COMPLEX MANIFOLDS AND UNITARY REPRESENTATIONS

Joseph A. Wolf

1. *Introduction*

Let G_0 be a locally compact group, $\hat{G}_0$ the set of its equivalence classes of irreducible unitary representations. Under certain circumstances one can work out some connections between $\hat{G}_0$ and "partially complex" homogeneous spaces X_0 of G_0, as follows. Let $\mathbb{E} \to X_0$ be a hermitian, G_0-homogeneous, "partially holomorphic" complex vector bundle. Then G_0 acts on the spaces $H_2^{0,q}(\mathbb{E})$ which consist of all measurable q-forms ω on X_0, values in $\mathbb{E}$, such that

(i) ω is harmonic and of type $(0,q)$, with finite L_2-norm, on almost all complex analytic pieces of X_0, and

(ii) the above L_2-norms give a square integrable function on the space of all complex analytic pieces of X_0.

Then, if matters are correctly arranged, the action of G_0 on $H_2^{0,q}(\mathbb{E})$ is an irreducible unitary representation $\pi_{\mathbb{E}}^q$. The immediate problems are the following.

1. Find all partially complex G_0-homogeneous spaces X_0 that carry partially holomorphic hermitian G_0-homogeneous complex vector bundles $\mathbb{E} \to X_0$ such that some $\pi_{\mathbb{E}}^q$ is irreducible and unitary.

2. Given X_0 as above, find all $\mathbb{E} \to X_0$ and q as above.

3. Given X_0, $\mathbb{E}$ and q as above, identify the class $[\pi_{\mathbb{E}}^q] \in \hat{G}_0$.

4. Describe the subset of $\hat{G}_0$ consisting of all classes $[\pi]$ containing an element of the form $\pi_{\mathbb{E}}^q$ above.

5. Given a class $[\pi]$ in that subset of $\hat{G}_0$, find all $(X_0, \mathbb{E}$, such that $\pi_{\mathbb{E}}^q \in [\pi]$.

If G_0 is a compact connected Lie group, these questions are settled by the Bott-Borel-Weil theorem ([2], [13]). If G_0 is a c nected semisimple Lie group with finite center, then the case of crete series representations has almost been settled by Harish-Ch [7], Narasimhan-Okamoto [16] and (principally) W. Schmid [18].

I will explore these questions for reductive real Lie groups only finitely many components in [22]. There it turns out that t subset of $\hat{G}_0$ consisting of the $[\pi_{\mathbb{E}}^q]$ is "almost" large enough to port Plancherel measure. Here I want to give a rough but self-contained description of the results of [22] for the case where G is a connected linear semisimple Lie group. That avoids technica problems with the center and the component group of G_0, with the cise character theory developed by Harish-Chandra ([8], [9]), and the still tentative matter of the various "complementary" series representations of G_0.

In section 2 we recall some results from [21] concerning G_0-orbits on complex flag manifolds $X = G/P$, where G is the comple fication of G_0. The spaces X_0 will be orbits $G_0(x_0) \subset X = G/P$, various choices of the parabolic subgroup $P \subset G$.

In section 3 we give a brief description of the realization discrete series representations of G_0 on spaces $H_2^{0,q}(\mathbb{E})$ where $\mathbb{E}$ a holomorphic vector bundle over an open G_0-orbit.

In section 4 we work out the geometric realization of princi series representations of G_0 on spaces $H_2^{0,0}(\mathbb{E})$ where $\mathbb{E}$ is a her mitian, partially holomorphic, G_0-homogeneous, vector bundle over closed G_0-orbit on X.

In section 5 we recall Harish-Chandra's construction of cert unitary representations $\pi_{\mu,\eta}$ of G_0, whose classes $[\pi_{\mu,\eta}] \in \hat{G}_0$ su the Plancherel measure there. The discrete and the principal series representations are the two extreme cases.

In section 6 we work out the geometric realization, for *most* of the classes $[\pi_{\mu,\eta}] \in \hat{G}_0$ mentioned just above, on spaces $H_2^{0,q}(\mathbb{E})$, where the bundle $\mathbb{E}$ sits over a measurable integrable G_0-orbit on X.

The incompleteness of our results on geometric realization of the $\pi_{\mu,\eta}$ comes down to some serious analytic problems in the geometric realization of discrete series representations whose lowest highest weight (on a maximal compact subgroup) fails to be "sufficiently" nonsingular. See [16] and [18].

2. *Real group orbits on complex flag manifolds*

The "partially complex" homogeneous spaces of semisimple groups G_0, mentioned in the Introduction, will be G_0-orbits on complex homogeneous spaces of the complexification of G_0. In this section 2 the pertinent results on those G_0-orbits are extracted from [21], omitting most proofs and structural results designed for applications ([22], [23]) more delicate than those that concern us here.

Let G be a complex connected semisimple Lie group. If $P \subset G$ is a complex Lie subgroup then the following conditions are equivalent.

(2.1a) the coset space $X = G/P$ is compact.

(2.1b) $X = G/P$ is a compact simply connected kaehler manifold.

(2.1c) $X = G/P$ is a projective algebraic variety.

(2.1d) $X = G/P$ is a closed G-orbit in a projective representation.

(2.1e) P contains a Borel subgroup of G.

In that case we define

(2.2a) P is a *parabolic subgroup* of G, and

(2.2b) $X = G/P$ is a *complex flag manifold* of G.

We use script for Lie algebras. Thus G denotes the Lie alge of G. If $P \subset G$ Lie subgroup then $P \subset G$ is the corresponding su algebra. We say that

(2.3) P is a *parabolic subalgebra* of G if P is a parabolic su group of G.

In that case P is the analytic subgroup of G for P.

We recall the structure of parabolic subgroups of G and para subalgebras of G. Choose

(2.4a) a Cartan subalgebra H of G and

(2.4b) a system Π of simple H-roots of G.

Given a subset $\Phi \subset \Pi$ we denote

(2.5a) Φ^r: all roots that are linear combinations of elements of Φ;

(2.5b) Φ^u: all positive roots not contained in Φ^r; and

(2.5c) $P_\Phi^r = H + \sum_{\Phi^r} G^\phi$, $P_\Phi^u = \sum_{\Phi^u} G^\phi$ and $P_\Phi = P_\Phi^r + P_\Phi^u$.

Then G has complex analytic subgroups

(2.5d) P_Φ^r for P_Φ^r, P_Φ^u for P_Φ^u, $P_\Phi = P_\Phi^r \cdot P_\Phi^u$ for P_Φ.

Now

(2.6a) P_Φ is a parabolic subgroup of G;

(2.6b) P_Φ has unipotent radical P_Φ^u, reductive part P_Φ^r;

(2.6c) Φ is a system of simple H-roots for P_Φ^r; and

(2.6d) every parabolic subgroup of G is conjugate to just one

Let $X = G/P$ complex flag manifold. As P is its own normali in G, we have a bijective correspondence $x \leftrightarrow P_x$ between X and the set of G-conjugates of P, given by

(2.7) $$P_x = \{g \in G : g(x) = x\}.$$

We will constantly use (2.7) without further reference.

Let G_0 be a real form of G. Thus G_0 is the real analytic subgroup of G for a real form G_0 of the Lie algebra G. Denote

(2.8) τ: complex conjugation of $\mathcal{G}$ over $\mathcal{G}_0$, G over G_0.

Now consider an orbit $G_0(x) \subset X = G/P$. The isotropy subgroup of G_0 at x is $G_0 \cap P_x$, which has Lie algebra

(2.9a) $\mathcal{G}_0 \cap \mathcal{P}_x = \mathcal{G}_0 \cap (\mathcal{P}_x \cap \tau \mathcal{P}_x)$ real form of $\mathcal{P}_x \cap \tau \mathcal{P}_x$.

The intersection of two Borel subgroups of G contains a Cartan subgroup; it follows that there exists

(2.9b) $H_0 \subset \mathcal{G}_0 \cap \mathcal{P}_x$ such that $H = H_0^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}$.

Now choose

(2.9c) Π simple H-root system, $\Phi \subset \Pi$, such that $\mathcal{P}_x = \mathcal{P}_\Phi$.

That done, we have

(2.9d) $$\mathcal{P}_x \cap \tau \mathcal{P}_x = \{H + \sum_{\Phi^r \cap \tau \Phi^r} \mathcal{G}^\phi\} + \{(\sum_{\Phi^r \cap \tau\Phi^u} + \sum_{\Phi^u \cap \tau\Phi^r} + \sum_{\Phi^u \cap \tau\Phi^u}) \mathcal{G}^\phi\}$$

where the second term in {braces} is the unipotent radical and the first term is its reductive complement.

The first consequence of (2.9) is the fact ([21], Theorem 2.6) that

(2.10) there are only finitely many G_0-orbits on X.

For there are only finitely many G_0-conjugacy classes of real Cartan subalgebras H_0; to each H_0 there are only finitely many systems Π of simple H-roots; to each (H_0,Π) there is just one set $\Phi \subset \Pi$ with P_Φ conjugate (in G) to P. In particular, by dimension,

(2.11) there are open G_0-orbits and closed G_0-orbits on X.

The second consequence of (2.9) is the fact ([21], Theorem 2.12) that

(2.12) $G_0(x)$ has real codimension $|\Phi^u \cap \tau\Phi^u|$ in X.

In particular,

(2.13) $G_0(x)$ is open in X if, and only if, $\Phi^u \cap \tau\Phi^u$ is empt

Let K be a maximal compact subgroup of G_0 and consider the C

decomposition

(2.14) $$\mathcal{G}_0 = \mathcal{K} + \mathcal{M}_0.$$

Every Cartan subalgebra of $\mathcal{G}_0$ is conjugate to one of the form

(2.15) $H_0 = H_T + H_V$; $H_T = H_0 \cap \mathcal{K}$, $H_V = H_0 \cap \mathcal{M}_0$.

Given H_0 in that form, the following conditions are equivalent (

Lemma 4.1).

(2.16a) H_T is a Cartan subalgebra of $\mathcal{K}$.

(2.16b) H_T contains a regular element of $\mathcal{G}$.

(2.16c) Some system Π of simple $H_0^{\mathbb{C}}$-roots has $\tau\Pi = -\Pi$.

Under conditions (2.16abc) we say that

(2.16d) H_0 is a *maximally compact* Cartan subalgebra of $\mathcal{G}_0$.

By careful application of (2.13) one can prove ([21], Theorem 4.5) that an orbit $G_0(x) \subset X$ is open if, and only if, there exist

(2.17a) a maximally compact Cartan subalgebra $H_0 \subset \mathcal{G}_0$ and

(2.17b) a simple $H_0^{\mathbb{C}}$-root system Π with $\tau\Pi = -\Pi$

such that

(2.17c) $\mathcal{P}_x = \mathcal{P}_\Phi$ for some subset $\Phi \subset \Pi$.

That criterion allows us to enumerate the open G_0-orbits as foll ([21], Theorem 4.9). Fix one open orbit $G_0(x)$ and H_0, Π and Φ as in (2.17). Denote

(2.18a) W_G: Weyl group of G relative to $H_0^{\mathbb{C}}$.

(2.18b) W_K: Weyl group of K for H_T, subgroup of W_G.

(2.18c) $W_G^{H_0} = \{w \in W_G : w(H_0) = H_0\}$.

