Homogenous Holomorphic Vector Bundles

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§1. Introduction. Let X be a complex manifold homogeneous under a Lie group G. Choose a base point $x_0 \in X$ and represent $X = G/H$ under $g(x_0) = gH$, where $H = \{g \in G : g(x_0) = x_0\}$. A vector bundle $\mathbf{V} \to X$ is called G-homogeneous if the action of G on X lifts to an action of G on $\mathbf{V}$ by bundle automorphisms. Homogeneous vector bundles are usually obtained as associated bundles $\mathbf{V}_\chi \to X$ to the principal H-bundle $G \to X$ where χ is a continuous representation of H on a finite dimensional complex vector space V. The real analytic structure of $\mathbf{V}_\chi \to X$ is easily described, but questions of existence and uniqueness of holomorphic vector bundle structures for $\mathbf{V}_\chi \to X$ have only been understood when X is an open G-orbit in a complex manifold G^C/L for which G^C is a complexification of G and L is a closed complex Lie subgroup. Here we drop the stringent requirement that G have a complexification, giving general criteria (Theorem 3.6) for $\mathbf{V}_\chi \to X$ to have a holomorphic structure, and in that case a parameterization (Theorem 5.11) of all such structures by a certain cohomology set. Applications will appear in [3] and [5].

§2. Real analytic structure. Here X is any homogeneous space G/H relative to a base point x_0, where G is a Lie group that is not assumed to act effectively on X. Let χ be a continuous representation of H on a (finite dimensional) vector space V. We recall the associated G-homogeneous vector bundle $\mathbf{V}_\chi \to X$. Its total space $G \times_\chi V$ consists of all equivalence classes $[g, v]$ of elements $(g, v) \in G \times V$ under

$$(2.1) \qquad (gh, v) \sim (g, \chi(h)v); \quad \text{all} \quad g \in G, h \in H, v \in V.$$

The projection $[g, v] \to g(x_0)$, so the fibre over $g(x_0)$ is $\{[g, v] : v \in V\} \cong V$, and G acts by $g' : [g, v] \to [g'g, v]$. If $U \subset X$ is open and we denote

$$(2.2) \qquad G^U = \{g \in G : g(x_0) \in U\},$$

then it follows that the sections of $\mathbf{V}_\chi$ over U are precisely the continuous maps $s : U \to G \times_\chi V$ of the form

$$(2.3) \qquad s(gx_0) = [g, f_s(g)], \qquad f_s : G^U \to V,$$

where f_s satisfies the "functional equation"

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(2.4) $$f_s(gh) = \chi(h)^{-1} f_s(g); \quad \text{all} \quad g \in G^U,\ h \in H.$$

Let $\mathfrak{G}$ denote the Lie algebra of G. Let $\mathfrak{H}$ be the subalgebra for H. If $\xi \in \mathfrak{G}$ we view ξ as a left-invariant vector field on G, *i.e.*

(2.5) $$(\xi \cdot f)(g) = \frac{d}{dt}\bigg|_{t=0} f(g \cdot \exp(t\xi))$$

whenever f is a differentiable vector valued function on a neighborhood of g in G.

$\dot{\chi}$ denotes the representation of $\mathfrak{H}$ on V that is the differential of χ: $\dot{\chi}(\xi)v = (d/dt)\,|_{t=0}\, \chi(\exp t\xi)v$.

2.6. Lemma. *Let $\tilde{U} \subset G$ be an open set whose intersection with every coset gH is connected. Let $U = \{gx_0 : g \in \tilde{U}\}$ open set in X. Then a function $f : \tilde{U} \to V$ is of the form $f_s\,|\tilde{U}$ for a continuously differentiable section s of $\mathbf{V}_\chi$ over U, if and only if f is continuously differentiable and*

(2.7) $$\xi \cdot f + \dot{\chi}(\xi) f = 0 \quad \text{on } \tilde{U}; \quad \text{all } \xi \in \mathfrak{H}.$$

Proof. If $f = f_s\,|\tilde{U}$ then $f(gh) = \chi(h)^{-1} f(g)$ for $h \in H$ near the identity $1 \in H$. Thus for $\xi \in \mathfrak{H}$,

$$0 = \frac{d}{dt}\bigg|_{t=0} \{f(g \cdot \exp t\xi) - \chi(\exp t\xi)^{-1} f(g)\} = \{\xi \cdot f + \dot{\chi}(\xi) f\}(g).$$

Conversely let f be a C^1 solution to (2.7) on $\tilde{U}$. Let $L(V)$ be the vector space of linear transformations of V, χ^* the representation of H on $L(V)$ by $\chi^*(h) : A \to \chi(h) \cdot A$, and $\mathbf{V}_{\chi^*} \to X$ the associated G-homogeneous vector bundle. Given $g' \in \tilde{U}$ we choose a neighborhood $W \subset U$ of $g'(x_0)$ that carries a C^1 local section of $\mathbf{V}_{\chi^*}$ represented by a function $F : G^W \to L(V)$ whose values all are nonsingular transformations of V. That is just a local trivialization of $\mathbf{V}_\chi$ around $g'(x_0)$. Now $F(gh) = \chi(h)^{-1} F(g)$ for all $g \in G^W$ and all $h \in H$. As above, now $\xi \cdot F + \dot{\chi}(\xi) F = 0$ on G^W for all $\xi \in \mathfrak{H}$.

Represent $f = FA$ on $\tilde{U} \cap G^W$. Thus $A : \tilde{U} \cap G^W \to V$ is the C^1 function defined by $A(g) = F(g)^{-1} f(g)$. As f satisfies (2.7) on $\tilde{U}$,

$$(\xi \cdot F)A + F(\xi \cdot A) + \dot{\chi}(\xi) FA = 0 \quad \text{on} \quad \tilde{U} \cap G^W; \quad \text{all} \quad \xi \in \mathfrak{H}.$$

Now $\xi \cdot F + \dot{\chi}(\xi) F = 0$ on G^W implies that $F(\xi \cdot A) = 0$ on $\tilde{U} \cap G^W$ for all $\xi \in \mathfrak{H}$. As the values of F are invertible that says

$$\xi \cdot A = 0 \quad \text{on} \quad \tilde{U} \cap G^W; \quad \text{all} \quad \xi \in \mathfrak{H}.$$

As A is continuously differentiable, and as each $\tilde{U} \cap gH$ is connected, it follows that

$$A(gh) = A(g) \quad \text{for} \quad h \in H \quad \text{and} \quad g, gh \in \tilde{U} \cap G^W.$$

Now we compute that, for g and gh in $\tilde{U} \cap G^W$,

$$f(gh) = F(gh)A(gh) = F(gh)A(g) = \chi(h)^{-1} F(g) A(g) = \chi(h)^{-1} f(g).$$

In particular, if $h \in H$ with $g'h \in \tilde{U}$, then

$$f(g'h) = \chi(h)^{-1}f(g').$$

We have proved that any C^1 solution f to (2.7) satisfies $f(gh) = \chi(h)^{-1}f(g)$ for $g \in \tilde{U}$ and $h \in H$ such that $gh \in \tilde{U}$. Now f extends to a C^1 function $f_s : G^U \to V$ by $f_s(gh) = \chi(h)^{-1}f(g)$ for $h \in H$ and $g \in \tilde{U}$, defining a C^1 section s of $\mathbf{V}_\chi$ over U such that $f = f_s |_{\tilde{U}}$. Q.E.D.

