A COMPATIBILITY CONDITION BETWEEN INVARIANT RIEMANNIAN METRICS AND LEVI-WHITEHEAD DECOMPOSITIONS ON A COSET SPACE

BY

JOSEPH A. WOLF[1]

0. **Introduction.** Let $M=G/K$ be an effective coset space of a connected Lie group by a compact subgroup. Then there may be many G-invariant riemannian metrics on M. But one expects the algebraic structure of the pair (G, K) to have a strong influence on the curvatures of M relative to any G-invariant riemannian metric. For example

(1) if G is semisimple with finite center and K is a maximal compact subgroup, then it is classical from symmetric space theory that all G-invariant riemannian metrics on M have every sectional curvature $\leqq 0$;

(2) if G is commutative then every G-invariant riemannian metric on M is flat; and

(3) if G is noncommutative and nilpotent then [7] every G-invariant riemannian metric on M has sectional curvatures of both signs.

Those results are proved by choosing an $\mathrm{ad}_G(K)$-stable complement $\mathfrak{M}$ to the Lie algebra $\mathfrak{K}$ of K inside the Lie algebra $\mathfrak{G}$ of G, and by performing calculations in $\mathfrak{M}$ and in $\mathfrak{G}$ in a manner justified by embedding G in the orthonormal frame bundle of M. But at certain crucial parts of those calculations one must have $\mathfrak{G}$ either semisimple or nilpotent. The idea in this paper is to create a setup in which the calculations can still be carried out, by requiring that the complement $\mathfrak{M}$ split as

$$\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{R})+(\mathfrak{M} \cap \mathfrak{L}) \text{ orthogonal direct sum,}$$

where

$$\mathfrak{G} = \mathfrak{R}+\mathfrak{L} \text{ is a Levi-Whitehead decomposition}$$

and

$$\mathfrak{M} \cap \mathfrak{R} \text{ contains the nilpotent radical of } \mathfrak{G}.$$

§2 is a study of the circumstances under which $\mathfrak{M}$ can be chosen, and the Levi-Whitehead decomposition $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$ can be chosen, so that M has such an orthogonal splitting. We describe those circumstances by the condition (2.2) that the invariant riemannian metric on M be "consistent" with $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$.

Given the consistency condition (2.2), our main result (Theorem 3.9) says that every unit vector $X \in \mathfrak{M} \cap \mathfrak{R}$, orthogonal to the nilpotent radical $\mathfrak{N}$ of $\mathfrak{G}$, is a

Received by the editors May 21, 1968.

[1] Research supported by an Alfred P. Sloan Research Fellowship and by National Science Foundation Grant GP-8008.

direction of negative mean curvature on M. Our applications (§4) essentially consist of observing that, if M has mean curvature $\geqq 0$ everywhere, then the consistency condition implies $\mathfrak{N}=(\mathfrak{M}\cap\mathfrak{N})$, i.e. $\mathfrak{R}=\mathfrak{N}+(\mathfrak{R}\cap\mathfrak{K})$ semidirect sum. The most striking of the applications is Theorem 4.4, which says:

Let M be a connected Riemannian manifold that has a solvable transitive group of isometries. Then the following conditions are equivalent.

(i) *M has mean curvature $\geqq 0$ everywhere.*

(ii) *M has every sectional curvature $\geqq 0$.*

(iii) *M has every sectional curvature zero.*

(iv) *M is isometric to the product of an euclidean space and a flat riemannian torus.*

Theorem 4.7 adds negative curvature conditions in case M has a transitive nilpotent group of isometries, extending the results of [7] to mean curvature.

1. **Definitions and notation.** $\mathfrak{G}$ is a real Lie algebra. We have the *nilpotent radical* $\mathfrak{N}$ and the *solvable radical* $\mathfrak{R}$, characteristic nilpotent and solvable ideals in $\mathfrak{G}$ defined by

$\mathfrak{N}$ is the union of the nilpotent ideals of $\mathfrak{G}$,

$\mathfrak{R}$ is the union of the solvable ideals of $\mathfrak{G}$.

The basic facts on $\mathfrak{N}$ and $\mathfrak{R}$ are the following.

(1.1) *If C is a fully reducible group of automorphisms of $\mathfrak{G}$, then there are C-invariant semisimple subalgebras $\mathfrak{L}\subset\mathfrak{G}$ that map isomorphically onto $\mathfrak{G}/\mathfrak{R}$ under the projection $\varphi\colon \mathfrak{G}\to\mathfrak{G}/\mathfrak{R}$, and any two such subalgebras are conjugate by an automorphism* $\mathrm{ad}_{\mathfrak{G}}(\exp n)$ *of $\mathfrak{G}$ where $n\in\mathfrak{N}$ is left fixed by every $c\in C$.*

The existence is due to G. D. Mostow [3, Corollary 5.2], and the conjugacy statement is the result [5, Theorem 4] of E. J. Taft. In general a semisimple subalgebra $\mathfrak{L}\subset\mathfrak{G}$ such that $\varphi\colon\mathfrak{L}\cong\mathfrak{G}/\mathfrak{R}$ is called a *Levi factor* of $\mathfrak{G}$, and the Levi factors of $\mathfrak{G}$ are just the maximal semisimple subalgebras.

(1.2) *If $\mathfrak{L}$ is a Levi factor of $\mathfrak{G}$, then $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$ semidirect sum, $\mathfrak{N}+\mathfrak{L}$ (semidirect) is an ideal in $\mathfrak{G}$, and the derived algebra* $[\mathfrak{G},\mathfrak{G}]\subset\mathfrak{N}+\mathfrak{L}$.

The first assertion is immediate and the second follows from $\mathfrak{N}\subset\mathfrak{R}$. For the third, one notes that $[\mathfrak{R},\mathfrak{R}]\subset\mathfrak{N}$ by Ado's Theorem and that $\mathrm{ad}_{\mathfrak{G}}(\mathfrak{L})$ normalizes $\mathrm{ad}_{\mathfrak{R}}(\mathfrak{R})$ in the derivation algebra of $\mathfrak{R}$.

