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## REMARK ON SIEGEL DOMAINS OF TYPE III<sup>1</sup>

JOSEPH A. WOLF

ABSTRACT. Bounded symmetric domains have standard realizations as "Siegel domains of type III." Pjateckiĭ-Šapiro has introduced a more restrictive notion of "Siegel domain of type III." Here we give a direct proof that those standard realizations satisfy the additional conditions of the new definition.

1. **Introduction.** In his book on the geometry and function theory of the classical domains [1], I. I. Pjateckiĭ-Šapiro introduced the very useful concepts of Siegel domains of types I, II and III for application to the theory of automorphic functions. Given an irreducible classical bounded symmetric domain  $D$  and an equivalence class  $\{B\}$  of boundary components of  $D$ , he worked out an *ad hoc* realization of  $D$  as a Siegel domain of type III with base  $B$ . Later, A. Korányi and I [4] worked out the type III Siegel domain realizations of all bounded symmetric domains in an intrinsic and classification free manner. At about the same time, Pjateckiĭ-Šapiro [2] revised his concept of type III Siegel domain for convenience of application. In this note I extract a few small pieces of [4] to show directly that the type III Siegel domains (old sense), that Korányi and I constructed in [4], all satisfy the revised conditions of Pjateckiĭ-Šapiro for Siegel domains of type III (new sense).

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I wish to thank Professor I. I. Pjateckiĭ-Šapiro for his comment on the manuscript. It turns out that the result had been obtained by Pjateckiĭ-Šapiro in somewhat more generality [3], using the detailed theory of bounded homogeneous domains. The present proof applies only to bounded symmetric domains, but it is somewhat more direct and elementary for that important case.

2. **Definitions.** Let  $U_R$  be a real vector space and  $\Omega \subset U_R$  a non-empty convex open cone that does not contain a straight line. This data defines a *Siegel domain of type I* (=tube domain) in the complex vector space  $U = U_R + iU_R = U_R \otimes C$ , which is

$$(S_1) \quad \{u \in U : \text{Im } u \in \Omega\}.$$

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Let  $V$  be a second complex vector space and  $F: V \times V \rightarrow U$  a map that is hermitian relative to complex conjugation of  $U$  over  $U_R$ . Suppose that  $F$  is positive definite in the sense that  $0 \neq v \in V$  implies  $0 \neq F(v, v) \in \bar{\Omega}$  (= closure of  $\Omega$ ). This data defines a *Siegel domain of type II* in  $U \oplus V$ , which is

$$(S_{II}) \quad \{(u, v) \in U \oplus V : \text{Im } u - F(v, v) \in \Omega\}.$$

Siegel domains of type I are the special case  $V=0$ .

There is no variation in these original definitions of Pjateckiĭ-Šapiro for Siegel domains of types I and II.

Let  $W$  be a third complex vector space,  $B \subset W$  a bounded domain containing the origin  $0$ , and  $w \mapsto F_w$  a smooth map of  $B$  into  $\text{Hom}_R(V \otimes_R V, U)$  such that

- (1)  $F_w = F_w^h + F_w^b$  where  $F_w^h$  is hermitian and  $F_w^b$  is  $C$ -bilinear,
- (2)  $F_0 = F$ , so  $F_0 = F_0^h$  and  $F_0^b = 0$ , and
- (3) if  $v \in V$  and either  $F_w(v, V) = 0$  or  $F_w(V, v) = 0$  then  $v = 0$ .

This data defines a *Siegel domain of type III* (old sense) in  $U \oplus V \oplus W$ , which is

$$(S_{III \text{ old}}) \quad \{(u, v, w) \in U \oplus V \oplus W : w \in B \text{ and } \text{Im } u - \text{Re } F_w(v, v) \in \Omega\}.$$

$B$  is its *base*. Siegel domains of type II are the case  $W=0$ , i.e.  $B=(0)$ .

Start again with the data  $(U_R, \Omega, V, F)$  for a Siegel domain of type II. That defines a complex vector space

$$(4) \quad W_{\text{univ}} = \{p: V \rightarrow V \text{ conjugate-linear: } F(pv, v') = F(pv', v) \text{ for all } v, v' \in V\}.$$

That vector space contains a bounded domain

$$(5) \quad B_{\text{univ}} = \{p \in W_{\text{univ}} : \text{if } 0 \neq v \in V \text{ then } 0 \neq F(v, v) - F(pv, pv) \in \bar{\Omega}\}.$$

If  $p \in B_{\text{univ}}$  then  $I+p$  is invertible, for  $(I+p)v=0$  implies  $F(v, v) = F(pv, pv)$ . Thus we have maps

$$(6) \quad L_p: V \times V \rightarrow U \quad \text{defined by} \quad L_p(v, v') = F(v, (I+p)^{-1}v') \quad \text{for } p \in B_{\text{univ}}.$$

