FINE STRUCTURE OF HERMITIAN SYMMETRIC SPACES

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1. Introduction

Let X_0 be a Hermitian symmetric space of noncompact type and X its compact dual. We express them as coset spaces of Lie groups

$$X_0 = G_0/K \qquad \text{and} \qquad X = G/P$$

where G_0 and G are the largest connected (compact-open topology) groups of complex analytic automorphisms. Several interesting facts then emerge.

1. G is a complex Lie group, complexification of the real Lie group G_0.
2. X is a projective algebraic variety.
3. Using $G_0 \subset G$, one can arrange $K = G_0 \cap P$, so X_0 has a natural embedding as an open G_0-orbit on X; there, every complex analytic automorphism of X_0 extends to an automorphism of X.
4. There is a natural complex Euclidean space $\mathfrak{m}^+ \subset X$, whose complement is a subvariety of lower dimension in X, such that

$$X_0 \subset \mathfrak{m}^+ \subset X$$

and the inclusion $X_0 \subset \mathfrak{m}^+$ is a canonical realization of X_0 as a bounded symmetric domain.

As a first illustration, consider the case where X_0 is the unit disk in the complex line $\mathbf{C}$,

$$X_0 = \{z \in \mathbf{C} : |z| < 1\}.$$

Then X is the Riemann sphere, $\mathfrak{m}^+ = \mathbf{C} = X - \{\infty\}$ embedded by stereographic projection, and X_0 is the open lower hemisphere of X. The groups are

$$G = \{\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}, \qquad \text{complex dimension } 3;$$

$$P = \{\pm\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : ad = 1\}, \qquad \text{complex dimension } 2;$$

$$G_0 = \{\pm\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\}, \qquad \text{real dimension } 3;$$

$$K = \{\pm\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : |a|^2 = 1\}, \qquad \text{real dimension } 1;$$

they act on X and X_0 by linear fractional transformations

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to (az + b)/(cz + d).$$

Under the action of G_0, X decomposes into two open orbits

$X_0 = G_0(0)$ lower hemisphere and $G_0(\infty)$ upper hemisphere

and one closed orbit $G_0(i)$, equator.

As a second illustration, consider the case where $X_0 = \{z \in \mathbf{C}^n : \|z\| < 1\}$ open unit ball in $\mathbf{C}^n$. Then X is the complex projective space $P^n(\mathbf{C})$ with the usual Fubini–Study metric of constant holomorphic sectional curvature, $\mathfrak{m}^+ = \mathbf{C}^n$ complement in X of the polar hyperplane to 0, and X_0 consists of all elements $x \in X$ of distance less than $\frac{1}{2}$(diameter of X) from 0. These facts are more easily seen by viewing X as the space of complex lines through the origin in $\mathbf{C}^{n+1}$ and choosing a basis $\{e_1, \ldots, e_{n+1}\}$, and a Hermitian form $\langle u,v \rangle = -\sum_{k=1}^{n} u^k\bar{v}^k + u^{n+1}\bar{v}^{n+1}$ on $\mathbf{C}^{n+1}$ relative to the basis. Then $\mathbf{C}^n = \mathfrak{m}^+$ injects to X by $z \to [z^1, z^2, \ldots, z^n, 1]$ and X_0 consists of all lines $x \in X$ that are positive definite under $\langle\ ,\ \rangle$. The groups are

G : complex general linear group of $\mathbf{C}^{n+1}$, modulo scalars;
P : subgroup of G preserving the line $[e_{n+1}]$;
G_0 : complex Lorentz group (for $\langle\ ,\ \rangle$), modulo scalars;

$$K : \{\lambda\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : \lambda \neq 0,\ A \in \mathbf{U}(n),\ \det A = a^{-1}\}/\{\lambda \cdot I\}.$$

They act on X through their linear action on $\mathbf{C}^{n+1}$. Again, X decomposes into two open G_0-orbits

$X_0 = G_0[e_{n+1}]$ positive definite lines and $G_0[e_1]$ negative definite lines,

and one closed G_0-orbit

$$G_0[e_1 + e_{n+1}] \text{ isotropic lines.}$$

Here, for $n > 1$, a new feature emerges. The open G_0-orbit $\neq X_0$ is concave and, by Hartogs' Theorem, carries no nonconstant holomorphic function. In fact it has compact subvarieties $gK[e_1]$, $g \in G_0$; for $K[e_1]$ is the polar hyperplane to $[e_{n+1}]$. Further there is a K-equivariant holomorphic fibration

$$\beta : G_0[e_1] \to K[e_1] \quad \text{by} \quad \beta[a^1, \ldots, a^{n+1}] = [a^1, \ldots, a^n, 0],$$

whose fibers are unit disks. That turns out [*17*] to compensate for the lack of holomorphic functions on the orbit.

In general we will see that if X_0 is an irreducible Hermitian symmetric space of noncompact type, and if its symmetric space rank is r, then there are precisely $\frac{1}{2}(r+1)(r+2)$ distinct G_0-orbits on X, $r+1$ of which are open and just one of which is closed. Each orbit turns out to decompose, in a G_0-equivariant manner, into complex submanifolds of X that we call the "holomorphic arc components" of the orbit. The open orbits, being complex manifolds, have just one holomorphic arc component; they turn out to be indefinite metric versions of X_0. The holomorphic arc components of the general orbit turn out to be lower dimensional versions of the open orbits. In a given orbit $G_0(x)$, x properly chosen within the orbit, each holomorphic arc component is fibered holomorphically over its maximal compact subvariety, and those fibrations fit together to form a real analytic bundle $G_0(x) \to K(x)$. These matters are the subject of Part II.

In Part I we develop the now-standard basic material of Hermitian symmetric spaces (Sections 2–4). We then apply that material (Sections 5 and 6) to obtain the G_0-orbit structure of the topological boundary of X_0 in X and the configuration of the holomorphic arc components of the boundary orbits. Those components are called the "boundary components" of X_0 in X. The boundary component theory is somewhat simplified here by systematic use of the restricted root system.

In Part II, we develop the holomorphic arc component theory and orbit configuration in general. Originally [*16*] I needed analytic machinery that went considerably beyond the theory of Hermitian symmetric spaces just in order to reduce these matters to the boundary component theory ([*7*], [*13*]) that Korányi and I had worked out. Here, Part I sheds a little more light on the boundary component theory, allowing development in Part II of the holomorphic arc component theory within the context of Hermitian symmetric spaces. That is a considerable simplification of the corresponding material of [*16*, Chapter III].

In Part III, we apply the general theory of Parts I and II to work out the cases where X_0 belongs to one of the four series of classical domains. Of course this is known [*9*] for the closure of X_0 in X, by *ad hoc* methods. Our method is easier, works for the two spaces X_0 that are not classical domains, and accommodates all of the G_0-orbits.

In this paper, we do not study realizations of X_0 as a Siegel domain of type I, II, or III; that is best done in [*7*], [*13*] and [*16*]. We do not study the function theory of X_0; it is the subject of a good number of papers in this volume. We do not consider arithmetic questions; that seems best done in [*1*]. Somewhat in the spirit of classical algebraic geometry, we just take a close look at a pretty subject.

Part I. Boundary Component Theory

We assume acquaintance with semisimple Lie groups and Riemannian or Hermitian symmetric spaces. The material of [*15*, Chapter *8*] is more than sufficient. Parts of the more basic material are reviewed in Section 2 in order to establish notation and terminology. Section 2 ends with a short proof of the Borel Embedding Theorem.

Sections 2–4 consist of preparatory material, divided so that readers with various degrees of familiarity with symmetric spaces can begin at the appropriate place. In general, however, the reader should at least skim through these sections for notation and terminology.

Section 3 is the first application of Harish-Chandra's maximal set Ψ of strongly orthogonal noncompact positive roots for the Lie algebra $\mathfrak{g}_0$ of a noncompact type Hermitian symmetric space X_0. We construct the polydisk of dimension rank (X_0) that sweeps out X_0 under the isotropy group, and use it to derive the Harish-Chandra Embedding Theorem.

Section 4 is the second application of Ψ. The Cayley transform and partial Cayley transforms are constructed, and the restricted root system is described. We do not prove the characterization of the restricted root system. Properties of the restricted root system are used to prove convexity of X_0.

In Section 5 we work out the G_0-orbit structure of the topological boundary of X_0 and also the boundary components. There is no essential change from Wolf–Korányi [*13*, Section 4].

The G_0-normalizers of the boundary components are worked out in Section 6. Originally [*13*] this was a direct, but complicated, business. Then it was observed [*1*] that those normalizers are just the maximal parabolic subgroups of G_0. Here we use the restricted root system to prove directly that the G_0-normalizers of the boundary components are the maximal parabolic subgroups of G_0, and use that fact to describe the normalizers precisely. The result is an improved version of [*13*, Sections 5 and 6] which allows further improvements in the material of Part II.

2. Borel Embedding

A *Riemannian symmetric space* is a connected Riemannian manifold Y

such that, given $y \in Y$, there is a (globally defined) isometry s_y that preserves y and has differential $-I$ on the tangent space to Y at y. Then y is an isolated fixed point of s_y, and the isometry s_y is called the *symmetry* of Y at y. The symmetry s_y is unique.

Given a complex manifold with Hermitian metric, one obtains a Riemannian manifold consisting of the underlying real differentiable manifold and the real part of the Hermitian metric. If this Riemannian manifold is symmetric, and if its symmetries are Hermitian isometries, then one says that the original complex manifold with Hermitian metric is a *Hermitian symmetric space.*

Let Y be a Hermitian symmetric space, $\tilde{Y} \to Y$ the universal covering space. The complex structure, Hermitian metric, and Riemannian symmetries, all lift from Y to $\tilde{Y}$. Thus $\tilde{Y}$ is a Hermitian symmetric space. Let

$$\tilde{Y} = Y_0 \times Y_1 \times \ldots \times Y_t$$

be the de Rham decomposition of the underlying Riemannian manifold, where Y_0 is a Euclidean space and the other Y_k are irreducible (not Euclidean, not locally isometric to a product of lower dimensional manifolds). Hermitian symmetric spaces are automatically Kaehlerian. Thus $\tilde{Y} = Y_0 \times Y_1 \times \ldots \times Y_t$ product of Hermitian symmetric spaces. We say that Y and $\tilde{Y}$ are

Euclidean if $\tilde{Y} = Y_0$;
irreducible if $\tilde{Y} = Y_1$;
strictly non-Euclidean if $\tilde{Y} = Y_1 \times \ldots \times Y_t$;
compact type if strictly non-Euclidean with each Y_k compact; and
noncompact type if strictly non-Euclidean with each Y_k noncompact.

In general [*14*, Lemma 1],

$Y = Y_0' \times Y_1 \times \cdots \times Y_t$ where Y_0' is the quotient of the complex Euclidean space Y_0 by a discrete group of translations.

In particular,

if Y is strictly non-Euclidean, then it is simply connected,

so

if Y is of compact or noncompact type, it is simply connected.

There is a duality between Hermitian symmetric spaces of compact and of noncompact type, given as follows. Let X be a Hermitian symmetric space of compact type, G_c its largest connected group of (Hermitian) isometries, $x_0 \in X$ and $K = \{g \in G_c : g(x_0) = x_0\}$ the isotropy subgroup there. Then $K \subset G_c$ are compact Lie groups and

$$X \cong G_c/K \qquad \text{under} \quad g(x_0) \leftrightarrow gK.$$

Let s be the symmetry to X at x_0 and decompose the Lie algebra of G_c by $\mathfrak{g}_c = \mathfrak{k} + \mathfrak{m}_c$, (± 1)-eigenspaces of ad(s). That gives another algebra

$$\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{m}_0, \qquad \mathfrak{m}_0 = i\mathfrak{m}_c,$$

with the same complexification as $\mathfrak{g}_c$. Passing to the group level, we get a Hermitian symmetric space $X_0 = G_0/K$ of noncompact type. If that construction is applied to X_0, one obtains X. We say

X_0 is the (noncompact) dual of X,

X is the (compact) dual of X_0.

This duality will be of constant high importance. Section 2 ends with a special embedding of X_0 in X such that the action of G_0 extends to X and $X_0 = G_0(x_0)$ open orbit.

In the notation above, G_c and G_0 have center reduced to $\{1\}$ and

$$\text{rank } G_c = \text{rank } K = \text{rank } G_0.$$

Further,

$$X \text{ is irreducible} \Leftrightarrow G_c \text{ is simple} \Leftrightarrow X_0 \text{ is irreducible}.$$

In the irreducible case, the fact that X and X_0 are Hermitian (not just Riemannian) symmetric is equivalent to the fact that K has nondiscrete center; then the center of K is a circle group which defines the almost complex structures of X and X_0.

We establish our notation for Hermitian symmetric spaces.

X_0 : Hermitian symmetric space of noncompact type. (2.1)

G_0 : largest connected group of isometries of X_0. (2.2)

x_0 : fixed "base point" in X_0. (2.3)

K : isotropy subgroup of G_0 at x_0. (2.4)

σ : Cartan involution ad(s) of G_0 where s is the symmetry of X_0 at x_0. (2.5)

$\mathfrak{k} \subset \mathfrak{g}_0$: Lie algebras of $K \subset G_0$. (2.6)

Then G_0 is a connected centerless semisimple Lie group, the center of K is a torus Z_K whose dimension is the number of simple direct factors of G_0, and K is the centralizer of Z_K in G_0. Also we have the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{m}_0, \text{ sum of } +1 \text{ and } -1 \text{ eigenspaces of } \sigma. \tag{2.7}$$

$$\mathfrak{g} = \mathfrak{g}_0^C = \mathfrak{k}^C + \mathfrak{m} \qquad \text{complexification where } \mathfrak{m} = \mathfrak{m}_0^C. \tag{2.8}$$

$$\mathfrak{g}_c = \mathfrak{k} + \mathfrak{m}_c \qquad \text{compact real form of } \mathfrak{g} \text{ where } \mathfrak{m}_c = i\mathfrak{m}_0. \tag{2.9}$$

G : adjoint group of $\mathfrak{g}$, so G_0 is the real analytic subgroup for $\mathfrak{g}_0$. (2.10)

G_c : real analytic subgroup of G for $\mathfrak{g}_c$, compact real form of G. (2.11)

X : G_c/K, Hermitian symmetric space, compact dual of X_0. (2.12)

$\mathfrak{t}$: Cartan subalgebra of $\mathfrak{k}$. (2.13)

As K is the centralizer of the torus Z_K, it has maximal rank in G_0, thus also in G_c. So $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g}_0$ and in $\mathfrak{g}_c$, and $\mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}$.

Δ : $\mathfrak{t}^C$-root system of $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{t}^C + \sum_{\Delta} \mathfrak{g}^{\varphi}$. (2.14)

Δ_K : compact roots, $\mathfrak{t}^C$-root system of $\mathfrak{k}^C$. (2.15)

Δ_M : noncompact roots, i.e., roots φ with $\mathfrak{g}^{\varphi} \subset \mathfrak{m}$. (2.16)

z : central element of $\mathfrak{k}$ such that $J = \mathrm{ad}(z)|_{\mathfrak{m}}$ is the almost-complex structure of X and X_0. (2.17)

$\mathfrak{m}^{\pm}$: $(\pm i)$-eigenspace of $J = \mathrm{ad}(z)|_{\mathfrak{m}}$. (2.18)

$\mathfrak{p}$: $\mathfrak{k}^C + \mathfrak{m}^-$, parabolic subalgebra of $\mathfrak{g}$ that is the sum of the nonnegative eigenspaces of $\mathrm{ad}(iz) : \mathfrak{g} \to \mathfrak{g}$. (2.19)

P : parabolic subgroup of G that is the complex analytic group for $\mathfrak{p}$. (2.20)

Borel Embedding Theorem. G_c is transitive on the complex coset space G/P with isotropy group $G_c \cap P = K$; thus

$$X = G/P \text{ coset space of complex Lie groups.}$$

Let x_c denote the identity coset $1 \cdot P \in G/P$. Then $G_0 \cap P = K$, so

$gK \to g(x_c)$ embeds X_0 holomorphically as an open G_0-orbit $G_0(x_c) \subset X$.

Proof. Note $\mathfrak{g}_c \cap \mathfrak{p} = \mathfrak{k}$, so $\dim_R G_c(x_c) = \dim_R \mathfrak{g}_c - \dim_R \mathfrak{k} = \dim_R \mathfrak{m}_c = \dim_R \mathfrak{m}^+ = \dim_R(G/P)$; thus $G_c(x_c)$ is open in G/P. Similarly $G_0(x_c)$ is open in G/P.

As G_c is compact, $G_c(x_c)$ is compact, so the latter is closed (as well as open) in the connected space G/P. Thus $G_c(x_c) = G/P$. Now $gK \to g(x_c)$ defines a complex analytic covering space $X = G_c/K \to G/P$. That gives G/P the structure of Hermitian symmetric space of compact type, so G/P is simply connected. Now $X \to G/P$ is a complex analytic diffeomorphism. Similarly $X_0 \to G_0(x_c)$ is a complex analytic diffeomorphism. Q.E.D.

3. Harish-Chandra Realization

Choose an ordering of the set Δ of $\mathfrak{t}^{\mathbf{C}}$-roots of $\mathfrak{g}$ such that

$$\mathfrak{m}^+ = \sum_{\varphi \in \Delta_M^+} \mathfrak{g}^\varphi \qquad \text{and} \qquad \mathfrak{m}^- = \sum_{\varphi \in \Delta_M^-} \mathfrak{g}^\varphi. \tag{3.1}$$

For example let $iy \in i\mathfrak{t}$ be in the interior of a Weyl chamber whose closure contains iz, and define the positive root system $\Delta^+ = \{\varphi \in \Delta : \varphi(iy) > 0\}$.

Two roots $\varphi, \psi \in \Delta$ are *strongly orthogonal*, denoted $\varphi \perp\!\!\!\perp \psi$, if neither of $\varphi \pm \psi$ is a root. In that case $\varphi \perp \psi$, ordinary orthogonality. We construct a maximal strongly orthogonal set of noncompact positive roots:

$$\Psi = \{\psi_1, \ldots, \psi_r\} \quad \text{where} \quad \psi_{j+1} \text{ is the lowest element of } \Delta_M^+ \text{ strongly orthogonal to each of } \{\psi_1, \ldots, \psi_j\}. \tag{3.2}$$

If $\varphi \in \Delta$ we have $h_\varphi \in i\mathfrak{t}$ defined by: $2\varphi(h)/\langle\varphi,\varphi\rangle = \langle h_\varphi, h\rangle$ $h \in \mathfrak{t}$. Choose root vectors $e_\varphi \in \mathfrak{g}^\varphi$ normalized by $[e_\varphi, e_{-\varphi}] = h_\varphi$. That choice can be made so that $\mathfrak{m}_0$ has a real basis consisting of all the

$$x_{\varphi,0} = e_\varphi + e_{-\varphi} \qquad \text{and} \qquad y_{\varphi,0} = i(e_\varphi - e_{-\varphi}), \qquad \varphi \in \Delta_M^+, \tag{3.3}$$

related to the almost-complex structure by

$$\begin{aligned} Jx_{\varphi,0} &= [z, x_{\varphi,0}] = y_{\varphi,0}; \\ Jy_{\varphi,0} &= [z, y_{\varphi,0}] = -x_{\varphi,0}; \ [x_{\varphi,0}, y_{\varphi,0}] = -2ih_\varphi. \end{aligned} \tag{3.4}$$

Then $\mathfrak{m}_c$ has a real basis consisting of all

$$x_\varphi = i(e_\varphi + e_{-\varphi}) \qquad \text{and} \qquad y_\varphi = -(e_\varphi - e_{-\varphi}), \qquad \varphi \in \Delta_M^+, \tag{3.5}$$

such that

$$Jx_\varphi = [z, x_\varphi] = y_\varphi; \quad Jy_\varphi = [z, y_\varphi] = -x_\varphi; \quad [x_\varphi, y_\varphi] = 2ih_\varphi. \tag{3.6}$$

From strong orthogonality of Ψ we have abelian subspaces $\mathfrak{a}_0 \subset \mathfrak{m}_0$,

$i\mathfrak{a}_0 = \mathfrak{a}_c \subset \mathfrak{m}_c = i\mathfrak{m}_0$, defined by

$$\mathfrak{a}_0 = \sum_{\Psi} x_{\psi,0}\mathbf{R} \qquad \text{and} \qquad \mathfrak{a}_c = \sum_{\Psi} x_{\psi}\mathbf{R}. \tag{3.7}$$

That provides closed abelian subgroups

$$A_0 = \exp \mathfrak{a}_0 \subset G_0 \qquad \text{and} \qquad A_c = \exp \mathfrak{a}_c \subset G_c. \tag{3.8}$$

If $\varphi \in \Delta$, we have 3-dimensional simple subalgebras

$$\mathfrak{g}[\varphi] = h_\varphi \mathbf{C} + \mathfrak{g}^{\varphi} + \mathfrak{g}^{-\varphi} \subset \mathfrak{g} \tag{3.9}$$

and their real forms

$$\mathfrak{g}_0[\varphi] = \mathfrak{g}_0 \cap \mathfrak{g}[\varphi] \qquad \text{and} \qquad \mathfrak{g}_c[\varphi] = \mathfrak{g}_c \cap \mathfrak{g}[\varphi]. \tag{3.10}$$

That defines 3-dimensional simple subgroups

$$G[\varphi] \text{ for } \mathfrak{g}[\varphi], \quad G_0[\varphi] \text{ for } \mathfrak{g}_0[\varphi], \quad \text{and } G_c[\varphi] \text{ for } \mathfrak{g}_c[\varphi]. \tag{3.11}$$

Similarly, if $\Gamma \subset \Psi$, then we have direct sums

$$\mathfrak{g}[\Gamma] = \sum_{\Gamma} \mathfrak{g}[\gamma], \quad \mathfrak{g}_0[\Gamma] = \sum_{\Gamma} \mathfrak{g}_0[\gamma], \quad \mathfrak{g}_c[\Gamma] = \sum_{\Gamma} \mathfrak{g}_c[\gamma] \tag{3.12}$$

and local direct products as their analytic groups

$$G[\Gamma] = \prod_{\Gamma} G[\gamma], \quad G_0[\Gamma] = \prod_{\Gamma} G_0[\gamma], \quad G_c[\Gamma] = \prod_{\Gamma} G_c[\gamma]. \tag{3.13}$$

Polydisk Theorem. If $\Gamma \subset \Psi$, then $G[\Gamma](x_0) = G_c[\Gamma](x_0)$ is a holomorphically embedded, totally geodesic submanifold of X that is a product of $|\Gamma|$ Riemann spheres, and $G_0[\Gamma](x_0)$ is a holomorphically embedded totally geodesic submanifold of X_0 that is the product of the $|\Gamma|$ lower hemispheres of $G_c[\Gamma](x_0)$. K exhausts X with the polysphere $G_c[\Psi](x_0)$ by $K \cdot G_c[\Psi](x_0) = X$. K exhausts X_0 with the polydisk $G_0[\Psi](x_0)$ by $K \cdot G_0[\Psi](x_0) = X_0$.

Proof. $G_c[\Gamma](x_0)$ is totally geodesic in X because $\sigma(\mathfrak{g}_c[\Gamma]) = \mathfrak{g}_c[\Gamma]$, holomorphically embedded in X because $J(\mathfrak{g}_c[\Gamma] \cap \mathfrak{m}_c) = \mathfrak{g}_c[\Gamma] \cap \mathfrak{m}_c$, Riemann polysphere because each $G_c[\gamma](x_0)$ is compact homogeneous, and of complex dimension 1. The same statements on $G_0[\Gamma](x_0)$ now follow.

For the exhaustion, it suffices to prove that

$$\mathfrak{a}_0 \quad \text{is a maximal abelian subspace of} \quad \mathfrak{m}_0, \tag{3.14}$$

$$\mathfrak{a}_c \quad \text{is a maximal abelian subspace of} \quad \mathfrak{m}_c. \tag{3.15}$$

For then standard symmetric space theory says

$$G_0 = KA_0K \quad \text{and} \quad G_c = KA_cK \tag{3.16}$$

so that

$$X_0 = G_0(x_0) = KA_0K(x_0) = KA_0(x_0) \subset KG_0[\Psi](x_0),$$
$$X = G_c(x_0) = KA_cK(x_0) = KA_c(x_0) \subset KG_c[\Psi](x_0).$$

Note that (3.14) and (3.15) are equivalent. Now suppose (3.15) false. Then we have $\xi \in \mathfrak{m}_0$, nonzero and orthogonal to $\mathfrak{a}_0$ such that $[\xi, \mathfrak{a}_0] = 0$. Thus there is a set $S \subset \Delta_M^+ - \Psi$ such that

$$0 \neq \xi = \textstyle\sum_S (s^\varphi x_{\varphi,0} + t^\varphi y_{\varphi,0}), \qquad \text{each } (s^\varphi)^2 + (t^\varphi)^2 > 0,$$

and, for every $\psi \in \Psi$,

$$0 = [x_{\psi,0}, \xi] = \textstyle\sum_S (s^\varphi [x_{\psi,0}, x_{\varphi,0}] + t^\varphi [x_{\psi,0}, y_{\varphi,0}])$$

which implies $\varphi \perp\!\!\!\perp \psi$. Thus maximality of Ψ forces (3.15). Q.E.D.

Note that $\mathfrak{m}^\pm$ are commutative subalgebras of $\mathfrak{g}$ consisting of nilpotent elements. That gives us complex analytic unipotent abelian subgroups of G, by

$$M^+ = \exp \mathfrak{m}^+ \quad \text{and} \quad M^- = \exp \mathfrak{m}^-. \tag{3.17}$$

Harish Chandra Embedding Theorem. The map $M^+ \times K^C \times M^- \to G$, given by $(m^+, k, m^-) \to m^+km^-$, is a complex analytic diffeomorphism onto a dense open subset of G that contains G_0. In particular,

$$\xi : \mathfrak{m}^+ \to X = G/P \quad \text{by} \quad \xi(m) = \exp(m)P$$

is a complex analytic diffeomorphism of $\mathfrak{m}^+$ onto a dense open subset of X that contains X_0. Furthermore, $\xi^{-1}(X_0)$ is a bounded domain in $\mathfrak{m}^+$.

Proof. Let $f : M^+ \times K^C \times M^- \to G$ be the map. It is holomorphic because M^+, K^C, and M^- are complex subgroups of G. Now we check that it is one to one. Suppose that

$$m_1^+ k_1 m_1^- = m_2^+ k_2 m_2^-.$$

Then $(m_1^+)^{-1} m_2^+ \in M^+ \cap K^{\mathbf{C}} M^-$. As it is unipotent, its entire 1-parameter subgroup of M^+ lies in $K^{\mathbf{C}} M^-$. As $\mathfrak{m}^+ \cap (\mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^-) = 0$, we conclude $m_1^+ = m_2^+$. Similarly $m_1^- = m_2^-$. Now $k_1 = k_2$. We have proved f one to one. We see f nonsingular by viewing the Lie algebras as left-invariant vector fields, so that, at (m^+, k, m^-),

$$f_*(\mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{m}^-) = \mathrm{ad}(k)\mathfrak{m}^+ + \mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^- = \mathfrak{m}^+ + \mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^- = \mathfrak{g}.$$

That also shows the image of f open in G.

Suppose $G = G[\psi]$. Then $G = \mathbf{SL}(2,\mathbf{C})/\{\pm I\}$ and we realize

$$e_\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_\psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular,

$$x_{\psi,0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad x_\psi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \exp(tx_{\psi,0}) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} 1 & \tanh t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh t} & 0 \\ 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh t & 1 \end{pmatrix} \\ &= \exp(\{\tanh t\} e_\psi) \cdot \exp(\{-\log \cosh t\} h_\psi) \cdot \exp(\{\tanh t\} e_{-\psi}) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \exp(tx_\psi) &= \begin{pmatrix} \cosh(it) & \sinh(it) \\ \sinh(it) & \cosh(it) \end{pmatrix} = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} \\ &= \begin{pmatrix} 1 & i \tan t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cos t} & 0 \\ 0 & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i \tan t & 1 \end{pmatrix}. \end{aligned} \tag{3.19}$$

Now observe

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \text{ real} \right\}.$$

k_θ will denote $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$,

so

$$\mathrm{ad}(k_\theta)e_\psi = e^{2i\theta}e_\psi.$$

Note also that

$$\xi(\mathfrak{m}^+) = \exp(e_\psi \mathbf{C})\cdot P = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\cdot P \quad : \quad z \in \mathbf{C}\right\}.$$

As $G_c = KA_cK$, so

$$X = \{k_\theta \exp(tx_\psi)\cdot P : \theta,t \text{ real}\},$$

we now have

$$\xi(\mathfrak{m}^+) = \{k_\theta\cdot\exp(tx_\psi)\cdot P : \theta,t \text{ real and } \cos t \neq 0\}.$$

That exhibits $\xi(\mathfrak{m}^+)$ as a dense open subset of X. And

$$X_0 = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\cdot P : |z| = \tanh t \text{ for some real } t\right\}$$
$$= \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\cdot P : |z| < 1\right\} = \xi(\{ze_\psi : |z| < 1\})$$

exhibits $\xi^{-1}X_0$ as a bounded domain in $\mathfrak{m}^+$.

Our assertions are proved for the case $G = G[\psi]$. They follow for the case $G = G[\Psi]$.

We go to the general case with the Polydisk Theorem. That says

$$X = \{k\cdot(\exp\sum_\Psi t_\psi x_\psi)\cdot P : k \in K \text{ and } t_\psi \text{ real}\}, \tag{3.20}$$

$$\xi(\mathfrak{m}^+) = \{k\cdot(\exp\sum_\Psi t_\psi x_\psi)\cdot P : k \in K,\ t_\psi \text{ real, } \cos t_\psi \neq 0\}, \tag{3.21}$$

$$X_0 = \{k\cdot\xi(\sum_\Psi z_\psi e_\psi) : k \in K \text{ and } |z_\psi| < 1\}. \tag{3.22}$$

Q.E.D.

4. Restricted Root Systems

Recall (3.7), (3.14), (3.15) the maximal abelian subspaces

$$\mathfrak{a}_0 = \sum_\Psi x_{\psi,0}\mathbf{R} \subset \mathfrak{m}_0 \qquad \text{and} \qquad \mathfrak{a}_c = i\mathfrak{a}_0 = \sum_\Psi x_\psi \mathbf{R} \subset i\mathfrak{m}_0 = \mathfrak{m}_c.$$

Strong orthogonality of Ψ, with (3.4) and (3.6), gives

$$[\mathfrak{a}_0, J\mathfrak{a}_0] = \sum_{\Psi} ih_\psi \mathbf{R} = [\mathfrak{a}_c, J\mathfrak{a}_c]. \tag{4.1}$$

Now

$$\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^- \quad \text{orthogonal direct sum} \quad \text{where} \tag{4.2}$$

$$\mathfrak{t}^- = [\mathfrak{a}_0, J\mathfrak{a}_0] \quad \text{and} \quad \mathfrak{t}^+ = \{\tau \in \mathfrak{t} : [\tau, \mathfrak{a}_0] = 0\}. \tag{4.3}$$

If $\Gamma \subset \Psi$, we define the *partial Cayley transform*

$$c_\Gamma = \prod_\Gamma c_\gamma, \qquad c_\gamma = \exp\left(\frac{\pi}{4} y_\gamma\right) \in G_c. \tag{4.4}$$

That gives Cartan subalgebras $\mathfrak{h}_\Gamma \subset \mathfrak{g}_0$ by

$$\mathfrak{h}_\Gamma = \mathfrak{g}_0 \cap \operatorname{ad}(c_\Gamma)\mathfrak{t}^{\mathbf{C}} = \mathfrak{t}^+ + \mathfrak{t}_\Gamma^- + \mathfrak{a}_\Gamma, \tag{4.5}$$

where

$$\mathfrak{t}_\Gamma^- = \sum_{\Psi - \Gamma} ih_\psi \mathbf{R} \quad \text{and} \quad \mathfrak{a}_\Gamma = \sum_\Gamma x_{\gamma,0}\mathbf{R}. \tag{4.6}$$

As $\mathfrak{h}_\phi = \mathfrak{t}$ maximally compact Cartan subalgebra of $\mathfrak{g}_0$, and $\mathfrak{h}_\Psi = \mathfrak{t}^+ + \mathfrak{a}_0$ maximally **R**-split Cartan subalgebra of $\mathfrak{g}_0$, it follows that every conjugacy class of Cartan subalgebras of $\mathfrak{g}_0$ is represented by one of the $\mathfrak{h}_\Gamma$.

