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A COMMUTATIVITY CRITERION FOR CLOSED SUBGROUPS OF COMPACT LIE GROUPS

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ABSTRACT. Let Γ be a closed subgroup of a compact Lie group G. If the identity component Γ_0 is commutative, and if the order of Γ/Γ_0 is prime to the order of the Weyl group of G, then it is shown that Γ is commutative. If G is a classical group this extends a theorem of Burnside on finite linear groups. If G is exceptional this gives some information on Cayley-Dickson algebras, Jordan algebras and the Cayley projective plane.

Let Γ be a complex linear group of degree n and finite order $|\Gamma|$. If every prime divisor p of $|\Gamma|$ satisfies $p>n$, then it is both standard and clear that the character of the representation $\Gamma \subset GL(n, \boldsymbol{C})$ is a sum of characters of degree 1, so Γ is a commutative group on $\leqq n$ generators. Here we extend that simple comment to a remark on subgroups of compact connected Lie groups:

THOEREM. *Let G be a compact connected Lie group and let W_G be its Weyl group. Let $\Gamma \subset G$ be a closed subgroup such that*

(i) *the identity component Γ_0 of Γ is commutative, and*

(ii) *the orders $|\Gamma/\Gamma_0|$ and $|W_G|$ are relatively prime.*

Then Γ is contained in a maximal torus subgroup T of G. In particular, Γ is commutative and Γ/Γ_0 can be generated by $\leqq \dim T - \dim \Gamma_0$ elements.

The special case $\Gamma_0 = \{1\}$, i.e. Γ finite, is:

COROLLARY 1. *Let G be a compact connected Lie group, W_G its Weyl group, and $\Gamma \subset G$ a finite subgroup such that the orders $|\Gamma|$ and $|W_G|$ are relatively prime. Then Γ is contained in a maximal torus subgroup T of G. In particular, Γ is commutative on $\leqq \dim T = \operatorname{rank} G$ generators.*

Let $\pi(W_G)$ denote the set of all prime divisors of $|W_G|$. G is locally isomorphic to a direct product of a torus and some simple groups G_i, and W_G is the direct product of the W_{G_i}, so $\pi(W_G)$ is the union of the $\pi(W_{G_i})$. To apply the theorem and its special case Corollary 1, now, one must know $\pi(W_G)$ when G is simple. For simple G, it is given as follows (cf. [**2**, §8.10]).

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Cartan class of G	$\pi(W_G)$
A_n, $n \geqq 1$	$\{\text{primes } p : p \leqq n+1\}$
B_n, C_n or D_n, $n \geqq 2$	$\{\text{primes } p : p \leqq n\}$
G_2 or F_4	$\{2, 3\}$
E_6	$\{2, 3, 5\}$
E_7 or E_8	$\{2, 3, 5, 7\}$

The result mentioned at the beginning of this note is the case $\Gamma_0 = \{1\}$ of part 1 of the following specialization of the theorem to classical linear groups.

COROLLARY 2. *Let Γ be a compact linear group of degree d. Suppose that its identity component Γ_0 is commutative. Let $\pi(\Gamma/\Gamma_0)$ denote the set of all prime divisors of $|\Gamma/\Gamma_0|$.*

1. *If $p > d$ for every $p \in \pi(\Gamma/\Gamma_0)$ then Γ is commutative and Γ/Γ_0 can be generated by $\leqq d - \dim \Gamma_0$ elements.*

2. *If Γ has a nonsingular symmetric or antisymmetric bilinear invariant, and if $p > \max\ (2, [d/2])$ for every $p \in \pi(\Gamma/\Gamma_0)$, then Γ is commutative and Γ/Γ_0 can be generated by $\leqq [d/2] - \dim \Gamma_0$ elements.*

One also has an interesting specialization of the theorem to groups of type G_2 or F_4. Here $\boldsymbol{F}$ denotes a subfield of the complex number field $\boldsymbol{C}$, and "compact" refers to the topology on matrices over $\boldsymbol{C}$ of appropriate degree.

COROLLARY 3. *Let Γ be a compact group, Γ_0 commutative, and* $2, 3 \notin \pi(\Gamma/\Gamma_0)$.

1. *If Γ is a group of automorphisms of a Cayley-Dickson algebra $\boldsymbol{A}$ over $\boldsymbol{F}$, then Γ is commutative and Γ/Γ_0 can be generated by $\leqq 2 - \dim \Gamma_0$ elements.*

2. *If Γ is a group of automorphisms of an exceptional simple Jordan algebra $\boldsymbol{J}$ over $\boldsymbol{F}$, then Γ is commutative and Γ/Γ_0 can be generated by $\leqq 4 - \dim \Gamma_0$ elements.*

3. *If Γ is a group of collineations of the* (*real*) *Cayley projective plane $\boldsymbol{P}$, then Γ is commutative, Γ/Γ_0 can be generated by $\leqq 4 - \dim \Gamma_0$ elements, and Γ has a fixed point on $\boldsymbol{P}$.*

PROOF OF THEOREM. If $|W_G|=1$ then G is a torus and the assertion is vacuous. Now suppose $|W_G|>1$. As W_G is generated by reflections it has even order, so Γ/Γ_0 has odd order, and the Feit-Thompson Theorem [1] proves Γ/Γ_0 solvable.