Now the *universal Siegel domain of type III* associated to  $(U_R, \Omega, V, F)$  is the domain in  $U \oplus V \oplus W_{\text{univ}}$  which is

$$(S_{\text{III univ}}) \quad \left\{ (u, v, p) \in U \oplus V \oplus W_{\text{univ}} : \right. \\ \left. p \in B_{\text{univ}} \text{ and } \text{Im } u - \text{Re } L_p(v, v) \in \Omega \right\}.$$

LEMMA 1. *If  $p \in B_{\text{univ}}$  then  $I - p^2$  is invertible and the map  $L_p$  satisfies (1), (2) and (3) with  $L_p^h$  and  $L_p^b$  given by*

$$(7) \quad \begin{aligned} L_p^h(v, v') &= F(v, (I - p^2)^{-1}v') \quad \text{and} \\ L_p^b(v, v') &= -F(v, (I - p^2)^{-1}pv'). \end{aligned}$$

*In particular, universal Siegel domains of type III are Siegel domains of type III in the old sense.*

PROOF. If  $(I - p^2)v = 0$  then  $F(v, v) = F(p^2v, v) = F(pv, pv)$  by (4), so  $v = 0$  by (5); that proves  $I - p^2$  invertible. Now  $(I - p^2)^{-1}(I - p) = (I + p)^{-1}$  by power series expansion in a neighborhood of 0 and then analytic continuation to  $B_{\text{univ}}$ . Given  $L_p^h$  and  $L_p^b$  as in (7) it follows that  $L_p = L_p^h + L_p^b$ . Now (1) and (2) are immediate, and (3) follows.  $\square$

Again let  $(U_R, \Omega, V, F)$  be the data for a Siegel domain of type II. Let  $W$  be a third complex vector space,  $B \subset W$  a bounded domain and  $\phi: B \rightarrow B_{\text{univ}}$  a holomorphic map with  $0 \in \phi(B)$ . This data defines a *Siegel domain of type III* (new sense) in  $U \oplus V \oplus W$  with base  $B$ , which is

$$(S_{\text{III new}}) \quad \left\{ (u, v, w) \in U \oplus V \oplus W : \right. \\ \left. w \in B \text{ and } \text{Im } u - \text{Re } L_{\phi(w)}(v, v) \in \Omega \right\}.$$

Lemma 1 says

LEMMA 2. *A Siegel domain of type III in the new sense is a Siegel domain of type III in the old sense for which  $F_w(v, v') = F(v, (I + \phi(w))^{-1}v')$ .*

**3. Siegel domain realizations.** We use the notation and conventions of [4] for our verification, even though they may be too complicated for other purposes. Now  $D$  is a bounded symmetric domain embedded in its antihomomorphic tangent space  $\mathfrak{p}^-$  by the method of Harish-Chandra.  $\Delta$  is the corresponding maximal set of strongly orthogonal noncompact positive roots,  $\Gamma$  is a subset of  $\Delta$ ,  $D_\Gamma$  is the subdomain of  $D$  whose maximal set of strongly orthogonal noncompact roots is  $\Gamma$ , and  $c_{\Delta-\Gamma}$  is the partial Cayley transform involving the elements of  $\Delta - \Gamma$ . The space  $\mathfrak{p}^- = U \oplus V \oplus W$  where

$U = \mathfrak{p}_{\Delta-\Gamma,1}^-$  is the  $(+1)$ -eigenspace of  $\text{ad}(c_{\Delta-\Gamma})^4$  on the subspace of  $\mathfrak{p}^-$  for  $\Delta-\Gamma$ ,

$V = \mathfrak{p}_2^{\Gamma-}$  is the  $(-1)$ -eigenspace of  $\text{ad}(c_{\Delta-\Gamma})^4$  on  $\mathfrak{p}^-$ , and

$W = \mathfrak{p}_{\Gamma}^-$  is the subspace of  $\mathfrak{p}^-$  for  $\Gamma$ , and is the ambient space of the domain  $D_{\Gamma}$ .

We also have

$U_R = \mathfrak{n}_1^{\Gamma-}$  real form of  $U = \mathfrak{p}_{\Delta-\Gamma,1}^-$  defined in [4, 6.1.3], and

$\Omega = \mathfrak{c}^{\Gamma}$  self dual cone in  $U_R = \mathfrak{n}_1^{\Gamma-}$  defined in [4, §7.1].