Let $M=G/K$ be a coset space of a Lie group by a closed subgroup. $\mathfrak{K}\subset\mathfrak{G}$ are the Lie algebras of $K\subset G$. An $\mathrm{ad}_G(K)$-invariant subspace $\mathfrak{M}\subset\mathfrak{G}$ such that $\mathfrak{G}=\mathfrak{M}+\mathfrak{K}$ (vector space direct sum), is called an *invariant complement* for K. If an invariant complement for K exists, then $M=G/K$ is called a *reductive coset space.*

K is called a *reductive subgroup* of G in case the group $\mathrm{ad}_G(K)$ of linear transformations of $\mathfrak{G}$ is fully reducible. If K is a reductive subgroup of G, then $\mathrm{ad}_G(K)\mathfrak{K}$

$=\mathfrak{K}$ implies that $M=G/K$ is a reductive coset space. The converse fails in the example

$$G = \boldsymbol{SL}(2, \boldsymbol{R}) \quad \text{and} \quad K = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \boldsymbol{Z}\right\}.$$

However, compact subgroups, and semisimple subgroups with only finitely many components, are reductive subgroups.

2. **The compatibility condition.** We can now define the compatibility conditions with which we will operate. $M=G/K$ is a coset space of a Lie group by a closed subgroup. $\mathfrak{L}$ is a Levi factor of $\mathfrak{G}$ and $\mathfrak{M}$ is an invariant complement for K. If

$$\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{R})+(\mathfrak{M} \cap \mathfrak{L}) \quad \text{and} \quad \mathfrak{R} = (\mathfrak{M} \cap \mathfrak{R})+(\mathfrak{K} \cap \mathfrak{R}) \tag{2.1}$$

then we say that $\mathfrak{L}$ *splits* $\mathfrak{M}$ and that $\mathfrak{M}$ is *split by the Levi-Whitehead decomposition* $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$. Suppose further that we have a G-invariant pseudo-riemannian metric ds^2 on M. Represent ds^2 by an $\mathrm{ad}_G(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{M}$. If

$$\begin{gathered} \mathfrak{M} = (\mathfrak{M} \cap \mathfrak{R})+(\mathfrak{M} \cap \mathfrak{L}), \\ \mathfrak{R} = (\mathfrak{M} \cap \mathfrak{R})+(\mathfrak{K} \cap \mathfrak{R}) \quad \text{and} \quad \langle \mathfrak{M} \cap \mathfrak{R}, \mathfrak{M} \cap \mathfrak{L}\rangle = 0, \end{gathered} \tag{2.2}$$

then we say that $\mathfrak{L}$ *splits* $\mathfrak{M}$ *orthogonally* and that ds^2 *is consistent with the Levi-Whitehead decomposition* $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$.

2.3. PROPOSITION. *Let K be a closed reductive subgroup of a Lie group G. Then for every* $\mathrm{ad}_G(K)$*-invariant Levi factor $\mathfrak{L}$ of $\mathfrak{G}$, there exists an invariant complement $\mathfrak{M}$ for K, such that $\mathfrak{L}$ splits $\mathfrak{M}$. If ds^2 is a G-invariant pseudo-riemannian metric on G/K, $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{R}$ is the projection, and the representations of K on $\mathfrak{R}/(\mathfrak{R} \cap \mathfrak{K})$ and $\mathfrak{L}/(\mathfrak{L} \cap \varphi^{-1}\varphi\mathfrak{K})$ are disjoint, then it is automatic that $\mathfrak{L}$ splits $\mathfrak{M}$ orthogonally and ds^2 is consistent with $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$.*

Proof. Mostow's result (1.1) provides an $\mathrm{ad}_G(K)$-invariant Levi factor $\mathfrak{L}$ of $\mathfrak{G}$. As K is reductive in G we have $\mathrm{ad}_G(K)$-invariant direct sum decompositions

$$\mathfrak{R} = \mathfrak{M}_1+(\mathfrak{R} \cap \mathfrak{K}) \quad \text{and} \quad \mathfrak{G}/\mathfrak{R} = \mathfrak{M}_2'+\varphi(\mathfrak{K}).$$

Now define

$$\mathfrak{M}_2 = \mathfrak{L} \cap \varphi^{-1}(\mathfrak{M}_2') \quad \text{and} \quad \mathfrak{M} = \mathfrak{M}_1+\mathfrak{M}_2.$$

Then $\mathrm{ad}_G(K)\mathfrak{M}_i=\mathfrak{M}_i$ so $\mathfrak{M}$ is an $\mathrm{ad}_G(K)$-invariant subspace of $\mathfrak{G}$ that satisfies (2.1). If $x \in \mathfrak{M} \cap \mathfrak{K}$ then $\varphi(x) \in \mathfrak{M}_2' \cap \varphi(\mathfrak{K})=0$ so $x \in \mathfrak{R}$; then $x \in \mathfrak{M}_1 \cap (\mathfrak{R} \cap \mathfrak{K})=0$ so $x=0$; thus $\mathfrak{M} \cap \mathfrak{K}=0$. On the other hand

$$\dim \mathfrak{R} = \dim \mathfrak{M}_1+\dim(\mathfrak{R} \cap \mathfrak{K})$$

and

$$\begin{aligned} \dim \mathfrak{G}/\mathfrak{R} &= \dim \mathfrak{M}_2'+\dim \varphi(\mathfrak{K}) = \dim \mathfrak{M}_2+\dim \varphi(\mathfrak{K}) \\ &= \dim \mathfrak{M}_2-\dim (\mathfrak{R} \cap \mathfrak{K})+\dim \mathfrak{K} \end{aligned}$$

so

$$\dim \mathfrak{G} = \dim \mathfrak{R}+\dim \mathfrak{G}/\mathfrak{R} = \dim \mathfrak{M}+\dim \mathfrak{K}.$$

Thus $\mathfrak{M}$ is a vector space complement to $\mathfrak{K}$ in $\mathfrak{G}$. Now $\mathfrak{M}$ is an invariant complement for K such that $\mathfrak{L}$ splits $\mathfrak{M}$.

