

THE AUTOMORPHISM GROUP OF A HOMOGENEOUS ALMOST COMPLEX MANIFOLD⁽¹⁾

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1. Introduction. Let M be a compact simply connected manifold of nonzero Euler characteristic that carries a homogeneous almost complex structure. We determine the largest connected group $A_0(M)$ of almost analytic automorphisms of M .

Our hypotheses represent M as a coset space G/K where G is a maximal compact subgroup of the Lie group $A_0(M)$ and K is a closed connected subgroup of maximal rank in G . In §2 we collect some information, decomposing $M = M_1 \times \cdots \times M_t$ as a product of “irreducible” factors along the decomposition of G as a product of simple groups; then every invariant almost complex structure or riemannian metric decomposes and every invariant riemannian metric is hermitian relative to any invariant almost complex structure; furthermore the decomposition is independent of G in a certain sense. In §3 we choose an invariant riemannian metric and determine the largest connected groups $H_0(M_i)$ of almost hermitian isometries of the M_i . Then $A_0(M)$ is determined in §4. There it is shown that $A_0(M) = A_0(M_1) \times \cdots \times A_0(M_t)$, that $A_0(M_i) = H_0(M_i)$ if the almost complex structure on M_i is not integrable, and that $A_0(M_i) = H_0(M_i)^C$ if the almost complex structure on M_i is induced by a complex structure. $A_0(M)$ thus is a centerless semisimple Lie group whose simple normal analytic subgroups are just the $A_0(M_i)$.

2. Decomposition. Let M be an effective coset space of a compact connected Lie group G by a connected subgroup K of maximal rank. In other words $M = G/K$ is compact, simply connected and of nonzero Euler characteristic; G is a compact centerless semisimple Lie group, rank $K = \text{rank } G$, and K contains no simple factor of G . Then

$$(2.1a) \quad G = G_1 \times \cdots \times G_t, \quad K = K_1 \times \cdots \times K_t \quad \text{and} \quad M = M_1 \times \cdots \times M_t$$

where

$$(2.1b) \quad G_i \text{ is simple,} \quad K_i = K \cap G_i \quad \text{and} \quad M_i = G_i/K_i.$$

G_i is a compact connected centerless simple Lie group, K_i is a connected subgroup of maximal rank, and $M_i = G_i/K_i$ is a simply connected effective coset space of nonzero Euler characteristic. The decomposition of M is unique up to order of the factors because it is determined by the decomposition of G .

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We call (2.1) the *canonical decomposition* of the coset space $M=G/K$. The factors $M_i=G_i/K_i$ are the *irreducible factors* of $M=G/K$. If there is just one irreducible factor, i.e. if G is simple, then we say that $M=G/K$ is *irreducible*.

2.2. PROPOSITION. *Let M be an effective coset space G/K where G is a compact connected Lie group and K is a connected subgroup of maximal rank. Let $M=M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible factors $M_i=G_i/K_i$.*

1. *The G -invariant almost complex structures J on M are just the $J_1 \times \cdots \times J_t$ where J_i is a G_i -invariant almost complex structure on M_i .*

2. *The G -invariant riemannian metrics ds^2 on M are just the $ds_1^2 \times \cdots \times ds_t^2$ where ds_i^2 is a G_i -invariant riemannian metric on M_i ; there each (M_i, ds_i^2) is an irreducible riemannian manifold, so*

$$(M, ds^2) = (M_1, ds_1^2) \times \cdots \times (M_t, ds_t^2)$$

is the de Rham decomposition.