In [4] following Lemma 7.2 one finds the following definitions, which we rewrite according to the “dictionary” just above. If  $w \in D_{\Gamma}$  then  $\mu(w): V \rightarrow V$  is the conjugate-linear map given by  $\mu(w)v = \text{ad}(w)\text{ad}(c_{\Delta-\Gamma})^2 \cdot \nu(v)$  where  $\nu$  is a certain complex conjugation, and  $\Lambda_w: V \times V \rightarrow U$  is map given by

$$\begin{aligned} \Lambda_w(v, v') &= -\frac{i}{2} [v, \text{ad}(c_{\Delta-\Gamma})^2 \nu(I + \mu(w))^{-1} v'] \\ &= -\frac{i}{2} \text{ad}(c_{\Delta-\Gamma})^2 \nu [\text{ad}(c_{\Delta-\Gamma})^2 \nu v, (I + \mu(w))^{-1} v']. \end{aligned}$$

In [4, §7] it is shown that

$$(8) \quad \begin{aligned} c_{\Delta-\Gamma}(D) &= \{(u, v, w) \in U \oplus V \oplus W: \\ &w \in D_{\Gamma} \text{ and } \text{Im } u - \text{Re } \Lambda_w(v, v) \in \Omega\} \end{aligned}$$

and that the  $\Lambda_w, w \in D_{\Gamma}$ , satisfy (1), (2) and (3), so that

$$(9) \quad c_{\Delta-\Gamma}(D) \text{ is a Siegel domain of type III (old sense) in } \mathfrak{p}^-.$$

**THEOREM.**  $\mu$  is a holomorphic map from  $D_{\Gamma}$  to the domain  $B_{\text{univ}}$  for the type II Siegel domain data  $(U_R, \Omega, V, \Lambda_0)$ . Thus  $c_{\Delta-\Gamma}(D)$  is a Siegel domain of type III (new sense) in  $\mathfrak{p}^-$ .

**PROOF.** Let  $W_{\text{univ}}$  and  $B_{\text{univ}}$  be defined as in (4) and (5) for the type II Siegel domain data  $(U_R, \Omega, V, \Lambda_0)$ . If  $w \in D_{\Gamma}$  then  $\mu(w)$  acts on  $V$  by: the complex conjugation  $\nu$ , then  $\text{ad}(c_{\Delta-\Gamma})^2$ , then  $\text{ad}(w)$ . Thus  $\mu$  is a holomorphic map of  $D_{\Gamma}$  into the space of conjugate-linear transformations of  $V$ . If  $v, v' \in V$  and  $w \in D_{\Gamma}$  then [4, Lemma 7.3 (iii)] says that  $\Lambda_0(v, \mu(w)v') = \Lambda_0(v', \mu(w)v)$ ; after complex conjugation of  $U$  over  $U_R$  this says that  $\mu(w) \in W_{\text{univ}}$ . Now  $\mu$  is a holomorphic map of  $D$  into  $W_{\text{univ}}$ .

Let  $w \in D_{\Gamma}$  and  $0 \neq v \in V$ . Define  $v' = (I - \mu(w)^2)v$  so  $0 \neq v' \in V$ . We compute

$$\begin{aligned}
\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) & \\
&= \Lambda_0(v, v) - \Lambda_0(\mu(w)^2v, v) \quad \text{because } \mu(w) \in W_{\text{univ}}, \\
&= \Lambda_0((I - \mu(w)^2)v, v) \\
&= \Lambda_0(v', (I - \mu(w)^2)^{-1}v') \quad \text{by definition of } v', \\
&= \Lambda_0\left(v', \sum_{n=0}^{\infty} \mu(w)^{2n}v'\right) \quad \text{by power series expansion,} \\
&= \sum_{n=0}^{\infty} \Lambda_0(\mu(w)^{2n}v', \mu(w)^{2n}v') \quad \text{because } \mu(w) \in W_{\text{univ}}.
\end{aligned}$$

If  $\mu(w)^{2n}v' \neq 0$  then  $0 \neq \Lambda_0(\mu(w)^{2n}v', \mu(w)^{2n}v') \in \bar{\Omega}$ . As  $\bar{\Omega}$  is convex, and is strictly convex at 0, now  $\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) \in \bar{\Omega}$  and  $\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) = 0$  if and only if  $\mu(w)^{2n}v' = 0$  for all  $n \geq 0$ . But  $v' \neq 0$ , i.e.  $\mu(w)^0v' \neq 0$ , so now

$$0 \neq \Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) \in \bar{\Omega}.$$

We have proved  $\mu(w) \in B_{\text{univ}}$ , completing the proof of the theorem.  $\square$

#### REFERENCES

1. I. I. Pjateckiĭ-Šapiro, *Geometry of classical domains and theory of automorphic functions*, Fizmatgiz, Moscow, 1961. MR 25 #231.
2. ———, *Arihmetic groups in complex domains*, Uspehi Mat. Nauk 19 (1964), no. 6 (120), 93–121 = Russian Math. Surveys 19 (1964), no. 6, 83–109. MR 32 #7790.
3. ———, *Automorphic functions and the geometry of classical domains*, Gordon and Breach, New York, 1969. (This is an English translation of [1] incorporating the theory of bounded homogeneous domains.) MR 40 #5908.
4. J. A. Wolf and A. Korányi, *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. 87 (1965), 899–939. MR 33 #229.

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