§3. Holomorphic structure. We now assume that X has a G-invariant complex structure. Thus $\mathfrak{G}$, hence also its complexification $\mathfrak{G}^C$, acts as a Lie algebra of holomorphic vector fields on X. If $\xi \in \mathfrak{G}^C$ then $\xi = (\text{Re } \xi) + (-1)^{1/2} \cdot (\text{Im } \xi)$ with Re ξ, Im $\xi \in \mathfrak{G}$; if $x \in X$ then $\xi_x = (\text{Re } \xi)_x + (-1)^{1/2}(\text{Im } \xi)_x$ holomorphic tangent vector at x. Now define

$$\mathfrak{L} = \{\xi \in \mathfrak{G}^C : \xi_{x_0} = 0\}. \tag{3.1}$$

Then $\mathfrak{L}$ is a complex subalgebra of $\mathfrak{G}^C$ such that

$$\mathfrak{G}^C = \mathfrak{L} + \bar{\mathfrak{L}} \quad \text{and} \quad \mathfrak{H}^C = \mathfrak{L} \cap \bar{\mathfrak{L}}. \tag{3.2}$$

Furthermore, by definition (3.1),

$$ad(h)\xi \in \mathfrak{L}; \quad \text{all} \quad h \in H, \xi \in \mathfrak{L}. \tag{3.3}$$

Now let χ be a continuous representation of H on a (finite dimensional) complex vector space V. By *extension of χ from H to $\mathfrak{L}$* we mean a representation λ of $\mathfrak{L}$ on V such that

$$\lambda |_{\mathfrak{H}} = \dot{\chi}, \quad i.e. \quad \lambda(\xi) = \dot{\chi}(\xi) \quad \text{for all} \quad \xi \in \mathfrak{H}; \tag{3.4}$$

and

$$\chi(h)\lambda(\xi)\chi(h)^{-1} = \lambda(ad(h)\xi) \quad \text{for all} \quad h \in H, \xi \in \mathfrak{L}. \tag{3.5}$$

If $\chi(H)$ is connected, in particular if H is connected, then (3.4) implies (3.5). Then if $\mathfrak{L}$ has an ideal $\mathfrak{L}^-$ complementary to $\mathfrak{H}^C$ we can construct an extension by

$$\lambda(\xi + \eta) = \dot{\chi}(\text{Re } \xi) + (-1)^{1/2}\dot{\chi}(\text{Im } \xi), \qquad \xi \in \mathfrak{H}^C \quad \text{and} \quad \eta \in \mathfrak{L}^-.$$

In general, however, χ need not have an extension from H to $\mathfrak{L}$.

3.6. Theorem. *The structures of G-homogeneous holomorphic vector bundle on $\mathbf{V}_\chi \to X$ are in one-one correspondence with the extensions λ of χ from H to $\mathfrak{L}$. The structure corresponding to λ is the one for which the holomorphic sections s over any open set $U \subset X$ are characterized by the following equation on G^U:*

$$\xi \cdot f_s + \lambda(\xi)f_s = 0; \quad \text{all} \quad \xi \in \mathfrak{L}. \tag{3.7}$$

We first suppose that λ is given and prove a sequence of lemmas. The basic existence theorem for the system (3.7) is

3.8. Lemma. *If $v \in V$ then there exist an open neighborhood U' of 1 in G and a solution $f' : U' \to V$ of (3.7) such that $f'(1) = v$.*

For continuity of exposition we postpone the proof of Lemma 3.8 to §6. We modify Lemma 3.8:

3.9. Lemma. *Let $g \in G$, $x = g(x_0) \in X$ and $v \in V$. Then there exist an open neighborhood U of x in X and a section s of $\mathbf{V}_\chi$ over U, such that f_s satisfies* (3.7) *and $f_s(g) = v$.*

Proof. In Lemma 3.8 we may cut U' down so that every $U' \cap g'H$, $g' \in G$, is connected. Now let $\tilde{U} = gU'$, and $f(gg') = f'(g')$ for all $g' \in U'$. Then f satisfies (3.7), so it satisfies (2.7), and now Lemma 2.6 provides a section s of $\mathbf{V}_\chi$ over $U = \{\tilde{g}(x_0) : \tilde{g} \in \tilde{U}\}$ such that $f = f_s \,|\tilde{U}$. Let $\xi \in \mathfrak{K}$, let $g'' \in G^U$ and express $g'' = \tilde{g}h$ with $\tilde{g} \in \tilde{U}$ and $h \in H$. Then we calculate that f_s satisfies (3.7) on G^U:

$$\{\xi \cdot f_s + \lambda(\xi)f_s\}(g'') = (\xi \cdot f_s)(\tilde{g}h) + \lambda(\xi)f_s(\tilde{g}h) = \chi(h)^{-1}(ad(h)\xi \cdot f_s)(\tilde{g})$$
$$+ \chi(h)^{-1}\chi(h)\lambda(\xi)\chi(h)^{-1}f_s(\tilde{g}) = \chi(h)^{-1}\{ad(h)\xi \cdot f + \lambda(ad(h)\xi)f\}(\tilde{g}) = 0.$$

Note the essential use of both (3.4) and (3.5). Q.E.D.

Proof of Theorem 3.6. Let $L(V)$ denote the vector space of linear transformations of V. Then χ defines a continuous representation χ^* of H on $L(V)$ by $\chi^*(h) : A \to \chi(h) \cdot A$. Let λ be an extension of χ from H to $\mathfrak{K}$. It defines an extension λ^* of χ^* from H to $\mathfrak{K}$ by $\lambda^*(\xi) : A \to \lambda(\xi) \cdot A$. We use λ^* to apply Lemma 3.9 to $\mathbf{V}_{\chi^*} \to X$. Given $x \in X$, choose $g_x \in G$ with $g_x(x_0) = x$, and then x has a neighborhood U_x such that there is a function $F_x : G^{U_x} \to L(V)$ with the properties

(i) $F_x(gh) = \chi(h)^{-1}F_x(g)$, all $h \in H$ and $g \in G^{U_x}$;

(ii) $\xi \cdot F_x + \lambda(\xi)F_x = 0$ on G^{U_x}, all $\xi \in \mathfrak{K}$;

(iii) $F_x(g_x) = I$ identity transformation of V.

We use property (iii) to refine the open cover $\{U_x\}$ of X to an open cover $\{U_\alpha\}$, and restrict certain of the F_x to functions F_α, such that

$$F_\alpha : G^{U_\alpha} \to GL(V) \qquad \text{and} \quad F_\alpha \quad \text{satisfies (i) and (ii).}$$

Note that $\{(U_\alpha, F_\alpha)\}$ specifies a local trivialization of $\mathbf{V}_\chi$. We must check that it is holomorphic.

Let $g(x_0) \in U_\alpha \cap U_\beta$. If $h \in H$ then $F_\alpha(gh)^{-1}F_\beta(gh) = F_\alpha(g)^{-1}\chi(h)\chi(h)^{-1}F_\beta(g) = F_\alpha(g)^{-1}F_\beta(g)$. Thus, given α and β, there is a well defined function

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V) \quad \text{with} \quad F_\beta(gh) = F_\alpha(gh) \cdot \phi_{\alpha\beta}(g(x_0)).$$

To check that the transition function $\phi_{\alpha\beta}$ is holomorphic at $g(x_0)$ we may left-translate by g^{-1}, *i.e.* may assume $g = 1$. Let T be an antihomomorphic tangent vector to X at x_0. Then $\bar{T}$ is a holomorphic tangent vector, hence of the form ζ_{x_0} for some $\zeta \in \mathfrak{G}^C$. As $\mathfrak{G}^C = \mathfrak{K} + \bar{\mathfrak{K}}$, and as $\xi_{x_0} = 0$ for $\xi \in \mathfrak{K}$, we may assume $\zeta \in \bar{\mathfrak{K}}$. Let Φ be the lift of $\phi_{\alpha\beta}$ to $G^{U_\alpha \cap U_\beta}$. Now

$$T(\phi_{\alpha\beta}) = \overline{\zeta_{x_0}}(\phi_{\alpha\beta}) = (\bar{\zeta}\cdot\Phi)(1) = [\bar{\zeta}\cdot(F_\alpha^{-1}F_\beta)](1)$$
$$= [-F_\alpha^{-1}(\bar{\zeta}\cdot F_\alpha)F_\alpha^{-1}F_\beta + F_\alpha^{-1}(\bar{\zeta}\cdot F_\beta)](1)$$

