma$ sends $N_x$ to $N_y$. The other assumption of (3) can be phrased: $T(N_z, N_z) \subset N_z$ for all $z \in N$. Lemma 3.4 (4) says $\Gamma = {}'\Gamma + \frac{1}{2}T$ where $\Gamma$ is the connection of $\phi$ and ${}'\Gamma$ is that of $ds^2$. Thus $\Gamma$-parallel displacement along $\gamma$ sends $N_x$ to $N_y$, i.e., $\phi_{yx}N_x = N_y$. Now (1) is derived from (3). q.e.d.

The following is stated separately for reference in § 8.

**6.4. Lemma.** *Let $(\tilde{M}, d\sigma^2)$ be a simply connected globally symmetric pseudo-riemannian manifold. If $\psi_U$ is an absolute parallelism on an open set $U \subset \tilde{M}$ consistent with $d\sigma^2|_U$, then there is a unique absolute parallelism $\psi$ on $\tilde{M}$ consistent with $d\sigma^2$ such that $\psi_U = \psi|_U$.*

*Proof.* Let $\{\eta_1, \cdots, \eta_n\}$ be a basis of the space of $\psi_U$-parallel fields on $U$. Each $\eta_i$ is a Killing vector field on $(U, d\sigma^2)$; by hypothesis on $(\tilde{M}, d\sigma^2)$ now $\eta_i$ has a unique extension $\xi_i$ to $\tilde{M}$, which is a Killing vector field of $(\tilde{M}, d\sigma^2)$. The $d\sigma^2(\xi_i, \xi_j) = d\sigma^2(\eta_i, \eta_j)$ are constant on $\tilde{M}$ by real analyticity. Now Corollary 4.15 provides an absolute parallelism $\psi$ on $\tilde{M}$ consistent with $d\sigma^2$ such that $\psi|_U = \psi_U$, and uniqueness also follows from Corollary 4.15. q.e.d.

Now we can describe the general situation.

**6.5. Theorem.** *Let $M$ be a connected manifold, $\phi$ an absolute parallelism on $M$, $ds^2$ a pseudo-riemannian metric $M$ consistent with $\phi$, and $(\tilde{M}, d\sigma^2)$ the simply connected globally symmetric space locally isometric to $(M, ds^2)$. Then there is an absolute parallelism $\tilde{\phi}$ on $\tilde{M}$ consistent with $d\sigma^2$, which has the following properties.*

(1) *Let $\mathfrak{p}$ be the* LTS *of $\phi$-parallel fields on $M$, and $\tilde{\mathfrak{p}}$ the one for $\tilde{M}$. Then every $x \in M$ has a neighborhood $U$ and an isometry*

$$h: (U, ds^2) \to (\tilde{U}, d\sigma^2),\ \tilde{U} \text{ open in } \tilde{M}\ ,$$

*such that $h$ sends $\phi|_U$ to $\tilde{\phi}|_{\tilde{U}}$,* i.e., *such that $h_*: \mathfrak{p} \to \tilde{\mathfrak{p}}$ well defined* LTS *isomorphism.*

(2) *If $(M, ds^2)$ is complete, then there is a pseudo-riemannian covering $\pi: (\tilde{M}, d\sigma^2) \to (M, ds^2)$ which sends $\tilde{\phi}$ to $\phi$,* i.e., *such that $\pi_*: \tilde{\mathfrak{p}} \cong \mathfrak{p}$. In that case the local isometries $h$ can be realized as local sections of the covering.*

(3) *Let $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ be a $ds^2$-orthogonal direct sum of ideals such that*

$$x \in M \text{ and } \xi, \eta \in \mathfrak{p}_i \text{ imply } [\xi, \eta]_x \in \{\zeta_x : \zeta \in \mathfrak{p}_i\}\ .$$