3. *Let J be a G -invariant almost complex structure on M . If ds^2 is a G -invariant riemannian metric, then it is the real part of a G -invariant almost hermitian (for J) metric h on M , and $h=h_1 \times \cdots \times h_t$ where h_i is a G_i -invariant almost hermitian (for J_i) metric on M_i and ds_i^2 is the real part of h_i .*

Proof. The Lie algebras decompose uniquely as direct sums $\mathfrak{G}=\mathfrak{K}+\mathfrak{M}$ and $\mathfrak{G}_i=\mathfrak{K}_i+\mathfrak{M}_i$, $\mathfrak{K}=\sum \mathfrak{K}_i$ and $\mathfrak{M}=\sum \mathfrak{M}_i$, with $[\mathfrak{K}, \mathfrak{M}] \subset \mathfrak{M}$ and $[\mathfrak{K}_i, \mathfrak{M}_i] \subset \mathfrak{M}_i$. Let Z be the center of K , so \mathfrak{K} is the centralizer of Z in \mathfrak{G} . Then $Z=Z_1 \times \cdots \times Z_t$ where Z_i is the center of K_i and \mathfrak{K}_i is the centralizer of Z_i in \mathfrak{G}_i .

π denotes the representation of K on \mathfrak{M} and π_i is the representation of K_i on \mathfrak{M}_i . Then $\pi=\pi_1 \oplus \cdots \oplus \pi_t$. Let $X=X_1 \cup \cdots \cup X_t$ be the set of nontrivial characters on Z such that

$$(2.3a) \quad \mathfrak{M}^C = \sum_X \mathfrak{M}_\chi \quad \text{and} \quad \mathfrak{M}_i^C = \sum_{X_i} \mathfrak{M}_\chi$$

where Z acts on \mathfrak{M}_χ by the character χ . Each \mathfrak{M}_χ is ad (K)-stable, so K acts on \mathfrak{M}_χ by a representation π_χ , and

$$(2.3b) \quad \pi^C = \sum_X \pi_\chi \quad \text{and} \quad \pi_i^C = \sum_{X_i} \pi_\chi.$$

The point [7, Theorem 8.13.3] is that

$$(2.3c) \quad \text{the } \pi_\chi \text{ are irreducible and mutually inequivalent.}$$

We transform the complex decomposition (2.3) to a real decomposition. Let $X=S \cup T$, $S=S_1 \cup \cdots \cup S_t$ and $T=T_1 \cup \cdots \cup T_t$ where S_i consists of the nonreal characters in X_i and T_i consists of the real ones. By *real partition* of X we mean a disjoint $X=S' \cup S'' \cup T$ where $S''=\bar{S}'$. If $\chi \in S_i$ then $\bar{\chi} \in S_i$; thus the real partition

induces real partitions $X_i = S'_i \cup S''_i \cup T_i$. If $|S| = 2n$ then X has 2^n real partitions. Now choose a real partition $X = S' \cup S'' \cup T$ and define

$$\begin{aligned} \chi \in S': K \text{ acts on } \mathfrak{M}_\chi^R &= \mathfrak{M} \cap (\mathfrak{M}_\chi + \mathfrak{M}_{\bar{\chi}}) \quad \text{by } \pi_\chi^R \\ \chi \in T: K \text{ acts on } \mathfrak{M}_\chi^R &= \mathfrak{M} \cap \mathfrak{M}_\chi \quad \text{by } \pi_\chi^R. \end{aligned}$$

Then (2.3abc) becomes

$$(2.4a) \quad \mathfrak{M} = \sum_{S'} \mathfrak{M}_\chi^R + \sum_T \mathfrak{M}_\chi^R \quad \text{and} \quad \mathfrak{M}_i = \sum_{S'_i} \mathfrak{M}_\chi^R + \sum_{T_i} \mathfrak{M}_\chi^R,$$

$$(2.4b) \quad \pi = \sum_{S'} \pi_\chi^R + \sum_T \pi_\chi^R \quad \text{and} \quad \pi_i = \sum_{S'_i} \pi_\chi^R + \sum_{T_i} \pi_\chi^R,$$

(2.4c) the π_χ^R are real-irreducible and mutually inequivalent.

Let A be the commuting algebra of π on \mathfrak{M} . By (2.4c), $A = \sum_{S'} C + \sum_T R$, for π_χ^R has commuting algebra C if $\chi \in S'$, R if $\chi \in T$. Invariant almost complex structures are in obvious correspondence with elements of square $-I$ of the commuting algebra, which now are seen to exist if and only if T is empty, and (1) follows. Similarly, the decomposition of ds^2 in (2), and the existence and decomposition of h in (3), are immediate.