But $\zeta \in \bar{\mathfrak{K}}$ implies $\bar{\zeta} \in \mathfrak{K}$, so that

$$\bar{\zeta}\cdot F_\alpha + \lambda(\bar{\zeta})F_\alpha = 0 = \bar{\zeta}\cdot F_\beta + \lambda(\bar{\zeta})F_\beta .$$

Now

$$T(\phi_{\alpha\beta}) = [F_\alpha^{-1}\lambda(\bar{\zeta})F_\alpha F_\alpha^{-1}F_\beta - F_\alpha^{-1}\lambda(\bar{\zeta})F_\beta](1) = 0.$$

That shows the transition function $\phi_{\alpha\beta}$ to be holomorphic.

We have shown that an extension λ of χ from H to $\mathfrak{K}$ provides $\mathbf{V}_\chi \to X$ with a G-invariant holomorphic vector bundle structure for which the local holomorphic trivializations, and thus the local holomorphic sections, are characterized by (3.7). Suppose that λ' is another extension that provides the same complex structure on $\mathbf{V}_\chi$. Then a section s of $\mathbf{V}_\chi$ over an open set $U \subset X$ has the property that f_s satisfies (3.7) with λ, if and only if s is holomorphic, if and only if f_s satisfies (3.7) with λ'. If $f : G^U \to V$ now $\xi\cdot f + \lambda(\xi)f = 0$ (all $\xi \in \mathfrak{K}$) if and only if $\xi\cdot f + \lambda'(\xi)f = 0$ (all $\xi \in \mathfrak{K}$). Let $v \in V$. Lemma 3.9 provides $U \ni x_0$ and $f : G^U \to V$ such that $f(1) = v$ and $\xi\cdot f + \lambda(\xi)f = 0$ (all $\xi \in \mathfrak{K}$). Now $\lambda'(\xi)v = \lambda'(\xi)f(1) = -(\xi\cdot f)(1) = \lambda(\xi)f(1) = \lambda(\xi)v$. Thus $\lambda = \lambda'$. Now our construction (3.7) is a one to one map from the set of all extenstions of χ from H to $\mathfrak{K}$, into the set of all G-invariant holomorphic vector bundle structures on $\mathbf{V}_\chi$.

Finally suppose that we start with a holomorphic structure on $\mathbf{V}_\chi$. Then $\mathfrak{G}^C$ acts as a Lie algebra of holomorphic vector fields on the total space $G \times_\chi V$ of $\mathbf{V}_\chi$, and $\mathfrak{K}$ is the subalgebra of fields tangent to the fibre V_{x_0} over x_0 . Let U be a neighborhood of x_0 and F', $F : G^U \to GL(V)$ holomorphic local trivializations of $\mathbf{V}_\chi$ over U. If $\xi \in \mathfrak{K}$ then $\xi_{x_0} = 0$ so $\xi_{x_0}(F^{-1}F') = 0$. Thus $(\xi\cdot F)F^{-1} = (\xi\cdot F')F'^{-1}$ at $1 \in G$. Define $\lambda(\xi) = [(\xi\cdot F)F^{-1}]$, independent of choice of F. Then $\lambda : \mathfrak{K} \to L(V)$ is linear and every local holomorphic section s of $\mathbf{V}_\chi$ satisfies (3.7). In the proof of Lemma 3.8, it is seen that the integrability condition for (3.7) is that the linear map λ be a Lie algebra representation; so λ is a representation of $\mathfrak{K}$ on V. Lemma 2.6 implies $\lambda|_{\mathfrak{H}} = \dot{\chi}$. If $\xi \in \mathfrak{K}$, $h \in H$ and $v \in V$, we choose a holomorphic section s of $\mathbf{V}_\chi$ over a neighborhood of x_0 such that $f_s(1) = v$; then

$$\chi(h)\lambda(\xi)\chi(h)^{-1}v = \chi(h)\lambda(\xi)f_s(h) = -\chi(h)[\xi\cdot f_s](h) = -[(ad(h)\xi)\cdot f_s](1)$$
$$= \lambda(ad(h)\xi)f_s(1) = \lambda(ad(h)\xi)v.$$

Thus λ is an extension of χ from H to $\mathfrak{K}$. Q.E.D.

We remark that we have arranged our proofs so that Theorem 3.6 remains true with $\dim_C V = \infty$, provided that Lemma 3.8 remains true. As will be seen in §6, the latter condition is that V be a Banach space and that $\lambda(\mathfrak{K})$ consist of bounded operators. In particular, $\dot{\chi}(\mathfrak{H})$ must consist of bounded operators.

That is very close to the finite dimensionality of V. For example, if H is compact, then $\dot{\chi}(\mathfrak{H})$ consists of bounded operators if and only if $\dot{\chi}$ has only finitely many inequivalent irreducible subrepresentations.

§4. Complementary ideal. In this section we give some applications of Theorem 3.6 involving particular types of extensions λ of χ from H to $\mathfrak{L}$.

4.1. Theorem. *Let X be a complex homogeneous space G/H, x_0 the base point, and*

$$\mathfrak{L} = \{\xi \varepsilon \mathfrak{G}^C : \xi_{x_0} = 0\}.$$

Suppose that $\mathfrak{L}$ has an ideal $\mathfrak{L}^-$ complementary to $\mathfrak{H}^C$ and invariant under $ad_G(H)$.

Let χ be a continuous representation of H on a complex vector space, $\mathbf{V}_\chi \to X$ the associated G-homogeneous complex vector bundle. Then $\mathbf{V}_\chi \to X$ has a G-homogeneous holomorphic vector bundle structure for which the holomorphic sections s over an open set $U \subset X$ are characterized by

$$\textit{section} \quad : \xi \cdot f_s + \dot{\chi}(\xi) f_s = 0 \qquad \textit{for all} \quad \xi \varepsilon \mathfrak{H};$$

$$\textit{holomorphic} : \xi \cdot f_s = 0 \qquad \textit{for all} \quad \xi \varepsilon \mathfrak{L}^-.$$

{For we define an extension λ of χ from H to $\mathfrak{L}$ by $\lambda(\mathfrak{L}^-) = 0$, and we then apply Theorem 3.6.}

If A is a complex semisimple Lie group, then *parabolic subgroup* of A means a complex Lie subgroup $B \subset A$ such that A/B is compact and connected. The coset spaces $F = A/B$, where A is a complex semisimple Lie group and B is a parabolic subgroup, are the *complex flag manifolds*. It is understood that F carries the A-invariant complex structure whose holomorphic functions lift to holomorphic functions on A.

4.2. Corollary. *Let $F = A/B$ complex flag manifold. Suppose $\mathfrak{A} = \mathfrak{G}^C$ and let $\bar{G}$ denote the real analytic subgroup of A for $\mathfrak{G}$. Let $\pi : G \to \bar{G}$ be a Lie group covering, so G acts on F by $g : f \to \pi(g)f$. Let $X = G(x_0)$ be an open G-orbit on F and represent $X = G/H$. Suppose that X has a G-invariant Radon measure (automatic if $\pi(H)$ is compact).*

If χ is a continuous complex representation of H, then $\mathbf{V}_\chi \to X$ has a natural structure of G-homogeneous holomorphic vector bundle.