(2.18d) $W^{H_0}_{P^r_x \cap \tau P^r_x} = \{w = \mathrm{ad}(g) \in W^{H_0}_G : g \in P^r_x \cap \tau P^r_x\}.$

Then the open G_0-orbits on X are in one to one correspondence with the elements of the double coset space

(2.18e) $$W_K \setminus W^{H_0}_G / W^{H_0}_{P^r_x \cap \tau P^r_x} .$$

Here we will be interested, for purposes of discrete series representations of G_0, in the case rank K = rank G_0. In that case the maximally compact Cartan subalgebras $H_0 = H_T + H_V$ of G_0 are *compact* in the sense that $H_0 = H_T \subset K$, so $\tau\phi = -\phi$ for every root, whence $\tau\Pi = -\Pi$ for every simple root system. Thus $G_0(x) \subset X$ is open if, and only if, $G_0 \cap P_x$ contains a compact Cartan subalgebra H_0 of G_0; in that case $P_x \cap \tau P_x = P^r_x$ reductive, and the open orbits are enumerated by $W_K \setminus W_G / W_{P^r_x}$.

It is for the nondiscrete series representations of G_0 that we need *partially* complex manifolds and *partially* holomorphic vector bundles. We go into that now.

Let D be a subset of a complex analytic space V. We see the extent to which D inherits complex analytic structure from V. For that, define *holomorphic arc in* D:

(2.19a) holomorphic map $f: \{z \in \mathbb{C} : |z| < 1\} \to V$ with image in D;

chain of holomorphic arcs in D:

(2.19b) sequence $\{f_1,\ldots,f_k\}$ of holomorphic arcs in D such that the image of f_j meets image of f_{j+1} for $1 \leqq j < k$; and

holomorphic arc component of D:

(2.19c) equivalence class of elements of D under the relation $v_1 \sim v_2$ if there is a chain $\{f_1,\ldots,f_k\}$ of holomorphic arcs in D with $v_1 \in$ image f_1 and $v_2 \in$ image f_k.

The main point is that the notion of holomorphic arc compone is intrinsic. Thus, for example, suppose that A is a group of hc morphic diffeomorphisms of V, that D is an A-stable subset, and t S is a holomorphic arc component of D. Define

(2.20a) $N_A(S) = \{a \in A : a(S) = S\}$, *normalizer of* S *in* A.

Then ([21], Lemma 8.2)

(2.20b) $N_A(S) = \{a \in A : a(S)$ meets $S\}$,

(2.20c) if D is an A-orbit then $N_A(S)$ is transitive on S,

(2.20d) if A is a Lie transformation group on V and D is an A-orbit, then $N_A(S)$ is a Lie subgroup of A, and $S \subset D \subset V$ are embedded real analytic submanifolds.

It does *not* follow that S is a complex analytic submanifold of V (cf. [21], Example 8.12).

We look at the holomorphic arc components of G_0-orbits on th complex flag $X = G/P$. It will be convenient to have the followi notation.

(2.21a) $S_{[x]}$: holomorphic arc component of $G_0(x) \subset X$ throug

(2.21b) $N_{[x],0}$: identity component of $\{g \in G_0 : gS_{[x]} = S_{[x]}\}$

(2.21c) $N_{[x],0}$ is the Lie algebra of $N_{[x],0}$ and $N_{[x]} = N^{\mathbb{C}}_{[x],}$

(2.21d) $N_{[x]}$ is the complex analytic subgroup of G for $N_{[x]}$.

The first step in studying the holomorphic arc components of $G_0(x)$ is the construction of a certain parabolic subalgebra $Q_{[x]} \subset N_{[x]}$ of G. Let $P_x = P_\Phi$ as in (2.9) and define

(2.22a) $$\delta_x = \sum_{\Phi^u \cap \tau\Phi^u} \phi = \tau\delta_x ,$$

(2.22b) $$Q_{[x]} = H + \sum_{\langle\phi,\delta_x\rangle \geqq 0} G^\phi = \tau Q_{[x]}.$$

Then ([21], Theorem 8.5)

(2.23a) $P_x^u \cap \tau P_x^u \subset Q_{[x]} \subset \{N_{[x]} \cap (P_x + \tau P_x)\}$, so

(2.23b) $N_{[x]}$ is a τ-stable parabolic subalgebra of G and

(2.23c) $N_{[x],0}$ is the identity component of the parabolic $N_{[x]} \cap G_0$ of G_0.

For the second step, define

(2.24a) $\Gamma^0 = \{\text{roots } \phi : -\phi \notin \Phi^u \cap \tau\Phi^u, \ \langle\phi,\delta_x\rangle < 0, \ \phi + \tau\phi \text{ not root}\}$;

(2.24b) $M_{[x]} = Q_{[x]} + \sum_{\Gamma^0} G^\phi$.

Then ([21], Proposition 8.7)

(2.24c) $Q_{[x]} \subset M_{[x]} \subset N_{[x]} \cap (P_x + \tau P_x)$.

The point of (2.24) is ([21], Theorem 8.9) that the following conditions are equivalent.

(2.25a) The holomorphic arc components of $G_0(x)$ are complex manifolds.

(2.25b) $N_{[x]} \subset P_x + \tau P_x$.

(2.25c) $N_{[x]} = M_{[x]}$ subspace of G defined in (2.24b).

(2.25d) $M_{[x]}$ is a subalgebra of G.

Let us agree to say that the orbit $G_0(x) \subset X$ is

(2.26) *partially complex* if its holomorphic arc components are complex submanifolds of X, i.e., if $S_{[x]}$ is a complex manifold;

(2.27) *flag type* if the $N_{[x']}(x') \subset G_0(x)$ are complex flag submanifolds of X, i.e., if $N_{[x]}(x)$ is a flag;

(2.28) *measurable* if its holomorphic arc components carry Radon

measures invariant under their normalizers, i.e., if S has an $N_{[x],0}$-invariant Radon measure;

(2.29) *integrable* if the distribution $x' \mapsto G_0 \cap (P_{x'} + \tau P_{x'})$ on $G_0(x)$ is integrable, i.e., if $P_x + \tau P_x$ is a subal bra of G.

The orbits with which we will be concerned are those that a partially complex, measurable and integrable. The criterion for partial complexity is (2.25), which is rather difficult to check. However, it is subsumed by the measurability criterion below, whi is rather more easily verified.

Let $P_x = P_\Phi$ as in (2.9) and denote

(2.30a) $V_x^+ = \sum_{\Phi^u \cap -\tau\Phi^u} G^\phi, \quad V_x^- = \tau V_x^+ = \sum_{-\Phi^u \cap \tau\Phi^u} G^\phi$ and $V_x = V_x^+ + V_x^-$.

Then ([21], Theorem 9.2) the orbit $G_0(x)$ is

(2.30b) measurable $\Leftrightarrow$ $N_{[x]} = (P_x \cap \tau P_x) + V_x$.

Further, if $G_0(x)$ is measurable then

(2.30c) $G_0(x)$ is partially complex and of flag type,

(2.30d) $G_0(x)$ is integrable if and only if $\tau P_x^r = P_x^r$, and

(2.30e) the invariant Radon measure on $S_{[x]}$ is the volume ele of an $N_{[x],0}$-invariant indefinite-kaehler metric.

From a slightly different starting point, suppose that τP_x^r Then ([21], Theorem 9.9) the following are equivalent.

(2.31a) $G_0(x)$ is measurable.

(2.31b) $G_0(x)$ is integrable.

(2.31c) $G_0(x)$ is partially complex and of flag type.

Under those circumstances,

(2.31d) $N_{[x]} = (P_x \cap \tau P_x) + V_x = Q_{[x]} = P_x + \tau P_x.$

In fact ([21], Corollary 9.11) integrability of $G_0(x)$ implies $N_{[x]} = P_x + \tau P_x = Q_{[x]}$, implies that $G_0(x)$ is partially complex and of flag type, and implies that $G_0(x)$ is measurable if and only if $\tau P_x^r = P_x^r$.

To understand the interplay (2.30) and (2.31) of these various global conditions on $G_0(x)$, we will examine them in the cases

(2.32a) $G_0(x)$ open in X,

(2.32b) $G_0(x)$ closed in X, and

(2.32c) $G_0(x)$ integrable in X.

Open orbits obviously are integrable, partially complex and of flag type. More precisely, if $G_0(x) \subset X$ is open then $P_x + \tau P_x = G = N_{[x]}$, so $S_{[x]} = G_0(x)$ and $N_{[x]}(x) = G(x) = X$.

Measurability of open orbits can be detected without the machinery of holomorphic arc components. The result ([21], Theorem 6.3) is that the following conditions on an orbit $G_0(x) \subset X$ are equivalent.

(2.33a) $G_0(x)$ is a measurable open orbit.

(2.33b) $G_0(x)$ has an invariant volume element.

(2.33c) $G_0(x)$ has an invariant indefinite-kaehler metric.

(2.33d) $G_0 \cap P_x$ is the centralizer of a (compact) torus.

(2.33e) $P_x \cap \tau P_x$ is reductive, i.e., $P_x \cap \tau P_x = P_x^r \cap \tau P_x^r$.

(2.33f)[1] $P_x \cap \tau P_x = P_x^r$, i.e., $\tau P_x^u = P_x^{-u}$, i.e., $G = P_x + \tau P_x^u$.

(2.33g) $P_x = P_\Phi$ with $\tau\Phi^r = \Phi^r$ and $\tau\Phi^u = -\Phi^u$.

---

[1] See footnote next page.

In particular (see discussion after (2.18e)),

(2.34) if rank K = rank G_0 then every open orbit it measur

The phenomenon (2.34) is not isolated. Denote

(2.35a)[1] if $P = P_\Phi$ then $P^- = P_\Phi^r + P_\Phi^{-u}$ *opposite* of P.

Then ([21], Theorem 6.7) the following conditions are equivalent

(2.35b) Some open G_0-orbit on X is measurable.

(2.35c) Every open G_0-orbit on X is measurable.

(2.35d) P^- and τP are ad(G)-conjugate.

Closed orbits are somewhat more tractable. The first main ([21], Theorem 3.3) is that

(2.36a) there is just one closed G_0-orbit on X,

(2.36b) every maximal compact subgroup of G_0 is transitive o

(2.36c) it is in the closure of every G_0-orbit, and

(2.36d) it is the lowest-dimensional G_0-orbit.

For example, if $X = G/P$ is a hermitian symmetric space of comp type under the compact real form of G, i.e. ([21], Lemma 9.22), P^u is abelian, and if there exists $x_0 \in X$ such that $G_0(x_0) \subset X$ the Borel embedding of the dual bounded symmetric domain, then i follows ([21], Corollary 3.9) that the closed G_0-orbit is the Bergman-Šilov boundary of $G_0(x_0)$ in X.

Minimal dimensionality (2.36d) of the closed orbit has an i teresting interpretation ([21], Theorem 3.6). Let $x \in X$ and $P_x = P_\Phi$ as in (2.9). Then $\dim_R G_0(x) \geqq \dim_{\mathbb{C}} X = \frac{1}{2}\dim_R X$, and t following conditions are equivalent.

---

[1] If $P_x = P_\Phi$ then $P_x^{-u} = P_\Phi^{-u}$ denotes $\sum_{\Phi^u} G^{-\phi}$, vector space complement to $P_x = P_\Phi$ in G composed of root spaces.