*Then $(\tilde{M}, d\sigma^2) = (M_1, ds_1^2) \times (M_2, ds_2^2)$ pseudo-riemannian product, and $\tilde{\phi} = \phi_1 \times \phi_2$ where $\phi_i$ is an absolute parallelism on $M_i$ consistent with $ds_i^2$,* *such that each of the local isometries h maps $\mathfrak{p}_i$ isomorphically onto the* LTS *of $\phi_i$-parallel vector fields on $M_i$. Conversely, any isometric product decomposition of a neighborhood of a point of M, which splits $\phi$, induces a $ds^2$-orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that $[\mathfrak{p}_i, \mathfrak{p}_i]_x \subset (\mathfrak{p}_i)_x$ for every $x \in M$.*

**Remark.** We claim neither local homogeneity of $(M, \phi, ds^2)$ nor global homogeneity of $(\tilde{M}, \tilde{\phi}, d\sigma^2)$. In fact that seems to be a delicate matter.

*Proof.* Fix $z \in M$, a normal coordinate neighborhood $V$ of $z$, and an isometry $f: (V, ds^2) \to (W, d\sigma^2)$ where $W$ is open in $\tilde{M}$. Then $f$ carries $\phi|_V$ to an absolute parallelism $\psi_W$ on $W$ consistent with $d\sigma^2|_W$. Lemma 6.4 says that $\psi_W$ extends uniquely to an absolute parallelism $\psi$ on $\tilde{M}$ consistent with $d\sigma^2$.

Let $x \in M$ with $x \neq z$. Choose a smooth curve $\gamma$ from $z$ to $x$ such that $\gamma$ has no self intersection and $\gamma \cap V$ is a geodesic arc. Choose a simply connected tubular neighborhood $T$ of $\gamma$ such that $T \cap V$ is a normal coordinate neighborhood of $z$. Then $f|_{T\cap V}$ extends uniquely to a differentiable map $g: T \to \tilde{M}$ which is locally an isometry. Let $U \subset T$ be a neighborhood of $x$ on which $g$ is an isometry, and $h = g|_U$. We will check that $(h, U)$ has the parallelism property of assertion (1).

Let $\xi \in \mathfrak{p}$. By construction of $\psi$ we have $\zeta \in \tilde{\mathfrak{p}}$ such that $f_*\xi_y = \zeta_{f(y)}$ for every $y \in V$. In particular, $g_*\xi_y = \zeta_{g(y)}$ for every $y \in T \cap V$. Let $\eta$ be the vector field on $T$, which is $g$-related to $\zeta$. Now $\eta$ and $\xi$ are Killing vector fields on the connected manifold $(T, ds^2)$, which coincide on the open subset $T \cap V$. Thus $\eta = \xi|_T$, i.e., $\xi|_T$ is $g$-related to $\zeta$. Hence $\xi|_U$ is $h$-related to $\zeta = f_*\xi$, and assertion (1) is proved.

Let $(M, ds^2)$ be complete, and define $\pi: W \to V$ to be the inverse of $f: V \to W$. Then $\pi$ continues to a pseudo-riemannian covering $(\tilde{M}, d\sigma^2) \to (M, ds^2)$, and the argument of (1) shows that $\pi_*: \tilde{\mathfrak{p}} \cong \mathfrak{p}$ is a well-defined LTS isomorphism, proving (2).

The LTS isomorphism $f_z: \mathfrak{p} \to \mathfrak{m}_z$ of Lemma 6.2 doubles lengths of tangent vectors, so it sends $ds^2$-orthogonal pairs to $b_z$-orthogonal pairs. If $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ is an orthogonal direct sum of ideals, then $\mathfrak{m}_z = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is a $b_z$-orthogonal direct sum of ideals where $\mathfrak{m}_i = f_z(\mathfrak{p}_i)$. The assertions of (3) now follow from Theorem 5.6 and Proposition 6.3. q.e.d.

We say that the absolute parallelism $\phi$ is of *reductive type* relative to $ds^2$, if the pair $(\mathfrak{p}, ds^2)$, viewed as an LTS with nondegenerate invariant bilinear form, is of reductive type. Thus Lemma 6.2 gives

**6.6. Lemma.** *$\phi$ is of reductive type relative to $ds^2$ if, and only if, $(M, ds^2)$ is of reductive type.*

Combining Theorems 5.9 and 6.5 we have the following description for parallelisms of reductive type.