{We may assume $x_0 = 1 \cdot B \varepsilon F$. Existence of the G-invariant measure says [4, Theorem 6.3] that $\mathfrak{B} = \mathfrak{H}^C + \mathfrak{B}^u$ where $\mathfrak{B}^u$ is an ideal stable under every automorphism that extends to $\mathfrak{A}$. Let $\mathfrak{L}^- = \mathfrak{B}^u$ and apply Theorem 4.1}.

4.3. Corollary. *Let X be a bounded symmetric domain, G a covering group of the group of holomorphic automorphisms of X. Represent $X = G/H$ and let χ be a continuous representation of H. Then $\mathbf{V}_\chi \to X$ has a natural structure of G-homogeneous holomorphic vector bundle.*

{For the Borel embedding of X in its compact dual provides the hypotheses of Corollary 4.2}.

A variation on Theorem 4.1 and its corollaries shows that our method includes the known method:

4.4. Theorem. *Let $X = G/H$ where G is a complex Lie group and H is a closed complex subgroup.*

1. $\mathfrak{G}^C = \mathfrak{G} \oplus \overline{\mathfrak{G}}$ *with* $\mathfrak{G} = \{(\xi_1, \xi_2) \in \mathfrak{G}^C : \xi_1 = \xi_2\}$ *and* $\mathfrak{H}^C = \mathfrak{H} \oplus \overline{\mathfrak{H}}$ *as real Lie algebras, and* $\mathfrak{L} = \mathfrak{H} \oplus \overline{\mathfrak{G}}$. *Thus* $\mathfrak{L}^- = \{(\xi_1, \xi_2) \in \mathfrak{G}^C : \xi_1 = 0\} \cong \mathfrak{G}$ *is a natural choice of $ad_G(H)$-invariant ideal in $\mathfrak{L}$ complementary to* $\mathfrak{H} \oplus 0 = \{(\xi_1, \xi_2) \in \mathfrak{G}^C : \xi_1 \in \mathfrak{H}$ *and* $\xi_2 = 0\}$.

2. *Let χ be a holomorphic representation of H on a complex vector space V. Then $\dot{\chi}$ maps $\mathfrak{H}^C$ by $\dot{\chi}(\xi_1, \xi_2) = \dot{\chi}(\xi_1)$. In particular there is a natural extension λ of χ from H to $\mathfrak{L}$ given by*

$$\lambda(\xi_1, \xi_2) = \dot{\chi}(\xi_1) \quad \textit{for} \quad (\xi_1, \xi_2) \in \mathfrak{L} = \mathfrak{H} \oplus \overline{\mathfrak{G}}. \tag{4.5}$$

3. *The product complex structure on $G \times V$ induces a complex structure on $G \times_\chi V$, and the latter is the complex structure on the total space for the G-homogeneous holomorphic vector bundle structure on $\mathbf{V}_\chi \to X$ defined by the extension λ of χ.*

Remark. Theorem 4.4 shows that the construction of Theorem 3.6 contains the usual construction (complex groups) as a special case.

Proof. That $\mathfrak{G}^C = \mathfrak{G} \oplus \overline{\mathfrak{G}} \supset \mathfrak{H} \oplus \overline{\mathfrak{H}} = \mathfrak{H}^C$ with scalar multiplication $\alpha(\xi_1, \xi_2) = (\alpha\xi_1, \bar{\alpha}\xi_2)$ and with complex conjugation given by $\overline{(\xi_1, \xi_2)} = (\xi_2, \xi_1)$, is standard. Then if we view the diagonal algebra $\mathfrak{G}$ as an algebra of real valued vector fields on X, and $\mathfrak{G} \oplus 0$ as an algebra of fields of type (1, 0), the assertion on $\mathfrak{L}$ is immediate and statement (1) follows.

For statement (2) let $(\xi_1, \xi_2) \in \mathfrak{H}^C$. It has complex conjugate (ξ_2, ξ_1), hence real and imaginary parts

$$\begin{aligned} 2Re(\xi_1, \xi_2) &= (\xi_1, \xi_2) + (\xi_2, \xi_1) = (\xi_1 + \xi_2, \xi_1 + \xi_2), \\ 2iIm(\xi_1, \xi_2) &= (\xi_1, \xi_2) - (\xi_2, \xi_1) = (\xi_1 - \xi_2, \xi_2 - \xi_1), \\ 2Im(\xi_1, \xi_2) &= (-i(\xi_1 - \xi_2), -i(\xi_1 - \xi_2)). \end{aligned}$$

Thus $\dot{\chi}$ is given on $\mathfrak{H}^C$ by $\dot{\chi}(\xi, \xi) = \dot{\chi}(\xi)$ and

$$\begin{aligned} \dot{\chi}(\xi_1, \xi_2) &= \dot{\chi}(Re(\xi_1, \xi_2)) + i\dot{\chi}(Im(\xi_1, \xi_2)) \\ &= \frac{1}{2}\dot{\chi}(\xi_1 + \xi_2, \xi_1 + \xi_2) + \frac{i}{2}\dot{\chi}(-i(\xi_1 - \xi_2), -i(\xi_1 - \xi_2)) \\ &= \frac{1}{2}\{\dot{\chi}(\xi_1) + \dot{\chi}(\xi_2) + \dot{\chi}(\xi_1) - \dot{\chi}(\xi_2)\} = \dot{\chi}(\xi_1). \end{aligned}$$

That proves (2).

Let f be a C^1 complex vector valued function on an open subset of G. Then f is holomorphic if and only if $\xi \cdot f = 0$ for every $\xi \in \mathfrak{G}^C$ of the form $\xi = (0, \xi_2)$. If f takes values in V, in particular if $f = f_s$ for some smooth local section s of $\mathbf{V}_\chi$, we note that the latter can be written

(i) $\xi \cdot f + \lambda(\xi) f = 0$ for all $\xi \in \mathfrak{L}^- = 0 \oplus \overline{\mathfrak{G}}$.

On the other hand, $f = f_s$ as above, says

(ii) $\xi \cdot f + \lambda(\xi) f = 0$ for all $\xi \in \mathfrak{H} \oplus 0$.

If $f = f_s$ for a local section s, now s is holomorphic relative to the complex structure on $G \times_\chi V$ induced by the product structure, if and only if

(iii) $\xi \cdot f + \lambda(\xi) f = 0$ for all $\xi \in \mathfrak{L} = \mathfrak{H} \oplus \overline{\mathfrak{G}}$.

Theorem 3.6 says that (iii) is the condition that s be holomorphic for the G-homogeneous holomorphic vector bundle structure on $\mathbf{V}_\chi \to X$ defined by the extension λ. That proves (3). Q.E.D.

4.6. Corollary. *Let $F = A/B$ complex flag manifold. Suppose $\mathfrak{A} = \mathfrak{G}^C$, let $\bar{G}$ be the analytic subgroup of A for $\mathfrak{G}$, and assume that $\overline{\mathfrak{G}}$ has a compact Cartan subgroup. Let $\pi : G \to \bar{G}$ covering group, $X = G(x_0)$ open orbit on F, and $H = \{g \in G : g(x_0) = x_0\}$ so $X = G/H$ homogeneous complex manifold.*

Let χ be a continuous representation of H on a complex vector space V that factors through $\pi(H)$. Then there is a unique holomorphic representation β of B on V such that $\chi = \beta \cdot \pi|_H$ and $\mathbf{V}_\chi \to X$ (holomorphic structure specified [See note] by Corollary 4.2) is the restriction to X of $\mathbf{V}_\beta \to F$ (holomorphic structure induced from $A \times V$).