(2.37a) $\dim_R G_0(x) = \dim_{\mathbb{C}} X.$

(2.37b) $\tau\Phi^u = \Phi^u$, i.e., $\delta_x = \sum_{\Phi^u} \phi$, i.e., $Q_{[x]} = P_x$.

(2.37c) View G as a linear algebraic group def/$\mathbb{R}$, with G_0 the topological identity component of the group G_R of real points; then P_x is def/$\mathbb{R}$, i.e., $\tau P_x = P_x$, i.e., $\tau P_x = P_x$.

(2.37d) $G_0(x)$ is the closed orbit, and some conjugate of P is def/$\mathbb{R}$.

(2.37e) X is a complex projective variety def/$\mathbb{R}$ in such a way that $G_0(x) = X_{\mathbb{R}}$.

In the hermitian symmetric case mentioned above, the various conditions of (2.37) all are equivalent to: $G_0(x_0)$ is a tube domain over a self-dual cone and $G_0(x)$ is its Bergman-Šilov boundary.

The second main fact ([21], Theorem 9.12) on closed orbits is:

(2.38) the closed G_0-orbit is measurable, partially complex, and of flag type.

That is the key to geometric realization of principal series representations of G_0. It allows characterization ([21], Corollary 9.16) of the closed orbit by the fact that its holomorphic arc components are compact and shows that those components are "tiny" complex flag manifolds; it also shows that the measure induced from the K-normalizer is invariant under the G_0-normalizer.

Finally we examine our global conditions for integrable orbits on the complex flag $X = G/P$. Let[2]

(2.39a) Q: τ-stable parabolic subalgebra of G

that is a candidate for $P_x + \tau P_x$ in the sense that

(2.39b) Q contains an ad(G)-conjugate of P.

[2] Of course one can start with Q and select $X = G/P$, i.e., select P, by means of (2.39b).

Then ([21], Theorem 7.10) there exist, and one can enumerate, orb $G_0(x) \subset X$ such that

(2.39c) $Q = P_x + \tau P_x.$

From our earlier discussion, these orbits satisfy

(2.40a) $N_{[x]} = Q_{[x]} = P_x + \tau P_x = Q,$

(2.40b) $G_0(x)$ is partially complex and of flag type,

(2.40c) $G_0(x)$ is measurable if, and only if, $\tau P^r_x = P^r_x$.

This completes our description of the "partially complex" ho geneous spaces of G_0 mentioned in the Introduction. There we al mentioned a notion of integration over the "space of complex ana pieces" of a partially complex G_0-homogeneous space. Thus we mu look at the space of holomorphic arc components of an orbit $G_0($

Consider an orbit $G_0(x) \subset X$. Then ([21], Theorem 8.15) the normalizers of its holomorphic arc components are specified by

(2.41a) $G_0 \cap N_{[x]} = \{g \in G_0 : gS_{[x]} = S_{[x]}\},$

so they are parabolic subgroups of G_0; that defines a G_0-equivar fibration

(2.41b) $\sigma\colon G_0(x) \to G_0/(G_0 \cap N_{[x]})$

of the orbit over its "space of holomorphic arc components", who

(2.41c) fibres: the holomorphic arc components of $G_0(x)$,

(2.41d) base: compact space $G_0/G_0 \cap N_{[x]} = K/K \cap N_{[x]}.$

Thus Haar measure of K induces a K-invariant Radon measure on th space $K/K \cap N_{[x]}$ of holomorphic arc components of $G_0(x)$. In pe we note that this space of holomorphic arc components is a real manifold.

3. *Open orbits and discrete series*

In this section we assume that G_0 has a compact Cartan subgroup, i.e., has rank equal to that of a maximal compact subgroup. For that is the condition (Harish-Chandra [8]) for the existence of discrete series representations of G_0.

$X = G/P$ is a complex flag manifold with $G = G_0^{\mathbb{C}}$. Suppose that we have an orbit

(3.1) $\quad X_0 = G_0(x_0) \subset X$ open with $G_0 \cap P_{x_0}$ compact.

Then we consider finite dimensional representations

(3.2) $\quad \lambda : G_0 \cap P_{x_0} \to GL(r, \mathbb{C})$ irreducible unitary.

Each such representation λ has a unique extension

(3.3) $\quad \lambda : P_{x_0} \to GL(r, \mathbb{C})$ holomorphic.

Thus we have associated G-homogeneous holomorphic vector bundles $\tilde{\pi} : \tilde{\mathbb{E}}_\lambda \to X$, with restrictions to X_0 that we denote

(3.4) $\quad \pi : \mathbb{E}_\lambda \to X_0$ $\quad G_0$-homogeneous, hermitian, holomorphic.

Consider the spaces

(3.5) $\quad A^{p,q}(\mathbb{E}_\lambda) = \{C^\infty\ (p,q)\text{-forms on } X_0 \text{ with values in } \mathbb{E}_\lambda\}$

and the maps

(3.6) $$\bar{\partial} : A^{p,q}(\mathbb{E}_\lambda) \to A^{p,q+1}(\mathbb{E}_\lambda).$$

Now the Dolbeault cohomology groups

(3.7) $\quad H^{0,q}(X_0, \mathbb{E}_\lambda) = \{\omega \in A^{0,q}(\mathbb{E}_\lambda) : \bar{\partial}\omega = 0\} / \bar{\partial} A^{0,q-1}(\mathbb{E}_\lambda)$

are G_0-modules isomorphic to the cohomology modules of the sheaf of germs of holomorphic sections of $\mathbb{E}_\lambda$. However we are looking for unitary representations so we must find "square integrable cohomology groups".

As the isotropy subgroup $G_0 \cap P_{x_0}$ of G_0 at x_0 was assumed pact (3.1) we have

(3.8) $\quad ds^2 : G_0$-invariant hermitian metric on X_0.

Also λ is unitary on $G_0 \cap P_{x_0}$ so, as mentioned in (3.4),

(3.9) $\quad h : G_0$-invariant hermitian metric on $\mathbb{E}_\lambda$.

Now we have the Hodge-Kodaira operators

(3.10) $\quad A^{p,q}(\mathbb{E}_\lambda) \xrightarrow{\#} A^{n-p,n-q}(\mathbb{E}_\lambda^*) \xrightarrow{\tilde{\#}} A^{p,q}(\mathbb{E}_\lambda), \quad n = \dim X$

They give the linear space

(3.11) $\quad A_2^{p,q}(\mathbb{E}_\lambda) = \{\alpha \in A^{p,q}(\mathbb{E}_\lambda) : \int \alpha \wedge \#\alpha < \infty\}$

which is a pre Hilbert space with inner product

(3.12) $$\langle \alpha, \beta \rangle = \int \alpha \wedge \#\beta .$$

Now denote

(3.13) $\quad L_2^{p,q}(\mathbb{E}_\lambda)$: Hilbert space completion of $A_2^{p,q}(\mathbb{E}_\lambda)$.

$\bar{\partial}$ is densely defined on $L_2^{p,q}(\mathbb{E}_\lambda)$, and

(3.14) $\quad \bar{\delta} = -\tilde{\#}\bar{\partial}\#$ is the formal adjoint of $\bar{\partial}$.

Now, as expected, we define

(3.15) $\quad \square = (\bar{\partial} + \bar{\delta})^2 = \bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}$ Kodaira-Hodge-Laplacian.

As ds^2 is positive definite and complete, the work of Andreotti-Vesentini [1] shows that

(3.16a) $\quad H_2^{p,q}(\mathbb{E}_\lambda) = \{\omega \in L_2^{p,q}(\mathbb{E}_\lambda) : \square\omega = 0\}$ square integra harmonic forms

is a closed subspace contained in the space $A_2^{p,q}(\mathbb{E}_\lambda)$ of smooth square integrable forms, and that

(3.16b) $\quad L_2^{p,q}(\mathbb{E}_\lambda) = \bar{\partial}L_2^{p,q-1}(\mathbb{E}_\lambda) \oplus \bar{\delta}L_2^{p,q+1}(\mathbb{E}_\lambda) \oplus H_2^{p,q}(\mathbb{E}_\lambda)$

orthogonal direct sum such that

(3.16c) $\quad \bar{\partial}L_2^{p,q-1} + H_2^{p,q}(\mathbb{E}_\lambda)$ is the kernel of $\bar{\partial}$,

(3.16d) $\quad \bar{\delta}L_2^{p,q+1} + H_2^{p,q}(\mathbb{E}_\lambda)$ is the kernel of $\bar{\delta}$.

Thus the "square integrable cohomology" is given by the square integrable harmonic forms.

The Hilbert space structure of $L_2^{p,q}(\mathbb{E}_\lambda)$ depends on the G_0-homogeneous bundle $\mathbb{E}_\lambda$ and the G_0-invariant hermitian metrics h and ds^2. Thus G_0 has a natural unitary action on $L_2^{p,q}(\mathbb{E}_\lambda)$. Consider the matrix coefficients

(3.17) $\quad f_{\alpha\beta}(g) = \langle g\alpha,\beta\rangle \;;\quad g \in G_0,\quad \alpha,\beta \in L_2^{p,q}(\mathbb{E}_\lambda).$

If T_0 denotes the holomorphic tangent space of X_0 at x_0, and if E_λ denotes the representation space of λ, then we may view

(3.18a) $\quad \alpha,\beta : G_0 \to E_\lambda^{p,q} = \Lambda^p T_0^* \otimes \Lambda^q \bar{T}_0^* \otimes E_\lambda$

with the appropriate transformation condition for right translation by elements of $G_0 \cap P_{x_0}$. The square integrability condition on α and β becomes

(3.18b) $\quad \alpha,\beta \in L_2(G_0) \otimes E_\lambda^{p,q}.$

G_0 acts on the first tensor factor. Thus, from the corresponding fact about the left regular representation of G_0, we have

(3.19) $\quad f_{\alpha\beta} \in L_1(G_0) \cap L_2(G_0)$ for $\alpha,\beta \in L_1^{p,q}(\mathbb{E}_\lambda) \cap L_2^{p,q}(\mathbb{E}_\lambda).$

Now define

(3.20) $\quad \pi_\lambda^{p,q}$: unitary representation of G_0 on $H_2^{p,q}(\mathbb{E}_\lambda)$.

Then (3.17) and (3.19) show, by L_1-approximation, that

(3.21) $\quad$ all matrix coefficients of $\pi_\lambda^{p,q}$ are in $L_2(G_0)$.

Recall that the following conditions are equivalent for an i reducible unitary representation π of G_0.

(3.22a) $\quad$ The class of π, element of $\hat{G}_0$, has positive Planchere measure.

(3.22b) $\quad$ π is a subrepresentation of the left regular represen tion of G_0.

(3.22c) $\quad$ π has a square integrable matrix coefficient.

(3.22d) $\quad$ All matrix coefficients of π are square integrable.

In that case π is said to be a *discrete series* representation of Thus the discrete series of G_0 is the subset of $\hat{G}_0$ consisting of classes $[\pi]$ of (necessarily finite) positive Plancherel mass. No (3.21) says

(3.23) $\quad$ if $\pi_\lambda^{p,q}$ is irreducible it is a discrete series repres tion of G_0.

The goal in geometric realization of discrete series is to exhibi every discrete series representation in the form $\pi_\lambda^{0,q}$. Schmid [1 has made considerable progress in that direction; one may reasona expect that he will carry it through, at least for L_1 representations.