**6.7. Theorem.** *Let $(M, \phi, ds^2)$ be a connected manifold with absolute parallelism and consistent pseudo-riemannian metric such that $\phi$ is of reductive type relative to $ds^2$. Then there exist simply connected globally symmetric pseudo-riemannian manifolds $(M_i, ds_i^2)$, $0 \leq i \leq t$, unique up to global isometry and* *permutation of* $\{1, 2, \cdots, t\}$, *and there exist absolute parallelism* $\phi_i$ *on* $M_i$ *consistent with* $ds_i^2$, *which have the following properties.*

1. $(M_0, ds_0^2)$ *is flat. If* $i > 0$ *then* $(M_i, ds_i^2)$ *is irreducible* (*strongest sense: infinitesimal holonomy*).

2. *Every* $x \in M$ *has a neighborhood* $U$ *and an isometry* $h: (U, ds^2) \to (\tilde{U}, d\sigma^2)$, $\tilde{U}$ *open in* $M_0 \times M_1 \times \cdots \times M_t$ *and* $d\sigma^2 = ds_0^2 \times ds_1^2 \times \cdots \times ds_t^2$, *such that* $h$ *sends* $\phi|_U$ *to* $(\phi_0 \times \phi_1 \times \cdots \times \phi_t)|_{\tilde{U}}$.

3. *If* $\phi$ *is complete*, i.e., *if* $(M, ds^2)$ *is complete, then there is a pseudo-riemannian covering*

$$\pi: (M_0, ds_0^2) \times (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2) \to (M, ds^2)$$

*which sends* $\phi_0 \times \phi_1 \times \cdots \times \phi_t$ *to* $\phi$.

The next two sections are a detailed examination of the possibilities for the $(M_i, \phi_i, ds_i^2)$ in Theorem 6.7.

## 7. The flat case

Let $(M, ds^2)$ be a flat connected pseudo-riemannian manifold. We will describe, locally in general and globally in the complete case, the absolute parallelisms $\phi$ on $M$ consistent with $ds^2$. Example (3.7) sets the pattern and shows that $\phi$ need not be the euclidean parallelism.

**7.1. Lemma.** *Let* $\phi$ *be an absolute parallelism on* $M$ *consistent with* $ds^2$, $\mathfrak{p}$ *the* LTS *of* $\phi$*-parallel vector fields on* $M$, *and* $T$ *the torsion tensor of* $\phi$. *Then* $T$ *is* $\phi$*-parallel*, $\mathfrak{p}$ *is a Lie algebra under Poisson bracket, and* $[\mathfrak{p}\mathfrak{p}\mathfrak{p}] = 0$.

*Proof.* According to Theorem 6.5 we may assume $(M, ds^2)$ to be complete and simply connected, and then let $\mathfrak{g}$ denote the Lie algebra of Killing vector fields of $(M, ds^2)$ and $\mathfrak{l}$ the subalgebra generated by $\mathfrak{p}$. Thus

$$\mathfrak{l} = \mathfrak{q} + \mathfrak{p} \quad \text{where} \quad \mathfrak{q} = [\mathfrak{p}, \mathfrak{p}] .$$

As $(M, ds^2)$ is flat, (4.7) says $[\mathfrak{p}\mathfrak{p}\mathfrak{p}] = 0$ so $[\mathfrak{q}, \mathfrak{p}] = 0$. Jacobi identity implies $[\mathfrak{q}, \mathfrak{q}] = 0$. Now $\mathfrak{q}$ is central in $\mathfrak{l}$. Thus $\mathfrak{q} = \mathfrak{l}'$ the derived algebra, and $ad(\mathfrak{l})$ is a commutative Lie algebra of linear transformations of $\mathfrak{l}$.