Let ds^2 and the inner product $\langle\ ,\ \rangle$ on $\mathfrak{M}$ be given. The representation of K on $\mathfrak{R}/(\mathfrak{R} \cap \mathfrak{K})$ is the representation $\mathrm{ad}_G|_K$ on $\mathfrak{M}_1$; the representation of K on $\mathfrak{L}/(\mathfrak{L} \cap \varphi^{-1}\varphi\mathfrak{K})$ is $\mathrm{ad}_G|_K$ on $\mathfrak{M}_2$. If those two are disjoint then necessarily $\langle \mathfrak{M}_1, \mathfrak{M}_2\rangle = 0$. Q.E.D.

We reformulate the metric portion of Proposition 2.3.

2.4. Proposition. *Let $M = G/K$ be a homogeneous pseudo-riemannian manifold with metric ds^2, where K is a reductive subgroup of G. Let $\mathcal{O}$ be the base point, $M_{\mathcal{O}}$ the tangent space at $\mathcal{O}$, χ the linear isotropy representation of K on $M_{\mathcal{O}}$, and $R_{\mathcal{O}}$ the subspace of $M_{\mathcal{O}}$ spanned by vector fields from elements of the solvable radical of $\mathfrak{G}$. Suppose that the representations of K induced by χ, on $R_{\mathcal{O}}$ and on $M_{\mathcal{O}}/R_{\mathcal{O}}$, are disjoint. Then $M_{\mathcal{O}} = R_{\mathcal{O}} + R_{\mathcal{O}}^{\perp}$ and ds^2 is consistent with every Levi-Whitehead decomposition $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$ for which* $\mathrm{ad}_G(K)\cdot \mathfrak{L} = \mathfrak{L}$.

For, in the notation of the proof of Proposition 2.3, $R_{\mathcal{O}}$ is spanned by the vector fields from $\mathfrak{M}_1$ while $R_{\mathcal{O}}^{\perp}$ is spanned by those from $\mathfrak{M}_2$.

In the riemannian case we will be able to arrange that $\mathfrak{M}_1 = \mathfrak{M} \cap \mathfrak{R}$ contain the nilpotent radical $\mathfrak{N}$. For that, we need a technical lemma.

2.5. Lemma. *In a connected nilpotent Lie group every compact subgroup is central.*

Proof. Let N be the connected nilpotent Lie group, $\pi: \tilde{N} \to N$ the universal Lie group covering, and Γ the kernel of π. Then Γ is a discrete central subgroup of $\tilde{N}$. Let C be a maximal compact subgroup of N and $\tilde{C} = \pi^{-1}(C)$. Then C is a torus group, $\tilde{C}$ is a simply connected commutative subgroup of $\tilde{N}$, and $\Gamma \subset \tilde{C}$ such that $C = \tilde{C}/\Gamma$ compact. As Γ is central in $\tilde{N}$, and as $\tilde{N}$ is nilpotent, now $\tilde{C}$ is central in $\tilde{N}$, so C is central in N. If E is any compact subgroup of N we have $n \in N$ such that $nEn^{-1} \subset C$, so $E \subset n^{-1}Cn = C$, proving E central in N. Q.E.D.

2.6. Proposition. *Let $M = G/K$ be an effective coset space of a connected Lie group by a compact subgroup. Then, for any* $\mathrm{ad}_G(K)$*-invariant Levi factor $\mathfrak{L}$ of $\mathfrak{G}$, there is an invariant complement $\mathfrak{M}$ for K such that*

$$(2.7)\quad \mathfrak{M} = (\mathfrak{M} \cap \mathfrak{R}) + (\mathfrak{M} \cap \mathfrak{L}), \quad \mathfrak{R} = (\mathfrak{M} \cap \mathfrak{R}) + (\mathfrak{K} \cap \mathfrak{R}), \quad \textit{and} \quad \mathfrak{N} \subset \mathfrak{M} \cap \mathfrak{R}.$$

In particular, if G is a group of isometries for a riemannian metric on M, and if the metric is consistent with the Levi-Whitehead decomposition $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$, then in addition we can choose $\mathfrak{M}$ so that $\mathfrak{M} \cap \mathfrak{R}$ has a subspace $\mathfrak{A}$ such that

$$(2.8)\quad \mathfrak{M} = \mathfrak{N} + \mathfrak{A} + (\mathfrak{M} \cap \mathfrak{L}), \quad \mathfrak{M} \cap \mathfrak{R} = \mathfrak{N} + \mathfrak{A}, \quad \textit{orthogonal direct sums.}$$

Proof. Following Proposition 2.3 we take $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$, invariant complement for K split by $\mathfrak{L}$, where $\mathfrak{M}_1$ is any $\mathrm{ad}_G(K)$-invariant complement to $\mathfrak{R} \cap \mathfrak{K}$ in $\mathfrak{R}$.

Let N be the analytic subgroup of G for $\mathfrak{N}$. Then N is closed in G, so $N \cap K$ is compact. Let T be a maximal compact subgroup of N. It contains $N \cap K$ and is central in N by Lemma 2.5. Thus T is unique, hence normal in G. As T is a torus and G is connected now T is central in G. Thus $N \cap K$ is central in G. But G acts effectively on M, so K contains no nontrivial normal subgroup of G. That proves $N \cap K = \{1\}$. In particular $\mathfrak{N} \cap \mathfrak{K} = 0$. Thus $\mathfrak{R} = \mathfrak{N} + \mathfrak{A} + (\mathfrak{R} \cap \mathfrak{K})$, $\mathrm{ad}_G(K)$-invariant direct sum, where $\mathfrak{A}$ is any invariant complement to $\mathfrak{N} + (\mathfrak{R} \cap \mathfrak{K})$. For (2.7) we just choose $\mathfrak{M}_1 = \mathfrak{N} + \mathfrak{A}$.