Note. *It is automatic here* [4, Corollary 6.4] *that X has a G-invariant Radon measure.*

Proof. $\mathfrak{B} = \mathfrak{H}^C \oplus \mathfrak{L}^- = \mathfrak{L}$ as in the proof of Corollary 4.2, with the holomorphic structure on $\mathbf{V}_\chi$ determined by the extension λ that annihilates $\mathfrak{L}^-$. As $\bar{G}$ has a compact Cartan subgroup and χ factors through $\pi|_H$, now $\lambda = \dot{\beta}$ where β represents B on V holomorphically and $\chi = \beta \cdot \pi|_H$. Now $\mathbf{V}_\chi \cong \mathbf{V}_\beta|_X$ as G-homogeneous complex vector bundles, and the correspondence is holomorphic by Theorem 4.4. Q.E.D.

§5. Galois cohomology. In this section we start with an arbitrary fixed extension λ of χ from H to $\mathfrak{L}$ and construct a noncommutative cohomology set $\mathbf{H}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^H_A$ that describes all A-equivalence classes of extensions of χ from H to $\mathfrak{L}$. There $\mathfrak{E} = \mathfrak{E}(V)$ endomorphism algebra of the representation space of χ and λ, and A is an arbitrary subgroup of the group of invertible elements of the commuting algebra $C(\chi) = \{a : V \to V \text{ linear: } a\chi(h) = \chi(h)a$ for all $h \in H\}$. The case $A = \{1\} = H$ has been studied by Nijenhuis and Richardson [2] from the viewpoint of deformations of λ.

Our cohomology set will be an equivariant Lie algebra version of a relative

Galois cohomology set. We recall the latter in order to motivate our definitions.

Let L be a group, $H \subset L$ a subgroup, and $\lambda : L \to GL(V)$ a representation on a vector space V. Then L acts on $GL(V)$ by ${}^l g = \lambda(l) g \lambda(l^{-1})$. By *1-cocycle* for L on $GL(V)$ in that action, one means a map $z : L \to GL(V)$ such that

$$z(l_1 l_2) = {}^{l_1}z(l_2) \cdot z(l_1) \quad \text{for all} \quad l_1, l_2 \in L.$$

The set of all such 1-cocycles is denoted $\mathbf{Z}^1_\lambda(L; GL(V))$. It is straightforward to check that a map $\mu : L \to GL(V)$ is a representation of L on V, if and only if $\mu(l) = z(l)^{-1}\lambda(l)$, all $l \in L$, with $z \in \mathbf{Z}^1_\lambda(L, GL(V))$. Then we denote $\mu = \lambda_z$, result of "twisting" λ by z. Note $\mu|_H = \lambda|_H$ if and only if $z(h) = I$ for all $h \in H$. Now the relative set

$$\mathbf{Z}^1_\lambda(L, H; GL(V)) = \{z \in \mathbf{Z}^1_\lambda(L; GL(V)) : z(H) = I\}$$

consists of all z such that $\lambda_z|_H = \lambda|_H$.

Cocycles $z, z' \in \mathbf{Z}^1_\lambda(L; GL(V))$ are *cohomologous* if there exists $a \in GL(V)$ such that

$$z'(l) = {}^l a \cdot z(l) \cdot a^{-1} \quad \text{for all} \quad l \in L.$$

It is straightforward to check that z and z' are cohomologous if and only if the representations λ_z and $\lambda_{z'}$ are equivalent. In particular this is an equivalence relation. The equivalence classes are the *1-cohomology classes* for L on $GL(V)$; they form a set denoted $\mathbf{H}^1_\lambda(L; GL(V))$; now the *cohomology set* $\mathbf{H}^1_\lambda(L; GL(V))$ parameterizes the equivalence classes of representations of L on V. The classes represented by elements $z \in \mathbf{Z}^1_\lambda(L, H; GL(V))$ form a *relative cohomology set* $\mathbf{H}^1_\lambda(L, H; GL(V))$. Note that if $z, z' \in \mathbf{Z}^1_\lambda(L, H; GL(V))$ are cohomologous by $a \in GL(V)$ then $a \in C(\lambda|_H)$. Thus $\mathbf{H}^1_\lambda(L, H; GL(V))$ parameterizes the representations of L on V that agree with λ on H, modulo equivalence by invertible elements of $C(\lambda|_H)$.

We now describe the equivariant Lie algebra version of the Galois cohomology set $\mathbf{H}^1_\lambda(L, H; GL(V))$.

Let $\mathfrak{L}$ be a Lie algebra, $\mathfrak{H}$ a subalgebra, and B a group that acts on $\mathfrak{L}$ by automorphisms preserving $\mathfrak{H}$. Suppose that B also acts by automorphisms on a Lie algebra $\mathfrak{E}$. Now fix an equivariant homomorphism $\lambda \in \mathrm{Hom}_B(\mathfrak{L}, \mathfrak{E})$.

By *equivariant 1-cocycle* from $\mathfrak{L}$ to $\mathfrak{E}$ (relative to λ) we mean a linear map $f : \mathfrak{L} \to \mathfrak{E}$ such that

$$f(b\xi) = bf(\xi) \quad \text{for} \quad b \in B, \quad \xi \in \mathfrak{L}; \tag{5.1}$$

and

$$[f\xi_1, f\xi_2] = -[\lambda\xi_1, f\xi_2] + [\lambda\xi_2, f\xi_1] + f[\xi_1, \xi_2] \quad \text{for} \quad \xi_1, \xi_2 \in \mathfrak{L}. \tag{5.2}$$

The set of all such cocycles forms a set denoted $\mathbf{Z}^1_\lambda(\mathfrak{L}; \mathfrak{E})^B$. The *relative* equivariant 1-cocycles form the subset

$$\mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B = \{f \in \mathbf{Z}^1_\lambda(\mathfrak{L}; \mathfrak{E})^B : f(\mathfrak{H}) = 0\}. \tag{5.3}$$

It is quite straightforward to check that

(5.4) $f \to \lambda + f$ bijects $\mathbf{Z}^1_\lambda(\mathfrak{L}; \mathfrak{E})^B$ onto $\mathrm{Hom}_B(\mathfrak{L}; \mathfrak{E})$,

and

(5.5) $\mu \in \mathrm{Hom}_B(\mathfrak{L}, \mathfrak{E})$ with $\mu|_\mathfrak{H} = \lambda|_\mathfrak{H}$ iff $\mu - \lambda \in \mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B$.

Now let A be a group of automorphisms of $\mathfrak{E}$ such that

(5.6) $ab = ba$ and $a(\lambda(\xi)) = \lambda(\xi)$ for all $a \in A, b \in B, \xi \in \mathfrak{H}$.

We say that cocycles $f, f' \in \mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B$ are *cohomologous* if there is an element $a \in A$ such that

(5.7) $f'(\xi) = a(\lambda(\xi) + f(\xi)) - \lambda(\xi)$ for $\xi \in \mathfrak{L}$.