It remains only to show the (M_i, ds_i^2) irreducible as riemannian manifolds in (2). That fact is known [3, §5.1], but in our present context we can give a short proof for the convenience of the reader. If (M_i, ds_i^2) reduces, then it is a riemannian product $M' \times M''$ because it is complete and simply connected, so we have an ad (K_i) -stable decomposition $\mathfrak{M}_i = \mathfrak{M}' \oplus \mathfrak{M}''$ with the properties

$$[\mathfrak{M}', \mathfrak{M}''] \subset \mathfrak{K}_i, \quad \mathfrak{M}'^C = \sum_{X'} \mathfrak{M}_x, \quad \mathfrak{M}''^C = \sum_{X''} \mathfrak{M}_x, \quad X_i = X' \cup X''.$$

Here X' and X'' are disjoint and self conjugate. If $\chi' \in X'$ and $\chi'' \in X''$ with $[\mathfrak{M}_{\chi'}, \mathfrak{M}_{\chi''}] \neq 0$, then $\chi'\chi'' = 1$ so $\chi' = \bar{\chi}'' \in X''$ which is absurd. Thus $[\mathfrak{M}', \mathfrak{M}''] = 0$, and it follows that the simple Lie algebra \mathfrak{G}_i is direct sum of ideals

$$\mathfrak{G}' = \{\mathfrak{K}_i \cap [\mathfrak{M}', \mathfrak{M}']\} + \mathfrak{M}' \quad \text{and} \quad \mathfrak{G}'' = \{\mathfrak{K}_i \cap [\mathfrak{M}'', \mathfrak{M}'']\} + \mathfrak{M}''.$$

That being absurd, irreducibility is proved. Q.E.D.

2.5. REMARK. In the notation of the proof of Proposition 2.2, M has a G -invariant almost complex structure if and only if $X = S$, and then those structures J correspond to the real partitions $X = S' \cup S''$ by: $\sum_{S'} \mathfrak{M}_\chi$ and $\sum_{S''} \mathfrak{M}_\chi$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J on \mathfrak{M}^C .

3. Almost hermitian isometries. Let M be a manifold with an almost hermitian metric h . Then $h = ds^2 + (-1)^{1/2}\omega$ where the riemannian metric ds^2 is the real part of h and $\omega(u, v) = ds^2(u, Jv)$ is the imaginary part; that determines the almost complex structure J . By *almost hermitian isometry* of (M, h) we mean a diffeomorphism that preserves h , i.e. that is a riemannian isometry of (M, ds^2) which preserves J .

Let $I(M)$ denote the (Lie) group of all isometries of (M, ds^2) , $H(M)$ the closed subgroup consisting of those isometries that preserve J . Then $H(M)$ is the (Lie) group of all almost hermitian isometries of (M, h) . In particular its identity component $H_0(M)$ is an analytic subgroup of the identity component $I_0(M)$ of $I(M)$. If $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product, then the de Rham decomposition says that $I_0(M)$ preserves each noneuclidean factor, so those factors are stable under $H_0(M)$.

Let $M = G/K$ as in Proposition 2.2. Let h be a G -invariant almost hermitian metric on M . The canonical decomposition induces $(M, h) = (M_1, h_1) \times \cdots \times (M_t, h_t)$ hermitian product where each (M_i, ds_i^2) , $ds_i^2 = \operatorname{Re} h_i$, is an irreducible noneuclidean riemannian manifold. Thus $H_0(M) = H_0(M_1) \times \cdots \times H_0(M_t)$, and $H(M)$ is generated by its subgroup $H(M_1) \times \cdots \times H(M_t)$ and permutations of mutually isometric (M_i, h_i) ; so its determination is more or less reduced to the case where $M = G/K$ is irreducible. There the result is