Suppose further that ds^2 is consistent with $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$. Then we have another choice, say $\mathfrak{M}^* = \mathfrak{M}_1^* + \mathfrak{M}_2^*$, of invariant complement for K, with $\langle \mathfrak{M}_1^*, \mathfrak{M}_2^* \rangle = 0$ and $\mathfrak{R} = \mathfrak{M}_1^* + (\mathfrak{R} \cap \mathfrak{K})$. If $\psi : \mathfrak{M} \cong \mathfrak{G}/\mathfrak{K}$ and $\psi^* : \mathfrak{M}^* \cong \mathfrak{G}/\mathfrak{K}$ are induced by the projection $\mathfrak{G} \to \mathfrak{G}/\mathfrak{K}$, now $\psi^{-1}\psi^* : \mathfrak{M}^* \to \mathfrak{M}$ is a linear isometry carrying $\mathfrak{M}_i^*$ to $\mathfrak{M}_i$. Thus $\langle \mathfrak{M}_1, \mathfrak{M}_2 \rangle = 0$, and we obtain (2.8) by choosing $\mathfrak{A}$ to be the orthocomplement of $\mathfrak{N}$ in $\mathfrak{M}_1$. Q.E.D.

We will view the space $\mathfrak{A}$ of (2.8) as the "gap" between nilpotent and solvable radicals of $\mathfrak{G}$, taken modulo $\mathfrak{K}$.

3. **Mean curvature along the gap between the nilpotent and solvable radicals.** We compute the mean curvature of a homogeneous riemannian manifold along a direction in the solvable radical complementary to the nilpotent radical. This is done by specializing the following general calculation to the case where the riemannian metric is consistent with a Levi-Whitehead decomposition.

3.1. LEMMA. *Let (M, ds^2) be a connected n-dimensional riemannian homogeneous space. Let $\mathcal{O} \in M$. Let G be a connected transitive group of isometries of M and let $M'_{\mathcal{O}}$ denote the subspace of $M_{\mathcal{O}}$ consisting of tangent vectors $Y_{\mathcal{O}}$ where Y is in the derived algebra $[\mathfrak{G}, \mathfrak{G}]$. If $X_{\mathcal{O}} \in M_{\mathcal{O}}$ is a unit vector orthogonal to $M'_{\mathcal{O}}$, then the mean curvature*

$$(3.2)\qquad \begin{aligned}(n-1)\boldsymbol{k}(X_{\mathcal{O}}) &= \sum_i \langle [\{[X, E_i]_{\mathfrak{K}} + \tfrac{1}{4}[X, E_i]_{\mathfrak{M}}\}, X]_{\mathfrak{M}}, E_i \rangle \\ &\quad - \tfrac{1}{2} \sum_i \|[X, E_i]_{\mathfrak{M}}\|^2 \\ &\quad - \tfrac{1}{4} \sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j \rangle \cdot \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle,\end{aligned}$$

where $\mathfrak{M} \subset \mathfrak{G}$ is an invariant complement to K, $X \in \mathfrak{M}$ represents $X_{\mathcal{O}}$, $\langle\ ,\ \rangle$ is the inner product on $\mathfrak{M}$ from ds^2, and $\{E_i\}$ is any orthonormal basis of $\mathfrak{M}$ containing X.

Proof. We follow the method of Nomizu [4], using the notation
$\alpha: \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ for the connection function,
$\mathscr{U}: \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ for the symmetric part of α,
$\mathscr{R}: \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ for the curvature tensor.
Now $(n-1)\boldsymbol{k}(X) = \sum_{i \neq i_0} K_i = -\sum_{i \neq i_0} \langle \mathscr{R}(X, E_i)X, E_i \rangle = -\sum_i \langle \mathscr{R}(X, E_i)X, E_i \rangle$ where

$\{E_i\}$ is an orthonormal basis of $\mathfrak{M}$, where $X = E_{i_0}$, and where K_i is the sectional curvature of the tangent 2-plane spanned by X and E_i. Thus

$$(n-1)k(X) = -\sum_i \langle \mathscr{R}(X, E_i)X, E_i\rangle. \tag{3.3}$$

Using [4, formulae 9.6 and 13.1] and correcting a misprint in the latter,

$$\mathscr{R}(X, E_i)X = \alpha(X, \alpha(E_i, X)) - \alpha(E_i, \alpha(X, X)) - \alpha([X, E_i]_{\mathfrak{M}}, X) - [[X, E_i]_{\mathfrak{K}}, X].$$

$$\alpha(S, T) = \tfrac{1}{2}[S, T]_{\mathfrak{M}} + \mathscr{U}(S, T).$$

$$\mathscr{U}(S, T) = -\tfrac{1}{2}\sum_j \{\langle [S, E_j]_{\mathfrak{M}}, T\rangle + \langle [T, E_j]_{\mathfrak{M}}, S\rangle\}E_j.$$

Our hypothesis on X and $X_{\mathscr{O}}$ is that $\langle X, [A, B]_{\mathfrak{M}}\rangle = 0$ for all $A, B \in \mathfrak{M}$. In particular

$$\mathscr{U}(X, S) = \mathscr{U}(S, X) = -\tfrac{1}{2}\sum_j \langle [X, E_j]_{\mathfrak{M}}, S\rangle E_j. \tag{3.4}$$

Thus $\alpha(X, X) = 0$. Substituting that into (3.3) we have

$$\begin{aligned}(n-1)k(X) = &-\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i\rangle \\ &+\sum_i \langle \alpha([X, E_i]_{\mathfrak{M}}, X), E_i\rangle \\ &+\sum_i \langle [[X, E_i]_{\mathfrak{K}}, X], E_i\rangle.\end{aligned} \tag{3.5}$$