The full *Cayley transform* is given by

$$c_\Psi = \prod_\Psi c_\psi \qquad \text{so} \qquad \operatorname{ad}(c_\Psi)\mathfrak{t}^{\mathbf{C}} = (\mathfrak{t}^+ + \mathfrak{a}_0)^{\mathbf{C}}. \tag{4.7}$$

Thus its dual map $\operatorname{ad}(c_\Psi)^*$ sends the $(\mathfrak{t}^+ + \mathfrak{a}_0)^{\mathbf{C}}$-root system of $\mathfrak{g}$ to the $\mathfrak{t}^{\mathbf{C}}$-root system Δ of $\mathfrak{g}$. Define

$$\mathfrak{a}_0\text{-root of } \mathfrak{g}_0 : \text{restriction to } \mathfrak{a}_0 \text{ of a } (\mathfrak{t}^+ + \mathfrak{a}_0)^{\mathbf{C}}\text{-root of } \mathfrak{g}. \tag{4.8}$$

As $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{m}_0$, the $\mathfrak{a}_0$-roots take real values on $\mathfrak{a}_0$, and they are the *real roots* of $\mathfrak{g}_0$. Now

$$\operatorname{ad}(c_\Psi)^* : \mathfrak{a}_0 \text{ roots} \to \{\varphi|_{i\mathfrak{t}^-} : \varphi \in \Delta\}. \tag{4.9}$$

Combinatorial arguments of Harish-Chandra [*5*] and C.C. Moore [*8*], which we omit here, result in

Restricted Root Theorem. Suppose that G is simple, i.e., that the symmetric space X_0 is irreducible. Let ρ denote restriction of roots from $\mathfrak{t}^C$ to $\mathfrak{t}^-$. Identify each element of $\Psi = \{\psi_1, \ldots, \psi_r\}$ with its ρ-image.

1. There are just two cases, as follows.

Case 1. $\rho(\Delta) \cup \{0\} = \{\pm\frac{1}{2}\psi_s \pm \frac{1}{2}\psi_t : 1 \leqq s, \ t \leqq r\}$. In that case, nonzero ρ-images of some subsets of Δ are given by

compact simple roots:	$\{\frac{1}{2}(\psi_{t+1} - \psi_t) : 1 \leqq t \leqq r - 1\}$,
compact positive roots:	$\{\frac{1}{2}(\psi_s - \psi_t) : 1 \leqq t < s \leqq r\}$,
noncompact positive roots:	$\{\frac{1}{2}(\psi_s + \psi_t) : 1 \leqq t \leqq s \leqq r\}$.

Case 2. $\rho(\Delta) \cup \{0\} = \{\pm\frac{1}{2}\psi_s \pm \frac{1}{2}\psi_t, \pm\frac{1}{2}\psi_t : 1 \leqq s, t \leqq r\}$. Then nonzero ρ-images from some subsets of Δ are given by

compact simple roots:	$\{\frac{1}{2}(\psi_{t+1} - \psi_t), 1 \leqq t \leqq r - 1\} \cup \{-\frac{1}{2}\psi_r\}$
compact positive roots:	$\{\frac{1}{2}(\psi_s - \psi_t) : 1 \leqq t < s \leqq r\} \cup \{-\frac{1}{2}\psi_t : 1 \leqq t \leqq r\}$
noncompact positive roots:	$\{\frac{1}{2}(\psi_s + \psi_t) : 1 \leqq t \leqq s \leqq r\} \cup \{\frac{1}{2}\psi_t : 1 \leqq t \leqq r\}$.

2. The ψ_t all have the same length.
3. The subgroup of the Weyl group of G that preserves Ψ induces all signed permutations $\psi_t \to \pm\psi_s$ of Ψ.

Using (4.9), the Restricted Root Theorem gives a description of the $\mathfrak{a}_0$-root system of $\mathfrak{g}_0$ in terms of Ψ.

We use the Restricted Root Theorem to express (3.22) in an invariant manner. Denote

$$\left.\begin{array}{ll} u \to \bar{u} & \text{complex conjugation of } \mathfrak{g} \text{ over } \mathfrak{g}_c, \\ \langle\ ,\ \rangle & \text{Killing form of } \mathfrak{g}, \text{ and} \\ (u,v) = -\langle u, \bar{v}\rangle & \text{positive definite Hermitian form on } \mathfrak{g}. \end{array}\right\} \tag{4.10}$$

If $u \in \mathfrak{g}$, we have the operator norm

$$\|\mathrm{ad}(u)\| = \sup\{|\mathrm{ad}(u)\cdot v| : v \in \mathfrak{g}, |v|^2 = (v,v) = 1\}. \tag{4.11}$$

Also let $\pi_0 : \mathfrak{m}^+ \to \mathfrak{m}_0$ be the projection,

$$\pi_0(u) = \tfrac{1}{2}(u - \bar{u}) \in \mathfrak{m}_0, \qquad \text{all} \quad u \in \mathfrak{m}^+. \tag{4.12}$$

Hermann Convexity Theorem. Let $D = \xi^{-1}(X_0)$, bounded symmetric domain in $\mathfrak{m}^+$ that is the image of X_0 under the Harish-Chandra Embedding. Then

$$D = \{u \in \mathfrak{m}^+ : \|\mathrm{ad}(\pi_0 u)\| < 1\},$$

unit ball in the (generally nondifferentiable) Banach space norm $\|\mathrm{ad}\pi_0 u\|$ of $\mathfrak{m}^+$. In particular D is convex in $\mathfrak{m}^+$.

Proof. Let $\mathfrak{a}^+ = \{\sum_\Psi b_\psi e_\psi : b_\psi \in \mathbf{C}\}$, so $\mathfrak{m}^+ = \mathrm{ad}(K)\mathfrak{a}^+$ and $D = \mathrm{ad}(K)(D \cap \mathfrak{a}^+)$. All norms are invariant under conjugation by elements of K. Thus we need only prove $D \cap \mathfrak{a}^+ = \{u \in \mathfrak{a}^+ : \|\mathrm{ad}(\pi_0 u)\| < 1\}$. Using (3.22), which says $D \cap \mathfrak{a}^+ = \{\sum_\Psi b_\psi e_\psi : |b_\psi| < 1\}$, it suffices to show

$$\text{if} \quad \{t_\psi\} \subset \mathbf{R}, \quad \text{then} \quad \|\mathrm{ad}(\tfrac{1}{2}\textstyle\sum_\Psi t_\psi x_{\psi,0})\| = \sup\{|t_\psi|\}. \tag{4.13}$$

Conjugating by the Cayley transform c_Ψ that says:

$$\text{if} \quad \{t_\psi\} \subset \mathbf{R}, \quad \text{then} \quad \|\mathrm{ad}(\tfrac{1}{2}\textstyle\sum_\Psi t_\psi h_\psi)\| = \sup\{|t_\psi|\}. \tag{4.14}$$

In terms of roots, the latter says, for $\{t_\psi\} \subset \mathbf{R}$,

$$|\mathrm{ad}(\tfrac{1}{2}\textstyle\sum t_\psi h_\psi)e_\varphi| \leqq \sup\{|t_\psi e_\psi|\} \quad \text{for all } \varphi \in \Delta \text{ with equality for at least one root } \varphi \in \Delta. \tag{4.15}$$

The equality of (4.15) comes from $\varphi = \psi_0$ where $|t_{\psi_0}| = \sup\{|t_\psi|\}$. For the inequality it suffices to note that, for every $\varphi \in \Delta$, either Ψ has just one element ψ not $\perp\varphi$ and it has $|\langle\psi,\varphi\rangle| = 2$, or Ψ has just two elements not $\perp\varphi$ and they have $|\langle\psi,\varphi\rangle| = 1$. That comes from the Restricted Root Theorem. Q.E.D.

5. Boundary Components

Suppose that V is a complex analytic space and $S \subset V$ is a subset. By *holomorphic arc* in S, we mean a holomorphic map

$$f : \{z \in \mathbf{C} : |z| < 1\} \to V \quad \text{with image in} \quad S. \tag{5.1}$$

By *chain of holomorphic arcs* in S, we mean a finite sequence $\{f_1, \ldots, f_k\}$ of holomorphic arcs in S such that the image of f_j meets the image of f_{j+1} for $1 \leqq j \leqq k - 1$. That provides an equivalence relation on S:

$$v_1 \sim v_2 \text{ iff } \exists \text{ chain } \{f_1, \ldots, f_k\} \text{ of holomorphic arcs in } S, \quad v_1 \in \text{Image } f_1 \quad \text{and} \quad v_2 \in \text{Image } f_k. \tag{5.2}$$

The equivalence classes are the *holomorphic arc components* of S in V.

If S is an open subset of V with topological boundary bd S, then the holomorphic arc components of bd S are the *boundary components* of S.

In this section, we describe the boundary components of the domain $X_0 = G_0(x_0) \subset X$, via convexity of the Harish-Chandra embedding of X_0 as a bounded domain in $\mathfrak{m}^+$. The result is joint work of Wolf and Korányi [*13*].

We must define certain subalgebras $\mathfrak{g}_\Gamma \subset \mathfrak{g}$ corresponding to the subsets Γ of our maximal set Ψ of strongly orthogonal noncompact positive roots. They will be normalized by semisimplicity and

$$\Gamma \text{ is the maximal } \perp\!\!\!\perp \text{ subset of } \Delta_M^+ \text{ for } \mathfrak{g}_\Gamma. \tag{5.3}$$

Then, in particular,

$$\mathfrak{g}_\Psi = \mathfrak{g} \qquad \text{and} \qquad \mathfrak{g}_\phi = 0. \tag{5.4}$$

For that, note that the centralizer of $\mathfrak{t}_\Gamma = \sum_{\Psi-\Gamma} ih_\psi \mathbf{R}$ is $\mathfrak{t}^{\mathbf{C}} + \sum_{\varphi\perp\Psi-\Gamma}\mathfrak{g}^\varphi$; that the centralizer of $\mathfrak{g}[\Psi - \Gamma] = \sum_{\Psi-\Gamma}(h_\psi\mathbf{C} + \mathfrak{g}^\psi + \mathfrak{g}^{-\psi})$ is $(\mathfrak{t}^+ + \mathfrak{t}_{\Psi-\Gamma})^{\mathbf{C}} + \sum_{\varphi\perp\Psi-\Gamma}\mathfrak{g}^\varphi$; and that those two centralizers are reductive subalgebras of $\mathfrak{g}$ with the same derived algebra. Now

$$\mathfrak{g}_\Gamma : \text{derived algebra of } \mathfrak{t}^{\mathbf{C}} + \sum_{\varphi\perp\Psi-\Gamma} \mathfrak{g}^\varphi. \tag{5.5}$$

It has real forms

$$\mathfrak{g}_{\Gamma,0} = \mathfrak{g}_0 \cap \mathfrak{g}_\Gamma \qquad \text{and} \qquad \mathfrak{g}_{\Gamma,c} = \mathfrak{g}_c \cap \mathfrak{g}_\Gamma. \tag{5.6}$$

The analytic groups G_Γ, $G_{\Gamma,0}$, and $G_{\Gamma,c}$ for $\mathfrak{g}_\Gamma$, $\mathfrak{g}_{\Gamma,0}$, and $\mathfrak{g}_{\Gamma,c}$ define symmetric subspaces

$$X_\Gamma = G_\Gamma(x_0) = G_{\Gamma,c}(x_0) \subset X; \qquad X_{\Gamma,0} = G_{\Gamma,0}(x_0) \subset X_0. \tag{5.7}$$

Boundary Orbit Theorem. Retain the notation $\xi : \mathfrak{m}^+ \to X$ of the Harish-Chandra Embedding Theorem. Denote $\mathfrak{m}_\Gamma^+ = \mathfrak{m}^+ \cap \mathfrak{g}_\Gamma$ and $\xi_\Gamma = \xi|_{\mathfrak{m}_\Gamma^+}$.

1. $X_\Gamma \subset X$ and $X_{\Gamma,0} \subset X_0$ are totally geodesic Hermitian symmetric subspaces.
2. $X_{\Gamma,0} \subset X_\Gamma$ is the Borel embedding of $X_{\Gamma,0}$ in its compact dual.
3. $\xi_\Gamma^{-1} : X_{\Gamma,0} \to \mathfrak{m}_\Gamma^+$ is the Harish-Chandra embedding of $X_{\Gamma,0}$ in its holomorphic tangent space at x_0.
4. The G_0-orbits on the topological boundary bd X_0 of X_0 in X are the

sets

$$G_0(c_{\Psi-\Gamma}x_0) = \bigcup_{k\in K} kc_{\Psi-\Gamma}X_{\Gamma,0}, \qquad \Gamma \subsetneqq \Psi.$$

5. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_q$ where the Δ_t are the root systems of the simple ideals of $\mathfrak{g}$. Let Γ, $\Sigma \subsetneqq \Psi$. Then $G_0(c_{\Psi-\Sigma}x_0)$ is in the closure of $G_0(c_{\Psi-\Gamma}x_0)$ if and only if $|\Sigma \cap \Delta_t| \leqq |\Gamma \cap \Delta_t|$ for $1 \leqq t \leqq q$.
6. Boundary orbits $G_0(c_{\Psi-\Gamma}x_0) = G_0(c_{\Psi-\Sigma}x_0)$ if and only if $|\Sigma \cap \Delta_t| = |\Gamma \cap \Delta_t|$ for $1 \leqq t \leqq q$.
7. $G_0(c_\Psi x_0)$ is the unique closed boundary orbit. It is a K-orbit and is the Bergman-Šilov boundary of X_0 in X.

Proof. (1) follows from stability of $\mathfrak{g}_\Gamma$ under both σ and ad(z), whence (2) is immediate. For (3) note that $\mathfrak{a}_\Gamma$ is maximal abelian in $\mathfrak{g}_{\Gamma,0} \cap \mathfrak{m}$.

The polydisk $G_0[\Psi](x_0)$ is the product of the lower hemispheres of the factors of the polysphere $G[\Psi](x_0)$. Thus its boundary is the product where each factor is a lower hemisphere or an equator. Now $G_0[\Psi](x_0)$ has boundary $\bigcup_\Gamma G_0[\Psi](c_{\Psi-\Gamma}x_0)$ where $\Gamma \subsetneqq \Psi$. So (4) follows from the Polydisk Theorem and the observation that $G_0(c_{\Psi-\Gamma}x_0) = K\cdot G_0[\Psi](c_{\Psi-\Gamma}x_0) = K\cdot G_0[\Gamma](c_{\Psi-\Gamma}x_0) = K\cdot G_{\Gamma,0}(c_{\Psi-\Gamma}x_0) = K\cdot c_{\Psi-\Gamma}G_{\Gamma,0}(x_0) = Kc_{\Psi-\Gamma}X_{\Gamma,0}$.

Let $\Sigma,\Gamma \subset \Psi$. If $|\Sigma \cap \Delta_t| \leqq |\Gamma \cap \Delta_t|$ for $1 \leqq t \leqq q$, then part (3) of the Restricted Root Theorem provides $k \in K$ such that $\mathrm{ad}(k)^*\Sigma \subset \Gamma$. Thus we may assume $\Sigma \subset \Gamma$. Now $\Sigma \subset \Gamma$ if and only if $G_0[\Psi](c_{\Psi-\Sigma}x_0)$ is in the closure of $G_0[\Psi](c_{\Psi-\Gamma}x_0)$. Applying K, the latter is equivalent to $G_0(c_{\Psi-\Sigma}x_0)$ being in the closure of $G_0(c_{\Psi-\Gamma}x_0)$. That proves (5); now (6) follows, as does the fact that $G_0(c_\Psi x_0)$ is the only closed boundary orbit.

$G_0(c_\Psi x_0) = Kc_\Psi X_{\phi,0} = K(c_\Psi x_0)$ shows it to be a K-orbit.

Let **B** denote the Banach algebra of functions continuous on the closure $X_0 \cup \mathrm{bd}X_0$ of X_0 in X and holomorphic on X_0 (point multiplication, sup norm). G acts on **B** by $g(f): x \to f(g^{-1}x)$. Let $f \in \mathbf{B}$ not constant on any irreducible factor of X_0, and x_f a maximum point of $|f(x)|$. Transforming by an element of K, we may assume x_f in the polysphere $G[\Psi](x_0)$. It is a maximum point of the restriction of f to the closure of the polydisk $G_0[\Psi](x_0)$, hence in the boundary $G_0[\Psi](c_\Psi x_0)$ of the polydisk. Thus $x_f \in G_0(c_\Psi x_0)$. By transitivity of G_0 on $G_0(c_\Psi x_0)$ now, the latter is the set of all maximum points of elements of **B**. Q.E.D.

Boundary Component Theorem. The boundary components of X_0 in X are just the sets $kc_{\Psi-\Gamma}X_{\Gamma,0}$ with $k \in K$ and $\Gamma \subsetneqq \Psi$. They are Hermitian symmetric spaces of noncompact type and rank given by

$$\operatorname{rank} kc_{\Psi-\Gamma}X_{\Gamma,0} = |\Gamma|. \tag{5.8}$$

Proof. The $kc_{\Psi-\Gamma}X_{\Gamma,0}$ are complex manifolds, so the boundary components are unions of such sets. Now we need only show that each $c_{\Psi-\Gamma}X_{\Gamma,0}$ is a boundary component.

We use Hermann Convexity of $\xi^{-1}X_0$ in $\mathfrak{m}^+$ to check that each $c_{\Psi-\Gamma}X_{\Gamma,0}$ is a boundary component. Define

$$\mathfrak{o}_\Gamma = \xi^{-1}c_{\Psi-\Gamma}x_0, \qquad \mathfrak{m}_\Gamma^+ = \mathfrak{g}_\Gamma \cap \mathfrak{m}^+, \qquad \mathfrak{e}_\Gamma = \mathfrak{o}_\Gamma + \mathfrak{m}_\Gamma^+. \tag{5.9}$$

Then $\mathfrak{e}_\Gamma$ is a complex affine subspace of $\mathfrak{m}^+$ that contains the boundary point $\mathfrak{o}_\Gamma$ of $\xi^{-1}X_0$. Looking at the polysphere we see that

$$\mathfrak{o}_\Gamma = \xi^{-1}\exp\Big(\frac{\pi}{4}\sum_{\Psi-\Gamma}(e_{-\psi} - e_\psi)\Big)x_0 = -\sum_{\Psi-\Gamma}e_\psi. \tag{5.10}$$

Recall the projection $\pi_0 : \mathfrak{m}^+ \to \mathfrak{m}_0$ from (4.12). Now

$$\pi_0(\mathfrak{o}_\Gamma + \mathfrak{m}_\Gamma^+) = \Big(\tfrac{1}{2}\sum_{\Psi-\Gamma}x_{\psi,0}\Big) + \mathfrak{m}_{\Gamma,0}; \qquad \mathfrak{m}_{\Gamma,0} = \mathfrak{m}_0 \cap \mathfrak{g}_\Gamma. \tag{5.11}$$

Let $\mathfrak{f}_\Gamma$ denote the orthocomplement of $\mathfrak{o}_\Gamma$ in $\mathfrak{m}^+$. The Restricted Root Theorem says that $\{v \in \pi_0(\mathfrak{o}_\Gamma + \mathfrak{f}_\Gamma) : \|\mathrm{ad}_\mathfrak{g}(v)\| \leqq 1\} = \frac{1}{2}\sum_{\Psi-\Gamma}x_{\psi,0} + \{w \in \mathfrak{m}_{\Gamma,0} : \|\mathrm{ad}_{\mathfrak{g}_\Gamma}(w)\| \leqq 1\}$. By the Hermann Convexity Theorem now

$$\overline{\xi^{-1}(X_0)} \cap (\mathfrak{o}_\Gamma + \mathfrak{f}_\Gamma) = \overline{\xi^{-1}(X_0)} \cap (\mathfrak{o}_\Gamma + \mathfrak{m}_\Gamma^+) = \overline{\xi^{-1}(c_{\Psi-\Gamma}X_{\Gamma,0})}. \tag{5.12}$$

Now let $U \subset \mathbf{C}$ unit disk and $l : U \to X$ a holomorphic arc in bd X_0 whose image meets $c_{\Psi-\Gamma}X_{\Gamma,0}$. $\lambda : U \to \mathfrak{m}^+$ is given by $\xi \cdot \lambda = l$. Let κ be the linear functional on $\mathfrak{m}^+$ such that the support hyperplane to $\xi^{-1}X_0$ at $\mathfrak{o}_\Gamma$ has equation Re $\kappa = 1$. Now $\kappa \cdot \lambda$ is a holomorphic function on U, whose real part is $\leqq 1$ and achieves its maximum value. Now $\kappa \cdot \lambda$ is constant, i.e., $\lambda(U) \subset \mathfrak{o}_\Gamma + \mathfrak{f}_\Gamma$. From (5.12) now λ takes values in the closure of $\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$. Were λ to take a value in the boundary of $\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$, the same maximum argument would keep all values of λ in that boundary. Thus $\lambda(U) \subset \xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$. Now $l(U) \subset c_{\Psi-\Gamma}X_{\Gamma,0}$. That proves that $c_{\Psi-\Gamma}X_{\Gamma,0}$ is a union of boundary components. As $c_{\Psi-\Gamma}X_{\Gamma,0}$ is a complex submanifold of X, we conclude that it is a boundary component of X_0.
Q.E.D.

Corollary 1. The boundary components of $\xi^{-1}X_0$ in $\mathfrak{m}^+$ are bounded symmetric domains

$$\xi^{-1}kc_{\Psi-\Gamma}X_{\Gamma,0} = \mathrm{ad}(k)\cdot\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0} \subset \mathrm{ad}(k)\,(\mathfrak{o}_\Gamma + \mathfrak{m}_\Gamma^+)$$

in Harish-Chandra Embedding.

{Combine (5.12) and the Boundary Component Theorem.}

Corollary 2. Decompose $X_0 = X_0^1 \times \cdots \times X_0^q$ product of irreducible symmetric spaces. Let X^t be the compact dual of X_0^t so $X = X^1 \times \ldots \times X^q$. Then the boundary components of X_0 in X are the products $Y_0^1 \times \cdots \times Y_0^q \neq X_0$ where Y_0^t is X_0^t or a boundary component of X_0^t.

{Everything decomposes as a product.}

Corollary 3. If X_0 is an irreducible Hermitian symmetric space, then its boundary components are irreducible Hermitian symmetric spaces of classical type.

Proof. Let W_Γ denote the subgroup of the Weyl group of G that preserves Ψ and acts trivially on $\Psi - \Gamma$. It induces every permutation of Γ by part 3 of the Restricted Root Theorem. That proves irreducibility of the Hermitian symmetric space $X_{\Gamma,0}$. The fact that it is of clasical type is contained in the classification theorem below. Q.E.D.

Corollary 4. A boundary component of a boundary component of X_0 is a boundary component of X_0.

{The boundary components of $c_{\Psi-\Gamma}X_{\Gamma,0}$ are the $kc_{\Psi-\Gamma}c_{\Gamma-\Sigma}X_{\Sigma,0}$ with $k \in K \cap G_{\Gamma,0}$ and $\Sigma \subsetneqq \Gamma$.}

X_0 is said to be of *tube type* if it is holomorphically equivalent to a tube domain over a self dual cone. That is equivalent to $c_\Psi^4 = 1$. Momentarily taking the latter as definition, we see that X_0 is of tube type if and only if each of its irreducible factors has restricted root system given by Case 1 of the Restricted Root Theorem. That situation persists under passage from $\mathfrak{g}_0$ to $\mathfrak{g}_{\Gamma,0}$. Thus:

Corollary 5. If X_0 is of tube type, then its boundary components are of tube type. If X_0 is irreducible and has a positive-dimensional boundary component of tube type, then X_0 is of tube type.

Suppose $\Sigma \subset \Gamma \subset \Psi$. Then $\mathrm{ad}(c_\Sigma)\mathfrak{g}_\Gamma = \mathfrak{g}_\Gamma$ and we define

$$\mathfrak{g}_\Gamma^\Sigma \quad \text{is the fixed point set of} \quad \mathrm{ad}(c_\Sigma^4) \quad \text{on} \quad \mathfrak{g}_\Gamma. \tag{5.13}$$

As c_Σ has order 1, 4, or 8, c_Σ^4 commutes with complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$, so we have

$$\mathfrak{g}_{\Gamma,0}^\Sigma = \mathfrak{g}_0 \cap \mathfrak{g}_\Gamma^\Sigma, \qquad \text{real form of} \quad \mathfrak{g}_\Gamma^\Sigma. \tag{5.14}$$

Now define

$$G_{\Gamma,0}^\Sigma : \text{analytic subgroup} \quad \text{of} \quad G_0 \quad \text{for} \quad \mathfrak{g}_{\Gamma,0}^\Sigma, \tag{5.15}$$

$$X_{\Gamma,0}^\Sigma = G_{\Gamma,0}^\Sigma(x_0), \text{ Hermitian symmetric subspace of } X_{\Gamma,0}. \tag{5.16}$$

Also observe

$$X_0 \qquad \text{is of tube type} \qquad \Leftrightarrow X_0 = X_{\Psi,0}^\Psi. \tag{5.17}$$

Lemma. If $\Gamma \subset \Psi$, then $[\mathfrak{g}_\Gamma^\Gamma, \mathfrak{g}_{\Psi-\Gamma}] = 0$, so we have

$$\mathfrak{g}_{\Psi,0}^\Gamma = \mathfrak{g}_{\Psi-\Gamma,0} \oplus \mathfrak{g}_{\Gamma,0}^\Gamma \oplus (\text{compact ideal}), \tag{5.18}$$

$$X_{\Psi,0}^\Gamma = X_{\Psi-\Gamma,0} \times X_{\Gamma,0}^\Gamma. \tag{5.19}$$

Proof. We may assume $\mathfrak{g}$ simple and use the Restricted Root Theorem. Let Δ_Γ^Γ and $\Delta_{\Psi-\Gamma}$ denote the respective root systems of $\mathfrak{g}_\Gamma^\Gamma$ and $\mathfrak{g}_{\Psi-\Gamma}$. Let $\alpha \in \Delta_\Gamma^\Gamma$ noncompact positive and $\beta \in \Delta_{\Psi-\Gamma}$ noncompact negative. Then

$$\alpha\big|_{\mathfrak{t}^-} = \tfrac{1}{2}(\gamma + \gamma') \qquad \text{for some} \quad \gamma, \gamma' \in \Gamma; \qquad \text{and}$$
$$\beta\big|_{\mathfrak{t}^-} = 0 \quad \text{or} \quad -\tfrac{1}{2}(\psi + \psi') \quad \text{or} \quad -\tfrac{1}{2}\psi \quad \text{for some} \quad \psi, \psi' \in \Psi - \Gamma.$$

Now $(\alpha + \beta)\big|_{\mathfrak{t}^-}$ is $\tfrac{1}{2}(\gamma + \gamma')$ or $\tfrac{1}{2}(\gamma + \gamma' - \psi - \psi')$ or $\tfrac{1}{2}(\gamma + \gamma' - \psi)$, so $\alpha + \beta$ is not a root. Similarly, if α is noncompact negative from Δ_Γ^Γ and β is noncompact positive from $\Delta_{\Psi-\Gamma}$, then $\alpha + \beta$ is not a root. This proves

$$[\mathfrak{m} \cap \mathfrak{g}_\Gamma^\Gamma, \mathfrak{m} \cap \mathfrak{g}_{\Psi-\Gamma}] = 0.$$

Now $\mathfrak{m} \cap \mathfrak{g}_{\Psi-\Gamma}$ generates the semisimple algebra $\mathfrak{g}_{\Psi-\Gamma}$, and $\mathfrak{g}_\Gamma^\Gamma = (\mathfrak{g}_\Gamma^\Gamma)' \oplus \mathfrak{l}$ where $\mathfrak{m} \cap \mathfrak{g}_\Gamma^\Gamma$ generates $(\mathfrak{g}_\Gamma^\Gamma)'$ and $\mathfrak{l} \subset \mathfrak{k}^C$. As $[\mathfrak{l}, \mathfrak{a}_{\Psi-\Gamma}] = 0$, we obtain $[\mathfrak{l}, \mathfrak{m} \cap \mathfrak{g}_{\Psi-\Gamma}] = 0$, and thus $[\mathfrak{l}, \mathfrak{g}_{\Psi-\Gamma}] = 0$. Now $[\mathfrak{g}_\Gamma^\Gamma, \mathfrak{g}_{\Psi-\Gamma}] = 0$.

Q.E.D.

We now describe the boundary components explicitly.

Classification of Boundary Components. Let $X_0 \subset X$ be the Borel embedding of an irreducible Hermitian symmetric space of noncompact type and rank r. For each integer m, $0 \leqq m < r$, there is just one G_0-equivalence class $\{X_{m,0}\}$ of boundary component of symmetric space rank m for X_0 in X; $X_{0,0}$ is a single point, and the $X_{m,0}$ are given by one of the following series with $X_0 = X_{r,0}$.

1. $X_{m,0} = \mathbf{SU}(m, m+k) / \mathbf{S}(\mathbf{U}(m) \times \mathbf{U}(m+k))$, $k \geqq 0$ fixed.
2. $X_{m,0} = \mathbf{SO}^*(4m) / \mathbf{U}(2m)$.
3. $X_{m,0} = \mathbf{SO}^*(4m+2) / \mathbf{U}(2m+1)$.
4. $X_{m,0} = \mathbf{Sp}(m ; \mathbf{R}) / \mathbf{U}(m)$.
5. $X_0 = \mathbf{SO}(2, n) / \mathbf{SO}(2) \times \mathbf{SO}(n)$, $n > 2$ fixed; here $r = 2$ and $X_{1,0}$ is the unit disk in $\mathbf{C}^1$.
6. $X_0 = \mathbf{E}_6 / \mathbf{SO}(10) \cdot \mathbf{SO}(2)$; here $r = 2$ and $X_{1,0}$ is the open unit ball in $\mathbf{C}^5$.
7. $X_0 = \mathbf{E}_7 / \mathbf{E}_6 \cdot \mathbf{SO}(2)$; here $r = 3$, $X_{2,0} = \mathbf{SO}(2,8)/\mathbf{SO}(2) \times \mathbf{SO}(8)$, and $X_{1,0}$ is the unit disk in $\mathbf{C}^1$.

Proof. That there is just one G_0-equivalence class $\{X_{m,0}\}$ of rank m boundary components, follows from the Boundary Component Theorem and the part of the Restricted Root Theorem that ensures that any two subsets of Ψ of the same size are equivalent under the restricted Weyl group. Thus $\{X_{m,0}\}$ is represented by any $c_\Gamma X_{\Psi-\Gamma,0}$ with $|\Psi - \Gamma| = m$.

Let $\Gamma = \{\alpha\}$ so $|\Psi - \Gamma| = r - 1$. Then $\text{ad}(c_\alpha)^4$ is an involutive inner automorphism of $\mathfrak{g}$, so its fixed point set $\mathfrak{g}_\Psi^{\{\alpha\}}$ is a symmetric subalgebra of maximal rank. The lemma says that $\mathfrak{g}_\Psi^{\{\alpha\}}$ has distinct ideals $\mathfrak{g}_{\Psi-\Gamma}$ and $\mathfrak{sl}(2, \mathbf{C}) \subset \mathfrak{g}_\Gamma^\Gamma$. The classification now is immediate from the Borel–de Siebenthal classification of maximal subalgebras of maximal rank in compact simple Lie algebras, and from the corollaries to the Boundary Component Theorem. Q.E.D.

Karpelevič defines boundary components, for noncompact type symmetric spaces, in terms of limiting behavior of geodesic rays. The following result says that his boundary components coincide with ours.

Geodesic Ray Theorem. Let $x \in X_0$ and F a boundary component of X_0 in X. Then there is a unique point $f \in F$ such that some geodesic ray of X_0 from x tends to f.

Proof. Some $g \in G_0$ carries x to x_0, and then some $k \in K$ carries $g(F)$ back to F; thus we may assume $x = x_0$. Applying an element of K, we may also assume $F = c_{\Psi-\Gamma} X_{\Gamma,0}$ for some $\Gamma \subset \Psi$.

Let $\{x_t\}_{0<t<\infty}$ be a geodesic ray in X_0 such that, in X, $\lim_{t\to\infty}\{x_t\} \in F$. We have $b \in \mathfrak{m}_0$ such that $x_t = \exp(tb)\cdot x_0$.

First suppose $b \in \mathfrak{a}_0$, so $b = \sum_\Psi b_\psi x_{\psi,0}$ with b_ψ real. Denote

$$\operatorname{sign}(b_\psi) = 1 \quad \text{if} \quad b_\psi > 0, \qquad 0 \quad \text{if} \quad b_\psi = 0, \qquad -1 \quad \text{if} \quad b_\psi < 0.$$

Now (3.18) says that $\lim_{t\to\infty}\{x_t\} = \xi(\sum \operatorname{sign}(b_\psi)e_\psi)$. As $\lim\{x_t\} \in c_{\Psi-\Gamma}X_{\Gamma,0}$, it follows that this limit is $c_{\Psi-\Gamma}(x_0) = \xi(\sum_{\Psi-\Gamma} e_\psi)$.