3.1. PROPOSITION. *Let M be an effective coset space G/K where G is a compact connected simple Lie group and K is a connected subgroup of maximal rank. Let h be a G -invariant almost hermitian metric on M , so $M = H_0(M)/B$ where $G \subset H_0(M)$ and $B \cap G = K$. If $G \neq H_0(M)$, then (M, h) is an irreducible hermitian symmetric space of compact type listed below.*

| Case | G | K | $H_0(M)$ | B | (M, h) |
|------|------------------------|----------------------|--------------------------|-----------------------|---|
| 1 | G_2 | $U(2)$ | $SO(7)$ | $SO(5) \times SO(2)$ | 5-dimensional complex quadric |
| 2 | $Sp(r)/\mathbb{Z}_2$ | $Sp(r-1) \cdot U(1)$ | $SU(2r)/\mathbb{Z}_{2r}$ | $U(2r-1)$ | complex projective $(2r-1)$ -space |
| 3 | $SO(2r+1)$ | $U(r)$ | $SO(2r+2)/\mathbb{Z}_2$ | $U(r+1)/\mathbb{Z}_2$ | unitary structures on \mathbb{R}^{2r+2} |
| 3' | $Spin(7)/\mathbb{Z}_2$ | $U(3)$ | $SO(8)/\mathbb{Z}_2$ | $SO(6) \cdot SO(2)$ | 6-dimensional complex quadric |

REMARK 1. In the exceptional cases above, K is not R -irreducible on the tangent space, so M has another G -invariant almost hermitian metric for which $G = H_0(M)$.

REMARK 2. The proof is easily reduced to the case where B is the centralizer of a toral subgroup of $H_0(M)$, and then the result can be extracted from [2, Table 5] and the Bott-Borel-Weil Theorem. But here it is convenient to reduce the proof to some classifications of Oniščik [4].

Proof. As M has nonzero Euler characteristic, B has maximal rank in $H_0(M)$, so $H_0(M)/B = G/K$ is one of the following entries in Oniščik's list [4, Table 7].

- (i) $A_{2n-1}/A_{2n-2} \cdot T = C_n/C_{n-1} \cdot T$ (our Case 2),
- (ii) $B_3/B_2 \cdot T = G_2/A_1 \cdot T$ (our Case 1),
- (iii) $B_3/D_3 = G_2/A_2$ (B_3 does not preserve J here),

- (iv) $D_{n+1}/A_n \cdot T = B_n/A_{n-1} \cdot T$ (our Case 3),
- (v) $D_4/D_3 \cdot T = B_3/A_2 \cdot T$ (our Case 3').

The assertions follow with the observation that $H_0(M)/B$ is an irreducible hermitian symmetric coset space of compact type in each of the admissible cases. Q.E.D.

4. Almost analytic automorphisms. Let M be a manifold with almost complex structure J . By *almost analytic automorphism* of M , we mean a diffeomorphism of M which preserves J . The set of all such diffeomorphisms forms a group $A(M)$. If M is compact, then [1] in the compact-open topology, $A(M)$ is a Lie transformation group of M . We denote its identity component by $A_0(M)$. If, further, we have an almost hermitian metric on M , then $H(M)$ is a compact subgroup of $A(M)$. That will be our main tool in studying $A(M)$.

4.1. THEOREM. *Let $M=G/K$ be a simply connected effective coset space of nonzero Euler characteristic where G is a compact connected Lie group. Let J be a G -invariant almost complex structure on M . Let $M=M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible coset spaces, and decompose $J=J_1 \times \cdots \times J_t$ where J_i is a G_i -invariant almost complex structure on M_i . Then*

1. $A_0(M)=A_0(M_1) \times \cdots \times A_0(M_t)$.
2. M has a G -invariant riemannian metric $ds^2=ds_1^2 \times \cdots \times ds_t^2$ for which $H_0(M)$ is a maximal compact subgroup of $A_0(M)$.
3. If J_i is integrable then $A_0(M_i)=H_0(M_i)^c$. If J_i is not integrable then $A_0(M)=H_0(M)$.

Proof. For the second statement, enlarge G to a maximal compact subgroup H of $A_0(M)$ and choose an H -invariant riemannian metric ds^2 on M . Then $ds^2=ds_1^2 \times \cdots \times ds_t^2$ as required, by Proposition 2.2, and $H=H_0(M)$ by construction.

We simplify notation for the proofs of the first and third statement by enlarging G to $H_0(M)$ and writing A for $A_0(M)$. That does not change the canonical decomposition of M , for the latter is the de Rham decomposition for ds^2 according to Proposition 2.2. Now $G/K=M=A/B$ where $G \subset A$ is a maximal compact subgroup and $K=G \cap B$.