In order to evaluate the right-hand side of (3.5) we define coefficients b_{jk} by $[X, E_j]_{\mathfrak{M}} = \sum_k b_{jk}E_k$. Then, using (3.4),

$$\begin{aligned}2\sum_i \langle [X, \mathscr{U}(E_i, X)]_{\mathfrak{M}}, E_i\rangle &= -\sum_{i,j} \langle [X, \langle [X, E_j]_{\mathfrak{M}}, E_i\rangle E_j]_{\mathfrak{M}}, E_i\rangle \\ &= -\sum_{i,j} \langle [X, E_j]_{\mathfrak{M}}, E_i\rangle^2 = -\sum_{i,j} b_{ji}^2 \\ &= -\sum_j \left(\sum_i b_{ji}^2\right) = -\sum_j \|[X, E_j]_{\mathfrak{M}}\|^2 \\ &= -\sum_i \|[X, E_i]_{\mathfrak{M}}\|^2 = \sum_i \langle [X, E_i]_{\mathfrak{M}}, [E_i, X]_{\mathfrak{M}}\rangle \\ &= +\sum_{i,j} \langle \{\langle [X, E_j]_{\mathfrak{M}}, [E_i, X]_{\mathfrak{M}}\rangle\}E_j, E_i\rangle \\ &= -2\sum_i \langle \mathscr{U}(X, [E_i, X]_{\mathfrak{M}}), E_i\rangle.\end{aligned}$$

In other words

$$-\tfrac{1}{2}\sum_i \langle [X, \mathscr{U}(E_i, X)]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{2}\sum_i \langle \mathscr{U}(X, [E_i, X]_{\mathfrak{M}}), E_i\rangle = 0. \tag{3.6}$$

Using (3.4) and (3.6) we compute

$$
\begin{aligned}
-\sum_i &\langle \alpha(X, \alpha(E_i, X)), E_i\rangle \\
&= -\tfrac{1}{4}\sum_i \langle [X, [E_i, X]_{\mathfrak{M}}]_{\mathfrak{M}}, E_i\rangle - \sum_i \langle \mathscr{U}(X, \mathscr{U}(E_i, X)), E_i\rangle \\
&\quad -\tfrac{1}{2}\sum_i \langle [X, \mathscr{U}(E_i, X)]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{2}\sum_i \langle \mathscr{U}(X, [E_i, X]_{\mathfrak{M}}), E_i\rangle \\
&= -\tfrac{1}{4}\sum_i \langle [X, [E_i, X]_{\mathfrak{M}}]_{\mathfrak{M}}, E_i\rangle - \sum_i \langle \mathscr{U}(X, \mathscr{U}(E_i, X)), E_i\rangle \\
&= -\tfrac{1}{4}\sum_i \langle [X, [E_i, X]_{\mathfrak{M}}]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{4}\sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j\rangle\langle [X, E_j]_{\mathfrak{M}}, E_i\rangle.
\end{aligned}
$$

That gives us the first summand of the right-hand side of (3.5):

$$
\begin{aligned}
-\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i\rangle &= -\tfrac{1}{4}\sum_i \langle [[X, E_i]_{\mathfrak{M}}, X]_{\mathfrak{M}}, E_i\rangle \\
&\quad -\tfrac{1}{4}\sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j\rangle\langle [X, E_j]_{\mathfrak{M}}, E_i\rangle.
\end{aligned} \tag{3.7}
$$

The second summand of the right-hand side of (3.5) is, again using (3.4),

$$
\begin{aligned}
\sum_i &\langle \alpha([X, E_i]_{\mathfrak{M}}, X), E_i\rangle \\
&= \tfrac{1}{2}\sum_i \langle [[X, E_i]_{\mathfrak{M}}, X]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{2}\sum_{i,j} \langle \{\langle [X, E_j]_{\mathfrak{M}}, [X, E_i]_{\mathfrak{M}}\rangle E_j\}, E_i\rangle \\
&= \tfrac{1}{2}\sum_i \langle [[X, E_i]_{\mathfrak{M}}, X]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{2}\sum_i \|[X, E_i]_{\mathfrak{M}}\|^2.
\end{aligned}
$$

Using (3.7) now, the sum of the first two summands of the right-hand side of (3.5) is

$$
\begin{aligned}
-\sum_i &\langle \alpha(X, \alpha(E_i, X)), E_i\rangle + \sum_i \langle \alpha([X, E_i]_{\mathfrak{M}}, X), E_i\rangle \\
&= +\tfrac{1}{4}\sum_i \langle [[X, E_i]_{\mathfrak{M}}, X]_{\mathfrak{M}}, E_i\rangle - \tfrac{1}{2}\sum_i \|[X, E_i]_{\mathfrak{M}}\|^2 \\
&\quad -\tfrac{1}{4}\sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j\rangle \cdot \langle [X, E_j]_{\mathfrak{M}}, E_i\rangle.
\end{aligned} \tag{3.8}
$$

Adding $\sum_i \langle [[X, E_i]_{\mathfrak{K}}, X]_{\mathfrak{M}}, E_i\rangle$ to both sides of (3.8), our assertion (3.2) follows from (3.5). Q.E.D.

We apply Lemma 3.1 to the gap between the nilpotent and solvable radicals of G.

3.9. Theorem. *Let (M, ds^2) be a riemannian homogeneous space, G a transitive Lie group of isometries, K the isotropy subgroup at a point $\mathcal{O} \in M$, and $\mathfrak{L}$ an* $\mathrm{ad}_G(K)$*-invariant Levi factor of $\mathfrak{G}$, such that ds^2 is consistent with the Levi-Whitehead decomposition $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$. Choose an invariant complement $\mathfrak{M} = \mathfrak{N} + \mathfrak{A} + (\mathfrak{M} \cap \mathfrak{L})$*

for K that satisfies (2.8). *Let $X \in \mathfrak{M}$ be a unit vector, $X_{\mathcal{O}} \in M_{\mathcal{O}}$ the corresponding unit tangent vector.*

1. *If $X \perp [\mathfrak{G}, \mathfrak{G}]_{\mathfrak{M}}$ then the mean curvature $k(X_{\mathcal{O}}) \leqq 0$, and $k(X_{\mathcal{O}}) = 0$ if and only if* (a) $X \in \mathfrak{N}$ *and* (b) $[X, \mathfrak{M}] = 0$.
2. *If $X \in \mathfrak{A}$ then $k(X_{\mathcal{O}}) < 0$.*