Now drop the assumption $b \in \mathfrak{a}_0$. Let $k \in K$ such that $\operatorname{ad}(k)b \in \mathfrak{a}_0$. We then know that $\lim\{k(x_t)\}$ is $c_{\Psi-\Gamma}(x_0) \in F$. As $\lim\{x_t\} \in F$, now $k(F) = F$. However $c_{\Psi-\Gamma}(x_0)$ is the point of $\xi^{-1}(F)$ of least $\mathfrak{m}^+$-distance from the origin, and k does not change that distance. Thus $k(c_{\Psi-\Gamma}(x_0)) = c_{\Psi-\Gamma}(x_0)$. Now $\lim\{x_t\} = k^{-1}\lim\{kx_t\} = c_{\Psi-\Gamma}(x_0)$. Q.E.D.

6. Boundary Groups

We have seen that X_0 has topological boundary

$$\operatorname{bd} X_0 = \bigcup_{\Gamma \subsetneqq \Psi} G_0(c_{\Psi-\Gamma}x_0) \tag{6.1}$$

where the boundary orbits are unions of boundary components

$$G_0(c_{\Psi-\Gamma}x_0) = \bigcup_{k\in K} kc_{\Psi-\Gamma}X_{\Gamma,0} \tag{6.2}$$

which in turn are given by

$$c_{\Psi-\Gamma}X_{\Gamma,0} = c_{\Psi-\Gamma}G_{\Gamma,0}(x_0) = G_{\Gamma,0}(c_{\Psi-\Gamma}x_0). \tag{6.3}$$

Here we will determine the normalizers of the boundary components and the structure of the space of boundary components of a given type. The results are in Wolf and Korányi [13], but the proofs here are a considerable simplification of [13].

Recall (3.3) $x_{\varphi,0} = e_\varphi + e_{-\varphi} \in \mathfrak{m}_0$ for $\varphi \in \Delta_M^+$. For each subset $\Phi \subset \Psi$, we define

$$x_{\Phi,0} = \sum_{\psi\in\Phi} x_{\psi,0} \in \mathfrak{a}_0. \tag{6.4}$$

We compute

$$\operatorname{ad}(c_\psi)^{-1}x_{\psi,0} = \operatorname{ad}(\exp(\tfrac{\pi}{4}(e_\psi - e_{-\psi})))\cdot(e_\psi + e_{-\psi})$$

in the model $e_\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{-\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. There

$$\exp\frac{\pi}{4}(e_\psi - e_{-\psi}) = \exp\frac{\pi}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

so

$$\mathrm{ad}[\exp \tfrac{\pi}{4}(e_\psi - e_{-\psi})]\cdot(e_\psi + e_{-\psi}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h_\psi.$$

Summing over the subset $\Phi \subset \Psi$, now

$$\mathrm{ad}(c_\Phi)^{-1} x_{\Phi,0} = \sum_{\psi \in \Phi} h_\psi. \tag{6.5}$$

Use the Restricted Root Theorem to compute the eigenspaces of $\mathrm{ad}(\sum_\Phi h_\psi)$ on $\mathfrak{g}$. Let ρ denote the restriction to $\mathfrak{t}^-$ as before. Now the $\mathfrak{t}^{\mathbb{C}}$-root system Δ is the disjoint union,

$$\Delta = E_2(\Phi) \cup E_1(\Phi) \cup E_0(\Phi) \cup E_{-1}(\Phi) \cup E_{-2}(\Phi), \tag{6.6}$$

where

$$E_{\pm 2}(\Phi) = \{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}(\psi + \psi') \quad \text{with} \quad \psi, \psi' \in \Phi\}; \tag{6.7}$$

$E_{\pm 1}(\Phi)$ *is the union of the three sets*:

$$\begin{aligned} &\{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}(\psi + \psi') \quad \text{with} \quad \psi \in \Phi,\ \psi' \in \Psi - \Phi\}, \\ &\{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}(\psi - \psi') \quad \text{with} \quad \psi \in \Phi,\ \psi' \in \Psi - \Phi\}, \\ &\text{and} \quad \{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}\psi \quad \text{with} \quad \psi \in \Phi\}; \end{aligned} \tag{6.8}$$

$E_0(\Phi)$ *is the union of*

$$\begin{aligned} &\{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}(\psi \pm \psi') \quad \text{with} \quad \psi, \psi' \in \Psi - \Phi\}, \\ &\{\delta \in \Delta : \rho(\delta) = \pm\tfrac{1}{2}\psi \quad \text{with} \quad \psi \in \Psi - \Phi\}, \\ &\{\delta \in \Delta : \rho(\delta) = 0\}, \quad \text{and} \\ &\{\delta \in \Delta : \rho(\delta) = \tfrac{1}{2}(\psi - \psi') \quad \text{with} \quad \psi, \psi' \in \Phi\}. \end{aligned} \tag{6.9}$$

Applying the Restricted Root Theorem to (6.6), we have

6.10. Lemma. The eigenvalues of $\mathrm{ad}(\sum_{\psi \in \Phi} h_\psi)$ on $\mathfrak{g}$ are ± 2, ± 1, and 0. The eigenspaces are given as follows.

$$\begin{aligned} (\pm 2)\text{-eigenspace:} \quad & \textstyle\sum_{\delta \in E_{\pm 2}(\Phi)} \mathfrak{g}^\delta \\ (\pm 1)\text{-eigenspace:} \quad & \textstyle\sum_{\delta \in E_{\pm 1}(\Phi)} \mathfrak{g}^\delta \\ 0\text{-eigenspace:} \quad & \mathfrak{t}^{\mathbb{C}} + \textstyle\sum_{\delta \in E_0(\Phi)} \mathfrak{g}^\delta. \end{aligned}$$

Combine Lemma 6.10 with (6.5) to obtain

6.11. Lemma. The eigenvalues of $\mathrm{ad}(x_{\Phi,0})$ on $\mathfrak{g}$ are ± 2, ± 1, and 0. The eigenspaces are given as follows.

$$
\begin{aligned}
(\pm 2)\text{-eigenspace:} &\quad \mathrm{ad}(c_\Phi)\cdot \textstyle\sum_{\delta\in E_{\pm 2}(\Phi)} \mathfrak{g}^\delta. \\
(\pm 1)\text{-eigenspace:} &\quad \mathrm{ad}(c_\Phi)\cdot \textstyle\sum_{\delta\in E_{\pm 1}(\Phi)} \mathfrak{g}^\delta. \\
0\text{-eigenspace:} &\quad \mathrm{ad}(c_\Phi)\cdot \{\mathfrak{t}^{\mathbf{C}} + \textstyle\sum_{\delta\in E_0(\Phi)} \mathfrak{g}^\delta\}.
\end{aligned}
$$

With the eigenspace decomposition (6.11) in mind, we define parabolic subalgebras $\mathfrak{n}_\Phi \subset \mathfrak{g}$ and $\mathfrak{n}_{\Phi,0} \subset \mathfrak{g}_0$ by

$$\mathfrak{n}_\Phi : \text{ sum of the nonpositive eigenspaces of } \mathrm{ad}(x_{\Phi,0}) \text{ on } \mathfrak{g}, \tag{6.12a}$$

$$\mathfrak{n}_{\Phi,0} : \text{ sum of the nonpositive eigenspaces of } \mathrm{ad}(x_{\Phi,0}) \text{ on } \mathfrak{g}_0. \tag{6.12b}$$

Then of course $\mathfrak{n}_{\Phi,0}$ is a real form of $\mathfrak{n}_\Phi$. Now define parabolic subgroups $N_\Phi \subset G$ and $N_{\Phi,0} \subset G_0$ by

$$N_\Phi \quad \text{is the complex analytic subgroup of } G \text{ for } \mathfrak{n}_\Phi, \tag{6.13a}$$

$$N_{\Phi,0} = N_\Phi \cap G_0, \quad \text{closed subgroup with Lie algebra } \mathfrak{n}_{\Phi,0}. \tag{6.13b}$$

Boundary Group Theorem. If $\Gamma \subsetneqq \Psi$, then the normalizer of the boundary component $c_{\Psi-\Gamma}X_{\Gamma,0}$ of X_0 in X is given by

$$N_{\Psi-\Gamma,0} = \{g \in G_0 : gc_{\Psi-\Gamma}X_{\Gamma,0} = c_{\Psi-\Gamma}X_{\Gamma,0}\}.$$

Furthermore,

(1) in general, $N_{\Psi-\Gamma,0}$ and $N_{\Psi-\Gamma',0}$ are conjugate in G_0 if, and only if, each $|\Gamma \cap \Delta_t| = |\Gamma' \cap \Delta_t|$ where the Δ_t are the root systems of the simple ideals of $\mathfrak{g}$;

(2) if G is simple, then the $N_{\Psi-\Gamma,0}$, $\Gamma \subsetneqq \Psi$, are maximal parabolic subgroups of G_0, and every maximal parabolic subgroup of G_0 is conjugate to one of them.

Proof. Let $N_0 = \{g \in G_0 : gc_{\Psi-\Gamma}X_{\Gamma,0} = c_{\Psi-\Gamma}X_{\Gamma,0}\}$, $\mathfrak{n}_0$ its Lie algebra, $\mathfrak{n} = \mathfrak{n}_0^{\mathbf{C}}$, and N the complex analytic subgroup of G for $\mathfrak{n}$.

The unipotent radical $\mathfrak{n}^u_{\Psi-\Gamma,0}$ of $\mathfrak{n}_{\Psi-\Gamma,0}$ is the sum of the (-2)-eigenspace and the (-1) eigenspace of $\mathrm{ad}(x_{\Psi-\Gamma,0})$ on $\mathfrak{g}_0$. We check $\mathfrak{n}^u_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$. Let δ be a root. If $\delta \in E_{-2}(\Psi - \Gamma)$, then (6.7) $\rho(\delta) = -\frac{1}{2}(\psi + \psi')$ with $\psi, \psi' \in \Psi - \Gamma$, so δ is a noncompact negative root. If $\delta \in E_{-1}(\Psi - \Gamma)$, then (6.8) there are three possibilities. If $\rho(\delta) = -\frac{1}{2}(\psi + \psi')$ with $\psi \in \Psi - \Gamma$ and

$\psi' \in \Gamma$, then δ is a noncompact negative root. If $\rho(\delta) = -\frac{1}{2}(\psi - \psi')$ with $\psi \in \Psi - \Gamma$ and $\psi' \in \Gamma$, then δ is a compact root. If $\rho(\delta) = -\frac{1}{2}\psi$ with $\psi \in \Psi - \Gamma$, then δ is a noncompact negative root. We have just shown, via Lemma 6.11, that $\mathfrak{n}^u_{\Psi-\Gamma} \subset \mathrm{ad}(c_{\Psi-\Gamma})(\mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^-)$, Lie algebra of the isotropy subgroup of G at $c_{\Psi-\Gamma}x_0$. Thus $\mathfrak{n}^u_{\Psi-\Gamma,0} \subset \mathfrak{g}_0 \cap \mathrm{ad}(c_{\Psi-\Gamma})(\mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^-)$, Lie algebra of the isotropy subgroup $G_0 \cap \mathrm{ad}(c_{\Psi-\Gamma})P$ of G_0 at $c_{\Psi-\Gamma}(x_0)$. Thus the analytic subgroup of G_0 for $\mathfrak{n}^u_{\Psi-\Gamma,0}$ preserves the holomorphic arc component $c_{\Psi-\Gamma}X_{\Gamma,0}$ of $G_0(c_{\Psi-\Gamma}x_0)$ through $c_{\Psi-\Gamma}(x_0)$. That completes our proof of $\mathfrak{n}^u_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$.

The reductive part $\mathfrak{n}^r_{\Psi-\Gamma,0}$ of $\mathfrak{n}_{\Psi-\Gamma,0}$ is the 0-eigenspace of $\mathrm{ad}(x_{\Psi-\Gamma,0})$ on $\mathfrak{g}_0$. We check $\mathfrak{n}^r_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$. Looking at (6.9), we see that

$$E_0(\Psi - \Gamma) = E_0'(\Psi - \Gamma) \cup E_0''(\Psi - \Gamma) \qquad \text{disjoint} \tag{6.14a}$$

where

$$E_0'(\Psi - \Gamma) = \{\delta \in E_0(\Psi - \Gamma) : \rho(\delta) = \tfrac{1}{2}(\psi - \psi');\ \psi, \psi' \in \Psi - \Gamma\}. \tag{6.14b}$$

That decomposes, via Lemma 6.11,

$$\mathfrak{n}^r_{\Psi-\Gamma} = \mathfrak{n}'_{\Psi-\Gamma} + \mathfrak{n}''_{\Psi-\Gamma} \qquad \text{vector space direct sum} \tag{6.15a}$$

where

$$\mathfrak{n}'_{\Psi-\Gamma} = \mathrm{ad}(c_{\Psi-\Gamma})\{\mathfrak{t}^{\mathbf{C}} + \sum_{\delta\in E_0'(\Psi-\Gamma)} \mathfrak{g}^{\delta}\} \tag{6.15b}$$

and

$$\mathfrak{n}''_{\Psi-\Gamma} = \mathrm{ad}(c_{\Psi-\Gamma}) \sum_{\delta\in E_0''(\Psi-\Gamma)} \mathfrak{g}^{\delta}. \tag{6.15c}$$

Every root $\delta \in E_0'(\Psi - \Gamma)$ is compact; thus $\mathfrak{n}'_{\Psi-\Gamma} \subset \mathrm{ad}(c_{\Psi-\Gamma})(\mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^-)$, and it follows as above that $\mathfrak{n}_0$ contains the real form $\mathfrak{g}_0 \cap \mathfrak{n}'_{\Psi-\Gamma}$ of $\mathfrak{n}'_{\Psi-\Gamma}$. If $\delta \in E_0''(\Psi - \Gamma)$, then $-\delta \in E_0''(\Psi - \Gamma)$ and $\mathfrak{g}^{\delta} + \mathfrak{g}^{-\delta}$ centralizes $\mathfrak{g}[\Psi - \Gamma]$; thus $\mathfrak{n}''_{\Psi-\Gamma} \subset \mathfrak{g}_\Gamma$, so the real form $\mathfrak{g}_0 \cap \mathfrak{n}''_{\Psi-\Gamma}$ of $\mathfrak{n}''_{\Psi-\Gamma}$ is contained in $\mathfrak{g}_{\Gamma,0} \subset \mathfrak{n}_0$. That completes our proof that $\mathfrak{n}^r_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$.

We have proved $\mathfrak{n}_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$.

We digress to prove assertion (2). Suppose that G is simple and $\Psi = \{\psi_1, \ldots, \psi_r\}$. Let $\Gamma_k = \{\psi_1, \psi_2, \ldots, \psi_k\}$ for $0 \leqq k \leqq r - 1$. Then the orthocomplement of $E_0(\Psi - \Gamma_k)$ in the real dual of $i\mathfrak{t}^-$ is the span of the element $\psi_{k+1} + \cdots + \psi_r$, by (6.9) and the Restricted Root Theorem. Thus the reductive part $N^r_{\Psi-\Gamma_k,0}$ has identity component whose connected center is the product of a torus and a 1-dimensional vector group. It follows that the $N_{\Psi-\Gamma_k,0}$ are maximal parabolic subgroups of G_0. Again by the Restricted Root Theorem, the $N_{\Psi-\Gamma_k,0}$ are mutually nonconjugate, so, there being r of them, they represent every conjugacy class of maximal parabolic subgroup of G_0. Now (2) is proved, and incidentally (1) follows without further argument.

We had proved $\mathfrak{n}_{\Psi-\Gamma,0} \subset \mathfrak{n}_0$. To prove them equal, it suffices to consider the case where G is simple. There $\mathfrak{n}_{\Psi-\Gamma,0}$ is a maximal subalgebra of $\mathfrak{g}_0$ by the fact (2) just proved. If $\mathfrak{n}_0 = \mathfrak{g}_0$, then G_0 preserves $c_{\Psi-\Gamma}X_{\Gamma,0}$, hence acts on it by analytic automorphisms. Rendering that action effective, we obtain $G_{\Gamma,0}$ as a homomorphic image of the simple Lie group G_0, so either $X_0 = X_{\Gamma,0}$ or $X_{\Gamma,0}$ is reduced to a point. As $X_{\Gamma,0}$ has rank $|\Gamma| < |\Psi|$ that is impossible. Thus $\mathfrak{n}_{\Psi-\Gamma,0} = \mathfrak{n}_0$.

We have just proved $\mathfrak{n}_0 = \mathfrak{n}_{\Psi-\Gamma,0}$. Thus N_0 and $N_{\Psi-\Gamma,0}$ have the same identity component. The parabolic subgroup $N_{\Psi-\Gamma,0}$ of G_0 is the normalizer in G_0 of that identity component. Now $N_0 \subset N_{\Psi-\Gamma,0}$ open subgroup, and to prove equality there we need only show that every topological component of $N_{\Psi-\Gamma,0}$ has an element that normalizes $c_{\Psi-\Gamma}X_{\Gamma,0}$.

Recall the Cartan involution σ of G_0 with fixed point set K. As $\sigma(x_{\Psi-\Gamma,0}) = -x_{\Psi-\Gamma,0}$ now

$$\sigma(\mathfrak{n}^r_{\Psi-\Gamma,0}) = \mathfrak{n}^r_{\Psi-\Gamma,0} \qquad \text{and} \qquad \mathfrak{g}_0 = \mathfrak{n}_{\Psi-\Gamma,0} + \sigma(\mathfrak{n}^u_{\Psi-\Gamma,0}). \tag{6.16a}$$

Let $N^u_{\Psi-\Gamma,0}$ denote the unipotent radical $\exp(\mathfrak{n}^u_{\Psi-\Gamma,0})$ of $N_{\Psi-\Gamma,0}$, so $N_{\Psi-\Gamma,0}$ is its G_0-normalizer. Now

$$N^r_{\Psi-\Gamma,0} = \{g \in G_0 : \mathrm{ad}(g)x_{\Psi-\Gamma,0} = x_{\Psi-\Gamma,0}\} \tag{6.16b}$$

is a σ-stable subgroup of G_0, with Lie algebra $\mathfrak{n}^r_{\Psi-\Gamma,0}$, such that

$$N_{\Psi-\Gamma,0} = N^r_{\Psi-\Gamma,0} \cdot N^u_{\Psi-\Gamma,0} \qquad \text{semidirect product.} \tag{6.16c}$$

In particular,

$$K \cap N_{\Psi-\Gamma,0} = K \cap N^r_{\Psi-\Gamma,0}$$
$$\text{and meets every component of } N_{\Psi-\Gamma,0}. \tag{6.17a}$$

Thus the remaining assertion $N_0 = N_{\Psi-\Gamma,0}$ of our theorem is reduced to the assertion that

$$\text{every element of } K \cap N^r_{\Psi-\Gamma,0} \text{ leaves } c_{\Psi-\Gamma}(x_0) \text{ fixed.} \tag{6.17b}$$

Let $k \in K \cap N^r_{\Psi-\Gamma,0}$. Recall (2.17) that $\mathfrak{k}$ has a central element z such that

$J = \mathrm{ad}(z)|_{\mathfrak{m}}$ is the almost complex structure of X_0. Now apply $\mathrm{ad}(\exp(\pi/2)z)$ to the identity $\mathrm{ad}(k)x_{\Psi-\Gamma,0} = x_{\Psi-\Gamma,0}$. According to (3.4), the result is

$$\mathrm{ad}(k)y_{\Psi-\Gamma,0} = y_{\Psi-\Gamma,0}.$$

Multiply by i and compare (3.3) and (3.5):

$$\mathrm{ad}(k)y_{\Psi-\Gamma} = y_{\Psi-\Gamma}.$$

Exponentiate $\pi/4$ times each side, using (4.4):

$$kc_{\Psi-\Gamma}k^{-1} = c_{\Psi-\Gamma}.$$

Thus

$$k(c_{\Psi-\Gamma}x_0) = c_{\Psi-\Gamma}(kx_0) = c_{\Psi-\Gamma}x_0.$$

That completes our proof of $N_0 = N_{\Psi-\Gamma,0}$. Q.E.D.

Corollary 1. Let F be a boundary component of X_0 in X. Then the G_0-normalizer of F is a parabolic subgroup of G_0.

{For $F = k \cdot c_{\Psi-\Gamma}X_{\Gamma,0}$ with $k \in K$ and $\Gamma \subsetneqq \Psi$, so its G_0-normalizer is $\mathrm{ad}(k)N_{\Psi-\Gamma,0}$.}

Corollary 2. Let F be a boundary component of X_0 in X. Then the G_0-normalizer of F is transitive on the space X_0.

{For if N_0 is a parabolic subgroup of G_0, then $KN_0 = G_0$.}

Corollary 3. The space of all boundary components of X_0 in X that are contained in a given G_0-orbit on the boundary, is the set of real points in a G-homogeneous projective variety defined over the rational number field.

{For the boundary orbit is $G_0(c_{\Psi-\Gamma}(x_0))$, $\Gamma \subsetneqq \Psi$, and its space of holomorphic arc components is $G_0/N_{\Psi-\Gamma,0} = (G/N_{\Psi-\Gamma})_{\mathbf{R}}$.}

Corollary 3 says that the space of boundary components $kc_{\Psi-\Gamma}X_{\Gamma,0}$, $k \in K$, of a given type Γ, is a "real flag manifold" $G_0/N_{\Psi-\Gamma,0}$. We take a careful look at the structure of that real flag manifold.

Boundary Flag Theorem. Let $\Gamma \subsetneqq \Psi$ and consider the fibration

$$\pi : G_0(c_{\Psi-\Gamma}x_0) \to G_0/N_{\Psi-\Gamma,0} \quad \text{by} \quad \pi(gc_{\Psi-\Gamma}x_0) = gN_{\Psi-\Gamma,0} \tag{6.18a}$$

of the corresponding boundary orbit over the space of boundary components of type Γ.

1. $K(c_{\Psi-\Gamma}x_0)$ meets each π-fiber (i.e., each boundary component contained in $G_0(c_{\Psi-\Gamma}x_0)$) in just one point, so it can be identified with the base of the fibration under $kN_{\Psi-\Gamma,0} \leftrightarrow k(c_{\Psi-\Gamma}x_0)$. Thus π is a fibration

$$\pi : G_0(c_{\Psi-\Gamma}x_0) \to K(c_{\Psi-\Gamma}x_0) \qquad K\text{-equivariant} \tag{6.18b}$$

such that

$$\pi^{-1}(kc_{\Psi-\Gamma}x_0) = kc_{\Psi-\Gamma}X_{\Gamma,0} \quad \text{boundary component through} \quad kc_{\Psi-\Gamma}x_0. \tag{6.18c}$$

2. The space $K(c_{\Psi-\Gamma}x_0) = G_0/N_{\Psi-\Gamma,0}$ of boundary components contained in the orbit $G_0(c_{\Psi-\Gamma}x_0)$, is fibered

$$\eta : K(c_{\Psi-\Gamma}x_0) \to K(c_{\Psi-\Gamma}^2x_0) \quad \text{by} \quad \eta(kc_{\Psi-\Gamma}x_0) = kc_{\Psi-\Gamma}^2x_0 \tag{6.19}$$

over the simply connected complex totally geodesic submanifold $K(c_{\Psi-\Gamma}^2x_0)$ of X, with fibers connected and totally geodesic in X. Thus the base of η is a compact Hermitian symmetric space and the η-fibers are compact Riemannian symmetric spaces.

3. The following conditions are equivalent:

(3a) The base $K(c_{\Psi-\Gamma}^2x_0)$ of η is a single point.

(3b) The total space $K(c_{\Psi-\Gamma}x_0)$ of η is totally geodesic in X.

(3c) The partial Cayley transform $c_{\Psi-\Gamma}$ has $c_{\Psi-\Gamma}^4 = 1$.

(3d) Decompose the boundary component $c_{\Psi-\Gamma}X_{\Gamma,0} = c_{\Psi_1-\Gamma_1}X_{\Gamma_1,0} \times \cdots \times c_{\Psi_q-\Gamma_q}X_{\Gamma_q,0}$ where the factors are the corresponding domains or boundary components of the irreducible factors of X_0. Then for $1 \leqq t \leqq q$, either $c_{\Psi_t-\Gamma_t}X_{\Gamma_t,0}$ is the t-factor of X_0, or that factor is of tube type and $c_{\Psi_t-\Gamma_t}X_{\Gamma_t,0}$ is a point on its Bergman–Šilov boundary.

Proof. Assertion (1) follows from (6.17) and the fact that K is transitive on the set of boundary components of X_0 that are in the orbit $G_0(c_{\Psi-\Gamma}x_0)$.

Recall the symmetry s of X_0 and X at x_0. Now $\sigma = \text{ad}(s)$ sends $c_{\Psi-\Gamma}$ to $c_{\Psi-\Gamma}^{-1}$ because each $y_\psi \in \mathfrak{m}$. Similarly the conjugation τ of G over G_0 sends $c_{\Psi-\Gamma}$ to $c_{\Psi-\Gamma}^{-1}$ because $y_\psi \in i\mathfrak{g}_0$. As each $c_\psi^8 = 1$, so $c_{\Psi-\Gamma}^8 = 1$, now

$$\sigma(c_{\Psi-\Gamma}^4) = c_{\Psi-\Gamma}^4 \quad \text{and} \quad \tau(c_{\Psi-\Gamma}^4) = c_{\Psi-\Gamma}^4. \tag{6.20}$$

Suppose that $k \in K$ leaves $c_{\Psi-\Gamma}(x_0)$ fixed. Then k commutes with the symmetry of X at $c_{\Psi-\Gamma}x_0$. That symmetry is $\mathrm{ad}(c_{\Psi-\Gamma})s = c^2_{\Psi-\Gamma}s$. Thus k commutes with $c^2_{\Psi-\Gamma}$. Now $\mathrm{ad}(k)$ leaves $\mathrm{ad}(c^2_{\Psi-\Gamma})z$ fixed. Thus $k(c^2_{\Psi-\Gamma}x_0) = c^2_{\Psi-\Gamma}x_0$. This proves $\eta : K(c_{\Psi-\Gamma}x_0) \to K(c^2_{\Psi-\Gamma}x_0)$ well defined.

The symmetry s' of X at $c^2_{\Psi-\Gamma}x_0$ is $\mathrm{ad}(c^2_{\Psi-\Gamma})s = c^2_{\Psi-\Gamma}sc^{-2}_{\Psi-\Gamma} = c^2_{\Psi-\Gamma}\cdot\sigma(c^{-2}_{\Psi-\Gamma})\cdot s = c^4_{\Psi-\Gamma}s$. Now, using (6.20),

$$\mathrm{ad}(s')|_K : K \to K \qquad \text{by} \qquad k \to \mathrm{ad}(c^4_{\Psi-\Gamma})k. \tag{6.21}$$

First, this shows that $K(c^2_{\Psi-\Gamma}x_0)$ is a (compact) totally geodesic submanifold of X. Second, it shows that the isotropy subgroup of K at $c^2_{\Psi-\Gamma}x_0$ contains the center of K. Thus $K(c^2_{\Psi-\Gamma}x_0)$ is a symmetric space of compact type.

The almost complex structure on X at $c^2_{\Psi-\Gamma}x_0$ is the transformation

$$J' = \mathrm{ad}(c^2_{\Psi-\Gamma})z : \mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{m} \to \mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{m}.$$

Let $v \in \mathfrak{k} \cap \mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{m}$, i.e., $v \in \mathfrak{k}$ such that (by (6.21)) $\mathrm{ad}(c^4_{\Psi-\Gamma})v = -v$.

$$\begin{aligned}\text{Now } \sigma[\mathrm{ad}(c^2_{\Psi-\Gamma})z,v] &= [\mathrm{ad}(\sigma(c_{\Psi-\Gamma})^2)\cdot\sigma z,\ \sigma v]\\ &= [\mathrm{ad}(c^{-2}_{\Psi-\Gamma})z,\ v] = [\mathrm{ad}(c^2_{\Psi-\Gamma})z,\ v], \text{ so } [\mathrm{ad}(c^2_{\Psi-\Gamma})z,\ v] \in \mathfrak{k}.\end{aligned}$$

In particular $K(c^2_{\Psi-\Gamma}x_0)$ is a complex submanifold of X. We have proved that $K(c^2_{\Psi-\Gamma}x_0)$ is a Hermitian symmetric subspace of compact type in X. It follows that $K(c^2_{\Psi-\Gamma}x_0)$ is simply connected. This completes the proof of the assertions of (2) concerning the base space of the fibration (6.19).

Let L be the isotropy subgroup of K at $c^2_{\Psi-\Gamma}x_0$. Thus $L(c_{\Psi-\Gamma}x_0) = \eta^{-1}(c^2_{\Psi-\Gamma}x_0)$, η-fiber. L is connected because $K(c^2_{\Psi-\Gamma}x_0) \cong K/L$ is simply connected. Now η has connected fibers, and the Lie algebra of L is given (6.21) by

$$\mathfrak{l} = \{v \in \mathfrak{k} : \mathrm{ad}(c^4_{\Psi-\Gamma})\, v = v\}.$$

Now observe

$$\tau\cdot\mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{l} = \mathrm{ad}(c^{-2}_{\Psi-\Gamma})\,(\tau\mathfrak{l}) = \mathrm{ad}(c^{-2}_{\Psi-\Gamma})\mathfrak{l} = \mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{l}$$

and

$$\sigma\cdot\mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{l} = \mathrm{ad}(c^{-2}_{\Psi-\Gamma})\,(\sigma\mathfrak{l}) = \mathrm{ad}(c^{-2}_{\Psi-\Gamma})\mathfrak{l} = \mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{l}.$$

Thus

$$\mathrm{ad}(c^2_{\Psi-\Gamma})\mathfrak{l} \subset \mathfrak{g}_0 \cap \mathfrak{k}^{\mathbb{C}} = \mathfrak{k}.$$

It follows that

$$\mathrm{ad}(\mathrm{ad}(c_{\Psi-\Gamma})s) : L \to L \qquad \text{by} \qquad l \to \mathrm{ad}(c_{\Psi-\Gamma}^2)l.$$

Now the space $L(c_{\Psi-\Gamma}x_0)$ is totally geodesic in X. That completes the proof of the assertions of (2) concerning the fibers of the fibration (6.19), thus completing the proof of (2).

In proving (3) we may, and do, assume G simple.

(3a) implies (3b) because the η-fibers (6.19) are totally geodesic in X.

Assume (3b). Then K is stable under conjugation by $\mathrm{ad}(c_{\Psi-\Gamma})s = c_{\Psi-\Gamma}^2 s$, hence under $\mathrm{ad}(c_{\Psi-\Gamma}^2)$. Now $\mathrm{ad}(c_{\Psi-\Gamma}^2)s$ is a central element of order 2 in K, hence necessarily equal to s. Thus $c_{\Psi-\Gamma}^2$ commutes with s, so $c_{\Psi-\Gamma}^4 = c_{\Psi-\Gamma}^2 \cdot \mathrm{ad}(s)c_{\Psi-\Gamma}^2 = c_{\Psi-\Gamma}^2 c_{\Psi-\Gamma}^{-2} = 1$. Now (3b) implies (3c).

(3c) implies (3a) by a glance at the symmetry to X at $c_{\Psi-\Gamma}(x_0)$.

Now (3a), (3b), and (3c) are equivalent.

(3d) implies (3c) rather trivially. Now assume (3c). Then (5.19), with roles of Γ and $\Psi - \Gamma$ interchanged, says $X_0 = X_{\Psi-\Gamma,0} \times X_{\Gamma,0}$. However we have assumed G simple, thus X_0 irreducible; so either Γ or $\Psi - \Gamma$ is empty. If $\Psi - \Gamma$ is empty, then $c_{\Psi-\Gamma}X_{\Gamma,0} = X_0$. If Γ is empty, then $c_{\Psi-\Gamma}x_0 = c_{\Psi-\Gamma}X_{\Gamma,0}$ point on the Šilov boundary, and X_0 is tube type by (3c). Now (3c) implies (3d), so (3c) and (3d) are equivalent. This completes the proof of (3). Q.E.D.