4. *Closed orbit and principal series*

We recall the construction of the "principal series" of unit representations of G_0. Denote

(4.1) $\quad K$: maximal compact subgroup of G_0,

(4.2) $\quad G_0 = KAN$: Iwasawa decomposition of G_0,

(4.3) $\quad MAN$: minimal parabolic subgroup of G_0.

In more detail, consider the Cartan involution σ of G_0 with fixed point set K. Choose

(4.4) A: maximal abelian subspace of (-1)-eigenspace of σ on G_0.

Then we have a finite set Δ_A of nonzero real linear functionals on A and a direct sum decomposition

(4.5a) $G_0 = Z_A + \sum_{\Delta_A} G_0^{\phi}$ where

(4.5b) Z_A is the centralizer of A in G_0 and

(4.5c) $G_0^{\phi} = \{u \in G_0 : [a,u] = \phi(a)u \text{ for } a \in A\} \neq 0$.

The elements of Δ_A are the "A-roots" or "restricted roots" of G_0. Every $\phi \in \Delta_A$ determines a hyperplane of A by

(4.6a) $\phi^{\perp} = \{a \in A : \phi(a) = 0\}$.

Now $A - \bigcup \phi^{\perp}$ is a finite union of convex cones on which every restricted root avoids the value 0. By *Weyl chamber* of A we mean a topological component of $A - \bigcup \phi^{\perp}$. Choose a Weyl chamber $\mathcal{D}$ of A. We define

(4.6b) $\Delta_A^+ = \{\phi \in \Delta_A : \phi > 0 \text{ on } \mathcal{D}\}$ positive restricted root system.

Now observe

(4.6c) $\Delta_A = \Delta_A^+ \cup \Delta_A^-$ disjoint where $\Delta_A^- = -\Delta_A^+$.

Finally, define

(4.7a) $N = \sum_{\Delta_A^+} G_0^{\phi}$ and $N^- = \sum_{\Delta_A^-} G_0^{\phi}$,

(4.7b) N: analytic subgroup of G_0 for N,

(4.7c) A: analytic subgroup of G_0 for A,

(4.7d) M: centralizer of A in K.

That explains (4.2) and (4.3). Note that $MA = M \times A$ is the

centralizer of A in G_0, that N is a unipotent group, and that M normalizes N.

Let λ be an irreducible complex representation of MAN. As is a nilpotent group normal in the irreducible group $\lambda(MAN)$ we $\lambda(N) = 1$ by Lie's theorem. So $\lambda = \mu \otimes \chi : man \mapsto \mu(m)\cdot\chi(a)$ w is an irreducible unitary representation of M and χ is a not-ne sarily-unitary character on A. Now define

(4.8a) $\rho = \frac{1}{2} \sum_{\Delta_A^+} (\dim G_0^{\phi})\cdot\phi$ linear form on A.

If μ is an irreducible unitary representation of M and η is a c valued linear form on A, we define

(4.8b) $\beta_{\mu,\eta}(man) = \mu(m)\cdot\exp((\rho + i\eta)(\log a))$,

representation of MAN on the representation space V_μ of μ. The discussion says that

(4.8c) the $\beta_{\mu,\eta}$ are the irreducible complex representations of MAN.

The reason for insertion of ρ in (4.8b) will come out in a momen

Given $\beta_{\mu,\eta}$ we have an associated complex vector bundle

(4.9a) $\mathbb{E}_{\mu,\eta} \to G_0/MAN = K/M$.

Its space of measurable sections is

(4.9b) $\Gamma(\mathbb{E}_{\mu,\eta}) = \{f: G_0 \to V_\mu$ measurable:
$f(gman) = \beta_{\mu,\eta}(man)^{-1}f(g)\}$

Note that a measurable section $f: G_0 \to V_\mu$ is specified by $f|_K$ for every $g \in G_0$ has decomposition $g = kan$, and $f(g) = \beta_{\mu,\eta}(an)^{-1}$ Let dk denote normalized Haar measure on K and (v,v') the unitary inner product on V_μ. Now we have the space of square integrable tions, (4.10a) $\Gamma_2(\mathbb{E}_{\mu,\eta}) = \{f \in \Gamma(\mathbb{E}_{\mu,\eta}) : \int_K (f(k),f(k))dk < \infty\}.$

It is a complex Hilbert space with inner product

(4.10b) $\langle f,f'\rangle = \int_K (f(k),f'(k))dk.$

G_0 acts on $\Gamma(\mathbb{E}_{\mu,\eta})$ by the algebraic representation

(4.11a) $[\tilde{\pi}_{\mu,\eta}(g)f](g') = f(g^{-1}g').$

We will check that

(4.11b) $\tilde{\pi}_{\mu,\eta}$ restricts to a bounded representation $\pi_{\mu,\eta}$ of G_0 on $\Gamma_2(\mathbb{E}_{\mu,\eta})$

and that

(4.11c) $\pi_{\mu,\eta}$ is unitary if and only if $\eta: A \to \mathbb{R}$ real.

For that we define

(4.12a) $\kappa_g: K \to K$ and $\alpha_g: K \to A$ by

(4.12b) $g^{-1}k \in \kappa_g(k)\cdot\alpha_g(k)\cdot N \subset KAN = G_0.$

If $f \in \Gamma(\mathbb{E}_{\mu,\eta})$, $g \in G_0$ and $k \in K$, now

(4.12c) $[\tilde{\pi}_{\mu,\eta}(g)f](k) = \exp((\rho + i\eta)\cdot\log \alpha_g(k))^{-1} f(\kappa_g(k)).$

The definition of ρ and the trace of A on N^- say, for $F \in L_1(K)$, that

(4.13) $\int_K F(k)dk = \int_K \exp(-2\rho \log \alpha_g(k))F(\kappa_g(k))dk.$

Let $f \in \Gamma_2(\mathbb{E}_{\mu,\eta})$ and $g \in G_0$, and set

$$F(k) = (f(k),f(k)) \quad\text{and}\quad F^g(k) = (f(g^{-1}k),f(g^{-1}k)).$$

Then (4.11a), (4.12c) and (4.13) give us

$$\int_K F^g(k) = \int_K |\exp(i\eta\cdot\log \alpha_g(k))|^{-2} \exp(-2\rho\cdot\log \alpha_g(k))F(\kappa_g(k))d$$

$$\leqq C_\eta(g)\int_K \exp(-2\rho\cdot\log \alpha_g(k))F(\kappa_g(k))dk$$

$$= C_\eta(g)\int_K F(k)dk,$$

where

$$C_\eta(g) = \sup_{k\in K} |\exp(i\eta\cdot\log \alpha_g(k))|^{-2}.$$

Thus $\tilde{\pi}_{\mu,\eta}(g)f \in \Gamma_2(\mathbb{E}_{\mu,\eta})$, so $\pi_{\mu,\eta}(g)$ is defined, and $\|\pi_{\mu,\eta}(g)\| \leqq C_\eta(g)$. A similar argument shows that $\pi_{\mu,\eta}$ is contin That proves (4.11b). For (4.11c) note that η is real if and onl $|\exp(i\eta\cdot\log \alpha_g(k))|^{-2} \equiv 1$, and that the latter holds precisely $\pi_{\mu,\eta}$ is unitary.

The *principal series* of unitary representations of G_0 is, b definition, the set of equivalence classes of unitary representat $\pi_{\mu,\eta}$ just defined. We have parameterized the principal series by $\hat{M} \times A'$ where $\hat{M}$ is the set of equivalence classes of irreducible tary representations of M and A' is the real dual space of A.

It is possible to see that two principal series representati $\pi_{\mu,\eta}$ and $\pi_{\mu',\eta'}$ are equivalent if, and only if, $(\mu',\eta') = (\mu\circ\mathrm{ad}(m), \eta\circ\mathrm{ad}(m))$ where $m \in K$ normalizes A (and thus also normalizes M). More precisely, let W_A be the "restricted Weyl gr i.e.,

(4.14a) $W_A = \{k \in K : \mathrm{ad}(k)A = A\}/M.$

Then W_A acts

(4.14b) on A' by $w(\eta) = \eta\circ\mathrm{ad}(k)$, $k \in w$;

(4.14c) on $\hat{M}$ by $w(\mu) = \mu\circ\mathrm{ad}(k)$, $k \in w$.

Now[3] [3]

(4.15) $\pi_{\mu,\eta} \sim \pi_{\mu',\eta'} \Longleftrightarrow (\mu',\eta') = (w\mu, w\eta)$, some $w \in W_A$.

Principal series representations are not automatically irreducible. Bruhat [3] has shown that

(4.16) if $(\mu,\eta) \neq (w\mu, w\eta)$ for $1 \neq w \in W_A$ then $\pi_{\mu,\eta}$ is irreducible.

A number of authors have worked out, for specific groups G_0, which principal series representations are irreducible; cf. Kunze and Stein [15], Knapp and Stein [12], Helgason [11], Wallach [19].

The *spherical principal series* consists of the classes of principal series representations $\pi_{\mu,\eta}$ such that $\mu = 1_M$ trivial representation of M. Irreducibility of all spherical principal series representations was proved by Parthasarathy, Ranga-Rao and Varadarajan [17] for complex G_0, by Kostant [14] in general. Their algebraic methods have been extended to the principal series by Wallach [19]. Also see Zelobenko [24].

Recall (4.10) that the representation space $\Gamma_2(\mathbb{E}_{\mu,\eta})$ is a certain space of L_2-functions on K with values in the representation space V_μ of μ. In other words $\Gamma_2(\mathbb{E}_{\mu,\eta}) \subset L_2(K) \otimes V_\mu$. We will need to identify it as a subspace. For that

(4.17a) if $\kappa \in \hat{K}$ then U_κ denotes its representation space.

Thus the Peter-Weyl theorem for K says

(4.17b) $$L_2(K) = \sum_{\kappa \in \hat{K}} U_\kappa \otimes U_\kappa^* .$$

In particular,

(4.17c) $$L_2(K) \otimes V_\mu = \sum_{\kappa \in \hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}$$

[3] There seems to be some trouble with the argument in [3], but it is fixed up in a forthcoming book by Garth Warner [20].

where the left action of K is κ on the U_κ-factor of each $U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}$. The right action of M on $U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}$ is $1_M \otimes \kappa^*|_M \otimes \mu$, so the M-fixed elements are given by $U_\kappa \otimes \{U_\kappa^* \otimes$ there. Now

(4.18) $$\Gamma_2(\mathbb{E}_{\mu,\eta}) = \sum_{\kappa \in \hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}^M \text{ as a } K\text{-module.}$$

Note that this K-module structure does not involve η. Here we re from (4.12) that the action of G_0 is

(4.19a) $$[\pi_{\mu,\eta}(g)f](k) = \exp((\rho + i\eta)(\log \alpha_g(k)))^{-1} f(\kappa_g(k)).$$

In other words,

(4.19b) $$\pi_{\mu,\eta}(g)f = \exp((\rho + i\eta)(\log \alpha_g(\cdot)))^{-1} \cdot (f \circ \kappa_g)(\cdot).$$

It is the composition with κ_g that mixes the K-summands $U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}^M$ of $\Gamma_2(\mathbb{E}_{\mu,\eta})$.

We describe the realization of a principal series representa $\pi_{\mu,\eta}$ on the closed orbit in a complex flag manifold.