If $\Gamma=\Gamma_0$ the assertion is vacuous. Now assume $|\Gamma/\Gamma_0|>1$. As Γ/Γ_0 is solvable it has a normal subgroup Δ/Γ_0 of prime index p. By induction on $|\Gamma/\Gamma_0|$, Δ is contained in a maximal torus subgroup S of G.

Let $Z_G(\Delta)$ denote the centralizer of Δ in G. Let K denote the identity component of $Z_G(\Delta)$. As S is connected, $\Delta\subset S\subset Z_G(\Delta)$ implies $\Delta\subset S\subset K$. Thus K is a closed connected subgroup of maximal rank in G, and Δ is central in K.

Γ normalizes Δ, thus $Z_G(\Delta)$, and thus normalizes K. Let $\gamma\in\Gamma-\Delta$. Now K has a maximal torus T that is stable under conjugation by γ. Note $\Delta\subset T$ because Δ is central in K. Now Γ normalizes T. Let $N_G(T)$ denote the normalizer of T in G, so $\Gamma\subset N_G(T)$, and represent $W_G=N_G(T)/T$. Then $N_G(T)\to W_G$ induces a homomorphism $\Gamma/\Gamma_0\to W_G$. As $|\Gamma/\Gamma_0|$ is prime to $|W_G|$, the image of $\Gamma/\Gamma_0\to W_G$ is trivial, so $\Gamma\subset T$. Q.E.D.

PROOF OF COROLLARY 1. Let $\Gamma_0=\{1\}$ in the theorem.

PROOF OF COROLLARY 2. We start with a compact group Γ in the general linear group $\boldsymbol{GL}(d, \boldsymbol{C})$, so we may conjugate and assume Γ in the unitary group $\boldsymbol{U}(d)$. The chart says $\pi(W_{\boldsymbol{U}(d)})=\{\text{primes } p:p\leqq d\}$ because $\boldsymbol{U}(d)$ is locally isomorphic to the product of the special unitary group $\boldsymbol{SU}(d)$ (type A_{d-1}) and a circle group. So the hypothesis of part 1 says that $|\Gamma/\Gamma_0|$ is prime to $|W_{\boldsymbol{U}(d)}|$, and the assertion of part 1 follows from the theorem.

Suppose that Γ has a nonsingular bilinear invariant β. If β is symmetric then $\Gamma\subset\boldsymbol{O}(d)$ orthogonal group. Under the hypothesis of part 2, Γ/Γ_0 has odd order, so $\Gamma\subset\boldsymbol{SO}(d)$. The latter is of type B_n for $d=2n+1$, type D_n for $d=2n$, so we have $\Gamma\subset G$ with rank $G=[d/2]$ and $|\Gamma/\Gamma_0|$ prime to $|W_G|$. If β is antisymmetric then $\Gamma\subset\boldsymbol{Sp}(d/2)$, symplectic group which is of type $C_{d/2}$, and again $\Gamma\subset G$ with rank $G=[d/2]$ and $|\Gamma/\Gamma_0|$ prime to $|W_G|$. The assertion of part 2 now follows from the theorem. Q.E.D.

PROOF OF COROLLARY 3. We write $\mathrm{Aut}(\cdot)$ for the automorphism group, $\boldsymbol{A}_C$ and $\boldsymbol{J}_C$ for the scalar extensions $\boldsymbol{A}\otimes_F\boldsymbol{C}$ and $\boldsymbol{J}\otimes_F\boldsymbol{C}$, $\boldsymbol{G}_2$ and $\boldsymbol{F}_4$ for the compact connected simple groups of types G_2 and F_4, and $\boldsymbol{G}_2^C$ and $\boldsymbol{F}_4^C$ for their complexifications.

In part 1, $\Gamma\subset\mathrm{Aut}(\boldsymbol{A})\subset\mathrm{Aut}(\boldsymbol{A}_C)=\boldsymbol{G}_2^C$, and the maximal compact subgroups of $\boldsymbol{G}_2^C$ are the conjugates of $\boldsymbol{G}_2$, so we may take $\Gamma\subset\boldsymbol{G}_2$. As

$\pi(W_{G_2}) = \{2, 3\}$ our assertion follows directly from the theorem.

In part 2, $\Gamma \subset \mathrm{Aut}(J) \subset \mathrm{Aut}(J_C) = \boldsymbol{F}_4^C$, so we may assume $\Gamma \subset \boldsymbol{F}_4$ as above, and the assertion follows directly from the theorem.

The collineation group of the Cayley plane $\boldsymbol{P}$ is a connected Lie group of type E_6 whose maximal compact subgroups are the conjugates of the elliptic group $\boldsymbol{F}_4$ of $\boldsymbol{P}$. Thus we may take $\Gamma \subset \boldsymbol{F}_4$, and the theorem says that Γ is contained in a maximal torus T of $\boldsymbol{F}_4$. As a homogeneous space, $\boldsymbol{P} = \boldsymbol{F}_4/\mathbf{Spin}(9)$, so T (and thus also Γ) has a fixed point. Q.E.D.

Remark 1. It would be preferable to avoid use of the powerful Feit-Thompson result [**1**].

Remark 2. Parts 1 and 2 of Corollary 3 remain valid when Γ is finite and F is an arbitrary field of characteristic zero.

References

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