Part II. Holomorphic Arc Component Theory

Suppose that X_0 is irreducible and of rank r. In Section 5, we saw that the topological boundary bd X_0 of X_0 in X is the union of r distinct G_0-orbits, say $G_0(x_k)$ for $1 \leqq k \leqq r$ where $x_k = c_\Gamma x_0$ and $|\Gamma| = k$, and that the holomorphic arc components of $G_0(x_k)$ are noncompact type irreducible Hermitian symmetric spaces of rank $r - k$ which are transitively permuted by K. In Section 6 we saw that the G_0-normalizers of these boundary components are just the maximal proper parabolic subgroups of G_0 and that $K(x_k)$ represented the space of boundary components contained in $G_0(x_k)$. In Part II, we extend those results to arbitrary G_0-orbits on X.

The G_0-orbit structure of X is worked out in Section 7, simplifying the methods of Takeuchi [*10*] and Wolf [*16*]. The idea is to use the whole polysphere $G[\Psi](x_0)$ rather than the polydisk $G_0[\Psi](x_0)$. The problem comes down to that of distinguishing open orbits, and we do that by using results of Section 6.

The holomorphic arc components of a G_0-orbit are worked out in Section 8. They turn out to be the K-translates of the open $G_{\Psi-\Gamma,0}$ -orbits on a certain translate of X_Γ. The method here eliminates a lot of machinery from my original method [*16*], relying on the results of Sections 5–7.

The general open G_0-orbits are indefinite metric versions of X_0 but may have compact subvarieties. See the example of the ball in $\mathbf{C}^n$ in the Introduction. Takeuchi [*10*] showed that an open G_0-orbit is holomorphically fibered over a certain maximal compact subvariety, and I showed [*16*] that the corresponding fibrations of the holomorphic arc components of a G_0-orbit fit together. That material is worked out in Section 9 with a few simplifications. For the full story, however, one should see Section 11 of [*16*].

7. Orbit Structure

The Boundary Orbit Theorem of Section 5 gives the G_0-orbit structure of the topological closure of X_0 in X. There the main trick was the use of the polydisk $G_0[\Psi](x_0)$. Here we use the whole polysphere $G[\Psi](x_0)$ to obtain the G_0-orbit structure of X. The result is in Takeuchi [*10*] and Wolf [*16*], but we avoid most of the heavy machinery used in the arguments of [*10*] and [*16*].

To see the idea, let X be a complex Grassmannian, consisting of all k-dimensional linear subspaces of $\mathbf{C}^n$. The group $\mathbf{GL}(n,\mathbf{C})$ of all invertible linear transformations of $\mathbf{C}^n$ acts on X. The kernel of the action consists of the scalar transformations. Dividing out that kernel, we obtain

$G = \mathbf{GL}(n,\mathbf{C})$ / (scalars) : complex group of automorphisms of X.

Consider the nondegenerate Hermitian form on $\mathbf{C}^n$ given by

$$b(u,v) = -\sum_{j=1}^{k} u^j\bar{v}^j + \sum_{j=k+1}^{n} u^j\bar{v}^j.$$

The subgroup of $\mathbf{GL}(n,\mathbf{C})$ preserving b is the indefinite unitary group usually denoted $\mathbf{U}^k(n)$ or $\mathbf{U}(k, n-k)$. Dividing out the kernel of its action on X, we obtain

$$G_0 = \mathbf{U}^k(n) \,/\, (\text{scalars}) : \text{real group of automorphisms of } X.$$

Witt's Theorem says: two k-dimensional subspaces of $\mathbf{C}^n$ are $\mathbf{U}^k(n)$-equivalent if, and only if, the restrictions of b to those subspaces have the same rank and same signature. That gives the G_0-orbit structure of X. The noncompact Hermitian symmetric space X_0 consists of the (using b) positive definite subspaces of $\mathbf{C}^n$; its boundary orbits are the sets consisting of all positive semidefinite subspaces on which b has rank t, $n - k - r \leqq t <$

$n - k$ where $r = \min(k, n - k)$ is the symmetric space rank of X_0 and X. The open G_0 orbits are the sets of k-planes on which b has a fixed nondegenerate signature; there are $r + 1$ of them. There are $\frac{1}{2}(r + 1)(r + 2)$ orbits in all.

We now return to the general case.

To minimize notation, we denote

$$\Delta_t,\ 1 \leqq t \leqq q : \text{root systems of the simple ideals of } \mathfrak{g}. \tag{7.1}$$

$$x_{\Gamma,\Sigma} = c_\Gamma c_\Sigma^2 x_0 \qquad \text{whenever} \quad \Gamma, \Sigma \subset \Psi. \tag{7.2}$$

Orbit Structure Theorem. The G_0-orbits on X are just the $G_0(x_{\Gamma,\Sigma})$ where Γ and Σ are disjoint subsets of Ψ. Furthermore, $G_0(x_{\Gamma',\Sigma'})$ is in the closure of $G_0(x_{\Gamma,\Sigma})$ if, and only if, for $1 \leqq t \leqq q$,

$$|(\Sigma' - \Gamma') \cap \Delta_t| \leqq |(\Sigma - \Gamma) \cap \Delta_t| \quad \text{and} \quad |(\Sigma \cup \Gamma) \cap \Delta_t| \leqq |(\Sigma' \cup \Gamma') \cap \Delta_t|.$$

In particular,

(i) $G_0(x_{\Gamma',\Sigma'}) = G_0(x_{\Gamma,\Sigma})$ if, and only if, for $1 \leqq t \leqq q$, $|(\Sigma' - \Gamma') \cap \Delta_t| = |(\Sigma - \Gamma) \cap \Delta_t|$ and $|\Gamma' \cap \Delta_t| = |\Gamma \cap \Delta_t|$;

(ii) the number of G_0-orbits on X is $\frac{1}{2}\sum_{t=1}^{q}(r_t + 1)(r_t + 2)$ where $r_t = |\Psi \cap \Delta_t|$;

(iii) $G_0(x_{\Gamma,\Sigma})$ is open in X if, and only if, Γ is empty; so there are $\sum_{t=1}^{q}(r_t + 1) = |\Psi| + q$ open orbits;

(iv) the boundary of a typical open orbit $G_0(x_{\phi,\Sigma})$ is the union of the $G_0(x_{\Gamma',\Sigma'})$, $\Sigma' - \Gamma' \subset \Sigma \subset \Sigma' \cup \Gamma'$;

(v) the Bergman–Šilov boundary $G_0(x_{\Psi,\phi})$ of X_0 in X is in the closure of every orbit and is the unique closed G_0-orbit on X.

Proof. Let φ be a noncompact positive root, $\mathfrak{g}[\varphi] = h_\varphi \mathbf{C} + \mathfrak{g}^\varphi + \mathfrak{g}^{-\varphi}$ three-dimensional complex simple algebra, $\mathfrak{g}_0[\varphi] = \mathfrak{g}_0 \cap \mathfrak{g}[\varphi]$ noncompact real form; $G[\varphi] \subset G$ and $G_0[\varphi] \subset G_0$ analytic subgroups for $\mathfrak{g}[\varphi]$ and $\mathfrak{g}_0[\varphi]$; $S[\varphi] = G[\varphi](x_0)$ Riemann sphere and $S_0[\varphi] = G_0[\varphi](x_0)$ lower hemisphere. Then the $G_0[\varphi]$-orbits on $S[\varphi]$ are

lower hemisphere:	$G_0[\varphi]\,(\exp((\pi/4)y_\varphi))^n x_0,$	$n \equiv 0 \bmod 4,$
equator	$G_0[\varphi]\,(\exp((\pi/4)y_\varphi))^n x_0,$	$n \equiv 1 \bmod 2,$
upper hemisphere:	$G_0[\varphi]\,(\exp((\pi/4)y_\varphi))^n x_0,$	$n \equiv 2 \bmod 4.$

Let $S[\Psi] = G[\Psi](x_0)$, product of the $|\Psi|$ Riemann spheres $G[\psi](x_0) = S[\psi]$, $\psi \in \Psi$. The $G_0[\Psi]$-orbits on $S[\Psi]$ are the products of the form

$\prod_{\psi\in\Psi} D_\psi$ where D_ψ is a $G_0[\psi]$-orbit on $S[\psi]$. Given such an orbit, define

$$\Gamma = \{\psi \in \Psi : D_\psi \text{ is the equator of } S[\psi]\}, \qquad \text{and}$$
$$\Sigma = \{\psi \in \Psi : D_\psi \text{ is the upper hemisphere of } S[\psi]\}.$$

Then $\Gamma, \Sigma \subset \Psi$ disjoint, and the orbit is $G_0[\Psi](x_{\Gamma,\Sigma})$.

Recall (3.8) and the corresponding decomposition $G_c = KA_cK$ of the compact real form of G. Now

$$X = G_c(x_0) = KA_cK(x_0) = KA_c(x_0) = K\cdot S[\Psi].$$

If $x \in X$, now $x = k(x')$ where $k \in K$ and $x' \in S[\Psi]$. We have disjoint Γ, $\Sigma \subset \Psi$ such that $G_0[\Psi](x') = G_0[\Psi](x_{\Gamma,\Sigma})$. Now $G_0(x) = G_0(x') = G_0\cdot G_0[\Psi](x') = G_0\cdot G_0[\Psi](x_{\Gamma,\Sigma}) = G_0(x_{\Gamma,\Sigma})$. This proves the first assertion of the Orbit Structure Theorem.

We now prove that an orbit $G_0(x_{\Gamma,\Sigma})$ is open in X if, and only if, $\Gamma = \phi$ empty set. First suppose $G_0(x_{\Gamma,\Sigma})$ open in X. Then $G_0(x_{\Gamma,\Sigma}) \cap S[\Psi]$ is an open $G_0[\Psi]$-invariant subset of $S[\Psi]$, hence a union of sets $G_0[\Psi](x_{\Gamma',\Sigma'})$ that are open in $S[\Psi]$. In particular $G_0[\Psi](x_{\Gamma,\Sigma})$ is open in $S[\Psi]$; thus each of its factors along the $S[\psi]$, $\psi \in \Psi$, is a hemisphere; that proves $\Gamma = \phi$ empty. Conversely consider an orbit $G_0(x_{\phi,\Sigma})$. The isotropy subgroup of G at $x_{\phi,\Sigma}$ is $\mathrm{ad}(c_\Sigma^2)P$, so G_0 has isotropy $G_0 \cap \mathrm{ad}(c_\Sigma^2)P$ there. That isotropy subgroup of G_0 has Lie algebra $\mathfrak{g}_0 \cap \mathrm{ad}(c_\Sigma^2)\mathfrak{p}$ which contains $\mathfrak{t} = \mathrm{ad}(c_\Sigma^2)\mathfrak{t}$ and is stable under the complex conjugation τ of $\mathfrak{g}$ over $\mathfrak{g}_0$. From the latter, $\mathfrak{g}_0 \cap \mathrm{ad}(c_\Sigma^2)\mathfrak{p} = \mathfrak{g}_0 \cap [\mathrm{ad}(c_\Sigma^2)\mathfrak{p} \cap \tau\, \mathrm{ad}(c_\Sigma^2)\mathfrak{p}]$, real form of

$$\mathrm{ad}(c_\Sigma^2)\mathfrak{p} \cap \tau\, \mathrm{ad}(c_\Sigma^2)\mathfrak{p} = \mathfrak{t}^C + \sum_{E\cap\tau^*E} \mathfrak{g}^\varphi \tag{7.3a}$$

where E is the set of all $\mathfrak{t}^C$-roots φ such that $\mathfrak{g}^\varphi \subset \mathrm{ad}(c_\Sigma^2)\mathfrak{p}$. As $\tau^*\varphi = -\varphi$ for every root φ, this says

$$\mathrm{ad}(c_\Sigma^2)\mathfrak{p} \cap \tau\, \mathrm{ad}(c_\Sigma^2)\mathfrak{p} = (\mathrm{ad}(c_\Sigma^2)\mathfrak{p})^r = \mathrm{ad}(c_\Sigma^2)\mathfrak{p}^r = \mathrm{ad}(c_\Sigma^2)\mathfrak{k}^C. \tag{7.3b}$$

Thus $\dim_R G_0(x_{\phi,\Sigma}) = \dim_R \mathfrak{g}_0 - \dim_R \mathfrak{g}_0 \cap \mathrm{ad}(c_\Sigma^2)\mathfrak{p}$
$= \dim_R \mathfrak{g}_0 - \dim_C \mathrm{ad}(c_\Sigma^2)\mathfrak{k}^C = \dim_R \mathfrak{g}_0 - \dim_R \mathfrak{k} = \dim_R X$, proving $G_0(x_{\phi,\Sigma})$ open in X.

Next, we prove that open orbits $G_0(x_{\phi,\Sigma'}) = G_0(x_{\phi,\Sigma})$ if, and only if, $|\Sigma' \cap \Delta_t| = |\Sigma \cap \Delta_t|$ for $1 \leqq t \leqq q$. For that we may, and do, assume G simple. Then if $|\Sigma'| = |\Sigma|$, the Restricted Root Theorem provides a Weyl group element $\mathrm{ad}(k)|_{\mathfrak{t}^c}$, $k \in K$, that carries Σ' to Σ; so $G_0(x_{\phi,\Sigma}) = G_0(x_{\phi,ad(k)^*\Sigma'}) = G_0k(x_{\phi,\Sigma'}) = G_0(x_{\phi,\Sigma'})$. Conversely suppose that $G_0(x_{\phi,\Sigma})$

$= G_0(x_{\Phi,\Sigma'})$. Then some $g \in G_0$ sends $x_{\Phi,\Sigma}$ to $x_{\Phi,\Sigma'}$. Replacing g by an element of $g \cdot [G_0 \cap \mathrm{ad}(c_{\Sigma'}^2)P]$ if necessary, we may also assume that $\mathrm{ad}(g)$ preserves $\mathrm{ad}(c_{\Sigma'}^2)\mathfrak{t} = \mathfrak{t}$. Now the computation (7.3) says

$$\mathrm{ad}(g)\ \mathrm{ad}(c_\Sigma^2)\mathfrak{k}^C = \mathrm{ad}(c_{\Sigma'}^2)\mathfrak{k}^C, \qquad \mathrm{ad}(g)\mathfrak{t} = \mathfrak{t}.$$

In other words, we have Weyl group elements $w_1, w_2 \in W_K$ such that

$$\mathrm{ad}(c_{\Sigma'}^2)|_{\mathfrak{t}} = w_1 \cdot \mathrm{ad}(c_\Sigma^2)|_{\mathfrak{t}} \cdot w_2.$$

Let z denote the central element (2.17) of $\mathfrak{k}$ that gives the almost complex structure of X_0 and X. If $\Phi \subset \Psi$, then $\mathrm{ad}(c_\Phi^2)z - z = \sum_\Phi i h_\varphi$, so $w_1(z) = z = w_2(z)$ says

$$-\sum_{\psi \in \Sigma'} h_\psi = i\{\mathrm{ad}(c_{\Sigma'}^2)z - z\} = i\{w_1 \cdot \mathrm{ad}(c_\Sigma^2) \cdot w_2 z - z\}$$
$$= i\{w_1 \cdot \mathrm{ad}(c_\Sigma^2)z - w_1 z\} = w_1(-\sum_{\psi \in \Sigma} h_\psi).$$

Now (6.12) and Lemma 6.10 say that the parabolic subgroups $N_{\Sigma',0}$ and $N_{\Sigma,0}$ of G_0 are conjugate, and then the Boundary Group Theorem ensures $|\Sigma'| = |\Sigma|$.

Note that we have proved assertion (iii) on open orbits in the Orbit Structure Theorem.

We return for a moment to the $G_0[\Psi]$-orbit structure of $S[\Psi]$. The equator of $S[\Psi]$ had descriptions $G_0[\Psi](c_\psi x_0) = G_0[\Psi]\,(c_\psi^3 x_0)$. If $\Gamma, \Sigma \subset \Psi$, it follows that

$$x_{\Gamma,\Sigma} = c_{\Gamma-\Sigma} \cdot c_{\Gamma\cap\Sigma}^3 \cdot c_{\Sigma-\Gamma}^2 x_0 \in G_0[\Psi]\,(x_{\Gamma,\Sigma-\Gamma}). \tag{7.4a}$$

In other words,

$$G_0[\Psi](x_{\Gamma,\Sigma}) = G_0[\Psi](x_{\Gamma,\Sigma-\Gamma}). \tag{7.4b}$$

If $\Gamma, \Sigma, \Gamma', \Sigma' \subset \Psi$, it follows that $G_0[\Psi](x_{\Gamma',\Sigma'})$ is in the closure of $G_0[\Psi](x_{\Gamma,\Sigma})$ if, and only if,

(i) $\Gamma \subset \Gamma'$

(ii) $(\Sigma - \Gamma) \subset (\Sigma' \cup \Gamma')$. and

(iii) $\{\Psi - (\Sigma \cup \Gamma)\} \subset \{[\Psi - (\Sigma' \cup \Gamma')] \cup \Gamma'\}$.

In other words,

$G_0[\Psi]\,(x_{\Gamma',\Sigma'})$ is in the closure of $G_0[\Psi]\,(x_{\Gamma,\Sigma})$ if and only if

$$(\Sigma' - \Gamma') \subset (\Sigma - \Gamma) \quad \text{and} \quad (\Sigma \cup \Gamma) \subset (\Sigma' \cup \Gamma'). \tag{7.5}$$

Now consider an open orbit $G_0(x_{\phi,\Sigma})$. Its intersection with $S[\Psi]$ is $G_0[\Psi]$ invariant and open in $S[\Psi]$, hence is the union of all sets $G_0[\Psi](x_{\phi,B})$ such that $G_0(x_{\phi,B}) = G_0(x_{\phi,\Sigma})$. Thus

$$G_0(x_{\phi,\Sigma}) \cap S[\Psi] = \bigcup_{\mathscr{B}} G_0[\Psi](x_{\phi,B}) \tag{7.6a}$$

where

$$\mathscr{B} = \{B \subset \Psi : |B \cap \Delta_t| = |\Sigma \cap \Delta_t| \text{ for } 1 \leqq t \leqq q\}. \tag{7.6b}$$

Now an orbit $G_0(x_{\Gamma',\Sigma'})$ is in the closure of $G_0(x_{\phi,\Sigma})$ if and only if $x_{\Gamma',\Sigma'}$ is in that closure. Thus (7.5) with $\Gamma = \phi$ gives us

$$\begin{aligned} G_0(x_{\Gamma',\Sigma'}) \quad &\text{is in the closure of} \quad G_0(x_{\phi,\Sigma}) \quad \text{iff some} \\ &B \in \mathscr{B} \quad \text{has} \quad (\Sigma' - \Gamma') \subset B \subset (\Sigma' \cup \Gamma'). \end{aligned} \tag{7.6c}$$

Now we reformulate (7.6) as

$$\begin{aligned} G_0(x_{\Gamma',\Sigma'}) \quad &\text{is in the closure of} \quad G_0(x_{\phi,\Sigma}) \quad \text{iff, for} \quad 1 \leqq t \leqq q, \\ &|(\Sigma' - \Gamma') \cap \Delta_t| \leqq |\Sigma \cap \Delta_t| \leqq |(\Sigma' \cup \Gamma') \cap \Delta_t|. \end{aligned} \tag{7.7}$$

Now consider arbitrary orbits $G_0(x_{\Gamma,\Sigma})$ and $G_0(x_{\Gamma',\Sigma'})$. From (7.7), G_0 $(x_{\Gamma,\Sigma})$ is in the closure of an open orbit $G_0(x_{\phi,\Phi})$ if and only if

$$|(\Sigma - \Gamma) \cap \Delta_t| \leqq |\Phi \cap \Delta_t| \leqq |(\Sigma \cup \Gamma) \cap \Delta_t|.$$

In particular

$$G_0(x_{\Gamma,\Sigma}) \quad \text{is in the closures of} \quad G_0(x_{\phi,\Sigma-\Gamma}) \quad \text{and} \quad G_0(x_{\phi,\Sigma\cup\Gamma}). \tag{7.8a}$$

Again using (7.7),

$$\begin{aligned} &\text{if } \ G_0(x_{\Gamma',\Sigma'}) \ \text{ is in the closure of } G_0(x_{\phi,\Sigma-\Gamma}), \\ &\text{then} \quad |(\Sigma' - \Gamma') \cap \Delta_t| \leqq |(\Sigma - \Gamma) \cap \Delta_t| \ \text{ for } \ 1 \leqq t \leqq q, \end{aligned} \tag{7.8b}$$

and

$$\begin{aligned} &\text{if} \quad G_0(x_{\Gamma',\Sigma'}) \quad \text{is in the closure of} \quad G_0(x_{\phi,\Sigma\cup\Gamma}) \\ &\text{then} \quad |(\Sigma \cup \Gamma) \cap \Delta_t| \leqq |(\Sigma' \cup \Gamma') \cap \Delta_t| \quad \text{for} \quad 1 \leqq t \leqq q. \end{aligned} \tag{7.8c}$$

If $G_0(x_{\Gamma',\Sigma'})$ is in the closure of $G_0(x_{\Gamma,\Sigma})$ then (7.8a) says that it is in the closures of $G_0(x_{\phi,\Sigma-\Gamma})$ and $G_0(x_{\phi,\Sigma\cup\Gamma})$, whence (7.8b) and (7.8c) say that it satisfies

$$|(\Sigma' - \Gamma') \cap \Delta_t| \leqq |(\Sigma - \Gamma) \cap \Delta_t| \quad \text{and}$$
$$|(\Sigma \cup \Gamma) \cap \Delta_t| \leqq |(\Sigma' \cup \Gamma') \cap \Delta_t| \quad \text{for} \quad 1 \leqq t \leqq q.$$

Conversely, if $G_0(x_{\Gamma',\Sigma'})$ satisfies those numerical conditions, then some $\mathrm{ad}(k)|_{\mathfrak{t}} \in W_K$ sends $\Sigma' - \Gamma'$ into $\Sigma - \Gamma$ and $\Sigma \cup \Gamma$ into $\Sigma' \cup \Gamma'$, so we may assume $(\Sigma' - \Gamma') \subset (\Sigma - \Gamma)$ and $(\Sigma \cup \Gamma) \subset (\Sigma' \cup \Gamma')$, whence (7.5) shows that $G_0(x_{\Gamma',\Sigma'})$ is in the closure of $G_0(x_{\Gamma,\Sigma})$. This proves the second assertion of the Orbit Structure Theorem.

Orbits $G_0(x_{\Gamma',\Sigma'}) = G_0(x_{\Gamma,\Sigma})$ if and only if each is in the closure of the other. Now that is the same as

$$|(\Sigma' - \Gamma') \cap \Delta_t| = |(\Sigma - \Gamma) \cap \Delta_t| \qquad \text{and}$$
$$|(\Sigma' \cup \Gamma') \cap \Delta_t| = |(\Sigma \cup \Gamma) \cap \Delta_t|$$

for $1 \leqq t \leqq q$, which is equivalent to the condition of the particular assertion (i). Now enumerate

$$\Psi \cap \Delta_t = \{\psi_{t,1}, \ldots, \psi_{t,r_t}\}.$$

Then the choices of (Γ,Σ) leading to distinct G_0-orbits are given by

$$\Gamma \cap \Delta_t = \{\psi_{t,1}, \ldots, \psi_{t,j}\}, \qquad 0 \leqq j \leqq r_t$$
$$\Sigma \cap \Delta_t = \{\psi_{t,j+1}, \ldots, \psi_{t,k}\}, \qquad j \leqq k \leqq r_t.$$

The number of such partitions is

$$\sum_{t=1}^{q} \sum_{j=0}^{r_t} (r_t + 1 - j) = \sum_{t=1}^{q} \{(r_t + 1)^2 - \sum_{j=0}^{r_t} j\} = \tfrac{1}{2} \sum_{t=1}^{q} (r_t + 1)(r_t + 2).$$

That proves the particular assertion (ii). Recall that we have already proved (iii). (iv) and (v) follow easily. Q.E.D.

8. Holomorphic Arc Components

The Orbit Structure Theorem of Section 7 says that the G_0-orbits on X are the $G_0(x_{\Gamma,\Sigma})$, where $x_{\Gamma,\Sigma} = c_\Gamma c_\Sigma^2 x_0$ and $\Gamma, \Sigma \subset \Psi$ are disjoint. It also gives the relative position of those orbits. Here we work out the decomposition of an arbitrary orbit $G_0(x_{\Gamma,\Sigma})$ into holomorphic arc components, due to Wolf [16]. This is done in Section 5 for the boundary orbits $G_0(x_{\Gamma,\phi})$, and the method of [16] is a rather complicated reduction to the boundary orbit case. Here we simplify the reduction, substituting some specific information about Hermitian symmetric spaces for the general theory [*16*, Chapter II] of holomorphic arc components.

Recall that the boundary components of $X_0 = G_0(x_0)$ in X are the holomorphic arc components of the topological boundary bd X_0. They are the

$$kG_{\Psi-\Gamma,0}(c_\Gamma x_0) = kG_{\Psi-\Gamma,0}(x_{\Gamma,\phi}); \qquad \Gamma \subsetneqq \Psi, \quad k \in K$$

as described in Section 5. Thus the boundary components of X_0 in X are the holomorphic arc components of G_0-orbits on bd X_0. The general idea of the reduction is to show that the holomorphic arc components of an arbitrary orbit $G_0(x_{\Gamma,\Sigma})$, Γ and Σ disjoint in Ψ, are the

$$kG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}), \qquad k \in K,$$

by comparing $G_0(x_{\Gamma,\Sigma}) = G_0(c_\Sigma^2 x_{\Gamma,\phi})$ with $G_0(x_{\Gamma,\phi})$.

Holomorphic Arc Component Theorem. Let Γ, $\Sigma \subset \Psi$ disjoint. Then the holomorphic arc components of $G_0(x_{\Gamma,\Sigma})$ are the $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$, $k \in K$. Furthermore, they have the following structure.

(1) $G_{\Psi-\Gamma}(x_{\Gamma,\Sigma}) = c_\Gamma X_{\Psi-\Gamma}$ complex totally geodesic submanifold of X that is a Hermitian symmetric space of compact type and rank $|\Psi - \Gamma|$.

(2) $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is an open $G_{\Psi-\Gamma,0}$-orbit on $G_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ and is an indefinite-Kaehler symmetric space of $G_{\Psi-\Gamma,0}$; the isotropy subgroups of $G_{\Psi-\Gamma,0}$ at $x_{\Gamma,\Sigma}$ and $x_{\Gamma,\phi}$ have complexifications isomorphic under $\mathrm{ad}(c_\Sigma^2)$.

(3) The G_0-normalizer of the holomorphic arc component $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is $N_{\Gamma,0}$, independent of Σ; the action of $N_{\Gamma,0}$ on $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ factors through the action of $G_{\Psi-\Gamma,0}$ there.

Proof. Using the definition of $x_{\Gamma,\Sigma}$, then $c_\Sigma \in G_{\Psi-\Gamma}$ (because Γ and Σ are disjoint), then the fact that c_Γ centralizes $G_{\Psi-\Gamma}$, and finally the definition of $X_{\Psi-\Gamma}$, we compute $G_{\Psi-\Gamma}(x_{\Gamma,\Sigma}) = G_{\Psi-\Gamma}(c_\Sigma^2 c_\Gamma x_0) = G_{\Psi-\Gamma}(c_\Gamma x_0) = c_\Gamma G_{\Psi-\Gamma}(x_0) = c_\Gamma X_{\Psi-\Gamma}$. That proves (1).

Apply the Orbit Structure Theorem of Section 7 to $c_\Gamma X_{\Psi-\Gamma}$. It shows that $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is an open $G_{\Psi-\Gamma,0}$ orbit there. Now (7.3) says that the complexified isotropy groups of $G_{\Psi-\Gamma,0}$ at $x_{\Gamma,\Sigma}$ and $x_{\Gamma,\phi}$ are $\mathrm{ad}(c_\Sigma^2)$-conjugate. As the boundary component $G_{\Psi-\Gamma,0}(x_{\Gamma,\phi})$ of X_0 in X is a Hermitian symmetric space of $G_{\Psi-\Gamma,0}$, the assertions of (2) on $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ follow.

We may, and do, assume G simple. Thus the normalizer $N_{\Gamma,0}$ of the boundary component $G_{\Psi-\Gamma,0}(x_{\Gamma,\phi})$ is a maximal parabolic subgroup of G_0. From its action on the boundary component,

$$N_{\Gamma,0} = G_{\Psi-\Gamma,0} \cdot Z_{\Gamma,0} \qquad \text{semidirect product,} \tag{8.1a}$$

$$Z_{\Gamma,0} \quad \text{leaving fixed every point of} \quad G_{\Psi-\Gamma,0}(x_{\Gamma,\phi}). \tag{8.1b}$$

Now by complex analyticity,

$$Z_{\Gamma,0} \quad \text{acts trivially on} \quad G_{\Psi-\Gamma}(x_{\Gamma,\phi}) = c_\Gamma X_{\Psi-\Gamma}. \tag{8.1c}$$

In particular $Z_{\Gamma,0}$ preserves $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$. Thus

$$N_{\Gamma,0} \subset \{g \in G_0 : gG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})\}.$$

If that inclusion is strict, then maximality of $N_{\Gamma,0}$ in G_0 implies G_0-stability of $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$, hence of its Zarisky closure $c_\Gamma X_{\Psi-\Gamma}$, and thus of the interior of bd $X_0 \cap c_\Gamma X_{\Psi-\Gamma}$. That interior being $G_{\Psi-\Gamma,0}(x_{\Gamma,\phi})$, its G_0-stability implies $\Gamma = \phi$, so then $G_{\Psi-\Gamma,0} = G_0 = N_{\Gamma,0}$. Now we have proved that

$$N_{\Gamma,0} = \{g \in G_0 : gG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})\}. \tag{8.2}$$

Thus (3) is proved, subject to completion of the proof that $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is a holomorphic arc component of $G_0(x_{\Gamma,\Sigma})$.

Let D be the unit disk in $\mathbf{C}$ and $f : D \to X$ a holomorphic arc in $G_0(x_{\Gamma,\Sigma})$. We will prove $f(D) \subset kG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ for some $k \in K$. For that it suffices to show that every $z \in D$ has a neighborhood U_z such that $f(U_z) \subset k_z G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ where $k_z \in K$; for then every compact connected subset of D has image in some $kG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$, and we exhaust D by an increasing sequence of compact connected subsets. Preceding f by a linear fractional transformation of D, we may assume the given $z \in D$ to be 0. Observe (8.2) that the G_0-normalizers of the $gG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ are real parabolic subgroups, hence self-normalizing in G_0, so the action of G_0 permutes them; thus we may follow f by an element of G_0 and assume $f(0) = x_{\Gamma,\Sigma}$. Now the proof that $f(D)$ is in some $kG_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ is reduced to: if $f(0) = x_{\Gamma,\Sigma}$ then $f(U) \subset G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ for some neighborhood U of 0 in D.

$f : D \to X$ is a holomorphic arc in $G_0(x_{\Gamma,\Sigma})$ such that $f(0) = x_{\Gamma,\Sigma}$. If $0 < r \leqq 1$, then

$$D_r = \{z \in \mathbf{C} : |z| < r\}, \qquad \text{so} \quad D = D_1. \tag{8.3}$$

We will check that there exist $r > 0$ and holomorphic maps

$$f_1 : D_r \to K^{\mathbf{C}} \qquad \text{and} \qquad f_2 : D_r \to N_\Gamma \tag{8.4a}$$

such that

$$f(z) = f_1(z) \cdot f_2(z) \cdot x_{\Gamma,\Sigma} \qquad \text{for} \quad z \in D_r, \tag{8.4b}$$

$$f_1(0) = 1 = f_2(0) \quad \text{and} \quad f_1(D) \subset K \cdot (K^{\mathbf{C}} \cap N_\Gamma). \tag{8.4c}$$

First recall that $g \to g(x_{\Gamma,\Sigma})$ defines a holomorphic fiber bundle $G \to X$. For r_1 sufficiently small, $f(D_{r_1})$ lies in a locally trivializing open set of X for the bundle. Thus the restriction of f lifts to a holomorphic map,

$$f' : D_{r_1} \to G \quad \text{with} \quad f'(0) = 1 \quad \text{and} \quad f(z) = f'(z) \cdot x_{\Gamma,\Sigma}.$$

Next observe $G_0 = KN_{\Gamma,0}$ as $N_{\Gamma,0}$ is parabolic in G_0. Thus $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{n}_{\Gamma,0}$, so $\mathfrak{g} = \mathfrak{k}^C + \mathfrak{n}_\Gamma$. Using canonical coordinates on G, that gives us neighborhoods U of 1 in K^C and V of 1 in N_Γ, and a nonsingular subvariety W through 1 in U, such that $W \times V$ is holomorphically equivalent to a neighborhood of 1 in G under $(w, v) \to wv$. For $r \leqq r_1$ and sufficiently small, $f'(D_r)$ lies in that neighborhood of 1 in G. Thus $z \in D_r$ gives unique factorization $f'(z) = f_1(z)f_2(z)$ with $f_1 : D_r \to W$ and $f_2 : D_r \to V$ holomorphic. Now f_1 and f_2 satisfy (8.4a), (8.4b), and $f_1(0) = 1 = f_2(0)$. But $f(D) \subset G_0(x_{\Gamma,\Sigma}) = KN_{\Gamma,0}(x_{\Gamma,\Sigma}) \subset KN_\Gamma(x_{\Gamma,\Sigma})$ implies $f'(D_r) \subset KN_\Gamma$. Thus $f_2(D_r) \subset N_\Gamma$ implies $f_1(D_r) \subset K^C \cap KN_\Gamma = K \cdot (K^C \cap N_\Gamma)$, completing the proof of (8.4c). Now (8.4) is proved.