$X = G/P$ is a complex flag manifold with $G = G_0^{\mathbb{C}}$. Now denc

(4.20a) $Y = G_0(y_0) \subset X$ closed orbit

and suppose that the holomorphic arc components $S_{[y]}$, $y \in Y$, ha real normalizers given by

(4.20b) $$MAN = \{g \in G_0 : gS_{[y_0]} = S_{[y_0]}\}.$$

Then $S_{[y_0]}$ is a complex flag manifold of $M^{\mathbb{C}}$ and has descriptions

(4.20c) $$S_{[y_0]} = M^{\mathbb{C}}/Q = M/L = MAN/LAN$$

where

(4.20d) $Q = M^{\mathbb{C}} \cap P_{y_0}$ has $Q^r = L^{\mathbb{C}}$ and $L = M \cap P_{y_0}$.

Here $M^{\mathbb{C}}$ and $L^{\mathbb{C}}$ denote the complexifications of the topological identity components of the compact groups M and L; the other components of M and L are represented by elements of $\exp(iA)$.

We need notation for representation spaces of irreducible unitary representations of L, M and K:

(4.21a) if $\nu \in \hat{L}$ then W_ν is its representation space,

(4.21b) if $\mu \in \hat{M}$ then V_μ is its representation space,

(4.21c) if $\kappa \in \hat{K}$ then U_κ is its representation space.

Let $\nu \in \hat{L}$ and consider the associated M-homogeneous holomorphic vector bundle

(4.22a) $\mathbb{W}_\nu \to M/L = S_{[y_0]}$.

Suppose that ν is chosen so that we can arrange, e.g. by means of the Borel-Weil Theorem, that it is related to a given fixed $\mu \in \hat{M}$ by the condition that the space of holomorphic sections

(4.22b) $H^0(S_{[y_0]}, \mathcal{O}(\mathbb{W}_\nu)) = V_\mu$ as an M-module.

We construct the representation $\pi_{\mu,\eta}$ as follows. Define

(4.23a) $\sigma_{\nu,\eta}$: irreducible representation of LAN on W_ν

(4.23b) by: $\sigma_{\nu,\eta}(lan) = \nu(l)\cdot\exp((\rho + i\eta)(\log a))$.

We have the associated G_0-homogeneous complex vector bundle

(4.23c) $\mathbb{W}_{\nu,\eta} \to G_0/LAN = G_0(y_0) = Y$,

which is holomorphic over every holomorphic arc component of Y because

(4.23d) $\mathbb{W}_{\nu,\eta}|_{S_{[y_0]}} = \mathbb{W}_\nu$.

Now consider

(4.24a) $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$: square integrable partially holomorphic sections.

It consists of all

(4.24b) $f\colon G_0 \to W_\nu$ measurable such that

(4.24c) $f(glan) = \sigma_{\nu,\eta}(lan)^{-1}f(g)$,

(4.24d) $f|_{gMAN}$ specifies a holomorphic section of $\mathbb{W}_{\nu,\eta}|_{S_{[gy}}$

(4.24e) $\int_K (f(k),f(k))_{W_\nu}\,dk < \infty$.

Our claim is that

(4.25) $\pi_{\mu,\eta}$ is the representation of G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$.

To prove (4.25) we realize $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$ as a space of functions on K. The square integrable sections of $\mathbb{W}_{\nu,\eta} \to Y = K/L$ are ju the L-fixed square integrable functions from K to W_ν as follows:

(4.26a) $\Gamma_2(\mathbb{W}_{\nu,\eta}) = \{L_2(K) \otimes W_\nu\}^L = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes W_\nu\}^L$.

The holomorphic square integrable sections are those also fixed under the unipotent radical Q^u of the group Q of (4.20d); thus

(4.26b) $\Gamma_2^h(\mathbb{W}_{\nu,\eta}) = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes W_\nu\}^{L\cdot Q}$.

Similarly, the space of (automatically square integrable) holomor sections of $\mathbb{W}_\nu \to S_{[y_0]} = M/L$ is

(4.27a) $\Gamma^h(\mathbb{W}_\nu) = \sum_{m\in\hat{M}} V_m \otimes \{V_m^* \otimes W_\nu\}^{L\cdot Q}$.

By hypothesis (4.22b) that is V_μ, so

(4.27b) $\dim_{\mathbb{C}}\{V_m^* \otimes W_\nu\}^{LQ}$ is 1 if $m = \mu$, 0 if $m \neq \mu$.

If we denote, for all $m \in \hat{M}$ and $\kappa \in \hat{K}$

$$n(m,\kappa): \text{ multiplicity of } m \text{ in } \kappa|_M ,$$

then (4.27b) allows us to re-write (4.26b) as

(4.28a) $$\Gamma_2^h(\mathbb{W}_{\nu,\eta}) = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \mathbb{C}^{n(\mu,\kappa)}.$$

As $n(\mu,\kappa) = \dim\{U_\kappa^* \otimes V_\mu\}^M$ by Schur's Lemma, (4.28a) says

(4.28b) $$\Gamma_2^h(\mathbb{W}_{\nu,\eta}) = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}^M \text{ as } K\text{-module.}$$

Comparing (4.12ab) and (4.24c) we see that the action $g: f \to f^g$ of G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$ is given by

(4.28c) $$f^g(k) = \exp((\rho + i\eta)\log \alpha_g(k))^{-1} f(\kappa_g(k)), \quad k \in K.$$

We use (4.28bc) to compute the distribution character of the representation of G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$. If ϕ is a C^∞ function of compact support on G_0 then, for $f \in \Gamma_2^h(\mathbb{W}_{\nu,\eta})$ and $k \in K$,

(4.29a) $$f^\phi(k) = \int_{G_0} \phi(g) f^g(k)dg$$
$$= \int_{G_0} \phi(g)\cdot\exp((\rho + i\eta)\log \alpha_g(k))^{-1}\cdot f(\kappa_g(k))dg.$$

Thus

(4.29b) $$\langle f^\phi, f\rangle = \int_K \int_{G_0} \phi(g)\cdot\exp((\rho+i\eta)\log \alpha_g(k))^{-1}\cdot(f(\kappa_g k), f(k))dg\, dk.$$

Now choose orthonormal bases

(4.30a) $\{u_\kappa^r\}$ of U_κ and $\{v_\kappa^s\}$ of $(U_\kappa^* \otimes V_\mu)^M$.

That defines an orthonormal basis

(4.30b) $\{f_\kappa^{rs}\}$ of $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$ by $f_\kappa^{rs} = u_\kappa^r \otimes v_\kappa^s$.

Now, for each $\kappa \in \hat{K}$, $k \in K$,

$$\sum_{r,s} (f_\kappa^{rs}(\kappa_g(k)), f_\kappa^{rs}(k)) = \sum_{r,s} ((\kappa_g(k))^{-1}\cdot u_\kappa^r, k^{-1}\cdot u_\kappa^r)$$
$$= n(\mu,\kappa) \sum_r (k\cdot\kappa_g(k)^{-1}u_\kappa^r, u_\kappa^r) = n(\mu,\kappa)\chi_\kappa(k\cdot\kappa_g(k$$

where χ_κ is the character of $\kappa \in \hat{K}$ viewed, as usual, as a func on K. Thus (4.29b) says that the representation of G_0 on $\Gamma_2^h(\mathbb{W}$ has distribution character $\chi_{\nu,\eta}: C_c^\infty(G_0) \to \mathbb{C}$ given by

(4.30c) $$\chi_{\nu,\eta}(\phi) = \int_K \int_{G_0} \sum_{\hat{K}} \phi(g)\cdot\exp((\rho+i\eta)\log \alpha_g(k))^{-1}\cdot$$
$$\cdot\, n(\mu,\kappa)\cdot\chi_\kappa(k\cdot\kappa_g(k)^{-1})$$

Recall that (4.18) shows the representation space $\Gamma_2(\mathbb{E}_{\mu,\eta})$ $\pi_{\mu,\eta}$ to have the same K-module structure (4.28b) as that of $\Gamma_2^h($ and (4.19) shows the action of G_0 there to be given by the same mula (4.28c) as that of the action of G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$. Thus con siderations (4.29) and (4.30) are valid for $\pi_{\mu,\eta}$, which conseque has distribution character given by

(4.31a) $$\chi_{\mu,\eta}(\phi) = \int_K \int_{G_0} \sum_{\hat{K}} \phi(g)\cdot\exp((\rho+i\eta)\log \alpha_g(k))^{-1}\cdot$$
$$\cdot\, n(\mu,\kappa)\cdot\chi_\kappa(k\cdot\kappa_g(k)^{-1}),$$

Thus, from (4.30c) and (4.31a),

(4.31b) $$\chi_{\nu,\eta} = \chi_{\mu,\eta}\ .$$

Now $\pi_{\mu,\eta}$ and the representation of G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$ are K-finite tary representations with the same global character. Fell's modification [4] of Harish-Chandra's ([6], Lemma 3) (cf. Warner [20], Theorem 4.5.8.1) says that

(4.31c) $\pi_{\mu,\eta}$ is infinitesimally equivalent to: G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$.

Extending Harish-Chandra's [5, Theorem 8], it also shows that infinitesimal equivalence is the same as unitary equivalence for K-finite unitary representations. Thus

(4.31d) $\pi_{\mu,\eta}$ is unitarily equivalent to: G_0 on $\Gamma_2^h(\mathbb{W}_{\nu,\eta})$.

Now (4.25) is proved.

The careful reader will note that our proof of (4.25) gives a geometric realization of the complementary series representations of G_0.

Two assumptions were made in the geometric realization just given for principal series representations. They are the hypothesis (4.20b) that MAN is the real normalizer of the holomorphic arc component $S_{[y_0]}$, and the hypothesis (4.22b) that the representation $\mu \in \hat{M}$ is obtained by holomorphic induction of Borel-Weil type from some $\nu \in \hat{L}$. We discuss them separately, observing that both can always be arranged if P is a Borel subgroup of G, and also if $P = (MAN)^{\mathbb{C}}$, complexification of the minimal parabolic subgroup of G_0.

The closed orbit $G_0(y_0) = Y \subset X$ is automatically measurable, hence partially complex and of flag type [21, Theorem 9.12]. Let τ denote complex conjugation of G over G_0 and $P^r_{y_0}$ the reductive part of P_{y_0} relative to the τ-stable Cartan subalgebra $H \subset G \cap P_{y_0}$ given by

(4.32a) $H = (T + A)^{\mathbb{C}}$, T Cartan subalgebra of M.

Let $N_{[y_0],0}$ be the Lie algebra of

(4.32b) $N_{[y_0],0} = \{g \in G_0 : gS_{[y_0]} = S_{[y_0]}\}$ identity component

and $N_{[y_0]}$ its complexification. Then [21, Theorem 9.2] and [21, Corollary 9.11] say

(4.32c) $\tau P^r_{y_0} = P^r_{y_0} \Leftrightarrow P_{y_0} + \tau P_{y_0}$ algebra $\Leftrightarrow N_{[y_0]} = P_{y_0} +$

From [21, (9.13b)] and the last paragraph of the proof of [21, T rem 9.12] we have

(4.32d) $N_{[y_0]} = (M + A + N)^{\mathbb{C}} \Leftrightarrow P_{y_0} \subset (M + A + N)^{\mathbb{C}} \Leftrightarrow N_{[y_0]} = P_{y_0} + \tau$

In summary, the hypothesis (4.20b) is satisfied if, and only if, conjugate to a subgroup of $(MAN)^{\mathbb{C}}$; i.e., precisely when there a equivariant holomorphic fibrations $G/B \to G/P \to G/(MAN)^{\mathbb{C}}$ for s Borel subgroup B of G. In that case the holomorphic arc componen of Y are the fibres of Y over the closed orbit on $G/(MAN)^{\mathbb{C}}$.