Choose a basis $\{\beta_1, \cdots, \beta_r\}$ of $ad(\mathfrak{l})$. As $\beta_1$ and $\beta_2$ are commuting linear transformations of $\mathfrak{l}$, there is a linear combination $\alpha_2$ of them such that $\alpha_2(\mathfrak{l}) = \beta_1(\mathfrak{l}) + \beta_2(\mathfrak{l})$. As $\alpha_2$ and $\beta_3$ commute, they have a linear combination $\alpha_3$ such that $\alpha_3(\mathfrak{l}) = \alpha_2(\mathfrak{l}) + \beta_3(\mathfrak{l}) = \beta_1(\mathfrak{l}) + \beta_2(\mathfrak{l}) + \beta_3(\mathfrak{l})$. Continuing, we get $\alpha_r \in ad(\mathfrak{l})$ such that $\alpha_r(\mathfrak{l}) = \beta_2(\mathfrak{l}) + \cdots + \beta_r(\mathfrak{l}) = [\mathfrak{l}, \mathfrak{l}] = \mathfrak{q}$. Choose $\xi \in \mathfrak{p}$ with $ad(\xi) = \alpha_r$; that is possible because $\mathfrak{l} = \mathfrak{p} + \mathfrak{q}$ and $ad(\mathfrak{q}) = 0$. Now $[\xi, \mathfrak{p}] = \mathfrak{q}$.

As $(M, ds^2)$ is flat, Theorem 4.3 says that $ds^2([\eta, \zeta], [\eta, \zeta]) = 0$ whenever $\eta, \zeta \in \mathfrak{p}$ span a plane on which $ds^2$ is nondegenerate. Such pairs $(\eta, \zeta)$ are dense in $\mathfrak{p} \times \mathfrak{p}$. By continuity now $ds^2([\eta, \zeta], [\eta, \zeta]) = 0$ for all $\eta, \zeta \in \mathfrak{p}$. If $\eta, \zeta \in \mathfrak{p}$, then

$$\begin{aligned} 0 &= ds^2([\xi, \eta + \zeta], [\xi, \eta + \zeta]) \\ &= ds^2([\xi, \eta], [\xi, \eta]) + 2ds^2([\xi, \eta], [\xi, \zeta]) + ds^2([\xi, \zeta], [\xi, \zeta]) \\ &= 2ds^2([\xi, \eta], [\xi, \zeta]) . \end{aligned}$$

As $[\xi, \mathfrak{p}] = \mathfrak{q}$, now $ds^2(\alpha, \beta) = 0$ for all $\alpha, \beta \in \mathfrak{q}$.

Choose a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathfrak{p}$, so the torsion tensor $T(\xi_i, \xi_j) = T^m_{ij}\xi_m = -[\xi_i, \xi_j]$. We have just seen $ds^2([\xi_i, \xi_j], [\xi_k, \xi_r]) = 0$. Thus

$$0 = ds^2(-T^m_{ij}\xi_m, -T^s_{kr}\xi_s) = T^m_{ij}T_{krm} = T^m_{ij}T_{mkr} .$$

Raising $r$ we get $T^m_{ij}T^r_{mk} = 0$.

As $(M, ds^2)$ is flat, its curvature tensor has $'R(\xi_i, \xi_j)\cdot\xi_k = 0$. Now (4.7) and the just-proved fact $T^m_{ij}T^r_{mk} = 0$ give

$$0 = [[\xi_i, \xi_j], \xi_k] = [-T^m_{ij}\xi_m, \xi_k] = T^m_{ij}T^r_{mk}\xi_r + \xi_k(T^m_{ij})\xi_m = \xi_k(T^m_{ij})\xi_m .$$

Thus the $T^m_{ij}$ are constants, i.e., $T$ is $\phi$-parallel, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. q.e.d.

We now work out the algebraic classification of the pairs $(\mathfrak{p}, ds^2)$ of Lemma 7.1.

Let $\mathfrak{w}$ be an $r$-dimensional vector space over a field $F$, let $\mathfrak{v} = \mathfrak{w}^*$ dual space, and let $\tau \in \Lambda^3(\mathfrak{v})$ alternating trilinear form on $\mathfrak{w}$. We use this data to define a $2r$-dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}(\tau, \mathfrak{w})$:

(7.2a) $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$ as vector space over $F$,

(7.2b) $\mathfrak{v}$ is central in $\mathfrak{g}$, i.e., $[\mathfrak{g}, \mathfrak{v}] = 0$, and

(7.2c) if $w_1, w_2 \in \mathfrak{w}$ then $[w_1, w_2] \in \mathfrak{v}$ defined by: $\langle [w_1, w_2], w\rangle = \tau(w_1, w_2, w)$ for $w \in \mathfrak{w}$.