Using (8.4) we will find a number s, $0 < s \leqq r$, such that $f(D_s) \subset G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$. Define

$$f^* : D_r \to X \quad \text{holomorphic by} \quad f^*(z) = f_1(z)f_2(z)x_{\Gamma,\phi}. \tag{8.5a}$$

From (8.4), we have

$$f^*(D_r) \subset KN_\Gamma(x_{\Gamma,\phi}) = KG_{\Psi-\Gamma}(x_{\Gamma,\phi}). \tag{8.5b}$$

As $G_{\Psi-\Gamma,0}(x_{\Gamma,\phi})$ is open in $G_{\Psi-\Gamma}(x_{\Gamma,\phi})$, and as $f^*(0) = x_{\Gamma,\phi}$, (8.5b) says that for s_1 sufficiently small $f^*(D_{s_1}) \subset KG_{\Psi-\Gamma,0}(x_{\Gamma,\phi}) = G_0(x_{\Gamma,\phi})$. Thus we have $0 < s_1 \leqq r$ such that

$$f^* : D_{s_1} \to X \quad \text{is a holomorphic arc in} \quad G_0(x_{\Gamma,\phi}). \tag{8.6a}$$

As $G_{\Psi-\Gamma,0}(x_{\Gamma,\phi})$ is a holomorphic arc component of $G_0(x_{\Gamma,\phi})$ now (8.6a) says that

$$f^*(D_{s_1}) \subset G_{\Psi-\Gamma,0}(x_{\Gamma,\phi}). \tag{8.6b}$$

Now, from (8.4c) and (8.5a), we see that

$$f_1(D_{s_1}) \subset K^C \cap N_\Gamma. \tag{8.6c}$$

Combine (8.4) with (8.6c); the result is that

$$f(D_{s_1}) \subset N_\Gamma(x_{\Gamma,\Sigma}) = G_{\Psi-\Gamma}(x_{\Gamma,\Sigma}). \tag{8.7a}$$

As $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is open in $G_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$, now for $s \leqq s_1$ small but positive, we have

$$f(D_s) \subset G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}). \tag{8.7b}$$

We have just proved that every holomorphic arc $f\colon D \to X$ in $G_0(x_{\Gamma,\Sigma})$ has image contained in $kG_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ for some $k \in K$. As the $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ are connected complex submanifolds of X by (1) and (2), it follows that they are the holomorphic arc components of $G_0(x_{\Gamma,\Sigma})$. That completes the proof of the Holomorphic Arc Component Theorem. Q.E.D.

Corollary. Let $\Gamma, \Sigma \subset \Psi$ disjoint. Then there is a K-equivariant fibration

$$\pi : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\phi})$$

whose fibers are the holomorphic arc components of $G_0(x_{\Gamma,\Sigma})$.

Proof. We combine the Holomorphic Arc Component Theorem for $G_0(x_{\Gamma,\Sigma})$ with the Boundary Flag Theorem for $G_0(x_{\Gamma,\phi})$. First we have

$$\pi : G_0(x_{\Gamma,\Sigma}) \to G_0/N_{\Gamma,0} \quad \text{by} \quad \pi(gx_{\Gamma,\Sigma}) = gN_{\Gamma,0},$$

fibration whose fibers are the holomorphic arc components of $G_0(x_{\Gamma,\Sigma})$. Then we identify the base of π with $K(x_{\Gamma,\phi})$, under $kN_{\Gamma,0} \leftrightarrow k(x_{\Gamma,\phi})$. Now π is given by

$$\pi(gx_{\Gamma,\Sigma}) = k(x_{\Gamma,\phi}) \qquad \text{where} \quad g \in kN_{\Gamma,0}, \quad k \in K.$$

Q.E.D.

9. Compact Subvarieties and Structure of an Orbit

The Holomorphic Arc Component Theorem of Section 8 describes the decomposition of an arbitrary orbit $G_0(x_{\Gamma,\Sigma}) \subset X$ as a fiber bundle over a real flag manifold $G_0/N_{\Gamma,0} = K(x_{\Gamma,\phi})$. The fibers, which are the holomorphic arc components of the orbit, are indefinite-Kaehler symmetric spaces isomorphic to the open orbit $G_{\Psi-\Gamma,0}(x_{\phi,\Sigma}) \subset X_{\Psi-\Gamma}$. Here we factor that fibration π as

$$\begin{array}{ccc} & \xrightarrow{\ \beta\ } & K(x_{\Gamma,\Sigma}) \\ G_0(x_{\Gamma,\Sigma}) & & \Big\downarrow \pi\cdot\beta^{-1} \\ & \xrightarrow[\ \pi\ ]{} & G_0/N_{\Gamma,0} = K(x_{\Gamma,\phi}) \end{array}$$

such that the restriction of β to a holomorphic arc component of $G_0(x_{\Gamma,\Sigma})$ is a holomorphic fiber bundle over a maximal compact subvariety of that arc component. The result is due to Takeuchi [*10*] for the case $\Gamma = \phi$ of open G_0-orbit, where the notion of a holomorphic arc component is irrelevant, without identification of the fiber. The general result is due to Wolf [*16*]; his identification of the β-fibers as certain bounded symmetric domains is important for applications [*17*] to the theory of unitary representations of groups locally isomorphic to G_0.

To formulate the description of $\beta : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\Sigma})$, Γ and Σ disjoint subsets of Ψ, we need some notation, part of which was established in Section 5. First, we have

$$\mathfrak{g}_{\Psi-\Gamma,0} = \mathfrak{g}_{\Psi-\Gamma} \cap \mathfrak{g}_0 \qquad \text{real form of} \quad \mathfrak{g}_{\Psi-\Gamma} \tag{9.1a}$$

stable under σ, so that

$$\mathfrak{g}_{\Psi-\Gamma,0} = \mathfrak{k}_{\Psi-\Gamma} + \mathfrak{m}_{\Psi-\Gamma,0} \qquad \text{vector space direct sum} \tag{9.1b}$$

where

$$\mathfrak{k}_{\Psi-\Gamma} = \mathfrak{k} \cap \mathfrak{g}_{\Psi-\Gamma} \qquad \text{and} \qquad \mathfrak{m}_{\Psi-\Gamma,0} = \mathfrak{m}_0 \cap \mathfrak{g}_{\Psi-\Gamma}. \tag{9.1c}$$

As $\mathrm{ad}(c_\Sigma^4)$ commutes with complex conjugation τ over $\mathfrak{g}_0$, commutes with σ, and preserves $\mathfrak{g}_{\Psi-\Gamma}$, now we define

$$\mathfrak{g}_{\Psi-\Gamma}^\Sigma = \{v \in \mathfrak{g}_{\Psi-\Gamma} : \mathrm{ad}(c_\Sigma^4)v = v\}; \tag{9.2a}$$

then

$$\mathfrak{g}_{\Psi-\Gamma,0}^\Sigma = \mathfrak{g}_0 \cap \mathfrak{g}_{\Psi-\Gamma}^\Sigma \qquad \text{is a real form of} \quad \mathfrak{g}_{\Psi-\Gamma}^\Sigma, \tag{9.2b}$$

and

$$\mathfrak{g}_{\Psi-\Gamma,0}^\Sigma = \mathfrak{k}_{\Psi-\Gamma}^\Sigma + \mathfrak{m}_{\Psi-\Gamma,0}^\Sigma \qquad \text{vector space direct sum} \tag{9.2c}$$

where

$$\mathfrak{k}_{\Psi-\Gamma}^\Sigma = \mathfrak{k} \cap \mathfrak{g}_{\Psi-\Gamma}^\Sigma \qquad \text{and} \qquad \mathfrak{m}_{\Psi-\Gamma,0}^\Sigma = \mathfrak{m}_0 \cap \mathfrak{g}_{\Psi-\Gamma}^\Sigma. \tag{9.2d}$$

Now denote the missing pieces of $\mathfrak{g}_{\Psi-\Gamma,0}$ by

$$\mathfrak{q}_{\Psi-\Gamma}^\Sigma = \{v \in \mathfrak{k}_{\Psi-\Gamma} : \mathrm{ad}(c_\Sigma^4)v = -v\}, \tag{9.3a}$$

$$\mathfrak{r}_{\Psi-\Gamma,0}^\Sigma = \{v \in \mathfrak{m}_{\Psi-\Gamma,0} : \mathrm{ad}(c_\Sigma^4)v = -v\}. \tag{9.3b}$$

That gives us symmetric decompositions under $\mathrm{ad}(c_\Sigma^4)$:

$$\mathfrak{k}_{\Psi-\Gamma} = \mathfrak{k}^\Sigma_{\Psi-\Gamma} + \mathfrak{q}^\Sigma_{\Psi-\Gamma} \quad \text{and} \quad \mathfrak{g}_{\Psi-\Gamma,0} = \mathfrak{g}^\Sigma_{\Psi-\Gamma,0} + (\mathfrak{q}^\Sigma_{\Psi-\Gamma} + \mathfrak{r}^\Sigma_{\Psi-\Gamma,0}). \tag{9.3c}$$

Also, as $\mathrm{ad}(c_\Sigma^2)$ preserves $\mathfrak{g}_{\Psi-\Gamma}$ and satisfies

$$\sigma\cdot\mathrm{ad}(c_\Sigma^2)\cdot\sigma^{-1} = \mathrm{ad}(c_\Sigma^{-2}) = \tau\cdot\mathrm{ad}(c_\Sigma^2)\cdot\tau^{-1},$$

we have

$$\mathrm{ad}(c_\Sigma^2)\mathfrak{q}^\Sigma_{\Psi-\Gamma} = i\mathfrak{r}^\Sigma_{\Psi-\Gamma,0} \quad \text{and} \quad \mathrm{ad}(c_\Sigma^2)\mathfrak{r}^\Sigma_{\Psi-\Gamma,0} = i\mathfrak{q}^\Sigma_{\Psi-\Gamma}. \tag{9.3d}$$

Finally, we denote

$$K_{\Psi-\Gamma}: \quad \text{analytic subgroup of } K \text{ for } \mathfrak{k}_{\Psi-\Gamma}, \tag{9.4a}$$

$$K^\Sigma_{\Psi-\Gamma}: \quad \text{analytic subgroup of } K \text{ for } \mathfrak{k}^\Sigma_{\Psi-\Gamma}, \tag{9.4b}$$

$$G_{\Psi-\Gamma,0}: \quad \text{analytic subgroup of } G_0 \text{ for } \mathfrak{g}_{\Psi-\Gamma,0}, \tag{9.4c}$$

$$G^\Sigma_{\Psi-\Gamma,0}: \quad \text{analytic subgroup of } G_0 \text{ for } \mathfrak{g}^\Sigma_{\Psi-\Gamma,0}. \tag{9.4d}$$

We collect some information on the groups (9.4).

9.5. Lemma. $K_{\Psi-\Gamma} = K \cap G_{\Psi-\Gamma,0}$ and is a maximal compact subgroup of $G_{\Psi-\Gamma,0}$.

Proof. $G_{\Psi-\Gamma,0}(x_0) \cong G_{\Psi-\Gamma,0}/(K \cap G_{\Psi-\Gamma,0})$ is the Hermitian symmetric space $X_{\Psi-\Gamma,0}$ of noncompact type. Thus $K \cap G_{\Psi-\Gamma,0}$ is connected and is a maximal compact subgroup of $G_{\Psi-\Gamma,0}$. Now $K_{\Psi-\Gamma} = K \cap G_{\Psi-\Gamma,0}$ because they are connected and they have the same Lie algebra. Q.E.D.

9.6. Lemma. $K^\Sigma_{\Psi-\Gamma} = K \cap G^\Sigma_{\Psi-\Gamma,0}$ and is a maximal compact subgroup of $G^\Sigma_{\Psi-\Gamma,0}$. Both $K^\Sigma_{\Psi-\Gamma}$ and $G^\Sigma_{\Psi-\Gamma,0}$ are stable under $\mathrm{ad}(c_\Gamma c_\Sigma^2)$. $K^\Sigma_{\Psi-\Gamma}$ is the isotropy group of $G^\Sigma_{\Psi-\Gamma,0}$ at x_0, and also at $x_{\Gamma,\Sigma}$. In particular

$$G^\Sigma_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = c_\Gamma c_\Sigma^2 G^\Sigma_{\Psi-\Gamma,0}(x_0) \cong G^\Sigma_{\Psi-\Gamma,0}/K^\Sigma_{\Psi-\Gamma}. \tag{9.7a}$$

Further

$$G^\Sigma_{\Psi-\Gamma,0}(x_0) = G_{\Psi-(\Gamma\cup\Sigma),0}(x_0) \times G^\Sigma_{\Sigma,0}(x_0), \tag{9.7b}$$

totally geodesic Hermitian symmetric subspace of noncompact type and rank $|\Psi - \Gamma|$ in X_0. Thus

$$G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = G_{\Psi-(\Gamma\cup\Sigma),0}(x_{\Gamma,\Sigma}) \times G^{\Sigma}_{\Sigma,0}(x_{\Gamma,\Sigma}), \tag{9.7c}$$

totally geodesic Hermitian symmetric subspace of noncompact type and rank $|\Psi - \Gamma|$ in $c_\Gamma c_\Sigma^2 X_0$.

Proof. In the Lemma following (5.17), we replace Ψ by $\Psi - \Gamma$ and Γ by Σ, thus obtaining (9.7b). That proves the isotropy subgroup $K \cap G^{\Sigma}_{\Psi-\Gamma,0}$ of $G^{\Sigma}_{\Psi-\Gamma,0}$ at x_0 to be connected and to be a maximal compact subgroup; now $K^{\Sigma}_{\Psi-\Gamma} = K \cap G^{\Sigma}_{\Psi-\Gamma,0}$ and is a maximal compact subgroup of $G^{\Sigma}_{\Psi-\Gamma,0}$.

As c_Γ centralizes $G_{\Psi-\Gamma}$, both $K^{\Sigma}_{\Psi-\Gamma}$ and $G^{\Sigma}_{\Psi-\Gamma,0}$ are trivially stable under $\mathrm{ad}(c_\Gamma)$.

As $c_\Sigma \in G_{\Psi-\Gamma}$, the latter is stable under $\mathrm{ad}(c_\Sigma)$. Thus $\mathfrak{g}_{\Psi-\Gamma}$ is stable under $\mathrm{ad}(c_\Sigma)$, hence also under $\mathrm{ad}(c_\Sigma^2)$, so $\mathfrak{g}^{\Sigma}_{\Psi-\Gamma}$ is $\mathrm{ad}(c_\Sigma^2)$-stable. On $\mathfrak{g}^{\Sigma}_{\Psi-\Gamma}$, $\mathrm{ad}(c_\Sigma^2)$ is equal to its own inverse, so

$$\mathrm{ad}(c_\Sigma^2) \text{ commutes with } \sigma \text{ and with } \tau \text{ on } \mathfrak{g}^{\Sigma}_{\Psi-\Gamma}.$$

That proves $\mathrm{ad}(c_\Sigma^2)$-stability of both $\mathfrak{k}^{\Sigma}_{\Psi-\Gamma}$ and $\mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0}$, hence of $K^{\Sigma}_{\Psi-\Gamma}$ and $G^{\Sigma}_{\Psi-\Gamma,0}$.

Now $K^{\Sigma}_{\Psi-\Gamma}$ and $G^{\Sigma}_{\Psi-\Gamma,0}$ are stable under $\mathrm{ad}(c_\Gamma)$ and $\mathrm{ad}(c_\Sigma^2)$, hence under $\mathrm{ad}(c_\Gamma c_\Sigma^2)$. As $K^{\Sigma}_{\Psi-\Gamma}$ was seen to be the isotropy subgroup of $G^{\Sigma}_{\Psi-\Gamma,0}$ at x_0, it also is the isotropy subgroup at $c_\Gamma c_\Sigma^2 x_0 = x_{\Gamma,\Sigma}$. (9.7a) follows. Now (9.7c) follows from (9.7a) and (9.7b). Q.E.D.

9.8. Lemma. $K^{\Sigma}_{\Psi-\Gamma}$ is the isotropy subgroup of $K_{\Psi-\Gamma}$ at $x_{\Gamma,\Sigma}$, and the orbit $K_{\Psi-\Gamma}(x_{\Gamma,\Sigma}) \cong K_{\Psi-\Gamma}/K^{\Sigma}_{\Psi-\Gamma}$ is a totally geodesic Hermitian symmetric subspace of compact type in X.

Proof. In the part of the Boundary Flag Theorem of Section 6, concerning the base space of the fibration (6.19), we replace Ψ by $\Psi - \Gamma$ and Γ by Σ, and then apply c_Γ; that shows $K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ to be a totally geodesic Hermitian symmetric subspace of compact type in X. Thus the isotropy subgroup of $K_{\Psi-\Gamma}$ at $x_{\Gamma,\Sigma}$ is connected and has Lie algebra equal to

$$\begin{aligned} &\{v \in \mathfrak{k}_{\Psi-\Gamma} : \mathrm{ad}(c_\Gamma c_\Sigma^2)\cdot\sigma\cdot\mathrm{ad}(c_\Gamma c_\Sigma^2)^{-1} v = v\} \\ &= \{v \in \mathfrak{k}_{\Psi-\Gamma} : \mathrm{ad}(c_\Gamma^2)\mathrm{ad}(c_\Sigma^4)\sigma v = v\} \\ &= \{v \in \mathfrak{k}_{\Psi-\Gamma} : \mathrm{ad}(c_\Sigma^4) v = v\} = \mathfrak{k}^{\Sigma}_{\Psi-\Gamma}. \end{aligned}$$

Thus $K^{\Sigma}_{\Psi-\Gamma}$ is the isotropy subgroup of $K_{\Psi-\Gamma}$ at $x_{\Gamma,\Sigma}$. Q.E.D.

9.9. Lemma. The isotropy subgroup of $G_{\Psi-\Gamma,0}$ at $x_{\Gamma,\Sigma}$ is the analytic subgroup with Lie algebra $\mathfrak{k}^{\Sigma}_{\Psi-\Gamma} + \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0} = \mathrm{ad}(c_\Gamma c^2_\Sigma)\mathfrak{k}^{C}_{\Psi-\Gamma} \cap \mathfrak{g}_{\Psi-\Gamma,0}$.

Proof. $G_{\Psi-\Gamma,0}$ has isotropy subgroup $K_{\Psi-\Gamma}$ at $c_\Gamma x_0 = x_{\Gamma,\phi}$. Statement (2) of the Holomorphic Arc Component Theorem now says that the isotropy subgroup of $G_{\Psi-\Gamma,0}$ at $x_{\Gamma,\Sigma}$ has Lie algebra

$$\begin{aligned}
&\mathrm{ad}(c^2_\Sigma)\mathfrak{k}^{C}_{\Psi-\Gamma} \cap \mathfrak{g}_{\Psi-\Gamma,0} \\
&\quad = \{\mathrm{ad}(c^2_\Sigma)\mathfrak{k}^{\Sigma C}_{\Psi-\Gamma} + \mathrm{ad}(c^2_\Sigma)\mathfrak{q}^{\Sigma C}_{\Psi-\Gamma}\} \cap \mathfrak{g}_{\Psi-\Gamma,0} \\
&\quad = \{\mathfrak{k}^{\Sigma C}_{\Psi-\Gamma} + \mathfrak{r}^{\Sigma C}_{\Psi-\Gamma,0}\} \cap \mathfrak{g}_{\Psi-\Gamma,0} = \mathfrak{k}^{\Sigma}_{\Psi-\Gamma} + \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}.
\end{aligned}$$

That algebra being σ-stable, there is a deformation retraction of $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ onto $K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$, which is simply connected by Lemma 9.8. Thus the isotropy subgroup of $G_{\Psi-\Gamma,0}$ at $x_{\Gamma,\Sigma}$ is connected; so it must be the analytic subgroup for $\mathfrak{k}^{\Sigma}_{\Psi-\Gamma} + \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}$. Q.E.D.

9.10. Lemma. Define $f: K_{\Psi-\Gamma} \times \mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0} \times \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0} \to G_{\Psi-\Gamma,0}$ by $f(k, v_1, v_2) = k\cdot\exp(v_1)\cdot\exp(v_2)$. Then f is a diffeomorphism onto $G_{\Psi-\Gamma,0}$.

Proof. $X_{\Psi-\Gamma,0} = G_{\Psi-\Gamma,0}(x_0)$ is a Riemannian symmetric space of noncompact type, and hence is a complete simply connected Riemannian manifold of nonpositive sectional curvature. $X^{\Sigma}_{\Psi-\Gamma,0} = G^{\Sigma}_{\Psi-\Gamma,0}(x_0)$ is a complete totally geodesic submanifold. Let $\mathfrak{N} \to X^{\Sigma}_{\Psi-\Gamma,0}$ be the normal bundle in $X_{\Psi-\Gamma,0}$; the fiber $\mathfrak{N}_x$ over $x \in X^{\Sigma}_{\Psi-\Gamma,0}$ consists of all real tangent vectors to $X_{\Psi-\Gamma,0}$ at x that are orthogonal to the tangent space of $X^{\Sigma}_{\Psi-\Gamma,0}$ there. Then as in the classical Cartan–Hadamard Theorem,

$$\mathrm{Exp}\colon \mathfrak{N} \to X_{\Psi-\Gamma,0} \text{ is a diffeomorphism} \tag{9.11a}$$

where Exp is defined from the Riemannian exponential map:

$$\text{if } \xi \in \mathfrak{N}_x, \qquad \text{then } \mathrm{Exp}(\xi) = \exp_x(\xi) \in X_{\Psi-\Gamma,0}. \tag{9.11b}$$

We will translate (9.11) into the statement of the Lemma.

Identifying the infinitesimal transvections of $X_{\Psi-\Gamma,0}$ at x_0 (i.e., the elements of $\mathfrak{m}_{\Psi-\Gamma,0}$) with the real tangent vectors which are their values at x_0, we have $\mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0} \perp \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}$ because they are orthogonal under the Killing form of $\mathfrak{g}_0$. Thus

$$\mathfrak{N}_{x_0} = \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}. \tag{9.12a}$$

It follows that

$$\operatorname{Exp} \mathfrak{N}_{x_0} = \{\exp(v)\cdot x_0 : v \in \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}\}. \tag{9.12b}$$

As $G^{\Sigma}_{\Psi-\Gamma,0}$ acts by isometries on $X_{\Psi-\Gamma,0}$ and that action preserves $X^{\Sigma}_{\Psi-\Gamma,0}$, now

$$\mathfrak{N}_{gx_0} = g_*\mathfrak{N}_{x_0} \quad \text{and} \quad \operatorname{Exp} g_*\mathfrak{N}_{x_0} = g\cdot \operatorname{Exp} \mathfrak{N}_{x_0} \quad \text{for} \quad g \in G^{\Sigma}_{\Psi-\Gamma,0}.$$

In other words,

$$\operatorname{Exp} \mathfrak{N}_{gx_0} = g\cdot\operatorname{Exp} \ \mathfrak{N}_{x_0} \qquad \text{for} \quad g \in G^{\Sigma}_{\Psi-\Gamma,0}. \tag{9.12c}$$

Every $x \in X^{\Sigma}_{\Psi-\Gamma,0}$ has unique expression $x = gx_0$ where $g \in \exp(\mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0}) \subset G^{\Sigma}_{\Psi-\Gamma,0}$. Combining this fact with (9.11) and (9.12), we see that

$$F'(u,v) = \exp(u)\cdot\exp(v)\cdot x_0$$

defines a diffeomorphism F' of $\mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0} \times \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}$ onto $X_{\Psi-\Gamma,0}$. Replacing the isometries $\exp(u)\cdot\exp(v)$ by their inverses, we obtain another map

$$F : \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0} \times \mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0} \to X_{\Psi-\Gamma,0} \qquad \text{diffeomorphism} \tag{9.13a}$$

defined by

$$F(v,u) = \exp(-v)\cdot\exp(-u)\cdot x_0. \tag{9.13b}$$

Note that $\exp(-v)\cdot\exp(-u)\cdot K_{\Psi-\Gamma}$ is the set of all elements of $G_{\Psi-\Gamma,0}$ that send x_0 to $F(v,u)$. Thus F "lifts" to a map

$$f' : \mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0} \times \mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0} \times K_{\Psi-\Gamma} \to G_{\Psi-\Gamma,0} \quad \text{diffeomorphism} \tag{9.14a}$$

defined by

$$f'(v,u,k) = \exp(-v)\cdot\exp(-u)\cdot k^{-1}. \tag{9.14b}$$

The Lemma follows on observing that

$$f(k,v_1v_2) = f'(v_2,v_1,k)^{-1}.$$

Q.E.D.

We remark that our proof of Lemma 9.10 is based on the same general idea as the corresponding proofs in [*10*] and [*16*].

Orbit Fibration Theorem. Let Γ and Σ be disjoint subsets of Ψ. If

$g \in G_{\Psi-\Gamma,0}$, use Lemma 9.10 to factor

$$g = g_K \cdot g_M \cdot g_R \tag{9.15a}$$

where

$$g_K \in K_{\Psi-\Gamma}, \qquad g_M \in \exp(\mathfrak{m}^{\Sigma}_{\Psi-\Gamma,0}), \qquad g_R \in \exp(\mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0}). \tag{9.15b}$$

If $k \in K$, define

$$\beta_k : k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) \to k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma}) \quad \text{by} \quad \beta_k(kgx_{\Gamma,\Sigma}) = kg_K x_{\Gamma,\Sigma}. \tag{9.16a}$$

Further define

$$\beta : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\Sigma}) \quad \text{by} \quad \beta|_{k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})} = \beta_k. \tag{9.16b}$$

1. $\beta_k : k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) \to k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ is a well defined holomorphic fiber bundle with

(1a) structure group: the connected reductive complex Lie group $K^{\Sigma\mathbb{C}}_{\Psi-\Gamma}$;

(1b) total space: the holomorphic arc component $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ of $G_0(x_{\Gamma,\Sigma})$ that contains $k(x_{\Gamma,\Sigma})$;

(1c) base space: the maximal compact subvariety $k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ of the total space, which is a totally geodesic Hermitian symmetric subspace of compact type in X; and

(1d) fiber over $kk'(x_{\Gamma,\Sigma})$, $k' \in K_{\Psi-\Gamma}$: the totally geodesic Hermitian symmetric subspace

$$\begin{aligned} kk' \cdot G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) &= kk' c_\Gamma c_\Sigma^2 \cdot G^{\Sigma}_{\Psi-\Gamma,0}(x_0) \\ &= kk' c_\Gamma c_\Sigma^2 \{G_{\Psi-(\Gamma\cup\Sigma),0}(x_0) \times G^{\Sigma}_{\Sigma,0}(x_0)\} \end{aligned}$$

of noncompact type and rank $|\Psi - \Gamma|$ in $kk' c_\Gamma c_\Sigma^2 X_0$.

2. Let $\nu_k : k_* \mathfrak{M}^{\Sigma}_{\Psi-\Gamma} \to k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ denote the holomorphic normal bundle in $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$. Then ν_k is the homogeneous holomorphic vector bundle over the compact Hermitian symmetric coset space $K_{\Psi-\Gamma}/K^{\Sigma}_{\Psi-\Gamma} \cong k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ defined by the representation of $K^{\Sigma}_{\Psi-\Gamma}$ on the holomorphic tangent space of $G^{\Sigma}_{\Psi-\Gamma,0}/K^{\Sigma}_{\Psi-\Gamma} \cong k \cdot G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$. The bundle β_k is holomorphically fiber-equivalent to the relatively compact tubular neighborhood of the zero-section of ν_k, whose intersection with the fiber over $kk'(x_{\Gamma,\Sigma})$, $k' \in K_{\Psi-\Gamma}$, is the image of the Harish-Chandra embedding of $G^{\Sigma}_{\Psi-\Gamma,0}/K^{\Sigma}_{\Psi-\Gamma}$ as a bounded symmetric domain in $\mathfrak{m}^{\Sigma\mathbb{C}}_{\Psi-\Gamma} \cap \mathrm{ad}(c_\Gamma c_\Sigma^2)\mathfrak{m}^+$.

3. $\beta : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\Sigma})$ is a well-defined real analytic fiber bundle whose fibers and structure group coincide with those of its restrictions β_k to the holomorphic arc components $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ of $G_0(x_{\Gamma,\Sigma})$. The bundle β is real analytically fiber-equivalent to the homorgeneous complex vector bundle over $K(x_{\Gamma,\Sigma})$, whose fiber is the holomorphic tangent space $\mathfrak{m}^{\Sigma C}_{\Psi-\Gamma} \cap \mathrm{ad}(c_\Gamma c_\Sigma^2)\mathfrak{m}^+$ to $G^\Sigma_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$.

Proof. The factorization (9.15) of $G_{\Psi-\Gamma,0}$ is smooth and well defined by Lemma 9.10. Fix $k \in K$. Now the map β_k of (9.16a) is smooth and well defined. Its total space $k \cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ is the holomorphic arc component of $G_0(x_{\Gamma,\Sigma})$ through $k(x_{\Gamma,\Sigma})$, by the Holomorphic Arc Component Theorem of Section 8, and the base space $k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ is a totally geodesic Hermitian symmetric subspace of compact type in X by Lemma 9.8. As $k \cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma}) \cong K_{\Psi-\Gamma}/K^\Sigma_{\Psi-\Gamma}$ by Lemma 9.8, now β_k is a real analytic fiber bundle with structure group $K^\Sigma_{\Psi-\Gamma}$. We extend the structure group to $K^{\Sigma C}_{\Psi-\Gamma}$.

Let $k' \in K_{\Psi-\Gamma}$. Then $\beta_k^{-1}(kk'x_{\Gamma,\Sigma})$ consists of all $kgx_{\Gamma,\Sigma}$ such that $g \in G_{\Psi-\Gamma,0}$ and $kg_Kx_{\Gamma,\Sigma} = kk'x_{\Gamma,\Sigma}$. The latter condition is $g_K(x_{\Gamma,\Sigma}) = k'(x_{\Gamma,\Sigma})$, which just says $g_K \in k'K^\Sigma_{\Psi-\Gamma}$ by Lemma 9.8. Thus the fiber over $kk'(x_{\Gamma,\Sigma})$ consists of all $kk'k_1g_Mg_Rx_{\Gamma,\Sigma}$ such that $k_1 \in K^\Sigma_{\Psi-\Gamma}$, $g_M \in \exp(\mathfrak{m}^\Sigma_{\Psi-\Gamma,0})$, and $g_R \in \exp(\mathfrak{r}^\Sigma_{\Psi-\Gamma,0})$. As $g_R(x_{\Gamma,\Sigma}) = x_{\Gamma,\Sigma}$ by Lemma 9.9, and as $k_1g_Mx_{\Gamma,\Sigma} = \mathrm{ad}(k_1)g_M \cdot x_{\Gamma,\Sigma} \in \exp(\mathfrak{m}^\Sigma_{\Psi-\Gamma,0}) \cdot x_{\Gamma,\Sigma}$ by Lemma 9.9, now

$$\beta_k^{-1}(kk'x_{\Gamma,\Sigma}) = kk'\exp(\mathfrak{m}^\Sigma_{\Psi-\Gamma,0}) \cdot x_{\Gamma,\Sigma}.$$

Lemma 9.6 now says that the β_k-fiber over $kk'(x_{\Gamma,\Sigma})$ is $kk' \cdot G^\Sigma_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = kk'c_\Gamma c_\Sigma^2 G^\Sigma_{\Psi-\Gamma,0}(x_0) = kk'c_\Gamma c_\Sigma^2\{G_{\Psi-(\Gamma\cup\Sigma),0}(x_0) \times G^\Sigma_{\Sigma,0}(x_0)\}$, totally geodesic Hermitian symmetric subspace of noncompact type and rank $|\Psi - \Gamma|$ in $kk'c_\Gamma c_\Sigma^2 X_0$.

To complete the proof of (1), it now suffices to show that β_k has holomorphic $K^{\Sigma C}_{\Psi-\Gamma}$-valued transition functions. For that, it suffices to prove (2).