That is an equivalence relation because, as is easily checked,

(5.8) $\begin{cases} f, f' \in \mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B \text{ are cohomologous if and only if} \\ \lambda + f, \lambda + f' \in \mathrm{Hom}_B(\mathfrak{L}, \mathfrak{E}) \text{ are } A\text{-equivalent.} \end{cases}$

The equivalence classes are the *equivariant relative* 1-*cohomology classes*. They form a cohomology set $\mathbf{H}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B_A$. In summary

5.9. Proposition. *The map $[f] \to (\lambda + f)_A$ bijects $\mathbf{H}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B_A$ onto the set of all A-equivalence classes of elements $\mu \in \mathrm{Hom}_B(\mathfrak{L}, \mathfrak{E})$ such that $\mu|_\mathfrak{H} = \lambda|_\mathfrak{H}$.*

Now suppose $\mathfrak{E} = \mathfrak{E}(V)$, Lie algebra of endomorphisms of a vector space V, and that A and B act on $\mathfrak{E}$ by conjugation through representations on V. Thus viewing elements $a \in A$ and $b \in B$ as linear transformations of V, we re-write

(5.1)′ $f(b\xi) = bf(\xi)b^{-1}$, $b \in B$ and $\xi \in \mathfrak{L}$;

(5.6)′ $aba^{-1}b^{-1}$ scalar and $a\lambda(\xi)a^{-1} = \lambda(\xi)$, $a \in A, b \in B, \xi \in \mathfrak{H}$;

(5.7)′ $f'(\xi) = (a\lambda(\xi) - \lambda(\xi)a)a^{-1} + af(\xi)a^{-1}$, $\xi \in \mathfrak{L}$.

Then Proposition 5.9 specializes to

5.10. Proposition. *$[f] \to (\lambda + f)_A$ bijects $\mathbf{H}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^B_A$ onto the set of all A-equivalence classes of B-equivariant representations μ of $\mathfrak{L}$ on V such that $\mu|_H = \lambda|_H$.*

We apply Proposition 5.10 to homogeneous vector bundles.

5.11. Theorem. *Let $X = G/H$ homogeneous complex manifold, $\mathfrak{L} \subset \mathfrak{G}^C$ the subalgebra defined by* (3.1), *and χ a continuous representation of H on a complex vector space V. Suppose that χ has an extension λ from H to $\mathfrak{L}$. Let $\mathfrak{E} = \mathfrak{E}(V)$.*

1. *Let H act on $\mathfrak{L}$ by ad_G, on V by χ. Then $f \to \lambda + f$ is a bijection of $\mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^H$ onto the set of all extensions of χ from H to $\mathfrak{L}$. In particular* (*Theorem* 3.6) *$\mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{E})^H$ parameterizes the set of all G-homogeneous holomorphic vector bundle structures on $\mathbf{V}_\chi \to X$.*

2. *Let A be the group of invertible elements in the commuting algebra of χ. Then two extensions of χ from H to $\mathfrak{L}$ are A-equivalent if and only if the corresponding holomorphic structures on $\mathbf{V}_\chi \to X$ are related by a G-bundle equivalence, inducing the identity transformation of X. In particular $\mathbf{H}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{G})^H_A$ parameterizes the set of all G-bundle equivalence classes of G-homogeneous holomorphic vector bundle strunctures on $\mathbf{V}_\chi \to X$.*

Proof. By definition, μ is an extension of χ from H to $\mathfrak{L}$ if and only if $\mu \in \mathrm{Hom}_H(\mathfrak{L}, \mathfrak{G})$ with $\mu\,|_{\mathfrak{H}} = \dot{\chi}$. Thus $\lambda \in \mathrm{Hom}_H(\mathfrak{L}, \mathfrak{G})$, and μ is an extension of χ from H to $\mathfrak{L}$ if and only if $\mu \in \mathrm{Hom}_H(\mathfrak{L}, \mathfrak{G})$ with $\mu\,|_{\mathfrak{H}} = \lambda\,|_{\mathfrak{H}}$. Now (5.5) shows that $f \to \lambda + f$ bijects $\mathbf{Z}^1_\lambda(\mathfrak{L}, \mathfrak{H}; \mathfrak{G})^H$ onto the set of extensions of χ from H to $\mathfrak{L}$. That proves (1).

Let $\beta : \mathbf{V}_\chi \to \mathbf{V}_\chi$ be a G-bundle equivalence. Then β commutes with the action of G and β is a linear transformation on each fibre. Thus β is specified by an invertible linear transformation α on the fibre over x_0 that commutes with every $h \in H$ on that fibre, and any such α specifies an equivalence β, by $\beta = g\alpha g^{-1}$ on the fibre over $g(x_0)$. Now β is specified by an arbitrary $a \in A$ via $\beta[g, v] = [g, a(v)]$. The first assertion of (2) is proved, and the second now follows from (1) and Proposition 5.10. Q.E.D.

Remark. If we could find a Lie group L with Lie algebra $\mathfrak{L}$, such that H is a subgroup of L that meets every component, then we could manage a version of Theorem 5.11 with ordinary equivariant relative Galois cohomology. But if such a group L exists, we more or less have G contained in its complexification, and that is far from the case when X is the unit disc and G is the universal covering group of $SL(2, R)$. Thus we need the Lie algebra version of the Galois cohomology.

§6. **Proof of Lemma 3.8.** We obtain Lemma 3.8 as a consequence of the following Cauchy–Kowalewski–Lie type theorem on local solutions of uniformly bounded first order quasi-linear systems. As we could not find the statement or proof in the literature we include them for the convenience of the reader.

6.1. Theorem. *Let $\mathfrak{L}$ be a complex Lie albegra of holomorphic vector fields on an open set $W \subset C^N$ such that*

$$\text{if} \quad w \in W \quad \text{and} \quad 0 \neq \xi \in \mathfrak{L} \quad \text{then} \quad \xi_w \neq 0.$$

Let V be a complex Banach space, $L(V)$ the algebra of bounded linear transformations of V, and $\lambda : \mathfrak{L} \to L(V)$ a linear map. Consider the system

$$\xi \cdot f + \lambda(\xi) f = 0, \quad \text{all} \quad \xi \in \mathfrak{L}, \tag{6.2}$$

for functions $f : W \to V$. Choose $w_0 \in W$. Then the following conditions are equivalent.

1. *λ is a Lie algebra representation, i.e. $\lambda[\xi_1, \xi_2] = \lambda(\xi_1)\lambda(\xi_2) - \lambda(\xi_2)\lambda(\xi_1)$ for all $\xi_1, \xi_2 \in \mathfrak{L}$.*

2. *Given $v \in V$ there is an open neighborhood W_0 of w_0 in W that carries a holomorphic solution $f : W_0 \to V$ of (6.2) such that $f(w_0) = v$.*

3. *There are complex local coordinates $z = (z', z'')$ on a neighborhood W_0^1 of w_0 in W such that $z(w_0) = 0$ and*

(i) $z' = (z^1, \cdots, z^n)$, $n = \dim_C \mathfrak{L}$, *and* $z'' = (z^{n+1}, \cdots, z^N)$;

(ii) *if $\xi \in \mathfrak{L}$ then*

$$\xi_w = \sum_{k=1}^{n} a_\xi^k(z(w)) \frac{\partial}{\partial z^k} \quad \text{with} \quad a_\xi^k(z) \quad \text{holomorphic.}$$

If $\phi(z'')$ is a holomorphic function on a neighbordood of 0 in C^{N-n} then there exist a neighborhood W_0 of w_0 in W_0^1 and a unique holomorphic function $f : W_0 \to V$ that satisfies (6.2), such that

$$\text{if} \quad w \in W_0 \quad \text{and} \quad z'(w) = 0 \quad \text{then} \quad f(w) = \phi(z''(w)).$$

Remark. The proof will be valid for a real Lie algebra of C^ω vector fields, a real Banach space, and a real analytic initial value function ϕ.

Lemma 3.8 follows from Theorem 6.1. Let G^C denote any complex Lie group with Lie algebra $\mathfrak{G}^C$. Choose an open set $\mathfrak{W} \subset \mathfrak{G}^C$ such that

(i) $W = \exp(\mathfrak{W})$ is a complex local coordinate neighborhood of 1 in G^C,

(ii) $\exp : \mathfrak{W} \to W$ is a diffeomorphism, and

(iii) $\exp_G$ maps $\mathfrak{G} \cap \mathfrak{W}$ diffeomorphically onto an open set $U \subset G$.