Antisymmetry of $[\ ,\ ]$ is obvious, and the Jacobi identity follows from the observation that

(7.3a) $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$, i.e., $\mathfrak{g}$ is abelian as a LTS,

(7.3b) $\mathfrak{g}$ is abelian as Lie algebra if and only if $\tau = 0$.

Observe also that $\mathfrak{g} = \mathfrak{g}(\tau, \mathfrak{w})$ has a natural nondegenerate symmetric bilinear form given by

(7.4a) $b(\mathfrak{v}, \mathfrak{v}) = b(\mathfrak{w}, \mathfrak{w}) = 0$ and $b\colon \mathfrak{v} \times \mathfrak{w} \to F$ by $b(v, w) = \langle v, w\rangle$ .

If $w_1, w_2, w_3 \in \mathfrak{w}$, then $b([w_1, w_2], w_3) = \tau(w_1, w_2, w_3) = \tau(w_2, w_3, w_1) = b([w_2, w_3], w_1) = b(w_1, [w_2, w_3])$. Thus

(7.4b) $b$ is a nondegenerate invariant bilinear form on $\mathfrak{g}$.

In the real case, $b$ has signature $(r, r)$. The first nontrivial example of the real case is (3.7).

**7.5. Proposition.** *The pairs* $(\mathfrak{g}, b)$, *such that* $\mathfrak{g}$ *is a Lie algebra with* $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$ *and* $b$ *is a nondegenerate invariant bilinear form on* $\mathfrak{g}$, *are just the* $(\mathfrak{g}_1 \oplus \mathfrak{z}_2, b_1 \oplus b_2)$ *where*

(i) $(\mathfrak{g}_1, b_1)$ *is constructed as in* (7.2) *and* (7.4a),

(ii) $\mathfrak{z}_2$ *is an abelian Lie algebra, and*

(iii) $b_2$ *is a nondegenerate symmetric bilinear form on* $\mathfrak{z}_2$.

*Proof.* Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the derived algebra. Hypothesis $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$ implies $\mathfrak{g}' \subset \mathfrak{z}$. Let $\perp$ denote $b$-orthogonality. As $b(\mathfrak{z}, \mathfrak{g}') = 0$ by invariance, now $\mathfrak{g}' \subset \mathfrak{z}^\perp$. Define

$$\mathfrak{z}_1 = \mathfrak{z} \cap \mathfrak{z}^\perp \text{ and } \mathfrak{z}_2 \text{ is a complement to } \mathfrak{z}_1 \text{ in } \mathfrak{z}.$$

The $\mathfrak{z}_i$ are ideals in $\mathfrak{g}$, $\mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_2$, $\mathfrak{g}' \subset \mathfrak{z}_1$, and $b$ is nondegenerate on $\mathfrak{z}_2$. As $\mathfrak{z}_2$ is an ideal, so is $\mathfrak{z}_2^\perp$. Define

$$\mathfrak{g}_1 = \mathfrak{z}_2^\perp \text{ so that } \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z}_2 \ .$$

Now $b = b_1 \oplus b_2$. Thus we need only check that $(\mathfrak{g}_1, b_1)$ is constructed as in (7.2) and (7.4a).

We have reduced the proof to the case $\mathfrak{g} = \mathfrak{g}_1$. Thus we may assume $\mathfrak{z} = \mathfrak{z} \cap \mathfrak{z}^\perp$, i.e., $\mathfrak{z} \subset \mathfrak{z}^\perp$. If $\mathfrak{z} \neq \mathfrak{z}^\perp$, then there is an element $w \in \mathfrak{z}^\perp$ not central in $\mathfrak{g}$, so we have $x \in \mathfrak{g}$ with $[w, x] \neq 0$, and nondegeneracy of $b$ provides $y \in \mathfrak{g}$ with $b([w, x], y) \neq 0$. Invariance of $b$ gives $b([w, x], y) = b(w, [x, y])$, and $[x, y] \in \mathfrak{g}' \subset \mathfrak{z} \perp w$. That is impossible. Thus $\mathfrak{z} = \mathfrak{z}^\perp$.