In order to discuss holomorphic tangent spaces, we introduce the notation

$$\mathfrak{m}^\pm_{\Psi-\Gamma} = \mathfrak{m}^C_{\Psi-\Gamma,0} \cap \mathfrak{m}^\pm, \tag{9.17a}$$

$$\mathfrak{m}^{\Sigma\pm}_{\Psi-\Gamma} = \mathfrak{m}^{\Sigma C}_{\Psi-\Gamma,0} \cap \mathfrak{m}^\pm, \tag{9.17b}$$

$$\mathfrak{r}^{\Sigma\pm}_{\Psi-\Gamma} = \mathfrak{r}^{\Sigma C}_{\Psi-\Gamma} \cap \mathfrak{m}^\pm, \tag{9.17c}$$

$$\mathfrak{q}^{\Sigma\pm}_{\Psi-\Gamma} = \mathrm{ad}(c_\Sigma^2)\mathfrak{r}^{\Sigma\pm}_{\Psi-\Gamma}. \tag{9.17d}$$

Thus we have

$$\mathfrak{m}^{\pm}_{\Psi-\Gamma} = \mathfrak{m}^{\Sigma\pm}_{\Psi-\Gamma} + \mathfrak{r}^{\Sigma\pm}_{\Psi-\Gamma} \tag{9.18a}$$

and

$$\mathrm{ad}(c_\Gamma c_\Sigma^2)\mathfrak{m}^{\pm}_{\Psi-\Gamma} = \mathrm{ad}(c_\Sigma^2)\mathfrak{m}^{\Sigma\pm}_{\Psi-\Gamma} + \mathfrak{q}^{\Sigma\pm}_{\Psi-\Gamma}. \tag{9.18b}$$

In the notation (9.17), the holomorphic tangent space to $G_{\Psi-\Gamma,0}(x_0)$ at x_0 is $\mathfrak{m}^{+}_{\Psi-\Gamma}$, and $\mathfrak{m}^{\Sigma+}_{\Psi-\Gamma}$ is the subspace that is holomorphic tangent space to $G^{\Sigma}_{\Psi-\Gamma,0}(x_0)$. Now Lemma 9.6, Lemma 9.8, and (9.18) say that, at $x_{\Gamma,\Sigma}$,

$G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ has holomorphic tangent space $\mathrm{ad}(c_\Sigma^2)\mathfrak{m}^{\Sigma+}_{\Psi-\Gamma} + \mathfrak{q}^{\Sigma+}_{\Psi-\Gamma}$, (9.19a)

$G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ has holomorphic tangent space $\mathrm{ad}(c_\Sigma^2)\mathfrak{m}^{\Sigma+}_{\Psi-\Gamma}$, (9.19b)

$K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ has holomorphic tangent space $\mathfrak{q}^{\Sigma+}_{\Psi-\Gamma}$. (9.19c)

Now the holomorphic normal space to $K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ in $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ at $x_{\Gamma,\Sigma}$ is $\mathrm{ad}(c_\Sigma^2)\mathfrak{m}^{\Sigma+}_{\Psi-\Gamma}$. That space is $K^{\Sigma C}_{\Psi-\Gamma}$-stable, the representation being that of $K^{\Sigma C}_{\Psi-\Gamma}$ on the holomorphic tangent space to $G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ at $x_{\Gamma,\Sigma}$. Now $\nu_k : k_*\mathfrak{M}^{\Sigma}_{\Psi-\Gamma} \to k\cdot K_{\Psi-\Gamma}(x_{\Gamma,\Sigma})$ is identified as stated in (2), and the total space of β_k is injected into $k_*\mathfrak{M}^{\Sigma}_{\Psi-\Gamma}$, as stated, by the Harish-Chandra embeddings of the $kk'G^{\Sigma}_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ into the $\mathrm{ad}(kk')$-images of the $\mathrm{ad}(c_\Sigma^2)\mathfrak{m}^{\Sigma+}_{\Psi-\Gamma} = \mathfrak{m}^{\Sigma C}_{\Psi-\Gamma,0} \cap \mathrm{ad}(c_\Gamma c_\Sigma^2)\mathfrak{m}^{+}$. Now (1) and (2) are proved.

We check that the map $\beta : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\Sigma})$ of (9.16b) is well-defined. In other words, given $k, k' \in K$ with $k\cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}) = k'\cdot G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$, we show that $\beta_k = \beta_{k'}$. For that, suppose

$$kg(x_{\Gamma,\Sigma}) = x = k'g'(x_{\Gamma,\Sigma}); \qquad k, k' \in K \qquad \text{and} \qquad g, g' \in G_{\Psi-\Gamma,0}.$$

Then $k^{-1}k'$ preserves the holomorphic arc component $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$ of $G_0(x_{\Gamma,\Sigma})$, i.e., $k^{-1}k' \in K \cap N_{\Gamma,0}$, so

$$k^{-1}k' = k_1k_2, \qquad k_1 \in K_{\Psi-\Gamma}, \qquad k_2 \quad \text{trivial on} \quad G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}).$$

Thus we may ignore k_2 and assume that

$$k' = kk_1 \qquad \text{with} \quad k_1 \in K_{\Psi-\Gamma}.$$

As in (9.15), we factor

$$g = g_K g_M g_R \qquad \text{and} \qquad g' = g'_K g'_M g'_R.$$

Using $g_R(x_{\Gamma,\Sigma}) = x_{\Gamma,\Sigma} = g'_R(x_{\Gamma,\Sigma})$, we now have

$$kg_Kg_Mx_{\Gamma,\Sigma} = x = kk_1g'_Kg'_Mx_{\Gamma,\Sigma}.$$

Thus $k_1g'_Kg'_M = g_Kg_Mh$ where $h \in G_{\Psi-\Gamma,0}$ leaves $x_{\Gamma,\Sigma}$ fixed. Lemma 9.9 says $h = h_Kh_R$ with $h_K \in K^{\Sigma}_{\Psi-\Gamma}$ and $h_R \in \exp(\mathfrak{r}^{\Sigma}_{\Psi-\Gamma,0})$. Now

$$(k_1g'_K)g'_M = g_Kg_Mh_Kh_R = (g_Kh_K)(h_K^{-1}g_Mh_K)h_R.$$

Uniqueness of the factorization (9.15) thus says

$$k_1g'_K = g_Kh_K, \qquad g'_M = h_K^{-1}g_Mh_K, \qquad 1 = h_R.$$

Thus we compute

$$\begin{aligned}\beta_k(x) &= kg_K(x_{\Gamma,\Sigma}) = kg_Kh_K(x_{\Gamma,\Sigma}) = kk_1g'_K(x_{\Gamma,\Sigma})\\ &= k'g'_K(x_{\Gamma,\Sigma}) = \beta_{k'}(x).\end{aligned}$$

Now β is proved well defined.

Now that β is well defined, it also is real analytic. If $x \in G_0(x_{\Gamma,\Sigma})$, then x and $\beta(x)$ are in the same holomorphic arc component of $G_0(x_{\Gamma,\Sigma})$. This proves that $\beta : G_0(x_{\Gamma,\Sigma}) \to K(x_{\Gamma,\Sigma})$ is a real analytic bundle whose fibers coincide with those of its restrictions β_k.

The structure group of β may be taken to be the isotropy subgroup of K at $x_{\Gamma,\Sigma}$. That isotropy group preserves the holomorphic arc component $G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma})$, hence is in $K \cap N_{\Gamma,0}$, so its every element has factorization

$$k = k_1k_2, \qquad k_1 \in K^{\Sigma}_{\Psi-\Gamma}, \qquad k_2 \quad \text{trivial on} \quad G_{\Psi-\Gamma,0}(x_{\Gamma,\Sigma}).$$

Now we may reduce the structure group of β to $K^{\Sigma}_{\Psi-\Gamma}$, then increase it to the structure group $K^{\Sigma\mathbb{C}}_{\Psi-\Gamma}$ of the restrictions β_k. Q.E.D.

Mostly for reference later, we write down the case $\Gamma = \phi$ of an open orbit for the Orbit Fibration Theorem.

Corollary. Every open G_0-orbit on X has a holomorphic fibration

$$\beta : G_0(x_{\phi,\Sigma}) \to K(x_{\phi,\Sigma}) \qquad \text{by} \quad \beta(gx_{\phi,\Sigma}) = g_Kx_{\phi,\Sigma} \tag{9.20a}$$

where

$$g = g_Kg_Mg_R; \qquad g_K \in K, \quad g_M \in \exp(\mathfrak{m}^{\Sigma}_{\Psi,0}), \quad g_R \in \exp(\mathfrak{r}^{\Sigma}_{\Psi,0}). \tag{9.20b}$$

This fibration has structure group, the connected reductive complex Lie group $K^{\Sigma\mathbb{C}}_{\Psi}$. The total space is the open orbit $G_0(x_{\phi,\Sigma})$; the base space is the maximal compact subvariety $K(x_{\phi,\Sigma})$ of $G_0(x_{\phi,\Sigma})$, which is a totally geodesic Hermitian symmetric subspace of compact type in X; and the fiber over $k(x_{\phi,\Sigma})$, $k \in K$, is the totally geodesic Hermitian symmetric subspace

$$k \cdot G^{\Sigma}_{\Psi,0}(x_{\phi,\Sigma}) = kc^2_{\Sigma} \cdot G^{\Sigma}_{\Psi,0}(x_0) = kc^2_{\Sigma}\{G_{\Psi-\Sigma,0}(x_0) \times G^{\Sigma}_{\Sigma,0}(x_0)\}$$

of noncompact type and equal rank $|\Psi|$ in $kc^2_{\Sigma}X_0$.

For further details on the bundles β_k and β see [*16*, pp. 1213–1223].

Part III. Examples: The Classical Domains

Hermitian symmetric spaces were first studied by É. Cartan [*4*], who classified them by means of his classification [*3*] of Riemannian symmetric spaces. See [*2*], [*12*], [*11*], and [*14*] for increasingly direct proofs of Cartan's classification. The result is that the irreducible Hermitian symmetric spaces $X = G_c/K$ of compact type are

(i) the Grassmannian $\mathbf{SU}(m+n) / [\mathbf{SU}(m+n) \cap \{\mathbf{U}(m) \times \mathbf{U}(n)\}]$ of n-dimensional linear subspaces of $\mathbf{C}^{m+n}$, $1 \leqq m \leqq n$;

(ii) the subvariety $\mathbf{Sp}(n)/\mathbf{U}(n)$ of $\mathbf{SU}(2n)/[\mathbf{SU}(2n) \cap \{\mathbf{U}(n) \times \mathbf{U}(n)\}]$ consisting of the n-dimensional linear subspaces of $\mathbf{C}^{2n}$ annihilated by a nondegenerate antisymmetric bilinear form, $n \geqq 1$;

(iii) the subvariety $\mathbf{SO}(2n)/\mathbf{U}(n)$ of that same Grassmannian consisting of the n-dimensional linear subspaces of $\mathbf{C}^{2n}$ annihilated by a nondegenerate symmetric bilinear form, $n \geqq 3$;

(iv) the nonsingular quadric hypersurface $\mathbf{SO}(n+2)/\mathbf{SO}(n) \times \mathbf{SO}(2)$ in complex projective $(n+1)$-space, $n \geqq 3$;

(v) an "exceptional" space $\mathbf{E}_7/\mathbf{E}_6 \cdot \mathbf{SO}(2)$; and

(vi) an "exceptional" space $\mathbf{E}_6/\mathbf{SO}(10) \cdot \mathbf{SO}(2)$.

In the next four sections we work out the material of Parts I and II for the four series just described. The method is: use the general theory, first to guess the matrix and then to check the guess. It's quite efficient. Without the general theory, however, most of the facts would be extremely difficult to see, much less to prove; thus the books of Hua L.-K. [*6*] and Pjateckĭi-Šapiro [*9*] remain all the more impressive.

10. Complex Grassmann Manifolds

We work out the G_0-orbit structure, the holomorphic arc component structure of an arbitrary G_0-orbit, and the orbit fibration, for the case where X is a complex Grassmannian. The reader should notice that our method is that of constant application of the general theory, and he might enjoy working out some *ad hoc* proofs of those applications. The less energetic reader should at least glance at Pjateckiĭ-Šapiro's *ad hoc* determination of the boundary components of X_0 in X for the case of the complex Grassmannian [9, Section 6].

Let X be the complex Grassmann manifold consisting of n-planes through the origin in complex number space $\mathbf{C}^{m+n}$ ($m \geqq 1$, $n \geqq 1$). If $\{v_1, \ldots, v_n\} \subset \mathbf{C}^{m+n}$ is linearly independent, then

$$v_1 \wedge \cdots \wedge v_n \in X \text{ is the linear span of } \{v_1, \ldots, v_n\}.$$

The complex general linear group

$$\mathbf{GL}(m+n, \mathbf{C}) = \{g : \mathbf{C}^{m+n} \to \mathbf{C}^{m+n} : g \text{ is linear and invertible}\}$$

acts on X by

$$g(v_1 \wedge \cdots \wedge v_n) = g(v_1) \wedge \cdots \wedge g(v_n).$$

An element $g \in \mathbf{GL}(m+n, \mathbf{C})$ acts trivially on X if, and only if, it is a scalar multiplication. Thus $\mathbf{GL}(m+n, \mathbf{C})$ induces the complex Lie transformation group

$$G = \mathbf{GL}(m+n, \mathbf{C}) \,/\, \{aI : 0 \neq a \in \mathbf{C}\}$$

of complex dimension $(m+n)^2 - 1$ on X.

We can almost normalize representing matrices for elements of G by choosing them in the complex special linear group

$$\mathbf{SL}(m+n, \mathbf{C}) = \{g \in \mathbf{GL}(m+n, \mathbf{C}) : \det g = 1\}.$$

That realizes G as the quotient,

$$G = \mathbf{SL}(m+n, \mathbf{C})/\{e^{2\pi i l/(m+n)} \cdot I : 0 \leqq l < m+n\}$$

of $\mathbf{SL}(m+n, \mathbf{C})$ by a cyclic group of order $m+n$. In particular, now, the Lie algebra $\mathfrak{g}$ of G consists of all complex $(m+n) \times (m+n)$ matrices of trace zero.

Choose a basis of $\mathbf{C}^{m+n}$ and denote it by $\{e_1, \ldots, e_{m+n}\}$. That gives us

$$x_0 = e_{m+1} \wedge \cdots \wedge e_{m+n} \in X \qquad \text{base point.}$$

The linear transformations of $\mathbf{C}^{m+n}$ will be written in matrix form relative to the basis $\{e_j\}$. Thus every linear transformation of $\mathbf{C}^{m+n}$ is represented by a complex matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where the blocks have size

$$A \text{ is } m \times m, \quad B \text{ is } m \times n, \quad C \text{ is } n \times m, \quad D \text{ is } n \times n.$$

In particular,

$$P = \{g \in G : g(x_0) = x_0\}$$

is represented by

$$\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : B = 0 \text{ and } (\det A)(\det D) = 1\right\}.$$

Thus P is a connected complex Lie subgroup of dimension $m^2 + n^2 + mn - 1$ in G, and

$$X = G/P \qquad \text{compact complex } (mn)\text{-manifold.}$$

G has maximal compact subgroup G_c which is its compact real form, given by

$$G_c \text{ is the image of the unitary group } \mathbf{U}(m+n) \text{ in } G.$$

The isotropy subgroup of G_c at x_0 is the group

$$K = G_c \cap P$$

which consists of all elements of G represented by matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \text{with} \quad A \in \mathbf{U}(m), \quad D \in \mathbf{U}(n), \quad (\det A)(\det D) = 1.$$

Now $G_c(x_0)$ is open in X because it has the same real dimension, closed in X because it is compact, so

$$X = G_c/K \qquad \text{compact presentation.}$$

This exhibits X as a Hermitian symmetric space of compact type; the symmetry at x_0 is the element of G_c represented by any

$$s = a\begin{pmatrix} -I_m & 0 \\ 0 & I_n \end{pmatrix}, \qquad (-1)^m a^{m+n} = 1.$$

Consider the Hermitian form on $\mathbf{C}^{m+n}$ given by

$$\langle u,v \rangle = -\sum_{j=1}^{m} u^j \bar{v}^j + \sum_{k=1}^{n} u^{m+k} \bar{v}^{m+k}.$$

That defines an indefinite unitary group

$$\mathbf{U}^m(m+n) = \mathbf{U}(m,n) = \{g \in \mathbf{GL}(m+n, \mathbf{C}) : \langle g(u), g(v)\rangle = \langle u,v\rangle, \text{ all } u, v \in \mathbf{C}^{m+n}\}$$

and an indefinite special unitary group

$$\mathbf{SU}^m(m+n) = \mathbf{SU}(m,n) = \mathbf{U}(m,n) \cap \mathbf{SL}(m+n, \mathbf{C}).$$

Those two groups have the same image

$$\mathbf{U}(m,n)/\{aI : |a| = 1\} = G_0 = \mathbf{SU}(m,n)/\{aI : a^{m+n} = 1\}$$

in G. That group G_0 is a real form of G, so

$$G_0^{\mathbf{C}} = G = G_c^{\mathbf{C}},$$

and it is readily checked that

$$G_0 \cap P = K \qquad \text{so} \qquad G_0/K \cong G_0(x_0) \qquad \text{open in } X.$$

This exhibits

$$X_0 = G_0(x_0) \subset X \qquad \text{open } G_0\text{-orbit}$$

as a Hermitian symmetric space of noncompact type, with the same symmetry s at x_0, inside its compact dual under Borel Embedding.

We work out the Harish-Chandra embedding of X_0 as a bounded domain in the holomorphic tangent space at x_0. In our block form matrix representations, e_j is the column vector, i.e., $(m + n) \times 1$ matrix, with all entries zero except for a 1 in the jth place. $\mathbf{SL}(m+n, \mathbf{C}) \to G$ has finite kernel, so we identify Lie algebras there; now G has Lie algebra

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \text{trace}(A) + \text{trace}(D) = 0\right\}.$$

Now

$$\mathfrak{k}^{\mathbf{C}} = \left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \text{trace}(A) + \text{trace}(D) = 0\right\}$$

and

$$\mathfrak{p} = \left\{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : \text{trace}(A) + \text{trace}(D) = 0\right\}.$$

Thus

$$\mathfrak{m}^- = \left\{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right\} \quad \text{and} \quad \mathfrak{m}^+ = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\right\}.$$

In particular,

$$Z \to \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \qquad \text{maps } (m \times n \text{ matrices}) \cong \mathfrak{m}^+.$$

Harish-Chandra's map $\xi : \mathfrak{m}^+ \to X$, given by $\xi(\tilde{Z}) = (\exp \tilde{Z})(x_0)$, now may be viewed as a map from the complex vector space of all $m \times n$ matrices. Compute

$$\exp \tilde{Z} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}.$$

Thus

$$\xi(Z) = (\exp \tilde{Z})(x_0) = v_1 \wedge \cdots \wedge v_n \quad \text{where} \quad (v_1, \ldots, v_n) = \begin{pmatrix} Z \\ I \end{pmatrix}.$$

The condition $\xi(Z) \in X_0$ is, by Witt's theorem, that $\langle \ , \ \rangle$ be positive definite on $v_1 \wedge \cdots \wedge v_n$, i.e., that the matrix

$$(\langle v_j, v_l \rangle) = I_n - {}^t Z \cdot \bar{Z}, \qquad {}^t Z = \text{transpose } (Z),$$

be positive definite, i.e., such that its complex conjugate $I_n - Z^* \cdot Z, Z^* = {}^t\bar{Z}$, be positive definite. Thus

$$\xi^{-1}(X_0) = \{m \times n \text{ matrices } Z : I_n - Z^* \cdot Z \gg 0\}.$$

For comparison with more classical work on that domain, we work out the action of G_0 on $\xi^{-1}(X_0)$. Let $g \in G_0$ be represented by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{SL}(m+n, \mathbf{C})$.

Then $g \cdot \exp \tilde{Z}$ is represented by $\begin{pmatrix} A & AZ + B \\ C & CZ + D \end{pmatrix}$, so

$$(g \cdot \exp \tilde{Z})(x_0) = w_1 \wedge \cdots \wedge w_n \qquad \text{where} \quad (w_1, \ldots, w_n) = \begin{pmatrix} AZ + B \\ CZ + D \end{pmatrix}.$$

The columns of $\begin{pmatrix} AZ + B \\ CZ + D \end{pmatrix}$ and $\begin{pmatrix} (AZ + B)(CZ + D)^{-1} \\ I \end{pmatrix}$ have the same span, the inverse there existing because $(g \cdot \exp \tilde{Z})(x_0) \in X_0$. Thus

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{implies} \quad g(Z) = (AZ + B)(CZ + D)^{-1}.$$

Finally, we note that the operator norm of the Hermann Convexity Theorem, i.e., the Banach space norm on the $m \times n$ matrices for which $\xi^{-1}(X_0)$ is the open unit ball, is given by

$$\|Z\|^2 = \max \{\text{eigenvalues of } Z^* \cdot Z\}.$$

The common rank of the symmetric spaces X and X_0 is

$$r = \min(m,n).$$

The partial Cayley transforms $c_j \in G_c$, $1 \leqq j \leqq r$, are given by

$$c_j(e_j) = \frac{1}{\sqrt{-2}}(ie_j + e_{m+j}), \qquad c_j(e_{m+j}) = \frac{1}{\sqrt{-2}}(e_j + ie_{m+j}),$$
$$c_j(e_k) = e_k \qquad \text{for} \quad j \neq k \neq m + j.$$

Now define

$$x_{s,t} = c_1 c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0, \qquad 0 \leqq s \leqq t \leqq r.$$

These points will play the role of the $x_{\Gamma,\Sigma}$.

To $x \in X$, we associate the triple (a, b, c) where $\langle\ ,\ \rangle$ has rank $a + b$ on x with a positive squares and b negative squares, and $c = n - (a + b)$ is the nullity of $\langle\ ,\ \rangle$ on X. The only restrictions are that a, b, c be integers and

$$a + b + c = n, \qquad 0 \leqq a \leqq n, \quad 0 \leqq b \leqq n, \quad 0 \leqq c \leqq r.$$

For example,

to $x_{s,t}$, we associate $(n - t, t - s, s)$.

Thus

each admissible triple is associated to just one of the $x_{s,t}$.

If $y \in G_0(x)$, one associates the same triple to x and y. If x and y give the same triple, then Witt's Theorem says $y \in G_0(x_{s,t})$. Now

the G_0-orbits on X are the $G_0(x_{s,t})$, $\qquad 0 \leqq s \leqq t \leqq r$.

In particular,

there are precisely $\frac{1}{2}(r + 1)(r + 2)$ G_0-orbits on X;
the $r + 1$ orbits $G_0(x_{0,t})$, $0 \leqq t \leqq r$, are the open orbits;
the orbit $G_0(x_{r,r})$ is the unique closed orbit.

Note that $\xi(\mathfrak{m}^+)$ consists of all $x \in X$ whose orthogonal projection to x_0 is surjective, i.e.,

$$\xi(\mathfrak{m}^+) = \{x \in X : e_1 \wedge \cdots \wedge e_m \wedge x \neq 0\}.$$

In other words, the complement

$$X - \xi(\mathfrak{m}^+) = \{x \in X : e_1 \wedge \cdots \wedge e_m \wedge x = 0\},$$

which exhibits it as a proper subvariety of X. Now consider the sets

$$D_{s,t} = \xi^{-1} G_0(x_{s,t}) \subset \mathfrak{m}^+.$$

Continuing to view $\mathfrak{m}^+$ as the space of all $m \times n$ complex matrices, we checked

$$D_{0,0} = \xi^{-1}(X_0) = \{Z \in \mathfrak{m}^+ : I_n - Z^* \cdot Z \gg 0\}.$$

Let us agree that an $n \times n$ Hermitian matrix H has "signature" (a, b, c) if it has a positive eigenvalues, b negative eigenvalues, and c zero eigenvalues. Then H and $\bar{H}$ have the same signature. Now, as in the proof of the characterization of $\xi^{-1}(X_0)$, we see that

$$D_{s,t} = \{Z \in \mathfrak{m}^+ : I_n - Z^* \cdot Z \text{ has signature } (n - t, t - s, s)\}.$$

Recall that $(n - t, t - s, s)$ is the triple associated to $x_{s,t}$. Thus

if $x \in \xi(\mathfrak{m}^+)$, say $x = \xi(Z)$, then the triple

associated to x is the signature of $I_n - Z^*Z$.

The topological boundary of X_0 in X is given by

$$\text{bd } X_0 = G_0(x_{1,1}) \cup G_0(x_{2,2}) \cup \cdots \cup G_0(x_{r,r}).$$

The boundary orbit $G_0(x_{s,s})$ is the union of those boundary components that are symmetric spaces of rank $r - s$. To obtain one boundary component of rank $r - s$, define, for $0 \leqq s \leqq r$,

$X_{(m-s,n-s)}$ is the set of all elements $x \in X$ such that

(i) x is contained in $e_{s+1} \wedge e_{s+2} \wedge \cdots \wedge e_{m+n}$

(ii) x contains $e_{m+1}, e_{m+2}, \ldots, e_{m+s}$.

To $x \in X_{(m-s,n-s)}$, we assign its interior product with $e_{m+1} \wedge \cdots \wedge e_{m+s}$. That gives an isomorphism of $X_{(m-s,n-s)}$ onto the complex Grassmannian of $(n-s)$-planes through the origin in the $\mathbf{C}^{(m-s)+(n-s)}$ with basis $\{e_{s+1}, \ldots, e_m; e_{m+s+1}, \ldots, e_{m+n}\}$. Let

$$X_{(m-s,n-s),0} = X_0 \cap X_{(m-s,n-s)}.$$

Then for $1 \leqq s \leqq r$,

$$c_1 c_2 \cdots c_s X_{(m-s,n-s),0} \text{ is a boundary component of rank } r-s$$

so

$$G_0(x_{s,s}) = \bigcup_{k \in K} k c_1 c_2 \cdots c_s X_{(m-s,n-s),0}.$$

By Witt's Theorem, two totally $\langle\, ,\rangle$-isotropic subspaces of $\mathbf{C}^{m+n}$ are $\mathbf{SU}(m,n)$-equivalent if, and only if, they have the same dimension. The possible dimensions are $\{0, 1, \ldots, r\}$. The span

$$V_s = (e_1 + ie_{m+1}) \wedge \cdots \wedge (e_s + ie_{m+s})$$

is an s-dimensional totally $\langle\ ,\ \rangle$-isotropic subspace of $\mathbf{C}^{m+n}$, and for $1 \leqq s \leqq r$,

$$N_{s,0} = \{g \in G_0 : g(V_s) = V_s\}$$

is a maximal parabolic subgroup of G_0. Further, for $1 \leqq s \leqq r$

$$N_{s,0} = \{g \in G_0 : gc_1 \cdots c_s X_{(m-s,n-s),0} = c_1 \cdots c_s X_{(m-s,n-s),0}\}.$$

We pull the boundary components of X_0 in X back, by ξ^{-1}, to the boundary components of

$$\xi^{-1}(X_0) = \{m \times n \text{ matrices } Z : I_n - Z^* \cdot Z \gg 0\}.$$

The result is that

$$\xi^{-1}(c_1 c_2 \cdots c_s X_{(m-s,n-s),0}) \text{ consists of all } Z = \begin{pmatrix} I_s & 0 \\ 0 & 'Z \end{pmatrix} \text{ where}$$

$$'Z \text{ is an } (m-s) \times (n-s) \text{ matrix such that } I_{n-s} - 'Z^* \cdot 'Z \gg 0.$$

Thus, as indeed we see from our discussion of $D_{s,s}$,

$$K \cdot \xi^{-1}(c_1 c_2 \cdots c_s X_{(m-s,n-s),0}) = \xi^{-1} G_0(x_{s,s}) \text{ consists of all } m \times n$$
$$\text{matrices } Z \text{ such that } I_n - Z^* \cdot Z \text{ has signature } (n-s, 0, s).$$

Here $K = \{\mathbf{U}(m) \times \mathbf{U}(n)\}/\{aI : |a| = 1\}$ acts by $(k_1,k_2) : Z \to k_1 Z k_2^{-1}$.

From the general theory, we now see that the holomorphic arc components of an arbitrary orbit $G_0(x_{s,t})$ are the

$$k \cdot \{c_1 c_2 \cdots c_s X_{(m-s,n-s)} \cap G_0(x_{s,t})\}, \qquad k \in K,$$

and that

$$N_{s,0} \text{ is the } G_0\text{-normalizer of } c_1 c_2 \cdots c_s X_{(m-s,n-s)} \cap G_0(x_{s,t}).$$

We describe the K-equivariant holomorphic fibration

$$\beta : G_0(x_{0,t}) \to K(x_{0,t}), \qquad 0 \leqq t \leqq r,$$

of an open orbit over its maximal compact subvariety. Decompose $\mathbf{C}^{m+n} = U \oplus V$ where $U = e_1 \wedge \cdots \wedge e_m$ and $V = e_{m+1} \wedge \cdots \wedge e_{m+n}$. K consists of all transformations of X represented by block matrices of shape

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \text{with} \quad A \in \mathbf{U}(m) \quad \text{and} \quad D \in \mathbf{U}(n).$$

Thus $x_{0,t} = e_1 \wedge \cdots \wedge e_t \wedge e_{m+t+1} \wedge \cdots \wedge e_{m+n}$ implies

$$K(x_{0,t}) = \{y \wedge z : y \text{ is a } t\text{-plane in } U, z \text{ is an } (n-t)\text{-plane in } V\}.$$

To accomodate $c_1 c_2 \cdots c_t$, we break $(m+n) \times (m+n)$ matrices into refined block form, four blocks by four, with diagonal blocks of indicated size:

$$\left(\begin{array}{c|c|c|c} t \times t & & & \\ \hline & (m-t) \times (m-t) & & \\ \hline & & t \times t & \\ \hline & & & (n-t) \times (n-t) \end{array}\right)$$

Let

$$\mathfrak{k}_1 \text{ correspond to } \mathfrak{k}_{\Psi-\Sigma}, \qquad \mathfrak{m}_1 \text{ to } \mathfrak{m}_{\Psi-\Sigma};$$
$$\mathfrak{k}_2 \text{ correspond to } \mathfrak{k}_{\Sigma}^{\Sigma}, \qquad \mathfrak{m}_2 \text{ to } \mathfrak{m}_{\Sigma}^{\Sigma}.$$

Then the support of representing matrices for the algebra corresponding to $\mathfrak{g}_{\Psi}^{\Sigma} \sim \mathfrak{g}_{\Psi-\Sigma} \oplus \mathfrak{g}_{\Sigma}^{\Sigma}$ is indicated by

$$\left(\begin{array}{c|c|c|c} \mathfrak{k}_2 & & \mathfrak{m}_2 & \\ \hline & \mathfrak{k}_1 & & \mathfrak{m}_1 \\ \hline \mathfrak{m}_2 & & \mathfrak{k}_2 & \\ \hline & \mathfrak{m}_1 & & \mathfrak{k}_1 \end{array}\right)$$

From this and the general theory we see that $\beta^{-1}(x_{0,t})$ is the product of the manifold

$$\{e_1 \wedge \cdots \wedge e_t \wedge z : z \text{ is a positive definite } (n-t)\text{-plane in } e_{t+1} \wedge \cdots \wedge e_m \wedge e_{m+t+1} \wedge \cdots \wedge e_{m+n}\}$$

corresponding to $G_{\Psi-\Sigma,0}(x_{\phi,\Sigma})$, with the manifold

$$\{y \wedge e_{m+t+1} \wedge \cdots \wedge e_{m+n} : y \text{ is a negative definite } t\text{-plane in } e_1 \wedge \cdots \wedge e_t \wedge e_{m+1} \wedge \cdots \wedge e_{m+t}\}$$

corresponding to $G^{\Sigma}_{\Sigma,0}(x_{\phi,\Sigma})$. By K-equivariance all β-fibers are described.