Finally we examine the hypothesis (4.22b) that a given $\mu \in \hat{}$ obtained from some $\nu \in \hat{L}$ by $H^0(M/L, \mathcal{O}(\mathbb{W}_\nu)) = V_\mu$. If M (and th also L) is connected, that is just the Borel-Weil Theorem. If $P_{y_0} = (MAN)^{\mathbb{C}}$ then $L = M$ and we may simply take $\nu = \mu$. In ge al, operating under the assumption (4.20b) discussed above, we pr ceed as follows. Denote

(4.33a) $F = L \cap \exp(iA)$, so $F = G_0 \cap \exp(iA)$, and

(4.33b) $L^0 \subset M^0$: topological identity components of $L \subset M$, s

(4.33c) $L = F \cdot L^0$ and $M = F \cdot M^0$, F finite central in M.

In (4.33) we are making essential use of the simplifying assumpti

$$G_0 \subset G, \text{ i.e., } G_0 \text{ is a linear group,}$$

of this talk. Now further denote

(4.33d) if $\nu \in \hat{L}$ then $\nu^0 = \nu|_{L^0}$; if $\mu \in \hat{M}$ then $\mu^0 = \mu|_{M^0}$.

Then ν^0 and μ^0 are irreducible by (4.33) and Schur's Lemma. Thus $M/L = M^0/L^0$ says that

(4.34) $H^0(M^0/L^0, \mathcal{O}(\mathbb{V}_{\nu^0}) = V_{\mu^0} \Rightarrow H^0(M/L, \mathcal{O}(\mathbb{V}_\nu)) = V_\mu$,

so (4.22b) is automatic. The matter is slightly more delicate if $G_0 \not\subset G$, e.g., for G_0 the 2-sheeted universal covering group of $SL(3,R)$, where F is replaced by $\mathrm{ad}_G^{-1}(\mathrm{ad}_G(L) \cap \exp_{\mathrm{ad}\, G}(i\mathfrak{A}))$ and can fail to be central in L.

5. *Harish-Chandra's decomposition of* $L_2(G_0)$

For convenience we recall Harish-Chandra's construction of the series of unitary representations of G_0 corresponding to a conjugacy class of Cartan subgroups. Warning: "Cartan subgroup" means centralizer of a Cartan subalgebra. Here Cartan subgroups are abelian because G_0 is linear, but they need not be connected.

We denote

(5.1a) H : Cartan subgroup of G_0;

(5.1b) σ : Cartan involution of G_0 with $\sigma(H) = H$;

(5.1c) K : fixed point set of σ, maximal compact subgroup of G_0.

Decompose the Lie algebra $\mathfrak{H}$ of H as

(5.2a) $\mathfrak{H} = \mathfrak{T}_H + \mathfrak{A}_H$ into (+1)- and (-1)-eigenspaces of σ.

That decomposes H as a direct product

(5.2b) $H = T_H \times A_H$; $T_H = H \cap K$ and $A_H = \exp \mathfrak{A}_H$.

Now as in (4.5) we have a finite set $\Delta_{\mathfrak{A}_H}$ of nonzero real linear functionals on $\mathfrak{A}_H$ and a direct sum decomposition

(5.3a) $\mathfrak{G}_0 = \mathfrak{Z}_H + \sum_\Delta \mathfrak{G}_0^\phi$, $\Delta = \Delta_{\mathfrak{A}_H}$, where

(5.3b) $\mathfrak{Z}_H$ is the centralizer of $\mathfrak{A}_H$ in $\mathfrak{G}_0$ and

(5.3c) $\mathfrak{G}_0^\phi = \{u \in \mathfrak{G}_0 : [a,u] = \phi(a)u$ for $a \in \mathfrak{A}_H\} \neq 0$.

$\Delta_{\mathfrak{A}_H}$ is the set of "$\mathfrak{A}_H$-roots" of $\mathfrak{G}_0$.

Every $\phi \in \Delta = \Delta_{\mathfrak{A}_H}$ defines a hyperplane $\phi^\perp = \{a \in \mathfrak{A}_H : \phi(a)$ and $\mathfrak{A}_H - \bigcup_\Delta \phi^\perp$ is a finite union of convex cones (the *Weyl ch* of $\mathfrak{A}_H$) that are its topological components. Given a Weyl chambe $\mathcal{D} \subset \mathfrak{A}_H$ we have the corresponding

(5.4a) $\Delta^+ = \{\phi \in \Delta : \phi > 0$ on $\mathcal{D}\}$, *positive* $\mathfrak{A}_H$*-root system.*

As before in (4.7), denote

(5.4b) $\mathfrak{N}_H = \sum_{\Delta^+} \mathfrak{G}_0^\phi$ and $\mathfrak{N}_H^- = \sum_{\Delta^+} \mathfrak{G}_0^{-\phi}$,

(5.4c) $N_H = \exp \mathfrak{N}_H$ unipotent analytic subgroup of G_0,

(5.4d) P_H : normalizer of N_H in G_0.

Then P_H is a parabolic subgroup of G_0 with unipotent radical N_H, whose reductive part, the centralizer of A_H in G_0, has H as Cart subgroup. Thus

(5.5a) $P_H^r = M_H \times A_H$ and $P_H^u = N_H$ where

(5.5b) A_H is the $\mathbb{R}$-split component of the center of P_H^r,

(5.5c) M_H has a compact Cartan subgroup T_H.

There $\mathfrak{P}_H^r = \mathfrak{Z}_H = \mathfrak{M}_H + \mathfrak{A}_H$ orthogonal direct sum under the Killin form of $\mathfrak{G}_0$.

Construction (5.4) of $P_H = M_H A_H N_H$ is based on the choice positive $\mathfrak{A}_H$-root system Δ^+, i.e., on the choice of positive Weyl chamber $\mathcal{D} \subset A_H$. As $\mathcal{D}$ ranges over the set of all positive Weyl chambers of A_H we thus obtain a collection $\{P_{H,\mathcal{D}}\}$ of parabolic subgroups of G_0, whose elements need not all be G_0-conjugate.

We say that two parabolic subgroups of G_0 are *associated* if their reductive parts are G_0-conjugate, i.e., if the split components of the centers of their reductive parts are G_0-conjugate. Given H, now the G_0-association class of P_H is independent of choice of the positive A_H-root system.

Let $Q \subset G_0$ be a parabolic subgroup. We say that Q is *cuspidal* if the derived group $[Q^r, Q^r]$ of the reductive part Q^r has a compact Cartan subgroup. From (5.5) we note that the P_H are cuspidal parabolic subgroups of G_0. In any case, we may conjugate Q in G_0 so that the split component A^Q of the center of Q^r has Lie algebra A^Q in the (-1)-eigenspace of σ, and then

(5.6a) $\quad Q = M^Q A^Q N^Q$ with $Q^r = M^Q \times A^Q$ and $N^Q = Q^u$,

such that the reductive group M^Q has compact center. Now

(5.6b) $\quad Q$ is cuspidal $\Leftrightarrow M^Q$ has a compact Cartan subgroup.

On the other hand,

(5.6c) $\quad M^Q$ has a compact Cartan $\Leftrightarrow \exists H$ with $A^Q = A_H$.

It follows that

(5.6d) $\quad Q$ is cuspidal $\Leftrightarrow Q$ is G_0-conjugate to some P_H.

In summary

(5.7) $\quad H \mapsto P_H$ induces a bijection from the set of all G_0-conjugacy classes of Cartan subgroups to the set of all G_0-association classes of cuspidal parabolic subgroups of G_0.

We note the two extreme cases of (5.7). If H is a compact Cartan subgroup of G_0 then

$$A_H = \{1\}, \quad N_H = \{1\} \quad \text{and} \quad M_H = G_0 \; ; \quad \text{so} \quad P_H = G_0.$$

If H is a maximally $\mathbb{R}$-split Cartan subgroup of G_0 then in the nc tion of section 4,

$$A_H = A, \quad N_H = N, \quad M_H = M; \quad \text{so} \quad P_H = MAN \quad \text{minimal parabolic.}$$

Another special case of interest is that (cf. [21, pp. 1190-1191 in which G_0 has just one conjugacy class of Cartan subgroups.

Define a real linear functional ρ_H on A_H by

(5.8a) $\quad \rho_H(\alpha) = \tfrac{1}{2} \sum_{\Delta^+} (\dim G_0^\phi)\cdot\phi$, so

(5.8b) $\quad A_H$ acts on N_H^- with trace $-2\rho_H$.

Now suppose that we have

(5.9a) $\quad \mu$: discrete series representation of M_H on V_μ,

(5.9b) $\quad \eta$: real linear functional on A_H.

Then we define

(5.10a) $\quad \beta_{\mu,\eta}$: irreducible representation of P_H on $E_{\mu,\eta} = V_\mu \otimes$

(5.10b) $\quad \beta_{\mu,\eta}(man) = \mu(m)\cdot\exp((\rho_H + i\eta)\log a)$.

The associated complex Hilbert space bundle

(5.11a) $\quad \mathbb{E}_{\mu,\eta} \to G_0/P_H = K/K \cap M_H$

has space of measurable sections

(5.11b) $\quad \Gamma(\mathbb{E}_{\mu,\eta}) = \{f: G_0 \to V_\mu$ measurable :

$$f(gman) = \beta_{\mu,\eta}(man)^{-1} f(\varepsilon$$

As in section 4, the subspace of square integrable sections is

(5.12a) $\Gamma_2(\mathbb{E}_{\mu,\eta}) = \{f \in \Gamma(\mathbb{E}_{\mu,\eta}) : \int_K (f(k),f(k))_{V_\mu}\, dk < \infty\}.$

It is a Hilbert space whose inner product

(5.12b) $\langle f,f'\rangle = \int_K (f(k),f'(k))_{V_\mu}\, dk$

is invariant under the action of G_0. As in the case of the principal series, that defines a unitary representation

(5.13) $\pi_{\mu,\eta}$: H-series representation of G_0 on $\Gamma_2(\mathbb{E}_{\mu,\eta})$.

This H-series is parameterized by $(\hat{M}_H)_{\text{discrete}} \times A_H'$; for [9, §11] the distribution character of $\pi_{\mu,\eta}$ does not depend on the choice of positive A_H-root system Δ^+ used in construction of P_H and ρ_H. Moreover, the Weyl group

(5.14a) $W_H = \{g \in G_0 : \text{ad}(g)A_H = A_H\}/M_HA_H$,

finite group acting on

(5.14b) $\hat{M}_H$ by $\mu \mapsto \mu \circ \text{ad}(g)$ and A_H' by $\eta \mapsto \eta \circ \text{ad}(g)$,

acts trivially on the H-series [9, §11] in the sense

(5.14c) if $w \in W_H$ then $\pi_{\mu,\eta}$ and $\pi_{w(\mu),w(\eta)}$ are equivalent.