Now we are down to the case $\mathfrak{z} = \mathfrak{z}^\perp$. Let $r = \dim \mathfrak{z}$. Then $\dim \mathfrak{g} = \dim \mathfrak{z} + \dim \mathfrak{z}^\perp = 2r$. Let $\mathfrak{w}$ be a vector space complement to $\mathfrak{z}$ in $\mathfrak{g}$ such that $\mathfrak{w} = \mathfrak{w}^\perp$. As $b$ pairs $\mathfrak{z}$ with $\mathfrak{w}$, it identifies $\mathfrak{z}$ with the dual space $\mathfrak{w}^*$. Define a trilinear form $\tau$ on $\mathfrak{w}$ by

$$\tau(w_1, w_2, w_3) = b([w_1, w_2], w_3) \ .$$

Then $\tau(w_1, w_2, w_3)$ is visibly antisymmetric in $(w_1, w_2)$, and is antisymmetric in $(w_2, w_3)$ because $b([w_1, w_2], w_3) = b(w_1, [w_2, w_3])$; now antisymmetry in $(w_1, w_3)$ follows. Thus $\tau \in \Lambda^3(\mathfrak{w}^*)$ and $\mathfrak{g} = \mathfrak{g}(\tau, \mathfrak{w})$ with $b$ given by (7.4a). q.e.d.

We combine Lemma 7.1 and Proposition 7.5 with the flat case of Theorem 3.8, as follows.

**7.6. Theorem.** *The triples* $(M, \phi, ds^2)$, *such that* $M$ *is connected,* $(M, ds^2)$ *is flat, and* $\phi$ *is a complete absolute parallelism on* $M$ *consistent with* $ds^2$, *are precisely the triples* $(D\backslash G, \lambda, d\sigma^2)$ *given as follows.*

1. $(\mathfrak{g}_1, b_1)$ *is a real Lie algebra of dimension* $2r$ *with nondegenerate invariant bilinear form of signature* $(r, r)$, *constructed as in* (7.2) *and* (7.4a).

2. $(\mathfrak{z}_2, b_2)$ *is an abelian real Lie algebra of dimension* $p + q$ *with a (nondegenerate) symmetric bilinear form of signature* $(p, q)$.

3. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z}_2$ *and* $b = b_1 \oplus b_2$.

4. *$G$ is the simply connected group with Lie algebra* $\mathfrak{g}$, *and $D$ is a discrete subgroup of $G$.*

5. *$d\sigma^2$ has signature $(p + r, q + r)$ and is induced by the bi-invariant pseudo-riemannian metric on $G$ defined by $b$.*

6. *$\lambda$ is induced by the absolute parallelism of left translation on $G$.*

One can separate the euclidean and noneuclidean parts of the parallelism $\lambda$ by observing, in the proof of Proposition 7.5, that $\mathfrak{g}$ can be "normalized" by the condition: the center $\mathfrak{z}_1$ of $\mathfrak{g}_1$ has $\mathfrak{z}_1 = \mathfrak{z}_1^{\perp}$ relative to $b_1$.

If $r \leq 2$ in Theorem 7.6, then the form $\tau$ which is used in the definition (7.2) of $\mathfrak{g}_1$ must vanish, so $\mathfrak{g}_1$ is abelian. In particular, if $n = \dim G$, then $G$ is abelian in case $d\sigma^2$ has signature $(n, 0)$ or $(0, n)$ (riemannian), $(n - 1, 1)$ or $(1, n - 1)$ (lorentzian), or $(n - 2, 2)$ or $(2, n - 2)$.

Finally note that the $(M_0, \phi_0, ds_0^2)$ of Theorem 6.7 are just the $(G, \lambda, d\sigma^2)$ of Theorem 7.6, i.e., the case where $D = \{1\}$ there. In particular, the automorphism group of $(M_0, \phi_0, ds_0^2)$ is transitive on $M_0$.

UNIVERSITY OF CALIFORNIA, BERKELEY