We check that A is a centerless semisimple Lie group. If L is a closed normal analytic subgroup of A with $G \cap L$ discrete, then $G \cdot L \subset A$ is effective on

$$(G \cdot L)/(K \cdot L) = M, \text{ so } L = \{1\}.$$

Let L be the radical of A : now A is semisimple. Let \mathfrak{L} be the orthocomplement of \mathfrak{G} in a maximal compactly embedded subalgebra of \mathfrak{A} : now A has finite center, so the centerless group G contains the center of A , so A is centerless.

Let A^α , $1 \leq \alpha \leq r$, be the simple normal analytic subgroups of A . So $A=A^1 \times \cdots \times A^r$ with A^α centerless simple. Now $G=G^1 \times \cdots \times G^r$, $K=K^1 \times \cdots \times K^r$ and $M=M^1 \times \cdots \times M^r$ where

$$G^\alpha = G \cap A^\alpha, \quad K^\alpha = K \cap G^\alpha, \quad M^\alpha = G^\alpha/K^\alpha.$$

If $\alpha \neq \beta$ then A^α acts trivially on M^β . For every $a \in A^\alpha$ centralizes the transitive transformation group G^β of M^β , hence induces some transformation \bar{a} of M^β that is trivial or fixed point free. As A^α is connected, \bar{a} is homotopic to 1 so its Lefschetz number is the (nonzero) Euler characteristic of M^β ; that shows $\bar{a}=1$. Now $M^\alpha = A^\alpha/B^\alpha$, $B^\alpha = B \cap A^\alpha$, with $B = B^1 \times \cdots \times B^r$.

According to Oniščik [5, Table 1] the only possibilities for $G^\alpha/K^\alpha = M^\alpha = A^\alpha/B^\alpha$, A^α noncompact, are given in the following table.

| A^α | $M^\alpha = G^\alpha/K^\alpha$ | Conditions |
|-------------------|--|--------------------------------------|
| $SL(2n, R)/Z_2$ | $SO(2n)/SO(2n_1) \times \cdots \times SO(2n_p)$ | $n = \sum n_i > 1$ |
| $SL(2n+1, R)$ | $SO(2n+1)/SO(2n_1) \times \cdots \times SO(2n_{p-1}) \times SO(2n_p+1)$ | $n = \sum n_i$ |
| $GL(n, Q)/Z_2$ | $Sp(n)/Sp(n_1) \times \cdots \times Sp(n_p) \times U(1)^q$ | $n = q + \sum n_i$ |
| $SO(1, 2n-1)/Z_2$ | $SO(2n-1)/SO(2n_1) \times \cdots \times SO(2n_p) \times U(m_1) \times \cdots \times U(m_q)$ | $n-1 = \sum n_i + \sum m_j$ |
| $E_{6,c_4}/Z_2$ | $Sp(4)/Sp(2) \times Sp(2)$ and $Sp(4)/[Sp(1)]^4$ | none |
| E_{6,F_4} | $F_4/Spin(9)$, $F_4/Spin(8)$, $F_4/U(4)$ and $F_4/[SU(2)]^4$ | none |
| $(G^\alpha)^C$ | G^α/K^α where K^α is the centralizer of a nontrivial toral subgroup of G^α | G^α compact centerless simple |

Note that G^α is simple except in Case 1 with $n=2$. There M^α is the product of two Riemann spheres, so A^α is the product of two copies of $SL(2, C)/Z_2$, contradicting the table entry for A^α . Thus we always have G^α simple, so each M^α is an M_i , and the first statement of our theorem is proved with $A^\alpha = A_0(M^\alpha)$.

Now we may, and do, assume M irreducible. Thus A and G are simple.