Now we can specify the map β as follows. Define unit quadrics

$$Q = \{u \in \mathbf{C}^{m+n} : \langle u,u \rangle = -1\} \quad \text{and} \quad S = \{v \in \mathbf{C}^{m+n} : \langle v,v \rangle = +1\}$$

and orthogonal projections

$$\pi_U : \mathbf{C}^{m+n} \to U \qquad \text{and} \qquad \pi_V : \mathbf{C}^{m+n} \to V.$$

Let $x \in G_0(x_{0,t})$. Thus x is an n-dimensional linear subspace of $\mathbf{C}^{m+n}$ on which $\langle\ ,\ \rangle$ has t negative eigenvalues and $n-t$ positive eigenvalues. Define $\{u_1, \ldots, u_t\} \subset x$ by: $u_0 = 0$ and, for $1 \leqq j \leqq t$, $u_j \in Q \cap x$ is a maximum point for the negative function

$$f_j : Q \cap x \cap \{u_0, \ldots, u_{j-1}\}^{\perp} \to \mathbf{R} \qquad \text{by} \qquad f_j(u) = \langle \pi_U(u), \pi_U(u) \rangle.$$

Then $u_1 \wedge \cdots \wedge u_t$ is the t-dimensional negative definite subspace of x that is "nearest" to U. Define $\{v_1, \ldots, v_{n-t}\} \subset x$ by: $v_0 = 0$ and, for $1 \leqq k \leqq n-t$, $v_k \in S \cap x$ is a minimum point for the positive function

$$h_k : S \cap x \cap \{v_0, \ldots, v_{k-1}\}^{\perp} \to \mathbf{R} \qquad \text{by} \quad h_k(v) = \langle \pi_V(v), \pi_V(v) \rangle.$$

Then $v_1 \wedge \cdots \wedge v_{n-t}$ is the $(n-t)$-dimensional positive definite subspace of x that is "nearest" to V. Now

$$x = (u_1 \wedge \cdots \wedge u_t) \wedge (v_1 \wedge \cdots \wedge v_{n-t}) \in G_0(x_{0,t})$$

and we have

$$\beta(x) = \pi_U(u_1 \wedge \cdots \wedge u_t) \wedge \pi_V(v_1 \wedge \cdots \wedge v_{n-t}) \in K(x_{0,t}).$$

Note that this construction simply imitates the proof of Lemma 9.10.

The K-equivariant partially holomorphic fibration

$$\beta : G_0(x_{s,t}) \to K(x_{s,t}), \qquad 0 \leqq s \leqq t \leqq r,$$

of an arbitrary orbit, is specified by its restrictions to the holomorphic arc components of the orbit. We have described the holomorphic arc components, and the restrictions β_k there iterate the open orbit case.

11. Manifolds Corresponding to the Siegel Half Planes

We work out the G_0-orbit structure, holomorphic arc components, and orbit fibration, for the case where $X_0 \subset X$ is the Borel Embedding of the "Siegel upper half plane" in its compact dual. That is the case where Harish-Chandra's map $\xi : \mathfrak{m}^+ \to X$ satisfies

$$\xi^{-1} : c_\Psi X_0 \cong \{n \times n \text{ matrices } Z : Z = {}^tZ \text{ and } Im\, Z \gg 0\}.$$

X is a subvariety in the complex Grassmann manifold of n-planes in $\mathbf{C}^{2n}$, given as follows. Let us fix

$$J : \text{nondegenerate antisymmetric bilinear form on } \mathbf{C}^{2n}.$$

Then

$$X = \{n\text{-planes } x \subset \mathbf{C}^{2n} : J(x, x) = 0\}.$$

A different choice of J would alter X only by an automorphism of the Grassmannian, for any two nondegenerate antisymmetric bilinear forms on $\mathbf{C}^{2n}$ are $\mathbf{GL}(2n, \mathbf{C})$-equivalent.

The complex symplectic group is

$$\mathbf{Sp}(n, \mathbf{C}) = \{g : \mathbf{C}^{2n} \to \mathbf{C}^{2n} \text{ linear} : g \text{ preserves } J\}.$$

The latter condition is

$$J(u,v) = J(gu,gv) \qquad \text{for all} \quad u, v \in \mathbf{C}^{2n}.$$

It implies $\det g = \pm 1$, so g is nonsingular and g^{-1} also preserves J. $\mathbf{Sp}(n,\mathbf{C})$ visibly preserves X and acts there by holomorphic diffeomorphisms. The center of $\mathbf{Sp}(n,\mathbf{C})$ is $\{\pm I_{2n}\}$, which is the kernel of its action on X, so

$$G = \mathbf{Sp}(n,\mathbf{C})/\{\pm I_{2n}\}$$

is a complex Lie transformation group on X.

Choose a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathbf{C}^{2n}$ in which

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \text{i.e,} \qquad J(u,v) = {}^tuJv \qquad \text{for} \quad u,v \in \mathbf{C}^{2n}.$$

In that basis, $\mathbf{Sp}(n,\mathbf{C})$ consists of all transformations with $(n \times n)$-block form matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

$$ {}^{t}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, $$

i.e.,

$$ {}^{t}A\cdot C = {}^{t}C\cdot A, \qquad {}^{t}B\cdot D = {}^{t}D\cdot B, \qquad \text{and} \qquad {}^{t}A\cdot D - {}^{t}C\cdot B = I. $$

The matrix form of J shows $e_{n+1} \wedge \cdots \wedge e_{2n} \in X$. Define

$$ x_0 = e_{n+1} \wedge \cdots \wedge e_{2n} \in X \qquad \text{base point.} $$

The isotropy subgroup

$$ P = \{g \in G : g(x_0) = x_0\} $$

now consists of all linear transformations of $\mathbf{C}^{2n}$ with matrix

$$ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \qquad \text{where} \quad {}^{t}A\cdot C = {}^{t}C\cdot A \quad \text{and} \quad {}^{t}A\cdot D = I. $$

Witt's Theorem proves G transitive on X. Thus

$$ X = G/P \qquad \text{compact complex } \tfrac{1}{2}n(n+1)\text{-manifold.} $$

The maximal compact subgroups of G are the conjugates of its compact real form

$$ G_c = \mathbf{Sp}(n)/\{\pm I_{2n}\}, $$

image in G of the unitary symplectic group

$$ \mathbf{Sp}(n) = \mathbf{Sp}(n,\mathbf{C}) \cap \mathbf{U}(2n) \subset \mathbf{SU}(2n). $$

The isotropy subgroup of G_c at x_0 is

$$ K = G_c \cap P; $$

it consists of all

$$ \pm\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \qquad \text{such that} \qquad A \in \mathbf{U}(n). $$

Now $G_c(x_0)$ is open in X by dimension, closed in X by compactness, so

$$ X = G_c/K \qquad \text{compact presentation.} $$

This exhibits X as the Hermitian symmetric space $\mathbf{Sp}(n)/\mathbf{U}(n)$ of compact type. The symmetry at x_0 is

$$s = \pm\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.$$

Consider the Hermitian form on $\mathbf{C}^{2n}$ given by

$$\langle u,v\rangle = -\sum_{j=1}^{n} u^j\bar{v}^j + \sum_{k=1}^{n} u^{n+k}\bar{v}^{n+k}\,.$$

It specifies the indefinite unitary group $\mathbf{U}(n,n)$, hence the group $\mathbf{Sp}(n,\mathbf{C}) \cap \mathbf{U}(n,n)$. Define

$$G_0 : \text{transformation group on } X \text{ from } \mathbf{Sp}(n,\mathbf{C}) \cap \mathbf{U}(n,n).$$

As $\mathbf{U}(n,n)$ is a real form of $\mathbf{GL}(2n,\mathbf{C})$, we can see that

$$G_0 \text{ is a noncompact real form of } G.$$

Furthermore

$$G_0 \cap P = K \qquad \text{so} \qquad G_0/K \cong G_0(x_0) \qquad \text{open in } X.$$

Thus

$$X_0 = G_0(x_0) \subset X \qquad \text{open } G_0\text{-orbit}$$

is the Hermitian symmetric space of noncompact type, dual to $G_c/K = X$ with the same symmetry s at x_0, inside X under Borel Embedding. Note now $X_0 \cong \mathbf{Sp}(n,\mathbf{R})/\mathbf{U}(n)$ and $G_0 \cong \mathbf{Sp}(n,\mathbf{R})/\{\pm I_{2n}\}$.

Now we come to the Harish-Chandra embedding of X_0 in its holomorphic tangent space. Identify the Lie algebra $\mathfrak{g}$ of G with the Lie algebra of $\mathbf{Sp}(n,\mathbf{C})$ under $G = \mathbf{Sp}(n,\mathbf{C})/\{\pm I\}$. Then

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^t\!\begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\cdot\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0\right\}.$$

In other words,

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tA = -D,\ {}^tB = B, \text{ and } {}^tC = C\right\}.$$

Thus

$$\mathfrak{p} = \left\{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : {}^tA = -D \text{ and } {}^tC = C\right\}$$

and

$$\mathfrak{m}^+ = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : {}^tB = B\right\}.$$

In particular,

$$Z \to \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \quad \text{maps} \quad (n \times n \text{ symmetric matrices}) \cong \mathfrak{m}^+.$$

We now view Harish-Chandra's map $\xi : \mathfrak{m}^+ \to X$, given by $\xi(Z) = (\exp \tilde{Z})(x_0)$, as a map from the complex vector space of all $n \times n$ symmetric matrices. As

$$\exp\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}$$

now

$$\xi(Z) = (\exp \tilde{Z})(x_0) = v_1 \wedge \cdots \wedge v_n \qquad \text{where} \quad (v_1, \ldots, v_n) = \begin{pmatrix} Z \\ I \end{pmatrix}.$$

By Witt's Theorem, $\xi(Z) \in X_0$ if and only if $\langle\ ,\ \rangle$ is positive definite on $v_1 \wedge \cdots \wedge v_n$. That condition is that $(\langle v_k, v_l \rangle) = I - {}^tZ \cdot \bar{Z}$ be positive definite. Thus

$$\xi^{-1}(X_0) = \{n \times n \text{ symmetric } Z : I - Z^* \cdot Z \gg 0\}.$$

If

$$g = \pm\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_0,$$

then

$$g \cdot \exp \tilde{Z} = \pm\begin{pmatrix} A & AZ + B \\ C & CZ + D \end{pmatrix},$$

which sends x_0 to the span of the columns of $\begin{pmatrix} AZ + B \\ CZ + D \end{pmatrix}$. As $(CZ + D)^{-1}$ exists, that is the same as the span of the columns of $\begin{pmatrix} (AZ+B)(CZ+D)^{-1} \\ I \end{pmatrix}$.

Thus G_0 acts on $\xi^{-1}(X_0)$ by

$$\pm\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \to (AZ + B)\cdot(CZ + D)^{-1},$$

as expected. Finally, note that the Banach norm on $\mathfrak{m}^+$, for which $\xi^{-1}(X_0)$ is the unit ball, is

$$\|Z\|^2 = \max\{\text{eigenvalues of } Z^*\cdot Z\}.$$

The symmetric spaces X_0 and X have rank n. The partial Cayley transforms $c_j \in G_c$, $1 \leqq j \leqq n$, are given by

$$c_j(e_j) = \frac{1}{\sqrt{-2}}(ie_j + e_{n+j}), \qquad c_j(e_{n+j}) = \frac{1}{\sqrt{-2}}(e_j + ie_{n+j}),$$
$$c_j(e_k) = e_k \qquad \text{for} \quad j \neq k \neq n + j,$$

because X has equal rank and is totally geodesic in the ambient Grassmann manifold. As before, define

$$x_{s,t} = c_1 c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0, \qquad 0 \leqq s \leqq t \leqq n,$$

same centers as for the ambient Grassmannian of n-planes in $\mathbf{C}^{2n}$. Again,

the G_0-orbits on X are the $G_0(x_{s,t})$, $\quad 0 \leqq s \leqq t \leqq n.$

In particular,

there are precisely $\frac{1}{2}(n + 1)(n + 2)$ G_0-orbits on X,

the $n + 1$ orbits $G_0(x_{0,t})$, $0 \leqq t \leqq n$, are the open orbits, and

the orbit $G_0(x_{n,n})$ is the unique closed orbit.

The orbit $G_0(x_{s,t})$ consists of all elements of X on which $\langle\ ,\ \rangle$ has $n - t$ positive eigenvalues, $t - s$ negative eigenvalues, and s eigenvalues zero. Again as in the ambient Grassmannian,

$$X - \xi(\mathfrak{m}^+) = \{x \in X : e_1 \wedge \cdots \wedge e_n \wedge x = 0\},$$

and

$$D_{s,t} = \xi^{-1}G_0(x_{s,t}) \subset \mathfrak{m}^+$$

consists of all $n \times n$ symmetric matrices Z such that the Hermitian matrix $I - Z^*\cdot Z$ has $n - t$ positive eigenvalues, $t - s$ negative eigenvalues, and s eigenvalues zero.

The topological boundary of X_0 in X is bd $X_0 = G_0(x_{1,1}) \cup G_0(x_{2,2}) \cup \cdots \cup G_0(x_{n,n})$. Here $G_0(x_{s,s})$ is the union of those boundary components that are symmetric spaces of rank $n - s$. For $0 \leqq s \leqq n$, define

$$X_{(n-s)} \text{ is the set of all elements } x \in X \text{ such that}$$
$$e_{n+1} \wedge \cdots \wedge e_{n+s} \subset x \subset e_{s+1} \wedge e_{s+2} \wedge \cdots \wedge e_{2n}.$$

The interior product with $e_{n+1} \wedge \cdots \wedge e_{n+s}$ maps $X_{(n-s)}$ isomorphically onto the subvariety of the complex Grassmannian of $(n - s)$-planes in $\mathbf{C}^{2(n+s)} = (e_{s+1} \wedge \cdots \wedge e_n) \wedge (e_{n+s+1} \wedge \cdots \wedge e_{2n})$ defined by $J(y,y) = 0$. The Borel-Embedded noncompact dual of $X_{(n-s)}$ is

$$X_{(n-s),0} = X_0 \cap X_{(n-s)}.$$

For $1 \leqq s \leqq n$,

$$c_1 c_2 \cdots c_s X_{(n-s),0} \text{ is a boundary component of rank } n - s,$$

so

$$G_0(x_{s,s}) = \bigcup_{k \in K} k c_1 c_2 \ldots c_s X_{(n-s),0}.$$

As in the case of the Grassmannian, the maximal parabolic subgroups of G_0 are the

$$N_{s,0} = \{g \in G_0 : g(V_s) = V_s\}$$

where the

$$V_s = (e_1 + ie_{n+1}) \wedge \cdots \wedge (e_s + ie_{n+s}), \qquad 0 \leqq s \leqq n,$$

represent the G_0-equivalence classes of totally $\langle\ ,\ \rangle$-isotropic J-isotropic subspaces of $\mathbf{C}^{2n}$; and

$$N_{s,0} = \{g \in G_0 : gc_1 \cdots c_s X_{(n-s),0} = c_1 \cdots c_s X_{(n-s),0}\}.$$

We locate the ξ^{-1}-images of the boundary components of X_0 in X; they are the boundary components of

$$\{n \times n \text{ symmetric matrices } Z : I - Z^*Z \gg 0\}$$

in the space of $n \times n$ symmetric matrices. The result is that

$$\xi^{-1}(c_1 c_2 \cdots c_s X_{(n-s),0}) \text{ consists of all } n \times n \text{ symmetric matrices}$$

$$Z = \begin{pmatrix} I_s & 0 \\ 0 & 'Z \end{pmatrix} \qquad \text{such that} \qquad I_{n-s} - 'Z^* \cdot 'Z \gg 0.$$

$$K = \{\pm \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in U(n)\} \text{ acts on } \mathfrak{m}^+ \text{ by} \pm \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : Z \to AZ\bar{A}^{-1} = A \cdot Z \cdot {}^t A.$$

Thus

$K \cdot \xi^{-1}(c_1 c_2 \cdots c_s X_{(n-s),0}) = \xi^{-1} G_0(x_{s,s})$ consists of all $n \times n$ symmetric matrices Z such that $I_n - Z^* \cdot Z$ is positive semidefinite and of rank $n - s$.

We know from general theory that the holomorphic arc components of an arbitrary orbit $G_0(x_{s,t})$ are the

$$k \cdot \{c_1 c_2 \cdots c_s X_{(n-s)} \cap G_0(x_{s,t})\}, \qquad k \in K,$$

and that

$$N_{s,0} \text{ is the } G_0\text{-normalizer of } c_1 c_2 \cdots c_s X_{(n-s)} \cap G_0(x_{s,t}).$$

We describe the K-equivariant holomorphic fibration

$$\beta : G_0(x_{0,t}) \to K(x_{0,t}), \qquad 0 \leqq t \leqq n,$$

of an open orbit over its maximal compact subvariety. As before,

$\mathbf{C}^{2n} = U \oplus V$ where $U = e_1 \wedge \cdots \wedge e_n$ and $V = e_{n+1} \wedge \cdots \wedge e_{2n}$.

As

$$x_{0,t} = (e_1 \wedge \cdots \wedge e_t) \wedge (e_{n+t+1} \wedge \cdots \wedge e_{2n}),$$

now

$$K(x_{0,t}) = \{x = y \wedge z : x \in X, y \text{ is } t\text{-plane in } U, z \text{ is } (n-t)\text{-plane in } V\}.$$

Our refined matrix block form is

$$\left(\begin{array}{c|c|c|c} t \times t & & & \\ \hline & (n-t) \times (n-t) & & \\ \hline & & t \times t & \\ \hline & & & (n-t) \times (n-t) \end{array}\right)$$

Let

$$\mathfrak{k}_1 \text{ correspond to } \mathfrak{k}_{\Psi-\Sigma}, \qquad \mathfrak{m}_1 \text{ to } \mathfrak{m}_{\Psi-\Sigma},$$
$$\mathfrak{k}_2 \text{ correspond to } \mathfrak{k}_\Sigma^\Sigma, \qquad \mathfrak{m}_2 \text{ to } \mathfrak{m}_\Sigma^\Sigma.$$

Then representing matrices for the algebra corresponding to $\mathfrak{g}_\Psi^\Sigma \sim \mathfrak{g}_{\Psi-\Sigma} \oplus \mathfrak{g}_\Sigma^\Sigma$ have support

$$\left(\begin{array}{c|c|c|c} \mathfrak{k}_2 & & \mathfrak{m}_2 & \\ \hline & \mathfrak{k}_1 & & \mathfrak{m}_1 \\ \hline \mathfrak{m}_2 & & \mathfrak{k}_2 & \\ \hline & \mathfrak{m}_1 & & \mathfrak{k}_1 \end{array}\right)$$

Thus $\beta^{-1}(x_{0,t})$ is the product of the manifold

$$\{e_1 \wedge \cdots \wedge e_t \wedge z : z \subset (e_{t+1} \wedge \cdots \wedge e_n) \wedge (e_{n+t+1} \wedge \cdots \wedge e_{2n}) \text{ is a positive definite } (n-t)\text{-plane with } J(z,z) = 0\}$$

corresponding to $G_{\Psi-\Sigma,0}(x_{\phi,\Sigma})$, with the manifold

$$\{y \wedge e_{n+t+1} \wedge \cdots \wedge e_{2n} : y \subset (e_1 \wedge \cdots \wedge e_t) \wedge (e_{n+1} \wedge \cdots \wedge e_{n+t}) \text{ is a negative definite } t\text{-plane with } J(y,y) = 0\}$$

corresponding to $G^{\Sigma}_{\Sigma,0}(x_{\phi,\Sigma})$. Via K-equivariance, this describes all β-fibers.

We can now describe the map β, as before, by a minimax construction from the unit quadrics

$$Q = \{u \in \mathbf{C}^{2n} : \langle u,u \rangle = -1\} \qquad \text{and} \qquad S = \{v \in \mathbf{C}^{2n} : \langle v,v \rangle = +1\}$$

and the orthogonal projections

$$\pi_U : \mathbf{C}^{2n} \to U \qquad \text{and} \qquad \pi_V : \mathbf{C}^{2n} \to V.$$

Let $x \in G_0(x_{0,t})$. In other words, x is an n-dimensional subspace of $\mathbf{C}^{2n}$, on which $\langle\ ,\ \rangle$ has t negative eigenvalues and $n-t$ positive eigenvalues, such that $J(x,x) = 0$. Within the ambient Grassmannian of X, we decompose

$$x = (u_1 \wedge \cdots \wedge u_t) \wedge (v_1 \wedge \cdots \wedge v_{n-t})$$

where $\{u_1, \ldots, u_t\} \subset Q \cap x$ are mutually orthogonal, spanning the t-dimensional negative definite subspace of x closest to U, and $\{v_1, \ldots, v_{n-t}\} \subset S \cap x$ are mutually orthogonal, spanning to the $(n-t)$-dimensional positive definite subspace of x closest to V. As $G_0(x_{0,t})$ is totally geodesic in the open subset $\mathbf{SU}(n,n)\,(x_{0,t})$ of the ambient Grassmannian, it follows that

$$\beta(x) = \pi_U(u_1 \wedge \cdots \wedge u_t) \wedge \pi_V(v_1 \wedge \cdots \wedge v_{n-t}) \in K(x_{0,t}).$$

As before, the K-equivariant partially holomorphic fibration

$$\beta : G_0(x_{s,t}) \to K(x_{s,t}), \qquad 0 \leqq s \leqq t \leqq n,$$

of an arbitrary orbit, is now specified by our knowledge of the holomorphic arc components and the open orbit case.

12. Subvarieties of Grassmannians Defined by Symmetric Forms

We work out the G_0-orbit structure, the holomorphic arc components, and the orbit fibrations, for the symmetric spaces corresponding to $X = \mathbf{SO}(2n)/\mathbf{U}(n)$. They differ from the Siegel Half Plane case in that J is replaced by a symmetric bilinear form. That change does not destroy the simplifying feature that X be totally geodesic in an ambient Grassmannian.

Fix a nondegenerate symmetric bilinear form S on $\mathbf{C}^{2n}$ and define

$$\tilde{X} = \{n\text{-planes } x \subset \mathbf{C}^{2n} : S(x,x) = 0\}.$$

Then $\tilde{X}$ is homogeneous under the complex orthogonal group

$$\mathbf{O}(2n,\mathbf{C}) = \{g : \mathbf{C}^{2n} \to \mathbf{C}^{2n} \text{ linear} : g \text{ preserves } S\}.$$

That orthogonal group has two topological components; the identity component is

$$\mathbf{SO}(2n,\mathbf{C}) = \{g \in \mathbf{O}(2n,\mathbf{C}) : \det g = +1\}$$

and the other component is

$$\{g \in \mathbf{O}(2n,\mathbf{C}) : \det g = -1\}.$$

From now on assume $n > 1$. Then the centers of $\mathbf{SO}(2n,\mathbf{C})$ and $\mathbf{O}(2n,\mathbf{C})$ are $\{\pm I_{2n}\}$, which is the common kernel of their actions on $\tilde{X}$, so

$$G = \mathbf{SO}(2n,\mathbf{C})/\{\pm I\} \qquad \text{and} \qquad \tilde{G} = \mathbf{O}(2n,\mathbf{C})/\{\pm I\}$$

are complex Lie transformation groups on $\tilde{X}$, of complex dimension $2n^2 - n$, with G connected and of index 2 in $\tilde{G}$.

Choose a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathbf{C}^{2n}$ in which

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \text{i.e.,} \qquad S(u,v) = \sum_{k=1}^{n} (u^k v^{n+k} + u^{n+k} v^k).$$

In that basis, using $n \times n$ matrix blocks,

$$\tilde{G} = \{\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tA\cdot C + {}^tC\cdot A = 0 = {}^tB\cdot D + {}^tD\cdot B, {}^tA\cdot D + {}^tC\cdot B = I\}$$

and

$$G = \{\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G} : \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1\}.$$

Now define

$$x_0 = e_{n+1} \wedge \cdots \wedge e_{2n} \in \tilde{X} \qquad \text{base point.}$$

The isotropy subgroup $P = \{g \in \tilde{G} : g(x_0) = x_0\}$ is

$$P = \{\pm\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : {}^tA\cdot C + {}^tC\cdot A = 0,\ {}^tA\cdot D = I\}.$$

If

$$\pm\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in P$$

we compute

$$\det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = (\det A)\,(\det D) = \det\,({}^tA\cdot D) = \det I = 1.$$

Now

$$P \subset G \qquad \text{so} \qquad \tilde{X} = \tilde{G}/P \qquad \text{has two components.}$$

Now define

$$X = G(x_0), \qquad \text{component of } \tilde{X} \text{ that contains } x_0.$$

Thus

$$X = G/P \qquad \text{compact complex } \tfrac{1}{2}n(n-1)\text{-manifold}$$

and

$$x_0 = e_{n+1} \wedge \cdots \wedge e_{2n} \in X \qquad \text{base point.}$$

The maximal compact subgroups of G are the conjugates of its compact real form

$$G_c = \{\mathbf{SO}(2n,\mathbf{C}) \cap \mathbf{U}(2n)\}/\{\pm I_{2n}\}.$$

That group obviously is compact. To check that it is a real form observe that

$$T = \frac{1}{\sqrt{2i}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix} \in \mathbf{U}(2n) \qquad \text{and} \qquad {}^tT\cdot T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

so that

$$T^{-1}\cdot\mathbf{SO}(2n)\cdot T = \mathbf{SO}(2n,\mathbf{C}) \cap \mathbf{U}(2n)$$

where $\mathbf{SO}(2n)$ denotes the group of all real orthogonal matrices of determinant 1 and degree $2n$. The isotropy subgroup of G_c at x_0 is

$$K = G_c \cap P = \{\pm\begin{pmatrix} A & 0 \\ 0 & \bar{A}\end{pmatrix} : A \in \mathbf{U}(n)\}.$$

Now $G_c(x_0)$ is open in X by dimension and closed in X by compactness, so

$$X = G_c/K \qquad \text{compact presentation.}$$

This exhibits X as the Hermitian symmetric space $\mathbf{SO}(2n)/\mathbf{U}(n)$ of compact type; the symmetry at x_0 is

$$s = \pm\begin{pmatrix} iI & 0 \\ 0 & -iI\end{pmatrix}.$$

We again use the Hermitian form on $\mathbf{C}^{2n}$ given by

$$\langle u,v\rangle = -\sum_{j=1}^{n} u^j\bar{v}^j + \sum_{k=1}^{n} u^{n+k}\bar{v}^{n+k}.$$

It specifies the indefinite unitary group $\mathbf{U}(n,n)$ and thus specifies $\mathbf{SO}(2n,\mathbf{C}) \cap \mathbf{U}(n,n)$. Define

$$G_0 : \text{transformation group on } X \text{ from } \mathbf{SO}(2n,\mathbf{C}) \cap \mathbf{U}(n,n).$$

Then

$$G_0 \text{ is a noncompact real form of } G$$

and

$$G_0 \cap P = K \qquad \text{so} \qquad G_0/K \cong G_0(x_0) \qquad \text{open in } X.$$

Thus

$$X_0 = G_0(x_0) \subset X \qquad \text{open } G_0\text{-orbit}$$

is the Hermitian symmetric space of noncompact type dual to $X = G_c/K$, with the same symmetry s at x_0, in X under Borel Embedding. In particular, $X_0 \cong \mathbf{SO}^*(2n)/\mathbf{U}(n)$ and $G_0 \cong \mathbf{SO}^*(2n)/\{\pm I_{2n}\}$.

Now we work out the Harish-Chandra embedding. Identify the Lie algebra $\mathfrak{g}$ of $G = \mathbf{SO}(2n,\mathbf{C})/\{\pm I\}$ with that of $\mathbf{SO}(2n,\mathbf{C})$. Thus

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix} : {}^t\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = 0\right\}.$$

That says

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : B + {}^tB = 0,\ C + {}^tC = 0, \text{ and } D = -{}^tA\right\}.$$

It follows that

$$\mathfrak{p} = \left\{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : C + {}^tC = 0, \text{ and } D = -{}^tA\right\}.$$

and

$$\mathfrak{m}^+ = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B + {}^tB = 0\right\}.$$

In particular,

$$Z \to \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \text{ maps } \quad (\text{antisymmetric } n \times n \text{ matrices}) \cong \mathfrak{m}^+.$$

We view Harish-Chandra's map $\xi : \mathfrak{m}^+ \to X$ as the map $\xi(Z) = (\exp \tilde{Z})(x_0)$ from the space of antisymmetric complex $n \times n$ matrices. Then

$$\xi(Z) = (\exp \tilde{Z})(x_0) = v_1 \wedge \ldots \wedge v_n \text{ where } (v_1, \ldots, v_n) = \begin{pmatrix} Z \\ I \end{pmatrix}$$

and we use Witt's Theorem to see that

$$\xi^{-1}(X_0) = \{n \times n \text{ antisymmetric } Z : I - Z^* \cdot Z \gg 0\}.$$

The same calculations as in Sections 10 and 11 show that G_0 acts on $\xi^{-1}(X_0)$ by

$$\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \to (AZ + B)(CZ + D)^{-1}$$

and that the Banach norm on $\mathfrak{m}^+$, for which $\xi^{-1}(X_0)$ is the unit ball, is

$$\|Z\|^2 = \max\{\text{eigenvalues of } Z^* \cdot Z\}.$$

The symmetric spaces X_0 and X have the same rank r given by

$$2r = n \quad \text{for } n \text{ even}, \qquad 2r + 1 = n \quad \text{for } n \text{ odd}.$$

Observe that the matrix

$$c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \text{ satisfies: } c^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

so we can check that

$$c \in \mathbf{SO}(4,\mathbf{C}) \cap \mathbf{U}(4) \qquad \text{and} \qquad c^4 = -I_4.$$

Now, the partial Cayley transforms $c_i \in G_c$, $1 \leqq j \leqq r = [n/2]$, are given by

$$c_j(e_{2j-1}) = \frac{1}{\sqrt{2}}(e_{2j-1} - e_{n+2j}), \qquad c_j(e_{2j}) = \frac{1}{\sqrt{2}}(e_{2j} + e_{n+2j-1});$$

$$c_j(e_{n+2j-1}) = \frac{1}{\sqrt{2}}(-e_{2j} + e_{n+2j-1}), \qquad c_j(e_{n+2j}) = \frac{1}{\sqrt{2}}(e_{2j-1} + e_{n+2j});$$

$$c_j(e_k) = e_k \qquad \text{for} \quad k \notin \{2j-1, 2j, n+2j-1, n+2j\}.$$

These do not coincide with the partial Cayley transforms on the ambient Grassmannian of n-planes in $\mathbf{C}^{2n}$; for although X is totally geodesic there, it has lower rank $r < n$. Now our centers for G_0-orbits are the

$$x_{s,t} = c_1 c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0, \qquad 0 \leqq s \leqq t \leqq r = [n/2],$$

and

$$\text{the } G_0\text{-orbits on } X \text{ are the } G_0(x_{s,t}), \qquad 0 \leqq s \leqq t \leqq r.$$

In particular,

there are precisely $\frac{1}{2}(r+1)(r+2)$ G_0-orbits on X,

the $r+1$ orbits $G_0(x_{0,t})$, $0 \leqq t \leqq r$, are the open ones, and

the orbit $G_0(x_{r,r})$ is the unique closed orbit.

The orbit $G_0(x_{s,t})$ consists of all elements $x \in X$ on which $\langle\ ,\ \rangle$ has $n - 2t$ positive eigenvalues, $2(t-s)$ negative eigenvalues, and $2s$ eigenvalues zero. As before,

$$X - \xi(\mathfrak{m}^+) = \{x \in X : e_1 \wedge \cdots \wedge e_n \wedge x = 0\},$$

and

$$D_{s,t} = \xi^{-1} G_0(x_{s,t}) \subset \mathfrak{m}^+$$

consists of all $n \times n$ antisymmetric matrices Z such that the Hermitian matrix $I - Z^* \cdot Z$ has $n - 2t$ positive eigenvalues, $2(t - s)$ negative eigenvalues, and $2s$ eigenvalues zero.