Proof. By choice of $\mathfrak{M}$ and by $[\mathfrak{G}, \mathfrak{G}] \subset \mathfrak{N} + \mathfrak{L}$ we have an orthogonal direct sum decomposition

$$\begin{aligned} \mathfrak{M} &= \mathfrak{N}' + \mathfrak{B} + (\mathfrak{M} \cap \mathfrak{L}), \quad \mathfrak{N}' + \mathfrak{B} = \mathfrak{M} \cap \mathfrak{N}, \quad \mathfrak{N}' \subset \mathfrak{N}, \\ [\mathfrak{G}, \mathfrak{G}]_{\mathfrak{M}} &= \mathfrak{N}' + (\mathfrak{M} \cap \mathfrak{L}), \quad \mathfrak{A} \subset \mathfrak{B}. \end{aligned} \tag{3.10}$$

Let $X \in \mathfrak{B}$. Then we have an orthonormal basis $\{E_i\}$ of $\mathfrak{M}$ containing X, such that each E_i is in $\mathfrak{N}'$, $\mathfrak{M} \cap \mathfrak{L}$ or $\mathfrak{B}$. We apply Lemma 3.1 with that basis.

Define coefficients by $[X, E_j]_{\mathfrak{M}} = \sum_k a_{jk} E_k$ and let $A = (a_{jk})$. Then

$$\langle [X, E_i]_{\mathfrak{M}}, E_j \rangle = a_{ij} \quad \text{and} \quad \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle = a_{ji}$$

so

$$\sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j \rangle \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle = \operatorname{trace}(A \cdot {}^tA).$$

Take polar decomposition $A = ST$ with S symmetric and T orthogonal. Then $A \cdot {}^tA = S \cdot {}^tS$. Let $S = (s_{ij})$, so

$$\operatorname{trace}(A \cdot {}^tA) = \operatorname{trace}(S \cdot {}^tS) = \sum_{i,j} s_{ij}^2 \geqq 0,$$

and note that $\sum s_{ij}^2 = 0$ if and only if $S = 0$, which is equivalent to $A = 0$. Thus

$$-\tfrac{1}{4} \sum_{i,j} \langle [X, E_i]_{\mathfrak{M}}, E_j \rangle \langle [X, E_j]_{\mathfrak{M}}, E_i \rangle \leqq 0 \tag{3.11}$$

with equality if and only if $[X, \mathfrak{M}]_{\mathfrak{M}} = 0$.

That takes care of the last summand of (3.2). For the first two summands we define

$$k_i = \langle [\{[X, E_i]_{\mathfrak{K}} + \tfrac{1}{4}[X, E_i]_{\mathfrak{M}}\}, X]_{\mathfrak{M}}, E_i \rangle - \tfrac{1}{2} \| [X, E_i]_{\mathfrak{M}} \|^2. \tag{3.12}$$

If $E_i \in \mathfrak{B}$ then $\langle [\mathfrak{G}, \mathfrak{G}]_{\mathfrak{M}}, \mathfrak{B} \rangle = 0$ implies $k_i = -\frac{1}{2} \| [X, E_i]_{\mathfrak{M}} \|^2$. If $E_i \in \mathfrak{M} \cap \mathfrak{L}$ then $[\mathfrak{G}, X]_{\mathfrak{M}} \subset [\mathfrak{G}, \mathfrak{N}]_{\mathfrak{M}} \subset (\mathfrak{M} \cap \mathfrak{N}) \perp (\mathfrak{M} \cap \mathfrak{L})$ implies $\langle [\mathfrak{G}, X]_{\mathfrak{M}}, E_i \rangle = 0$ so

$$k_i = -\tfrac{1}{2} \| [X, E_i]_{\mathfrak{M}} \|^2.$$

Thus

$$\text{if } E_i \in \mathfrak{B} + (\mathfrak{M} \cap \mathfrak{L}) \text{ then } k_i = -\tfrac{1}{2} \| [X, E_i]_{\mathfrak{M}} \|^2 \leqq 0. \tag{3.13}$$

If $E_i \in \mathfrak{N}'$ then $E_i \in \mathfrak{N}$ so $[X, E_i] \in \mathfrak{N} \subset \mathfrak{M}$. Thus

$$\text{if } E_i \in \mathfrak{N}' \text{ then } k_i = -\tfrac{1}{4} \langle \operatorname{ad}(X)^2 E_i, E_i \rangle - \tfrac{1}{2} \| [X, E_i] \|^2. \tag{3.14}$$

As $(\operatorname{ad} X)\mathfrak{N}' \subset \mathfrak{N}'$ we can stipulate that, for numbers $\{\lambda_b\}$ such that $\{\lambda_b, \bar{\lambda}_b\}$ are the eigenvalues of $(\operatorname{ad} X)|_{\mathfrak{N}'}$, each $E_i \in \mathfrak{N}'$ is contained in the sum of the subspaces of $\mathfrak{N}'^C$ on which (for some $b = b_i$) $\operatorname{ad} X - \lambda_b$ and $\operatorname{ad} X - \bar{\lambda}_b$ are nilpotent. That stipulation made, $\|[X, E_i]\|^2 \geqq |\lambda_b|^2$ and $|\langle \operatorname{ad}(X)^2 E_i, E_i\rangle| \leqq |\lambda_b|^2$. So (3.14) implies

$$\text{if } E_i \in \mathfrak{N}' \text{ then } k_i \leqq -\tfrac{1}{4}|\lambda_b|^2 \leqq 0. \tag{3.15}$$

Combining (3.13) and (3.15) we have $\sum_i k_i \leqq 0$. Adding that inequality to (3.11), and applying Lemma 3.1, we conclude

$$\boldsymbol{k}(X_{\mathcal{O}}) \leqq 0 \text{ with equality if and only if } [X, \mathfrak{M}]_{\mathfrak{M}} = 0. \tag{3.16}$$