We identify U with a closed subset of W by $\exp_G(\xi) \to \exp(\xi) \in W$.

$\mathfrak{L}$ is a Lie algebra of left invariant holomorphic vector fields on W, and also a Lie algebra of complex valued vector fields on U. Those realizations are related by

$$(6.3) \qquad (\xi \cdot f)|_U = \xi \cdot (f|_U) \quad \text{for} \quad \xi \in \mathfrak{L}.$$

$\lambda : \mathfrak{L} \to L(V)$ is given as a Lie algebra representation. Let $w_0 = 1 \in W$ and choose $v \in V$. Theorem 6.1 provides an open set W_0, $1 \in W_0 \subset W$, and a holomorphic solution $f : W_0 \to V$ to (6.2), such that $f(1) = v$. Let $U' = U \cap W_0$ and $f' = f|_{U'}$. Then (6.3) shows that $f' : U' \to V$ is a solution of (3.7) such that $f'(1) = v$. Thus Theorem 6.1 implies Lemma 3.8. Q.E.D.

Proof of Theorem 6.1. If (3) holds, then (2) follows with the choice $\phi(z'') = v$ for all z''.

Assume (2) and let $f : W_0 \to V$ be a holomorphic solution to (6.2) such that $f(w_0) = v$. Then

$$\begin{aligned}\lambda[\xi_1, \xi_2]v &= \lambda[\xi_1, \xi_2]f(w_0) = -([\xi_1, \xi_2]\cdot f)(w_0) \\ &= -(\xi_1\cdot(\xi_2\cdot f))(w_0) + (\xi_2\cdot(\xi_1\cdot f))(w_0) \\ &= (\xi_1\cdot(\lambda(\xi_2)f))(w_0) - (\xi_2\cdot(\lambda(\xi_1)f))(w_0)\end{aligned}$$

$$= \lambda(\xi_2)(\xi_1 \cdot f)(w_0) - \lambda(\xi_1)(\xi_2 \cdot f)(w_0)$$
$$= \{-\lambda(\xi_2)\lambda(\xi_1) + \lambda(\xi_1)\lambda(\xi_2)\}f(w_0)$$
$$= \{\lambda(\xi_1)\lambda(\xi_2) - \lambda(\xi_2)\lambda(\xi_1)\}v.$$

As $v \in V$ is arbitrary that shows λ to be a Lie algebra representation. Thus (2) implies (1).

We go on to prove that (1) implies (3). The complex Frobenius Theorem (*cf.* [1], p. 323) provides a complex local coordinate neighborhood (W_0^1, z) of w_0 in W such that $z(w_0) = 0$ and $dz^i(\xi) = 0$ for $n < i \leqq N$ and $\xi \in \mathfrak{L}$. That is where we use the hypothesis that $\mathfrak{L}$ is a Lie algebra of holomorphic vector fields on W such that $\xi_w \neq 0$ for $w \in W$ and $0 \neq \xi \in \mathfrak{L}$. Now $z = (z', z'')$ as in (i) of (3), and every $\xi \in \mathfrak{L}$ has expression $\xi = \sum_{k=1}^n a_\xi^k(z)(\partial/\partial z^k)$, with $a_\xi^k(z)$ holomorphic in z and linear in ξ. Choose a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathfrak{L}$. Now

$$\xi_i = \sum_{k=1}^n a_i^k(z) \frac{\partial}{\partial z^k}, \; a_i^k \text{ holomorphic}, \quad \det (a_i^k) \neq 0. \tag{6.4}$$

Let (b_k^l) denote the inverse matrix, so

$$\sum_{k=1}^n a_i^k(z) b_k^l(z) = \delta_i^l, \; b_k^l \text{ holomorphic}. \tag{6.5}$$

Now we re-write our system (6.2) as

$$D_l f + \sum_{i=1}^n b_l^i \lambda_i f = 0, \qquad D_l = \frac{\partial}{\partial z^l}, \lambda_i = \lambda(\xi_i); 1 \leqq l \leqq n. \tag{6.6}$$

We use multi-indices $\pi = (p_1, \cdots, p_N)$, $\rho = (r_1, \cdots, r_N)$ and $\sigma = (s_1, \cdots, s_N)$ with integral entries $\geqq 0$. ϵ_l denotes the multi-index with δ_i^l in the i^{th} place. As usual,

z^π denotes $(z^1)^{p_1}(z^2)^{p_2} \cdots (z^N)^{p_N}$,

D^π denotes $D_1^{p_1}D_2^{p_2} \cdots D_N^{p_N}$,

$|\pi| = \sum p_i$ and $\pi! = (p_1!)(p_2!) \cdots (p_N!)$.

Now expand the holomorphic functions $b_l^i(z)$ by

$$b_l^i(z) = \sum_\rho b_{l;\rho}^i z^\rho. \tag{6.7}$$

Suppose that we have a formal V-valued power series

$$f(z) = \sum_\pi a_\pi z^\pi, \qquad a_\pi \in V, \tag{6.8}$$

with formal initial conditions

$$f(0, z'') = \phi(z'') \quad \text{formal } V\text{-valued power series}. \tag{6.9}$$

Then the equation (6.6) says formally that

(6.10) $$\sum_{\pi} a_\pi p_l z^{\pi-\epsilon_l} + \sum_{i=1}^{n} \sum_{\rho} \sum_{\pi} b^i_{l;\rho} \lambda_i(a_\pi) z^{\pi+\rho} = 0.$$

If we replace π by $\sigma + \epsilon_l$ in (6.10) and equate coefficients we obtain

(6.11) $$a_{\sigma+\epsilon_l} = -\frac{1}{s_l+1} \sum_{i=1}^{n} \sum_{\pi+\rho=\sigma} b^i_{l;\rho}\lambda_i(a_\pi); \qquad 1 \leqq l \leqq n.$$

That recursion formula in turn implies (6.2) in any subdomain of W_0^1 on which the series (6.8) is absolutely convergent.

We use (6.9) and (6.11) to determine the coefficients a_π in (6.8). Suppose that all a_π with $|\pi| \leqq M$ are determined; we determine the $a_{\sigma+\epsilon_l}$, $1 \leqq l \leqq N$, with $|\sigma + \epsilon_l| = M + 1$. If $n < l \leqq N$ then either all $s_i = 0$ for $1 \leqq i \leqq n$ and (6.9) determines $a_{\sigma+\epsilon_l}$, or there exists $k \leqq n$ with $s_k \geqq 1$ and we write $\sigma + \epsilon_l = \sigma' + \epsilon_k$ to throw the problem back to the case $l \leqq n$. If $1 \leqq l \leqq n$ then (6.11) determines $a_{\alpha+\epsilon_l}$ in terms of the already-known a_π with $|\pi| \leqq M$.

Now all a_π are determined. Some, however, are determined in several ways. To see that our recursive construction of the a_π is not ambiguous, we must check that

$$\text{if} \quad 1 \leqq k < l \leqq n \quad \text{and} \quad \tau + \epsilon_k = \sigma + \epsilon_l \quad \text{then}$$

$$\frac{1}{t_k+1} \sum_{i=1}^{n} \sum_{\mu+\nu=\tau} b^i_{k;\nu}\lambda_i(a_\mu) = \frac{1}{s_l+1} \sum_{i=1}^{n} \sum_{\pi+\rho=\sigma} b^i_{l;\rho}\lambda_i(a_\pi).$$

In other words, from the viewpoint of $\tau - \epsilon_l = \nu = \sigma - \epsilon_k$, we must check that if a formal power series (6.8) satisfies

(a) $D_l f + \sum_{i=1}^{n} b^i_l \lambda_i f = 0 = D_k f + \sum_{j=1}^{n} b^j_k \lambda_j f$

with $1 \leqq k < l \leqq n$, then it automatically satisfies

(b) $D_k(D_l f) = D_l(D_k f)$.