4.2. LEMMA. *The invariant almost complex structure J is integrable if and only if $A=G^C$. In that case B is a complex parabolic subgroup of A and J is induced either from the natural complex structure on A/B or from the conjugate structure.*

Proof of lemma. Let J be integrable; we check $\mathfrak{G}^C \subset \mathfrak{A}$. For if $\xi \in \mathfrak{G}$ and ξ^* denotes the holomorphic vector field induced on M , then $J(\xi^*)$ is holomorphic. Thus \mathfrak{G}^C acts on M by $\xi + i\eta \rightarrow \xi^* + J(\eta^*)$, and this action integrates to G^C because M is compact; that shows $G^C \subset A$ so $\mathfrak{G}^C \subset \mathfrak{A}$.

Let $\mathfrak{A} = \mathfrak{G}^C$. As \mathfrak{A} is its own normalizer in \mathfrak{G} because it has maximal rank, \mathfrak{B} is its own normalizer in \mathfrak{A} , so B is an R -algebraic subgroup of A . Thus A has an Iwasawa decomposition GSN with $B = KSN$. As $\mathfrak{A} = \mathfrak{G}^C$, the group S^C is a complex Cartan subgroup of A , so N is a complex unipotent subgroup. Now $K^C S^C N$ is the complex group generated by B and it has intersection K with G ; thus $M = A/B \rightarrow A/K^C S^C N = G/K$ is trivial so B is a complex subgroup of A . As A/B is compact now B is a complex parabolic subgroup.

Decompose $B=B^r \cdot B^u$ into reductive and unipotent parts. Let Z be the identity component of the center of B^r , complex subtorus of S^C . Let D be the set of characters $\chi \neq 1$ on Z that are restrictions of positive roots, so $\mathfrak{B}^u = \sum_D \mathfrak{A}_\chi$. Define $\mathfrak{B}^{-u} = \sum_{-\chi} \mathfrak{A}_{-\chi}$ so that \mathfrak{A} is the direct sum of its subspaces \mathfrak{B}^r , \mathfrak{B}^u and \mathfrak{B}^{-u} . $\mathfrak{G} \cap (\mathfrak{B}^u + \mathfrak{B}^{-u})$ represents the real tangent space of M , and $\mathfrak{B}^u + \mathfrak{B}^{-u}$ represents the complexified tangent space. If $\pm \chi \in D$, then \mathfrak{A}_χ is an irreducible representation space of B^r , so J acts on \mathfrak{A}_χ either as $\sqrt{-1}$ or as $-\sqrt{-1}$. Let \mathfrak{L}^+ (resp. \mathfrak{L}^-) denote the image in $\mathfrak{A}/\mathfrak{B}$ of the \mathfrak{A}_χ , $-\chi \in D$, on which J acts as $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then $\text{ad}(\mathfrak{B}) \cdot \mathfrak{L}^\pm \subset \mathfrak{L}^\pm$ by invariance of J under B . If ν is the restriction to Z of the highest root, then $\mathfrak{A}/\mathfrak{B} = \sum_{n \geq 0} \text{ad}(\mathfrak{B})^n \cdot (\mathfrak{A}_{-\nu} \text{ mod } \mathfrak{B})$, because \mathfrak{A} is simple, so $\mathfrak{A}/\mathfrak{B}$ is the one of \mathfrak{L}^+ or \mathfrak{L}^- into which $\mathfrak{A}_{-\nu}$ maps. Thus either J acts on \mathfrak{B}^{-u} as $\sqrt{-1}$ and the natural complex structure of A/B induces J , or J acts on \mathfrak{B}^{-u} as $-\sqrt{-1}$ and the natural structure induces $-J$. In either case J is integrable.

In general suppose $\mathfrak{G}^C \subset \mathfrak{A}$. Then $M=G^C/B \cap G^C$ is a complex flag manifold on which A is the largest connected group of analytic automorphisms. Thus A is a centerless complex semisimple group, hence the complexification of its maximal compact subgroup G .

Lemma 4.2 is proved.

4.3. LEMMA. *If B^C is parabolic in A^C , then J is integrable so $A=G^C$.*

Proof of lemma. J is an element of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B})$ on $\mathfrak{A}/\mathfrak{B}$. Thus it induces an element J^C of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B}^C)$ on $\mathfrak{A}^C/\mathfrak{B}^C$. Now suppose B^C parabolic in A^C , so $M^C=A^C/B^C$ is compact and of positive Euler characteristic with invariant almost complex structure J^C .