If $[X, \mathfrak{M}]_{\mathfrak{M}} = 0$ then $[X, \mathfrak{M}] \subset \mathfrak{K}$. As $\mathfrak{N} \subset \mathfrak{M}$ and $[X, \mathfrak{N}] \subset \mathfrak{N}$ it follows that $[X, \mathfrak{N}] = 0$. Then $\mathfrak{S} = X\boldsymbol{R} + \mathfrak{N}$ is a nilpotent subalgebra of $\mathfrak{G}$. But $X \in \mathfrak{R}$ and $[\mathfrak{R}, \mathfrak{R}] \subset \mathfrak{N}$, so $\mathfrak{S}$ is a nilpotent ideal in $\mathfrak{R}$. As $\mathfrak{N}$ is the maximal nilpotent ideal of $\mathfrak{R}$ it follows that $X \in \mathfrak{N}$. This fact and (3.16) imply the first statement of Theorem 3.9. If $X \in \mathfrak{A}$ then $X \notin \mathfrak{N}$, so $\boldsymbol{k}(X_{\mathcal{O}}) < 0$. That completes the proof of Theorem 3.9. Q.E.D.

4. **Application to manifolds of nonnegative mean curvature.** We first apply Theorem 3.9 to homogeneous riemannian manifolds.

4.1. THEOREM. *Let (M, ds^2) be a connected homogeneous riemannian manifold, G a transitive Lie group of isometries, K an isotropy subgroup, and $\mathfrak{L}$ an* $\operatorname{ad}_G(K)$-*invariant Levi factor of $\mathfrak{G}$ such that ds^2 is consistent with the Levi-Whitehead decomposition $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$.*

1. *If (M, ds^2) has mean curvature $\geqq 0$ everywhere, then the solvable radical $\mathfrak{R}$ and the nilpotent radical $\mathfrak{N}$ of $\mathfrak{G}$ satisfy $\mathfrak{R} = \mathfrak{N} + (\mathfrak{K} \cap \mathfrak{R})$ semidirect sum.*

2. *If (M, ds^2) has mean curvature > 0 everywhere, then the derived group $[G, G]$ of G is transitive on M.*

Proof. If (M, ds^2) has mean curvature $\geqq 0$ everywhere, then, in the notation (2.8), Theorem 3.9 says $\mathfrak{A} = 0$, so $\mathfrak{M} \cap \mathfrak{R} = \mathfrak{N}$; thus $\mathfrak{R} = (\mathfrak{M} \cap \mathfrak{R}) + (\mathfrak{K} \cap \mathfrak{R}) = \mathfrak{N} + (\mathfrak{K} \cap \mathfrak{R})$. If further (M, ds^2) has mean curvature > 0 everywhere, then Theorem 3.9 says $\mathfrak{M} = [\mathfrak{G}, \mathfrak{G}]_{\mathfrak{M}}$, so the derived group $G' = [G, G]$ has an open orbit $G'(\mathcal{O}) \subset M$. As $G'(\mathcal{O})$ is complete and M is connected, $G'(\mathcal{O}) = M$, so G' is transitive on M. Q.E.D.

4.2. COROLLARY. *Let (M, ds^2) be a connected riemannian homogeneous manifold, G a transitive Lie group of isometries of M, $\mathcal{O} \in M$, and K the isotropy subgroup of G at $\mathcal{O}$. Let $R_{\mathcal{O}}$ denote the subspace of the tangent space $M_{\mathcal{O}}$ consisting of vectors $Y_{\mathcal{O}}$ where Y is contained in the solvable radical $\mathfrak{R}$ of $\mathfrak{G}$. Suppose that the linear isotropy representation of K splits into disjoint representations on $R_{\mathcal{O}}$ and $R_{\mathcal{O}}^{\perp}$.*

1. *If (M, ds^2) has mean curvature $\geqq 0$ everywhere, then $\mathfrak{R}$ is related to the nilpotent radical $\mathfrak{N}$ of $\mathfrak{G}$ by $\mathfrak{R} = \mathfrak{N} + (\mathfrak{K} \cap \mathfrak{R})$.*

2. *If (M, ds^2) has mean curvature >0 everywhere, then the derived group of G is transitive on M.*

Proof. Let $\mathfrak{L}$ be any $\mathrm{ad}_G(K)$-invariant Levi factor of $\mathfrak{G}$. Proposition 2.4 says that ds^2 is consistent with the Levi-Whitehead decomposition $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$. Our assertions now follow from Theorem 4.1. Q.E.D.

In order to apply Theorem 4.1 to the case of a transitive solvable group of isometries, we must first prove the following lemma about a simply transitive nilpotent group of isometries. Note that the lemma extends the positive curvature portion of [7].

4.3. Lemma. *Let (N, ds^2) be a connected nilpotent Lie group with a left invariant riemannian metric. Then the following conditions are equivalent.*

(i) *(N, ds^2) has mean curvature $\geqq 0$ everywhere.*

(ii) *(N, ds^2) has every sectional curvature zero.*

(iii) *N is commutative.*

Proof. As (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) trivially we need only check that (i) $\Rightarrow$ (iii). So assume that (N, ds^2) has mean curvature $\geqq 0$ everywhere. In the context of Theorem 3.9,

$$G = N, \quad K = \{1\}, \quad \mathfrak{L} = 0, \quad \mathfrak{M} = \mathfrak{N},$$

and consistency of ds^2 with $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}$ is automatic. Now Theorem 3.9 says that there is no noncentral element $X \in \mathfrak{N}$ such that $X \perp [\mathfrak{N}, \mathfrak{N}]$. But nilpotence of $\mathfrak{N}$ implies that in the lower central series

$$\mathfrak{N} = \mathfrak{N}_0 \supset \mathfrak{N}_1 \supset \cdots \supset N_s \supsetneqq \mathfrak{N}_{s+1} = 0, \quad \mathfrak{N}_{n+1} = [\mathfrak{N}, \mathfrak{N}_k],$$

any vector space complement to $\mathfrak{N}_1=[\mathfrak{N}, \mathfrak{N}]$ generates $\mathfrak{N}$. Let $[\mathfrak{N}, \mathfrak{N}]^\perp$ be the complement. As it consists of central elements of $\mathfrak{N}$ (our application of Theorem 3.9), it must be all of $\mathfrak{N}$. Thus N is commutative. Q.E.D.