The topological boundary of X_0 in X is $G_0(x_{1,1}) \cup G_0(x_{2,2}) \cup \cdots \cup G_0(x_{r,r})$ where $G_0(x_{s,s})$ is the union of those boundary components that are symmetric spaces of rank $r - s$. For $0 \leqq s \leqq r$, define

$$X_{(r-s)} \text{ is the set of all elements } x \in X \text{ such that}$$

$$e_{n+1} \wedge \cdots \wedge e_{n+2s} \subset x \subset e_{2s+1} \wedge e_{2s+2} \wedge \cdots \wedge e_{2n}.$$

Interior product with $e_{n+1} \wedge \cdots \wedge e_{n+2s}$ maps $X_{(r-s)}$ isomorphically onto the component of $e_{n+2s+1} \wedge \cdots \wedge e_{2n}$ in the two-component subvariety of the complex Grassmannian of $(n - 2s)$-planes in $\mathbf{C}^{2n-4s} = (e_{2s+1} \wedge \cdots \wedge e_n) \wedge (e_{n+2s+1} \wedge \cdots \wedge e_{2n})$ defined by $S(y,y) = 0$. The Borel-Embedded noncompact dual of $X_{(r-s)}$ is

$$X_{(r-s),0} = X_0 \cap X_{(r-s)}.$$

For $1 \leqq s \leqq r$,

$$c_1 c_2 \cdots c_s X_{(r-s),0} \text{ is a boundary component of rank } r - s,$$

so

$$G_0(x_{s,s}) = \bigcup_{k \in K} k c_1 c_2 \cdots c_s X_{(r-s),0}.$$

The even dimensional G_0-equivalence classes of totally S-isotropic $\langle\ ,\ \rangle$-isotropic subspaces of $\mathbf{C}^{2n}$ are represented by the $2s$-dimensional spaces V_s, $0 \leqq s \leqq r$, defined by

$$V_s = \{(-e_2 + e_{n+1}) \wedge (e_1 + e_{n+2})\} \wedge \cdots \wedge \{(-e_{2s} + e_{n+2s-1}) \wedge (e_{2s-1} + e_{n+2s})\}.$$

The maximal parabolic subgroups of G_0 are the

$$N_{s,0} = \{g \in G_0 : g(V_s) = V_s\}$$

and

$$N_{s,0} = \{g \in G_0 : g c_1 c_2 \cdots c_s X_{(r-s),0} = c_1 c_2 \cdots c_s X_{(r-s),0}\}.$$

We locate the ξ^{-1}-images of the boundary components of X_0 in X; they are the boundary components of

$$\{n \times n \text{ antisymmetric matrices } Z\colon I - Z^*\cdot Z \gg 0\}$$

in the space of $n \times n$ antisymmetric matrices:

$\xi^{-1}(c_1c_2 \cdots c_sX_{(r-s),0})$ consists of all $n \times n$ antisymmetric matrices $Z = \begin{pmatrix} J_{2s} & 0 \\ 0 & 'Z \end{pmatrix}$, where

$$J_{2s} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \quad \text{and} \quad I_{n-2s} - 'Z^* \cdot 'Z \gg 0.$$

$K = \left\{\pm\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}: A \in \mathrm{U}(n)\right\}$ acts on $\mathfrak{m}^+$ by

$$\pm\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}: Z \to AZ\bar{A}^{-1} = A\cdot Z\cdot {}^t A.$$

Thus $K\cdot\xi^{-1}(c_1c_2 \cdots c_sX_{(r-s),0}) = \xi^{-1}G_0(x_{s,s})$ consists of all $n \times n$ antisymmetric matrices Z such that $I_n - Z^*\cdot Z$ is positive semidefinite and of rank $n - 2s$.

The general theory ensures that the holomorphic arc components of an arbitrary orbit $G_0(x_{s,t})$ are the

$$k\{c_1c_2 \cdots c_sX_{(r-s)} \cap G_0(x_{s,t})\}, \qquad k \in K,$$

and that their normalizers are:

$N_{s,0}$ is the G_0-normalizer of $c_1c_2 \cdots c_sX_{(r-s)} \cap G_0(x_{s,t})$.

We describe the K-equivariant holomorphic fibration

$$\beta : G_0(x_{0,t}) \to K(x_{0,t}), \qquad 0 \leqq t \leqq r = [n/2],$$

of an open orbit over its maximal compact subvariety. As before, $\mathbf{C}^{2n} = U \oplus V$ where $U = e_1 \wedge \cdots \wedge e_n$ and $V = e_{n+1} \wedge \cdots \wedge e_{2n}$. Now

$$x_{0,t} = (e_1 \wedge \cdots \wedge e_{2t}) \wedge (e_{n+2t+1} \wedge \cdots \wedge e_{2n})$$

gives us

$K(x_{0,t}) = \{x = y \wedge z : x \in X,\ y \text{ is } 2t\text{-plane in } U,\ z \text{ is } (n-2t)\text{-plane in } V\}$.

We use refined block form $2n \times 2n$ matrices divided as follows.

$$\left(\begin{array}{c|c|c|c} 2t \times 2t & & & \\ \hline & (n-2t)\times(n-2t) & & \\ \hline & & 2t \times 2t & \\ \hline & & & (n-2t)\times(n-2t) \end{array}\right)$$

As before, let

$$\mathfrak{k}_1 \text{ correspond to } \mathfrak{k}_{\Psi-\Sigma}, \qquad \mathfrak{m}_1 \text{ to } \mathfrak{m}_{\Psi-\Sigma};$$
$$\mathfrak{k}_2 \text{ correspond to } \mathfrak{k}_\Sigma^\Sigma, \qquad \mathfrak{m}_2 \text{ to } \mathfrak{m}_\Sigma^\Sigma.$$

Then matrices representing the algebra corresponding to $\mathfrak{g}_\Psi^\Sigma \sim \mathfrak{g}_{\Psi-\Sigma} \oplus \mathfrak{g}_\Sigma^\Sigma$ have support

$$\left(\begin{array}{c|c|c|c} \mathfrak{k}_2 & & \mathfrak{m}_2 & \\ \hline & \mathfrak{k}_1 & & \mathfrak{m}_1 \\ \hline \mathfrak{m}_2 & & \mathfrak{k}_2 & \\ \hline & \mathfrak{m}_1 & & \mathfrak{k}_1 \end{array}\right)$$

Thus $\beta^{-1}(x_{0,t})$ is the product of the manifold

$$\{e_1 \wedge \cdots \wedge e_{2t} \wedge z : z \subset (e_{2t+1} \wedge \cdots \wedge e_n) \wedge (e_{n+2t+1} \wedge \cdots \wedge e_{2n})$$
$$\text{is a positive definite } (n-2t)\text{-plane with } S(z,z) = 0\}$$

corresponding to $G_{\Psi-\Sigma,0}(x_{\phi,\Sigma})$, with the manifold

$$\{y \wedge e_{n+2t+1} \wedge \cdots \wedge e_{2n} : y \subset (e_1 \wedge \cdots \wedge e_{2t}) \wedge (e_{n+1} \wedge \cdots \wedge e_{n+2t})$$
$$\text{is a negative definite } 2t\text{-plane with } S(y,y) = 0\}$$

corresponding to $G_{\Sigma,0}^\Sigma(x_{\phi,\Sigma})$. By means of K-equivariance, this describes all β-fibers.

As X is totally geodesic in the ambient Grassmannian of n-planes in $\mathbf{C}^{2n}$, and as the center $x_{0,t}$ of an open G_0-orbit on X is the center of an open $\mathbf{SU}(n,n)$-orbit on the Grassmannian, we now can use the minimax construction of Section 10 to describe the map β. Denote the unit quadrics

$$Q = \{u \in \mathbf{C}^{2n} : \langle u,u \rangle = -1\} \qquad \text{and} \qquad L = \{v \in \mathbf{C}^{2n} : \langle v,v \rangle = +1\}$$

and the orthogonal projections

$$\pi_U : \mathbf{C}^{2n} \to U \qquad \text{and} \qquad \pi_V : \mathbf{C}^{2n} \to V.$$

If $x \in G_0(x_{0,t})$, we construct, as in Section 10,

$$\{u_1, \ldots, u_{2t}\} \subset Q \cap x \quad \text{mutually orthogonal,}$$
$$\{v_1, \ldots, v_{n-2t}\} \subset L \cap x \quad \text{mutually orthogonal,}$$

with

$$x = (u_1 \wedge \cdots \wedge u_{2t}) \wedge (v_1 \wedge \cdots \wedge v_{n-2t}) \in G_0(x_{0,t}).$$

That construction is such that $u_1 \wedge \cdots \wedge u_{2t}$ is the negative definite $2t$-dimensional subspace of x nearest to U and $v_1 \wedge \cdots \wedge v_{n-2t}$ is the positive definite $(n - 2t)$-dimensional subspace of x nearest to V. Thus, as described in the first sentence of this paragraph,

$$\beta(x) = \pi_U(u_1 \wedge \cdots \wedge u_{2t}) \wedge \pi_V(v_1 \wedge \cdots \wedge v_{n-2t}) \in K(x_{0,t}).$$

Now, in the case of an arbitrary orbit $G_0(x_{s,t})$, the K-equivariant partially holomorphic fibration

$$\beta : G_0(x_{s,t}) \to K(x_{s,t}), \qquad 0 \leqq s \leqq t \leqq r,$$

is specified because we know the holomorphic arc components and the open orbit case.

13. The Complex Quadrics

In this section we work out the G_0-orbit structure, holomorphic arc components, and orbit fibration for the remaining classical series of irreducible Hermitian symmetric spaces. That is the case where X is a quadric in complex projective space $P^{n+1}(\mathbf{C})$, $n > 2$, which is not contained in a hyperplane. Here matters are rather more complicated than in Sections 11 and 12 because X is not totally geodesic in its ambient Grassmannian $P^{n+1}(\mathbf{C})$, and because G_0 is properly contained in its real algebraic hull.

Given an integer $n \geqq 0$, the complex projective space of complex dimension $n + 1$ is

$$P^{n+1}(\mathbf{C}) : \text{complex Grassmannian of lines through 0 in } \mathbf{C}^{n+2}.$$

Fix a nondegenerate symmetric bilinear form S on $\mathbf{C}^{n+2}$ and define

$$X = \{x \in P^{n+1}(\mathbf{C}) : S(x,x) = 0\}.$$

That is the complex quadric. It is homogeneous under the complex orthogonal group $\mathbf{O}(n + 2, \mathbf{C})$ associated to S. Thus X carries complex Lie transformation groups

$$\tilde{G} = \mathbf{O}(n + 2, \mathbf{C})/\{\pm I\} \qquad \text{and} \qquad G = \mathbf{SO}(n + 2, \mathbf{C})\cdot\{\pm I\}/\{\pm I\}$$

of complex dimension $\frac{1}{2}(n + 1)(n + 2)$, with G connected. If n is odd, so $\det(-I) = -1$, then $G = \tilde{G}$. If n is even, then G has index 2 in $\tilde{G}$.

As we do not use exterior products in expressing elements of X by their bases, we adopt the convention

$$\text{if } 0 \neq v \in \mathbf{C}^{n+2} \qquad \text{then } [v] \in P^{n+1}(\mathbf{C}) \text{ is the span } v\mathbf{C}.$$

Now choose a basis $\{e_1, \ldots, e_{n+2}\}$ of $\mathbf{C}^{n+2}$ in which $S(u,v) = \sum_{k=1}^{n+2} u^k v^k$. Then

$$\tilde{G} = \{\pm M : M \text{ is a complex } (n+2) \times (n+2) \text{ matrix and } {}^t M \cdot M = I\}.$$

Now choose base point

$$x_0 = [e_{n+1} + ie_{n+2}] \in X.$$

The isotropy subgroup $\tilde{P} = \{g \in \tilde{G} : g(x_0) = x_0\}$ is

$$\tilde{P} = \{\pm\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G} : B = (B', iB'),\ D = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ia - b = c + id\}$$

where the matrix blocks are of size given by

A is $n \times n$, B is $n \times 2$, B' is $n \times 1$, C is $2 \times n$, D is 2×2.

As $\tilde{P}$ contains the transformation $\pm\begin{pmatrix} -1 & 0 \\ 0 & I_{n+1} \end{pmatrix}$ represented by a matrix of determinant -1, it meets every component of $\tilde{G}$. Thus

$$X = G(x_0) \cong G/P \qquad \text{connected where} \quad P = G \cap \tilde{P}.$$

The maximal compact subgroups of G are the conjugates of its compact real form

$$G_c = \{\mathbf{SO}(n + 2, \mathbf{C}) \cap \mathbf{U}(n + 2)\} \cdot \{\pm I\}/\{\pm I\}.$$

Here note that

$$\mathbf{SO}(n + 2, \mathbf{C}) \cap \mathbf{U}(n + 2) = \mathbf{SO}(n + 2)$$

where the real special orthogonal group

$$\mathbf{SO}(n + 2) = \{g \in \mathbf{SO}(n + 2, \mathbf{C}) : g \text{ preserves real span of } \{e_1, \ldots, e_{n+2}\}\}.$$

The isotropy subgroup of G_c at x_0 is

$$K = G_c \cap P = \left\{\pm\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathbf{SO}(n) \text{ and } D \in \mathbf{SO}(2)\right\}.$$

Now dimension and compactness of $G_c(x_0)$ gives us

$$X = G_c/K \qquad \text{compact presentation.}$$

That exhibits X as the Hermitian symmetric space $\mathbf{SO}(n+2)/\mathbf{SO}(n) \times \mathbf{SO}(2)$ of compact type. The symmetry at x_0 is

$$s = \pm\begin{pmatrix} I_n & 0 \\ 0 & -I_2 \end{pmatrix}$$

We use the indefinite Hermitian form on $\mathbf{C}^{n+2}$ given by $\langle u,v\rangle = -\sum_{k=1}^{n} u^k\bar{v}^k + (u^{n+1}\bar{v}^{n+1} + u^{n+2}\bar{v}^{n+2})$. That defines the indefinite unitary group $\mathbf{U}(n,2)$ and thus the intersections

$$\mathbf{O}(n,2) = \mathbf{O}(n+2,\mathbf{C}) \cap \mathbf{U}(n,2) \qquad \text{and}$$

$$\mathbf{O}^+(n,2) = \mathbf{SO}(n+2,\mathbf{C}) \cap \mathbf{U}(n,2).$$

The indefinite orthogonal group $\mathbf{O}(n,2)$ has 4 topological components and $\mathbf{O}^+(n,2)$ has 2 components. Define

$\mathbf{SO}(n,2)$: common identity component of $\mathbf{O}(n,2)$ and $\mathbf{O}^+(n,2)$;

that is our indefinite special orthogonal group. Now define

G_0 : connected transformation group on X from $\mathbf{SO}(n,2)$.

Then

$$G_0 \text{ is a noncompact real form of } G$$

and

$$G_0 \cap P = K \qquad \text{so} \qquad G_0/K \cong G_0(x_0) \qquad \text{open in } X.$$

Now

$$X_0 = G_0(x_0) \subset X \qquad \text{open } G_0\text{-orbit}$$

is the Hermitian symmetric space of noncompact type dual to X, inside X by Borel Embedding. Note $X_0 \cong \mathbf{SO}(n,2) / \mathbf{SO}(n) \times \mathbf{SO}(2)$.

We come to the Harish-Chandra embedding. The Lie algebra $\mathfrak{g}$ of G is identified with the Lie algebra of $\mathbf{SO}(n+2, \mathbf{C})$. Thus, in block form,

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tA = -A,\ {}^tD = -D,\ {}^tB = -C\right\}.$$

Now

$$\mathfrak{p} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tA = -A,\ {}^tB = -C,\ B = (B', iB'),\ D = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right\}$$

and

$$\mathfrak{m}^+ = \left\{\begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix} : B = (iB'', B'') \text{ where } B'' \text{ is } n \times 1\right\}.$$

Now, viewing the elements of $\mathbf{C}^n$ as column vectors ($n \times 1$ matrices),

$$Z \to \tilde{Z} = \begin{pmatrix} 0 & Z' \\ -{}^tZ' & 0 \end{pmatrix}, \qquad Z' = (iZ, Z), \qquad \text{maps } \mathbf{C}^n \cong \mathfrak{m}^+.$$

View Harish-Chandra's map $\xi : \mathfrak{m}^+ \to X$ as a map from $\mathbf{C}^n$. Now compute

$$Z' \cdot {}^tZ' = (iZ, Z)\begin{pmatrix} i\,{}^tZ \\ {}^tZ \end{pmatrix} = -Z \cdot {}^tZ + Z \cdot {}^tZ = 0$$

and

$${}^tZ' \cdot Z' = \begin{pmatrix} i\,{}^tZ \\ {}^tZ \end{pmatrix}(iZ, Z) = \begin{pmatrix} -{}^tZ \cdot Z & i\,{}^tZ \cdot Z \\ i\,{}^tZ \cdot Z & {}^tZ \cdot Z \end{pmatrix};$$

thus also

$$Z' \cdot ({}^tZ' \cdot Z') = (iZ, Z)\begin{pmatrix} -{}^tZ \cdot Z & i\,{}^tZ \cdot Z \\ i\,{}^tZ \cdot Z & {}^tZ \cdot Z \end{pmatrix} = 0.$$

That gives us

$$\tilde{Z} \cdot \tilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & Z'' \end{pmatrix} \text{where} \quad Z'' = \begin{pmatrix} {}^tZ \cdot Z & -i\,{}^tZ \cdot Z \\ -i\,{}^tZ \cdot Z & -{}^tZ \cdot Z \end{pmatrix}$$

and

$$(\tilde{Z})^3 = 0, \qquad \text{so} \quad (\tilde{Z})^k = 0 \quad \text{for} \quad k \geqq 3.$$

Now

$$\exp \tilde{Z} = \sum_{k=0}^{\infty} \frac{1}{k!} (\tilde{Z})^k = \begin{pmatrix} I & Z' \\ -{}^tZ' & I + \frac{1}{2} Z'' \end{pmatrix}.$$

As $\xi(Z) = (\exp \tilde{Z})(x_0)$ and $x_0 = [e_{n+1} + ie_{n+2}]$ now

$$\xi(Z) = [v] \text{ where } v = \begin{pmatrix} 2\,i\,Z \\ 1 + {}^tZ\cdot Z \\ i(1 - {}^tZ\cdot Z) \end{pmatrix}.$$

In order to locate $\xi^{-1}(X_0) \subset \mathfrak{m}^+$, we consider the open set

$$\Omega = \{Z \in \mathbf{C}^n : \xi(Z) = [v] \text{ with } \langle v,v \rangle > 0\}.$$

The formula for $\xi(Z)$ says

$$\Omega = \{Z \in \mathbf{C}^n : 1 + |{}^tZ\cdot Z|^2 - 2Z^*\cdot Z > 0\}.$$

Thus

$$\Omega = \{Z \in \mathbf{C}^n : (1 - Z^*\cdot Z)^2 > (Z^*\cdot Z)^2 - |{}^tZ\cdot Z|^2\}.$$

As $Z^*\cdot Z \geqq |{}^tZ\cdot Z| \geqq 0$, we take positive square roots, thus expressing

$$\Omega = \Omega_1 \cup \Omega_2 \qquad \text{disjoint union}$$

where Ω_1 is the nonempty bounded domain star-shaped from 0 given by

$$\Omega_1 = \{Z \in \mathbf{C}^n : 1 - Z^*\cdot Z > [(Z^*\cdot Z)^2 - |{}^tZ\cdot Z|^2]^{1/2}\}$$

and Ω_2 is the nonempty unbounded domain star shaped from ∞ given by

$$\Omega_2 = \{Z \in \mathbf{C}^n : Z^*\cdot Z - 1 > [(Z^*\cdot Z)^2 - |{}^tZ\cdot Z|^2]^{1/2}\}.$$

From Witt's Theorem, $\xi^{-1}(X_0)$ is the topological component of Ω containing 0, thus is Ω_1:

$$\begin{aligned} \xi^{-1}(X_0) &= \{Z \in \mathbf{C}^n : 1 - Z^*\cdot Z > [(Z^*\cdot Z)^2 - |{}^tZ\cdot Z|^2]^{1/2}\} \\ &= \{Z \in \mathbf{C}^n : 1 + |{}^tZ\cdot Z|^2 - 2Z^*\cdot Z > 0 \text{ and } 1 - Z^*\cdot Z > 0\}. \end{aligned}$$

Recall that G_0 is the image in G of the identity component of $\mathbf{O}^+(n,2) = \mathbf{SO}(n+2, \mathbf{C}) \cap \mathbf{U}(n,2)$. Thus we have a subgroup of G in which G_0 has index 2, given by

$$G_{\mathbf{R}} = G_0 \cup c^2 G_0, \qquad c^2 = \pm \begin{pmatrix} -1 & & & \\ & I_{n-1} & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

The notation c^2 comes from $c = c_\Psi$ in the Cayley transform theory. Here the point is that $\xi^{-1}(c^2 X_0) = \Omega_2$ and that $c^2 G_0(x_0) = G_0(c^2 x_0)$ open G_0-orbit on X. This explains the occurrence of Ω_2. Compare with the special case $n = 1$.

The action of G_0 on $\xi^{-1}(X_0)$ is also a bit of a mess. Denote

$$q : \mathbf{C}^n \to \mathbf{C} \quad \text{by} \quad q(Z) = {}^tZ \cdot Z; \qquad \text{so } |q(Z)| < 1 \qquad \text{on} \quad \xi^{-1}(X_0).$$

Now the formula for $\xi(Z)$ says that

$$\text{if} \quad q = q(Z),\ |q| \neq 1, \qquad \text{then} \quad \xi(Z) = [z] \quad \text{where} \quad z = \begin{pmatrix} \dfrac{2}{1-q} Z \\ -i\dfrac{1+q}{1-q} \\ 1 \end{pmatrix}.$$

Now we invert $\xi : \xi^{-1}(X_0) \to X_0$. First note from our new formula for ξ that

$$\text{if } [v] \in X_0, \text{ say } v = \begin{pmatrix} V \\ v_1 \\ v_2 \end{pmatrix}, \text{ then } v_2 \neq 0 \neq (iv_1/v_2) + 1.$$

Using that notation and the identity $p = -i(1+q)/(1-q) \Rightarrow q = (ip-1)/(ip+1)$, the new formula for ξ says $\xi^{-1}[v] = \{(1-q)/2v_2\} V$ where $q = \dfrac{i(v_1/v_2) - 1}{i(v_1/v_2) + 1}$. Substituting q into $\xi^{-1}[v]$ now

$$\text{if} \quad [v] \in X_0, \qquad v = \begin{pmatrix} V \\ v_1 \\ v_2 \end{pmatrix}, \qquad \text{then } \xi^{-1}[v] = \frac{1}{v_2 + iv_1} V.$$

The action of G_0 on $\xi^{-1}(X_0)$ is, of course,

$$g(Z) = \xi^{-1} g \xi(Z).$$

If we write

$$g = \pm\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2iZ \\ 1 + q(Z) \\ i - iq(Z) \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

the inversion formula for ξ says

$$g(Z) = \frac{1}{(1,i)\,(CZ_1 + DZ_2)}\,(AZ_1 + BZ_2).$$

The symmetric spaces X_0 and X are of rank $r = 2$ provided $n \geqq 2$, $r = 1$ in the degenerate case $n = 1$ where X is the Riemann sphere $P^1(\mathbf{C})$. If $n = 2$, then $X = P^1(\mathbf{C}) \times P^1(\mathbf{C})$; if $n > 2$, then X is irreducible.

X_0 is of tube type. Except for the reducible case ($n = 2$), now $G_0(c_\Psi^2 x_0) = c_\Psi^2 X_0$ is the only orbit $\neq X_0$ but $\cong X_0$. As $[e_{n+1} + ie_{n+2}] = x_0$, that says

$$c_\Psi^2 X_0 = G_0([e_{n+1} - ie_{n+2}]) \supset \xi(\Omega_2).$$

That consideration suggests the condition, which defines the partial Cayley transform in the case $n = 1$ (where $r = 1$ and $c_\Psi = c_1$), that

$$c_\Psi : e_1 \to -e_{n+1}, \qquad e_{n+1} \to e_1, \qquad e_k \to e_k \qquad \text{for} \quad 1 \neq k \neq n+1.$$

Note that this definition implies that

$$c_\Psi^2 \text{ represents the nonidentity component of } G_{\mathbf{R}},$$

i.e., that

$$\Omega_2 = \xi^{-1} G_0(c_\Psi^2 x_0) = \xi^{-1} c_\Psi^2 X_0.$$

If $n > 1$, so $r = 2$, we define the partial Cayley transforms $c_1, c_2 \in G_c$ such that $c_\Psi = c_1 c_2$ by

$$c_1 = \exp\frac{\pi}{4}\begin{pmatrix} 0 & B_1 \\ -{}^tB_1 & 0 \end{pmatrix}, \qquad {}^tB_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

and

$$c_2 = \exp\frac{\pi}{4}\begin{pmatrix} 0 & B_2 \\ -{}^tB_2 & 0 \end{pmatrix}, \qquad {}^tB_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus

$$c_1(e_1) = \frac{1}{\sqrt{2}}(e_1 - e_{n+1}), \qquad c_1(e_2) = \frac{1}{\sqrt{2}}(e_2 - e_{n+2}),$$

$$c_1(e_{n+1}) = \frac{1}{\sqrt{2}}(e_1 + e_{n+1}), \qquad c_1(e_{n+2}) = \frac{1}{\sqrt{2}}(e_2 + e_{n+2}),$$

$$\text{and} \qquad c_1(e_k) = e_k \qquad \text{for} \quad 2 < k \leqq n;$$

and

$$c_2(e_1) = \frac{1}{\sqrt{2}}(e_1 - e_{n+1}), \qquad c_2(e_2) = \frac{1}{\sqrt{2}}(e_2 + e_{n+2}),$$

$$c_2(e_{n+1}) = \frac{1}{\sqrt{2}}(e_1 + e_{n+1}), \qquad c_2(e_{n+2}) = \frac{1}{\sqrt{2}}(-e_2 + e_{n+2}),$$

$$\text{and} \qquad c_2(e_k) = e_k \qquad \text{for} \quad 2 < k \leqq n.$$

Note that the c_j have order 4 if $n = 2$, order 8 if $n > 2$.

We now assume $n > 2$. Then X is irreducible and of rank $r = 2$, so there are six orbits $G_0(x_{s,t})$, $0 \leqq s \leqq t \leqq 2$ with centers as follows.

$x_{0,0} =$	x_0	$= [e_{n+1} + ie_{n+2}]$	positive
$x_{0,1} =$	$c_1^2 x_0$	$= [e_1 + ie_2]$	negative
$x_{0,2} =$	$c_1^2 c_2^2 x_0$	$= [e_{n+1} - ie_{n+2}]$	positive
$x_{1,1} =$	$c_1 x_0$	$= [e_1 + ie_2 + e_{n+1} + ie_{n+2}]$	isotropic
$x_{1,2} =$	$c_1 c_2^2 x_0$	$= [e_1 - ie_2 - e_{n+1} + ie_{n+2}]$	isotropic
$x_{2,2} =$	$c_1 c_2 x_0$	$= [e_1 + ie_{n+2}]$	isotropic

The six orbits are the

3 open orbits $G_0(x_{0,0})$, $G_0(x_{0,1})$, and $G_0(x_{0,2})$,

2 intermediate orbits $G_0(x_{1,1})$ and $G_0(x_{1,2})$,

1 closed orbit $G_0(x_{2,2})$.

Now

$$\{Z \in \mathbf{C}^n : 1 + |{}^tZ \cdot Z| - 2Z^* \cdot Z > 0\} = \xi^{-1}(G_0(x_{0,0}) \cup G_0(x_{0,2})),$$
$$\{Z \in \mathbf{C}^n : 1 + |{}^tZ \cdot Z| - 2Z^* \cdot Z < 0\} = \xi^{-1}(G_0(x_{0,1})), \quad \text{and}$$
$$\{Z \in \mathbf{C}^n : 1 + |{}^tZ \cdot Z| - 2Z^* \cdot Z = 0\} = \xi^{-1}(G_0(x_{1,1}) \cup G_0(x_{1,2}) \cup G_0(x_{2,2})).$$

The topological boundary of X_0 in X is $G_0(x_{1,1}) \cup G_0(x_{2,2})$. There $G_0(x_{1,1})$ is the union of those boundary components that are symmetric spaces of rank 1 (in fact, unit disks in $\mathbf{C}^1$) and $G_0(x_{2,2})$ is the Bergman–Šilov boundary. Define compact totally geodesic submanifolds of X by $X_{(0)} = \{x_0\}$ point, $X_{(2)} = X$, and

$$X_{(1)} = \{[a(e_1 + ie_2) + b(e_{n+1} + ie_{n+2})] : (a,b) \neq (0,0)\}$$
$$= \{x \in X : x \subset (e_1 + ie_2) \wedge (e_{n+1} + ie_{n+2})\}.$$

Note that $X_{(1)}$ is a Riemann sphere. The Borel Embedded noncompact dual of $X_{(1)}$ is

$$X_{(1),0} = X_0 \cap X_{(1)} = \{[z(e_1 + ie_2) + (e_{n+1} + ie_{n+2})] : |z| < 1\}.$$

Now

$\{c_1 c_2 x_0\}$ and $c_1 X_{(1),0}$ are boundary components of X_0

and

$$G_0(x_{2,2}) = K \cdot c_1 c_2 x_0 \quad \text{and} \quad G_0(x_{1,1}) = K \cdot c_1 X_{(1),0}.$$

The maximal parabolic subgroups of G_0 are $N_{0,0} = G_0$,

$$N_{1,0} = \{g \in G_0 : g \text{ preserves the 2-plane } (e_1 + ie_2) \wedge (e_{n+1} + ie_{n+2})\},$$
$$N_{2,0} = \{g \in G_0 : g \text{ preserves the line } [e_1 + ie_{n+2}]\}.$$

The boundary components of X_0 have normalizers

$$N_{1,0} = \{g \in G_0 : gc_1 X_{(1),0} = c_1 X_{(1),0}\},$$
$$N_{2,0} = \{g \in G_0 : gc_1 c_2 x_0 = c_1 c_2 x_0\}.$$

We locate the ξ^{-1}-images of the boundary components of X_0 in X; they are the boundary components of

$$\{Z \in \mathbf{C}^n : 1 - Z^* \cdot Z > 0 \quad \text{and} \quad 1 + |{}^tZ \cdot Z|^2 - 2Z^* \cdot Z > 0\}$$

in $\mathbf{C}^n$:

$$\xi^{-1}(c_1 X_{(1),0}) = \{Z \in \mathbf{C}^n : {}^tZ = (z, iz, 0, \ldots, 0) \text{ with } \operatorname{Im} z < 0\}$$

and

$$\xi^{-1}(c_1 c_2 x_0) = {}^t(-i, 0, \ldots, 0).$$

$K = \{\pm\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathbf{SO}(n) \text{ and } D \in \mathbf{SO}(2)\}$ acts on $\mathfrak{m}^+$ by

$$\pm\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : Z \to W \quad \text{where} \quad (iW, W) = A \cdot (iZ, Z) \cdot D^{-1};$$

thus

$$\text{if} \quad D = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \text{then} \quad \pm\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : Z \to e^{i\theta} AZ.$$

Now

$$K \cdot \xi^{-1}(c_1 c_2 x_0) = \xi^{-1} G_0(x_{2,2})$$
$$= \{Z \in \mathbf{C}^n : Z^* Z = 1 \text{ and } e^{i\theta} Z \in \mathbf{R}^n \text{ for some } \theta\}$$

and

$$K \cdot \xi^{-1}(c_1 X_{(1),0}) = \xi^{-1} G_0(x_{1,1})$$
$$= \{Z \in \mathbf{C}^n : Z = U + iV \text{ with } U, V \in \mathbf{R}^n, U^* U = V^* V \neq 0 = U^* V\}.$$

The general theory ensures that the holomorphic arc components of $G_0(x_{s,t})$ are the $k\cdot\{c_1 \cdots c_s X_{(2-s)} \cap G_0(x_{s,t})\}$, $k \in K$, and that $N_{s,0}$ is the normalizer of $c_1 \cdots c_s X_{(2-s)} \cap G_0(x_{s,t})$.

We describe the K-equivariant holomorphic fibrations

$$\beta : G_0(x_{0,t}) \to K(x_{0,t}), \qquad t = 0, 1, 2,$$

of an open orbit over its maximal compact subvariety. If $t = 0$ or $t = 2$, then $K(x_{0,t})$ is reduced to a point; thus

$$\text{if} \quad t = 0 \quad \text{or} \quad t = 2, \qquad \text{then} \quad \beta(gx_{0,t}) = x_{0,t}.$$

Now suppose $t = 1$. We note

$$x_{0,1} = [e_1 + ie_2]$$

Denoting real span by Re,

$$K(x_{0,1}) = \{[u+iv] : u,v \in \mathrm{Re}(e_1, \ldots, e_n), \langle u,u\rangle = \langle v,v\rangle = -1, \langle u,v\rangle = 0\}.$$

Note also that

$$G_0(x_{0,1}) = \{[z_1(u + iv) + z_2(e_{n+1} + ie_{n+2})] : u,v \in \mathrm{Re}(e_1, \ldots, e_n), \langle u,u\rangle = \langle v,v\rangle = -1, \langle u,v\rangle = 0, |z_1| < 1, |z_2| < 1\}.$$

Now

$$\beta[z_1(u + iv) + z_2(e_{n+1} + ie_{n+2})] = [u + iv]$$

and the β-fibers are products of two unit disks.

The holomorphic arc components of $G_0(x_{1,1})$ and $G_0(x_{1,2})$ are unit disks; their maximal compact subvarieties are reduced to points. Thus the K-equivariant partially holomorphic fibrations

$$\beta : G_0(x_{1,t}) \to K(x_{1,t}), \qquad t = 1, 2,$$

are given by

$$\beta(k\cdot\{c_1 X_{(1)} \cap G_0(x_{1,t})\}) = k(x_{1,t}).$$

Finally, of course, $\beta: G_0(x_{2,2}) \to K(x_{2,2})$ has fiber reduced to a point.

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