An H-series representation is a finite sum of irreducible representations. Harish-Chandra has an irreducibility criterion that is tentative in the sense that he has not written down all the details. The criterion: choose A for a minimal parabolic subgroup MAN of G_0, such that $A_H \subset A$; extend η to A by zero on the orthocomplement of A_H; suppose then that η is not orthogonal to any A-root; in that case $\pi_{\mu,\eta}$ is irreducible.

If H is compact then the H-series is the discrete series. is maximally $\mathbb{R}$-split then the H-series is the principal series. Harish-Chandra's decomposition [9, §12] of $L_2(G_0)$ says that the ducible H-series representations, where H runs over a system of resentatives of the conjugacy classes of Cartan subgroups of G_0 a subset of the unitary dual $\hat{G}_0$ whose complement has Plancherel ure zero.

6. *Integrable orbits and intermediate series*

Fix a Cartan subgroup $H \subset G_0$. We describe the geometric r zation of an H-series representation $\pi_{\mu,\eta}$ on a partially holomo cohomology space over an appropriate G_0-orbit in a complex flag fold.

$X = G/P$ is a complex flag manifold with $G_0^{\mathbb{C}} = G$. Suppose we have an orbit

(6.1a) $X_H = G_0(x_H) \subset X$ measurable and integrable

whose holomorphic arc components have real normalizers specified

(6.1b) $P_H = M_HA_HN_H = \{g \in G_0 : gS_{[x_H]} = S_{[x_H]}\}.$

Then $S_{[x_H]}$ is an open M_H-orbit on a complex flag manifold $M_H^{\mathbb{C}}(x$ $= M_H^{\mathbb{C}}/Q_H$ of $M_H^{\mathbb{C}}$, so

(6.1c) $S_{[x_H]} = M_H/L_H = P_H/L_HA_HN_H$ where

(6.1d) L_H contains a compact Cartan subgroup of M_H.

We now add the condition that

(6.1e) L_H is compact.

Before proceeding with the realization of $\pi_{\mu,\eta}$ we note the c cumstances under which the arrangement (6.1) is possible. From

(6.1a) and (6.1b) we have [21, Corollary 9.11] that P_H is related to the isotropy subgroup P_{x_H} of G at x_H, via the complex conjugation τ of G over G_0, as follows.

(6.2a) $$P_{x_H} + \tau P_{x_H} = P_H^{\mathbb{C}} \quad\text{and}\quad \tau P_{x_H}^r = P_{x_H}^r .$$

That implies that $P_H^{\mathbb{C}}$ has unipotent and reductive parts

$$(P_H^{\mathbb{C}})^u = N_H^{\mathbb{C}} = P_{x_H}^u \cap \tau P_{x_H}^u \quad\text{and}$$

$$(P_H^{\mathbb{C}})^r = (M_H + A_H)^{\mathbb{C}} = P_{x_H}^r + (P_{x_H}^{-u} \cap \tau P_{x_H}^u) + (P_{x_H}^u \cap \tau P_{x_H}^{-u}).$$

Thus

(6.2b) $$N_H^{\mathbb{C}} = P_{x_H}^u \cap \tau P_{x_H}^u , \quad L_H^{\mathbb{C}} + A_H^{\mathbb{C}} = P_{x_H}^r , \quad\text{and}$$

(6.2c) $$M_H^{\mathbb{C}} = L_H^{\mathbb{C}} + (P_{x_H}^{-u} \cap \tau P_{x_H}^u) + (P_{x_H}^u \cap \tau P_{x_H}^{-u}).$$

In particular, the complex flag on which $S_{[x_H]}$ is an open M_H-orbit is

(6.2d) $$M_H^{\mathbb{C}}(x_H) = M_H^{\mathbb{C}}/Q_H , \quad Q_H^r = L_H^{\mathbb{C}}, \quad Q_H^u = P_{x_H}^u \cap \tau P_{x_H}^{-u} .$$

Now we can prescribe the construction of $x_H \in X = G/P$ so that (6.1) holds:

(i) Let $L_H \subset M_H$ be the Lie algebra of a compact subgroup of M_H that is the centralizer there of a torus. This is always possible, e.g., by choice of L_H as the Lie algebra of a compact Cartan subgroup of M_H.

(ii) Let $Q_H \subset M_H^{\mathbb{C}}$ parabolic subgroup such that $Q_H^r = L_H^{\mathbb{C}}$ and $M_H^{\mathbb{C}} = L_H^{\mathbb{C}} + Q_H^u + \tau Q_H^u$ direct sum. Let $x_H \in M_H^{\mathbb{C}}/Q_H$ denote the identity coset, so L_H is the isotropy subalgebra of M_H at x_H, and $M_H(x_H)$ is a measurable open orbit on $M_H^{\mathbb{C}}/Q_H$ with compact isotropy group L_H.

(iii) Define $P^u = N_H^{\mathbb{C}} + Q_H^u$ unipotent subalgebra of G. I fine P to be the normalizer of P^u in G. Observe that

$$P = (L_H + A_H)^{\mathbb{C}} + (N_H^{\mathbb{C}} + Q_H^u),$$

so it contains a Cartan subalgebra $H^{\mathbb{C}}$ of G and it has unipoten ical P^u. Thus [21, Lemma 7.3] P is a parabolic subalgebra of G. Let P be the corresponding parabolic subgroup of G and define

(iv) Observe $G_0 \cap P = L_H A_H N_H$, $\tau P^r = P^r$, and $P + \tau P = (M_H + A_H + N_H)^{\mathbb{C}} = P_H^{\mathbb{C}}$. Thus we identify $x_H = 1 \cdot Q_H \in M_H^{\mathbb{C}}/Q_H$ $1 \cdot P \in G/P = X$, still having $M_H^{\mathbb{C}}(x_H) = M_H^{\mathbb{C}}/Q_H$, and [21, Corol 9.11] says that (6.1a) and (6.1b) hold. For (6.1e) recall that $M_H \cap Q_H = L_H$ compact.

Conversely, (6.2) shows that this construction of $x_H \in X = G/P$ vides all $x_H \in X = G/P$ for which (6.1) is valid.

We go on to the realization of the H-series representation on the orbit $X_H = G_0(x_H) \subset X = G/P$ that satisfies (6.1).

Our notation for representation spaces of irreducible unit representations of L_H, M_H and K is

(6.3a) if $\nu \in \hat{L}_H$ then W_ν is its representation space,

(6.3b) if $\mu \in \hat{M}_H$ then V_μ is its representation space,

(6.3c) if $\kappa \in \hat{K}$ then U_κ is its representation space.

Fix a *discrete* series representation $\mu \in \hat{M}_H$. If $\nu \in \hat{L}_H$ we sider the associated M_H-homogeneous holomorphic vector bundle

(6.4a) $$\mathbb{W}_\nu \to M_H/L_H = M_H(x_H) = S_{[x_H]}.$$

Suppose that we can find $\nu \in \hat{L}_H$, and an integer $q \geq 0$, such the space of square integrable harmonic forms

(6.4b) $H_2^{0,q}(\mathbb{W}_\nu) = V_\mu$ as a unitary M_H-representation space.

We realize $\pi_{\mu,\eta}$ on partially holomorphic cohomology as follows. Define

(6.5a) $\sigma_{\nu,\eta}$: irreducible representation of $L_H A_H N_H$ on W_ν

(6.5b) by : $\sigma_{\nu,\eta}(lan) = \nu(l)\cdot\exp((\rho_H + i\eta)(\log a))$.

The associated G_0-homogeneous complex vector bundle

(6.5c) $\mathbb{W}_{\nu,\eta} \to G_0/L_H A_H N_H = G_0(x_H) = X_H$

is holomorphic over every holomorphic arc component of X_H in X because

(6.5d) $\mathbb{W}_{\nu,\eta}|_{S_{[x_H]}} = \mathbb{W}_\nu$.

The holomorphic tangent space of $S_{[x_H]}$ is given at x_H, in the notation (6.2d), by

(6.6a) $T_{x_H} = Q_H^{-u} = P_{x_H}^{-u} \cap \tau P_{x_H}^{u}$.

There $A_H N_H$ acts trivially, and the action of L_H is unitary because it is the restriction of the adjoint action of G. Thus the anti-holomorphic cotangent space to $S_{[x_H]}$ at x_H is also given by

(6.6b) $\bar{T}^*_{x_H} = Q_H^{-u}$ with $L_H A_H N_H$ acting there by

(6.6c) $A_H N_H$ acts trivially, L_H acts through ad_G.

Now consider the space

(6.7a) $\Gamma^{0,q}(\mathbb{W}_{\nu,\eta})$: partially-$(0,q)$-forms in $\mathbb{W}_{\nu,\eta}$.

It consists of all

(6.7b) $f: G_0 \to W_\nu \otimes \Lambda^q Q_H^{-u}$ measurable, such that

(6.7c) $f(glan) = [\sigma_{\nu,\eta}(lan) \otimes \Lambda^q \mathrm{ad}_G(l)]^{-1} f(g)$ identically.

Note the consequences of the definition (6.7abc):

(6.7d) $f|_{kP_H}$ is a $(0,q)$-form on $S_{[kx_H]}$ with values in $\mathbb{W}_{\nu,\eta}|$

Thus we have a space

(6.8a) $L_2^{0,q}(\mathbb{W}_{\nu,\eta})$: square integrable partially-$(0,q)$-forms in $\mathbb{W}_{\nu,\eta}$

consisting of all $f \in \Gamma^{0,q}(\mathbb{W}_{\nu,\eta})$ such that

(6.8b) the $(0,q)$-form $f|_{kP_H}$ has finite L_2-norm $\|f_k\|$ a.e. in

(6.8c) $k \mapsto \|f_k\|$ is measurable and $\int_K \|f_k\|^2 dk < \infty$.

$L_2^{0,q}(\mathbb{W}_{\nu,\eta})$ is a Hilbert space with inner product

(6.9a) $\langle f,f'\rangle = \int_K (f|_{kP_H}, f'|_{kP_H})dk = \int_K dk \int_{M_H} (f(km), f'(km))dm$.

The operator $\bar\partial$ along the holomorphic arc components is the restriction of the $\bar\partial$-operator of X, so it commutes with the action of K, X_H and $\mathbb{W}_{\nu,\eta}$. Now, applying (3.16) to every holomorphic arc component of our orbit,

(6.9b) $L_2^{0,q}(\mathbb{W}_{\nu,\eta}) = \bar\partial L_2^{0,q-1}(\mathbb{W}_{\nu,\eta}) \oplus \bar\partial L_2^{0,q+1}(\mathbb{W}_{\nu,\eta}) \oplus H_2^{0,q}(\mathbb{W}$

orthogonal direct sum of closed G_0-invariant subspaces, where

(6.9c) $H_2^{0,q}(\mathbb{W}_{\nu,\eta}) = \{f \in L_2^{0,q}(\mathbb{W}_{\nu,\eta})$: each $f|_{kP_H}$ is harmonic

is our space of square integrable partially-harmonic-$(0,q)$-forms in $\mathbb{W}_{\nu,\eta}$.

We assert that the H-series representation

(6.10) $\pi_{\mu,\eta}$ is unitarily equivalent to: G_0 on $H_2^{0,q}(\mathbb{W}_{\nu,\eta})$.

That will be our geometric realization of $\pi_{\mu,\eta}$.