We compute, using (a), that

$$D_k(D_l f) = \sum_{i,j=1}^{n} b^i_l b^j_k \lambda_i \lambda_j f - \sum_{i=1}^{n} D_k(b^i_l)\lambda_i f,$$

$$D_l(D_k f) = \sum_{i,j=1}^{n} b^i_l b^j_k \lambda_j \lambda_i f - \sum_{i=1}^{n} D_l(b^i_k)\lambda_i f.$$

Thus to show that (a) implies (b) we need only prove

(6.12) $$\sum_{i,j=1}^{n} b^i_l b^j_k [\lambda_i, \lambda_j] = \sum_{i=1}^{n} \{D_k(b^i_l) - D_l(b^i_k)\}\lambda_i.$$

If we apply $\sum_{k,l=1}^{n} a^l_p a^k_q$ to the left side of (6.12) we obtain $[\lambda_p, \lambda_q]$. If we apply it to the right side of (6.12) then we obtain

$$\sum_{i,k,l} a^l_p a^k_q D_k(b^i_l)\lambda_i - \sum_{i,k,l} a^l_p a^k_q D_l(b^i_k)\lambda_i$$

$$= \sum_{i,l} a_p^l \xi_q(b_l^i)\lambda_i - \sum_{i,k} a_q^k \xi_p(b_k^i)\lambda_i$$

$$= \sum_{i,k} \xi_p(a_q^k) b_k^i \lambda_i - \sum_{i,l} \xi_q(a_p^l) b_l^i \lambda_i$$

$$= \lambda(\sum_{i,k} \xi_p(a_q^k) b_k^i \xi_i - \sum_{i,l} \xi_q(a_p^l) b_l^i \xi_i) \quad \text{at each point}$$

$$= \lambda(\sum_m \xi_p(a_q^m)\, D_m - \sum_m \xi_q(a_p^m)\, D_m) \quad \text{at each point}$$

$$= \lambda[\xi_p , \xi_q] = [\lambda\xi_p , \lambda\xi_q] = [\lambda_p , \lambda_q].$$

Thus (6.12) is just the hypothesis that λ be a Lie algebra representation. Now we have proved that (6.9) and (6.11) determine the coefficients a_π in (6.8), completely and without ambiguity. Thus we have constructed a formal solution (6.8) to (6.6) with initial condition (6.9).

Now we assume that the initial condition series (6.9) represents a holomorphic function, and we conclude that the series (6.8) just constructed also represents a holomorphic function. Choose $r > 0$ such that

$$\text{the } b_i^j(z) \text{ are holomorphic on } |z^k| < 2r,\ 1 \leqq k \leqq N,$$

$$\phi(z'') \text{ is holomorphic on } |z^k| < 2r, \quad n < k \leqq N.$$

Then for any multi-index π, and any $\pi' = (0, \cdots, 0, p_{n+1}, \cdots, p_N) \neq 0$,

$$|(D^\pi b_i^j)(0)| \leqq \frac{M\pi!}{r^{|\pi|}} \quad \text{and} \quad ||(D^{\pi'}\phi)(0)|| \leqq \frac{M\pi'!}{r^{|\pi'|-1}}$$

for some $M > 0$. Increase M if necessary so that $M \geqq \max(||\lambda_1||, \cdots, ||\lambda_n||, 1)$, let $P = nM^2$, and then increase M again if necessary so that $r = 1/P$. Finally adjust the norm on V so that $||\phi(0)|| \leqq 1$. Now we must find a number $c > 0$ such that

$$(6.13) \qquad ||a_\pi|| \leqq (P/c)^{|\pi|}, \text{ all multi-indices } \pi.$$

For then on the polydisc $|z^k| \leqq trc$, $0 \leqq t < 1$,

$$\sum_\pi ||a_\pi z^\pi|| \leqq \sum_\pi ||a_\pi||\, t^{|\pi|} r^{|\pi|} c^{|\pi|}$$

$$\leqq \sum_\pi t^{|\pi|} = \left(\sum_{q=0}^{\infty} t^q\right)^N = \left(\frac{1}{1-t}\right)^N < \infty,$$

so f converges uniformly and absolutely on that polydisc. Thus the series (6.8) will represent a holomorphic function on the polydisc $|z^k| < rc$, and that will give the neighborhood W_0 carrying the required solution f to (6.2).

To prove (6.13) we first find an integer $A \geqq 1$ such that

$$(6.14) \qquad (A+1)^{N-1} \leqq A^N.$$

If $N = 1$ we may take A to be any integer $\geqq 1$. If $N > 1$ we note that log $(A + 1)/\log(A)$ decreases to 1 as $A \to \infty$, and we take A sufficiently large so that $N/(N - 1) \geqq \log(A + 1)/\log(A)$. Then we have (6.14). That done, note $\sum_{p=0}^{s} (A + 1)^p < (A + 1)^{s+1}/A$ by induction on s, and apply (6.14) to see

$$\sum_{\pi+\rho=\sigma} (A + 1)^{|\pi|} = \sum_{p_1=0}^{s_1} \sum_{p_2=0}^{s_2} \cdots \sum_{p_N=0}^{s_N} (A + 1)^{p_1+\cdots+p_N}$$
$$= \prod_{i=1}^{N} \left(\sum_{p_i=0}^{s_i} (A + 1)^{p_i} \right) < \prod_{i=1}^{N} \frac{1}{A} (A + 1)^{s_i+1}$$
$$= A^{-N}(A + 1)^{N-1}(A + 1)^{1+\Sigma s_i} \leqq (A + 1)^{|\sigma|+1}.$$

That gives us

$$(6.15) \qquad \sum_{\pi+\rho=\sigma} c^{-|\pi|} \leqq c^{-|\sigma|-1} \quad \text{where} \quad c = \frac{1}{A + 1}, \quad 0 < c \leqq 1.$$

Now we check that the number $c > 0$ of (6.15) satisfies (6.13). If $\pi = 0$ then (6.13) says $\|\phi(0)\| \leqq 1$, which was arranged. If $\pi = (0, \cdots, 0, p_{n+1}, \cdots, p_N) \neq 0$ then

$$\|a_\pi\| = \left\| \frac{1}{\pi!} (D^\pi \phi)(0) \right\| \leqq \frac{M}{r^{|\pi|-1}} \leqq P^{|\pi|} \leqq (P/c)^{|\pi|}$$

where the last inequality uses $c \leqq 1$. To prove (6.13) by induction on $|\pi|$ we recall the construction (6.11) and compute

$$\|a_{\sigma+\epsilon_l}\| = \left\| \frac{-1}{s_l + 1} \sum_{i=1}^{n} \sum_{\pi+\rho=\sigma} b^i_{l;\rho} \lambda_i(a_\pi) \right\|$$
$$\leqq \sum_{i=1}^{n} \sum_{\pi+\rho=\sigma} |b^i_{l;\rho}| \cdot \|\lambda_i\| \cdot \|a_\pi\|$$
$$\leqq n \sum_{\pi+\rho=\sigma} \frac{M}{r^{|\rho|}} \cdot M \cdot (P/c)^{|\pi|} \quad \text{induction step}$$
$$= \Big(\sum_{\pi+\rho=\sigma} c^{-|\pi|} \Big) P^{|\sigma+\epsilon_l|} \leqq (P/c)^{|\sigma+\epsilon_l|}$$

where the last inequality is (6.15). Thus (6.13) is proved, and that completes the proof that (1) implies (3) in Theorem 6.1. Now Theorem 6.1 is proved.

Q.E.D.

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