If A is complex then $A=G^C$ and Lemma 4.2 says that J is integrable. Thus we may assume A not complex so that A^C is simple. Then Lemma 4.2 says that J^C is integrable, and in fact that either J^C or $-J^C$ is induced by the natural complex structure on A^C/B^C . Replace J by $-J$ if necessary; that does not alter integrability of J , but it replaces J^C by $-J^C$, allowing us to assume J^C induced by the natural complex structure of A^C/B^C .

Decompose $B=B^r \cdot B^u$ into reductive and unipotent parts, so $\mathfrak{B}=\mathfrak{B}^r+\mathfrak{B}^u$ and $\mathfrak{A}=\mathfrak{B}+\mathfrak{B}^{-u}$ where $\mathfrak{B}^{\pm u}$ are subalgebras normalized by \mathfrak{B}^r . Let \mathfrak{B}^{-u} represent the real tangent space to M . Note that J^C acts on $(\mathfrak{B}^{-u})^C$ as $\sqrt{-1}$. That contradicts our arrangement that the action of J^C on $(\mathfrak{B}^{-u})^C$ is induced by the action of J on \mathfrak{B}^{-u} . Thus A cannot be noncomplex. Lemma 4.3 is proved.

We complete the proof of Theorem 4.1. As in the second paragraph of the proof of Lemma 4.2, B is a real algebraic subgroup of A , so there is a semidirect product decomposition $B=B^r \cdot B^u$ into reductive and unipotent parts. If $\text{rank } B^r < \text{rank } A$, then any Cartan subalgebra of \mathfrak{A} has an element ξ not contained in any isotropy subalgebra of \mathfrak{A} on M so it induces a nonvanishing vector field ξ^* on M . The

existence of a nonvanishing vector field ξ^* says that M has Euler characteristic zero. That contradiction proves $\text{rank } B' = \text{rank } A$.

Let σ be the Cartan involution of \mathfrak{A} with fixed point set \mathfrak{G} and let $\mathfrak{A} = \mathfrak{G} + \mathfrak{P}$ be the Cartan decomposition. We may assume $\sigma(\mathfrak{B}') = \mathfrak{B}'$, so $\mathfrak{B}' = \mathfrak{K} + (\mathfrak{P} \cap \mathfrak{B}')$. That gives compact real forms

$$\mathfrak{A}_c = \mathfrak{G} + \sqrt{-1}\mathfrak{P} \quad \text{and} \quad \mathfrak{B}'_c = \mathfrak{K} + \sqrt{-1}(\mathfrak{P} \cap \mathfrak{B}').$$

Let A_c denote the centerless group with Lie algebra \mathfrak{A}_c and let B'_c be the analytic subgroup for \mathfrak{B}'_c . Then $\text{rank } B'_c = \text{rank } B' = \text{rank } A = \text{rank } A_c$ tells us that $X = A_c/B'_c$ is a compact simply connected manifold of positive Euler characteristic. If $A = G$ then $B = B' = K$, so $A_c = G$ and $B'_c = K$, whence $X = M$.

As in the second paragraph of the proof of Lemma 4.2 we have Iwasawa decompositions $A = GSN$ and $B = KSN$. Choose a torus subgroup $T \subset K$ such that $H = T \times S \subset B'$ is a Cartan subgroup of A . Let Δ be the root system. Now $\Delta = D \cup E \cup -E$ disjoint, and $\mathfrak{A} = \mathfrak{B}' + \mathfrak{B}^u + \mathfrak{B}^{-u}$ direct, where

$$\mathfrak{B}' = \mathfrak{H} + \mathfrak{A} \cap \left\{ \sum_D \mathfrak{A}_\phi \right\}, \quad \mathfrak{B}^u = \mathfrak{A} \cap \left\{ \sum_E \mathfrak{A}_\phi \right\}, \quad \mathfrak{B}^{-u} = \mathfrak{A} \cap \left\{ \sum_E \mathfrak{A}_{-\phi} \right\}.$$