Looking at $L_2^{0,q}(\mathbb{W}_{\nu,\eta})$ as a K-module, we see that it is given by

(6.11a) $L_2^{0,q}(\mathbb{W}_{\nu,\eta}) = \{L_2(K) \otimes [L_2(M_H) \otimes W_\nu \otimes \Lambda^q Q_H^{-u}]^{L_H}\}^J$

where

(6.11b) $J = K \cap M_H$ maximal compact subgroup of M_H.

From (3.16) and (6.9bc), now

(6.11c) $H_2^{0,q}(\mathbb{W}_{\nu,\eta}) = \{L_2(K) \otimes H_2^{0,q}(\mathbb{W}_\nu)\}^J$ as K-module.

From the hypothesis (6.4) that $H_2^{0,q}(\mathbb{W}_\nu) = V_\mu$, and the Peter-Weyl decomposition of $L_2(K)$, we see that (6.11c) can be re-written as

(6.12) $H_2^{0,q}(\mathbb{W}_{\nu,\eta}) = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}^J$ as K-module.

Note that η does not occur in the K-module structure of $H_2^{0,q}(\mathbb{W}_{\nu,\eta})$.

If $g \in G_0$ we have $\kappa_g: K \to K$ defined by $g^{-1}k \in \kappa_g(k)AN$. Now $AN \subset MAN \subset M_H A_H N_H$. Thus we have

(6.13a) $\kappa_g: K \to K$, $\mu_g^H: K \to AN \cap M_H$ and $\alpha_g^H: K \to A_H \subset A$

such that

(6.13b) $g^{-1}k \in \kappa_g(k)\cdot\mu_g^H(k)\cdot\alpha_g^H(k)\cdot N_H$.

Viewing $f \in L_2^{0,q}(\mathbb{W}_{\nu,\eta})$ as a function from K to $L_2(M_H) \otimes W_\nu \otimes \Lambda^q Q_H^{-u}$ as in (6.11a), it follows from (6.7c) that the image of f under $g \in G_0$ is given by

(6.13c) $f^g(k) = [\lambda(\mu_g^H(k)) \otimes \exp((\rho_H + i\eta)\log\alpha_g^H(k)) \otimes 1]^{-1} f(\kappa_g(k))$

where λ is the left regular representation of M_H on $L_2(M_H)$. Th pothesis $H_2^{0,q}(\mathbb{W}_\nu) = V_\mu$ says that, on restriction of f to the space $H_2^{0,q}(\mathbb{W}_{\nu,\eta})$, and on then viewing f as a function from K t as in (6.12),

(6.14) $f^g(k) = \exp((\rho_H + i\eta)\log\alpha_g^H(k))^{-1}\cdot\mu(\mu_g^H(k))^{-1}\cdot f(\kappa_g(k))$

The representation space $\Gamma_2(\mathbb{E}_{\mu,\eta})$ of our H-series represe tion $\pi_{\mu,\eta}$ has K-module structure $\{L_2(K) \otimes V_\mu\}^J$, which we write the Peter-Weyl decomposition of $L_2(K)$ as

(6.15) $\Gamma_2(\mathbb{E}_{\mu,\eta}) = \sum_{\kappa\in\hat{K}} U_\kappa \otimes \{U_\kappa^* \otimes V_\mu\}^J$ as K-module.

Further, from (5.11b) and (6.12ab) we see that the action of G_0 $\Gamma_2(\mathbb{E}_{\mu,\eta})$ under (6.15) is

(6.16) $[\pi_{\mu,\eta}(g)f](k) = \exp((\rho_H + i\eta)\log\alpha_g^H(k))^{-1}\cdot\mu(\mu_g^H(k))^{-1}\cdot f(\kappa$

Finally, denote the multiplicity

(6.17a) $n(\mu,\kappa) = \dim\{U_\kappa^* \otimes V_\mu\}^J$.

As $\kappa \in \hat{K}$ has finite degree we have

(6.17b) $\kappa|_J = \sum_{j\in\hat{J}} n_1(j,\kappa)\cdot j$ with $\sum_{j\in\hat{J}} n_1(j,\kappa) < \infty$.

As J is a maximal compact subgroup of M_H and $\mu \in \hat{M}_H$ we have

(6.17c) $\mu|_J = \sum_{j\in\hat{J}} n_2(j,\mu)\cdot j$ with each $n_2(j,\mu) < \infty$.

Now compute, using Schur's Lemma and the fact that (6.17b) is a finite sum,

(6.17d) $n(\mu,\kappa) = \sum_{j\in\hat{J}} n_1(j,\kappa)\cdot n_2(j,\mu) < \infty$.

In summary, we have checked that

(6.18) $$\dim\{U_\kappa^* \otimes V_\mu\}^J < \infty.$$

Compare (6.12) with (6.15) and (6.14) with (6.16). In view of (6.18) it follows, as in the case of the principal series, that $\pi_{\mu,\eta}$ has the same distribution character as the representation of G_0 on $H_2^{0,q}(\mathbb{W}_{\nu,\eta})$. Both representations being K-finite and unitary, now they are infinitesimally equivalent, and thus unitarily equivalent. That completes the proof of (6.10).

I have indications of, but no pattern for, a "complementary series" of representations of G_0 associated to each nondiscrete H-series. The proof of (6.10) is designed so that it will automatically give geometric realization for any such "complementary H-series" representations of G_0.

Recall the main hypothesis (6.4) in the geometric realization of $\pi_{\mu,\eta}$: that the discrete series representation $\mu \in \hat{M}_H$ could be realized on the space $H_2^{0,q}(\mathbb{W}_\nu)$ of square integrable harmonic $(0,q)$-forms in the M_H-homogeneous holomorphic vector bundle $\mathbb{W}_\nu \to M_H/L_H = S_{[x_H]}$ associated to some $\nu \in \hat{L}_H$.

Making essential use of our simplifying assumption that $G_0 \subset G$, i.e., that G_0 is a linear group, we see that the finite group

(6.19a) $$F_H = K \cap \exp(iA_H)$$

is central in M_H and contained in L_H. Denote

(6.19b) $L_H^0 \subset M_H^0$: topological identity components of $L_H \subset M_H$;

(6.19c) $L_H^\dagger = F_H L_H^0$ and $M_H^\dagger = F_H M_H^0$.

Harish-Chandra pointed out to me that, looking at the character of a discrete series representation $\mu^\dagger \in M_H^\dagger$, one sees that the induced representation μ of M_H is irreducible; and also that one obtains all

discrete series representations μ of M_H in this way. Same situ for $L_H^\dagger$ and L_H.

As $S_{[x_H]}$ is connected and $F_H \subset L_H$, the subgroups $L_H^\dagger \subset L_H$ $M_H^\dagger \subset M_H$ have the same index r.

Let $\mu \in \hat{M}_H$ discrete. Now

(6.20a) $\mu|_{M_H^\dagger} = \mu_1 \oplus \ldots \oplus \mu_r$, $\mu_j \in \hat{M}_H^\dagger$ discrete, distinct,

such that

(6.20b) any of the μ_j induces μ.

Further note

(6.20c) $\mu_j|_{M_H^0} = \mu_j^0 \otimes \chi_\mu$, $\mu_j^0 \in \hat{M}_H^0$ discrete, $\chi_\mu \in \hat{F}_H$.

Similarly, if $\nu \in \hat{L}_H$, then

(6.21a) $\nu|_{L_H^\dagger} = \nu_1 \oplus \ldots \oplus \nu_r$, $\nu_j \in \hat{L}_H^\dagger$ distinct,

such that

(6.21b) any of the ν_j induces ν, and

(6.21c) $\nu_j|_{L_H^0} = \nu_j^0 \otimes \chi_\nu$, $\nu_j^0 \in \hat{L}_H^0$, $\chi_\nu \in \hat{F}_H$.

From $S_{[x_H]} = M_H/L_H = M_H^\dagger/L_H^\dagger = M_H^0/L_H^0$ we now have

(6.22a) if $H_2^{0,q}(\mathbb{W}_{\nu_j^0}) = V_{\mu_j^0}$ then $H_2^{0,q}(\mathbb{W}_{\nu_j}) = V_{\mu_j}$, and

(6.22b) if some $H_2^{0,q}(\mathbb{W}_{\nu_j}) = V_{\mu_j}$ then $H_2^{0,q}(\mathbb{W}_\nu) = V_\mu$.

Thus the question of validity of (6.4) is reduced to the case of connected linear Lie groups with compact center, whence it furthe reduces to the case of connected linear semisimple Lie groups. Tl was discussed in section 3.

References

1. A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami operator on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 81-130.

2. R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2) 66 (1957), 203-248.

3. F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France 84 (1956), 97-205.

4. J. M. G. Fell, book on harmonic analysis, in preparation.

5. Harish-Chandra, *Representations of a semisimple Lie group on a Banach space*, I, Trans. Amer. Math. Soc. 75 (1953), 185-243.

6. ————, *The Plancherel formula for complex semisimple Lie groups*, Trans. Amer. Math. Soc. 76 (1954), 485-528.

7. ————, *Representations of semisimple Lie groups*, IV, Amer. J. Math. 77 (1955), 743-777; V, ibid. 78 (1956), 1-41; VI, ibid. 78 (1956), 564-628.

8. ————, *Discrete series for semisimple Lie groups*, II, Acta Math. 116 (1966), 1-111.

9. ————, *Harmonic analysis on semisimple Lie groups*, Bull. Amer. Math. Soc. 76 (1970), 529-551.

10. S. Helgason, *Applications of the Radon transform to representations of semisimple Lie groups*, Proc. Nat. Acad. Sci. U. S. A. 63 (1969), 643-647.

11. ————, *A duality for symmetric spaces with applications to group representations*, to appear.

12. A. W. Knapp and E. M. Stein, *Singular integrals and the principal series*, Proc. Nat. Acad. Sci. U. S. A. 63 (1969), 281-284; II, ibid., to appear.

13. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) 74 (1961), 329-387.

14. ————, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. 75 (1969), 627-642.

15. R. A. Kunze and E. M. Stein, *Uniformly bounded representations*, III, Amer. J. Math. 89 (1967), 385-442.

16. M. S. Narasimhan and K. Okamoto, *An analogue of the Borel-Weil-Bott theorem for hermitian symmetric pairs of noncompact type*, Ann. of Math. (2) 91 (1970), 486-511.

17. K. R. Parthasarathy, R. Ranga-Rao and V. S. Varadarajan, *Representations of complex semisimple Lie groups and Lie algebras*, Ann. of Math. (2) 85 (1967), 383-429.

18. W. Schmid, *On a conjecture of Langlands*, Ann. of Math.,

19. N. R. Wallach, *Cyclic vectors and irreducibility for principal series representations*, to appear.

20. Garth Warner, *Harmonic analysis on semisimple Lie groups*, Springer-Verlag, Berlin-Heidelberg-New York, to appear about 1971.

21. J. A. Wolf, *The action of a real semisimple group on a complex flag manifold. I: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. 75 (1969), 1121-1237.

22. ————, II: *Unitary representations on partially holomorphic cohomology spaces*, to appear.

23. ————, III: *Induced representations based on hermitian symmetric spaces*, in preparation.

24. D. P. Zelobenko, *Analysis of irreducibility in the class of elementary representations of a complex semisimple Lie group*, Math. USSR - Izv. 2 (1968), 105-128.

Rutgers University, New Brunswick, New Jersey 08903 and

University of California, Berkeley, California 94720