Observe that σ interchanges \mathfrak{B}^u and \mathfrak{B}^{-u} . For $\mathfrak{B}^u \subset N$ because $N = N' \cdot \mathfrak{B}^u$ where $B' = KSN'$, and the dual space of \mathfrak{S} has an ordering such that

$$\mathfrak{A}^C = \sum_{\phi|_\mathfrak{S} > 0} \mathfrak{A}_\phi, \quad \text{and} \quad \phi|_\mathfrak{S} > 0 \quad \text{iff} \quad \sigma\phi|_\mathfrak{S} < 0.$$

View the invariant almost complex structure J of M as an element of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B})$ on $\mathfrak{A}/\mathfrak{B}$, hence in the commuting algebra of $\text{ad}(\mathfrak{B}')$ on $\mathfrak{B}^{-u} \cong \mathfrak{A}/\mathfrak{B}$; then extend J to an element J' of square $-I$ in the commuting algebra of $\text{ad}(\mathfrak{B}')$ on $\mathfrak{B}^u + \mathfrak{B}^{-u}$ by the formula

$$J'(\xi + \eta) = \sigma J(\sigma\xi) + J(\eta) \quad \text{where } \xi \in \mathfrak{B}^u, \eta \in \mathfrak{B}^{-u}.$$

Now J' is an A -invariant σ -invariant almost complex structure on A/B' , so [6, Proposition 7.7] it defines an A_c -invariant σ -invariant almost complex structure on A_c/B'_c . We have proved that $X = A_c/B'_c$ has an invariant almost complex structure.

Suppose $A \neq G$. Note that [6, Theorem 4.10] eliminates lines 5 and 6 of the Oniščik table above, so either $A = G^C$ or A is absolutely simple and of classical type. Suppose $A \neq G^C$ so A_c is simple and of classical type. Then [6, Theorem 4.10] shows that B'_c is the centralizer of a torus in A_c . Let \mathfrak{Z}_c denote the center of \mathfrak{B}'_c . Then $\sigma(\mathfrak{B}'_c) = \mathfrak{B}'_c$ implies $\sigma(\mathfrak{Z}_c) = \mathfrak{Z}_c$, so $\mathfrak{Z}_c = \mathfrak{U} + (-1)^{1/2}\mathfrak{B}$ with $\mathfrak{U} \subset \mathfrak{K}$ and $\mathfrak{B} \subset \mathfrak{P} \cap \mathfrak{B}'$. Now \mathfrak{B}' has center $\mathfrak{Z} = \mathfrak{U} + \mathfrak{B} \subset \mathfrak{T} + \mathfrak{S} = \mathfrak{H}$, and \mathfrak{B}' is the centralizer of \mathfrak{Z} in \mathfrak{A} . We order the root system Δ so that a root $\phi > 0$ whenever $\phi|_\mathfrak{Z} \neq 0$ and $\phi|_\mathfrak{S} > 0$. Then \mathfrak{B}^C contains the Borel subalgebra $\mathfrak{H}^C + \sum_{\phi > 0} \mathfrak{A}_\phi$ of \mathfrak{A}^C for that ordering, so \mathfrak{B}^C is a parabolic subalgebra of \mathfrak{A}^C . Then Lemma 4.3 says $A = G^C$. We have proved that $A \neq G$ implies $A = G^C$.

If J is integrable then Lemma 4.2 says $A = G^C$. If J is not integrable then Lemma 4.2 says $A \neq G^C$, so we cannot have $A \neq G$, and that forces $A = G$. Theorem 4.1 is proved. Q.E.D.

4.3. REMARK. Theorem 4.1 extends the scope of [8, Theorem 17.4(3)], but that result remains incomplete because, as remarked at the end of [8, §17], it is not known whether

$$A_0(E_6/\text{ad } SU(3)) \text{ is } E_6 \text{ rather than } E_6^C$$

or whether

$$\begin{aligned} A_0(SO(n^2-1)/\text{ad } SU(n)) &\text{ is } SO(n^2-1) \text{ rather than} \\ SO(n^2-1, C), \quad SL(n^2-1, R), \quad \text{or} \quad SO(1, n^2-1). \end{aligned}$$

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