Now we have a general result on the curvature of riemannian solvmanifolds.

4.4. Theorem. *Let (M, ds^2) be a connected riemannian manifold that has a solvable transitive group of isometries. Then the following conditions are equivalent.*

(i) *(M, ds^2) has mean curvature $\geqq 0$ everywhere.*

(ii) *(M, ds^2) has every sectional curvature $\geqq 0$.*

(iii) *(M, ds^2) has every sectional curvature zero.*

(iv) *(M, ds^2) is isometric to the product of an euclidean space and a flat riemannian torus.*

Proof. As (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) trivially we need only check that (i) $\Rightarrow$ (iv). So assume that (M, ds^2) has mean curvature $\geqq 0$.

G denotes the closure of a solvable transitive group of isometries of (M, ds^2) in the full group of isometries. So G is a solvable transitive Lie group of isometries. Let K be an isotropy subgroup. $\mathfrak{G}$ is its own solvable radical $\mathfrak{R}$, so the $\mathrm{ad}_G(K)$-invariant Levi factor $\mathfrak{L}=0$, and ds^2 is consistent with $\mathfrak{G}=\mathfrak{R}+\mathfrak{L}=\mathfrak{R}$. Our invariant complement $\mathfrak{M}$ for K satisfying (2.8) now has the form $\mathfrak{M}=\mathfrak{N}+\mathfrak{A}$, and Theorem 3.9 says $\mathfrak{A}=0$.

Let N be the analytic subgroup of G for $\mathfrak{N}$. Now we have an open orbit $N(\mathcal{O})\subset M$. As $N(\mathcal{O})$ is complete and M is connected, the two are equal. Thus N is transitive on M. As G acts effectively on M, also N acts effectively, so Lemma 2.5 says $K\cap N=\{1\}$. That proves N simply transitive on M. Lemma 4.3 says that (M, ds^2) is flat and N is commutative. It follows [6, Théorème 4] that (M, ds^2) is the product of an euclidean space and a flat riemannian torus. Q.E.D.

4.5. COROLLARY. *Let (M, ds^2) be a connected riemannian Einstein manifold that has a solvable transitive group of isometries. Then either (M, ds^2) has vanishing Ricci tensor and is isometric to the product of an euclidean space with a flat riemannian torus, or (M, ds^2) has negative definite Ricci tensor.*

Proof. The Einstein homogeneous hypothesis says that (M, ds^2) has constant mean curvature, say $\boldsymbol{k}$. If $\boldsymbol{k}\geqq 0$ then Theorem 4.4 says that $(R_{ij})\equiv 0$ and that (M, ds^2) is the product of an euclidean space with a flat riemannian torus. If $\boldsymbol{k}<0$ then (R_{ij}) is negative definite. Q.E.D.

For examples of the latter case of Corollary 4.5, let (M, ds^2) be a noncompact irreducible riemannian symmetric space, G the largest connected group of isometries, K an isotropy subgroup, and $G=NAK$ an Iwasawa decomposition. Then $S=NA$ is a simply transitive solvable Lie group of isometries of (M, ds^2), and (M, ds^2) is a connected riemannian Einstein manifold with negative definite Ricci tensor. G. Jensen [2] has shown that this example is essentially exhaustive in dimensions $\leqq 4$.

The following lemma is similar to results of G. Jensen [2].

4.6. LEMMA. *Let G be a Lie group, let ds^2 be a left invariant riemannian metric on G, and let X be a nonzero central element of the Lie algebra $\mathfrak{G}$. Then the mean curvature $\boldsymbol{k}(X)\geqq 0$, and $\boldsymbol{k}(X)=0$ if and only if X is orthogonal to the derived algebra of $\mathfrak{G}$.*

Proof. We use the notation of the proof of Lemma 3.1. Note $\mathfrak{M}=\mathfrak{G}$. We take X to be a unit vector and $\{E_i\}$ to be an orthonormal basis of $\mathfrak{G}$ that contains X. Then (3.3) holds. As X is central in $\mathfrak{G}$, the analog of (3.4) is

$$\mathscr{U}(S, X) = \mathscr{U}(X, S) = -\tfrac{1}{2}\sum_j \langle [S, E_j], X\rangle E_j.$$

We still have $\mathscr{U}(X, X)=0$, so $\alpha(X, X)=0$ and (3.5) holds. But $[X, E_i]=0$ simplifies (3.5) to

$$\begin{aligned}(n-1)\boldsymbol{k}(X) &= -\sum_i \langle \alpha(X, \alpha(E_i, X)), E_i\rangle \\ &= \tfrac{1}{2}\sum_{i,j} \langle \alpha(X, \langle [E_i, E_j], X\rangle E_j), E_i\rangle \\ &= \tfrac{1}{2}\sum_{i,j} \langle [E_i, E_j], X\rangle \cdot \langle \alpha(X, E_j), E_i\rangle \\ &= -\tfrac{1}{4}\sum_{i,j,k} \langle [E_i, E_j], X\rangle\langle\langle [E_j, E_k], X\rangle E_k, E_i\rangle \\ &= -\tfrac{1}{4}\sum_{i,j} \langle [E_i, E_j], X\rangle\langle [E_j, E_i], X\rangle \\ &= \tfrac{1}{4}\sum_{i,j} \langle [E_i, E_j], X\rangle^2.\end{aligned}$$

Thus $\boldsymbol{k}(X)\geqq 0$, and $\boldsymbol{k}(X)=0$ if and only if each $\langle [E_i, E_j], X\rangle=0$, which is equivalent to $\langle [\mathfrak{G}, \mathfrak{G}], X\rangle=0$. Q.E.D.

We now combine Lemmas 4.3 and 4.6, extending our calculations [7] from sectional curvature to mean curvature, and sharpening Theorem 4.4 in the case of a nilpotent group. After hearing the result, G. Jensen gave another proof of Theorem 4.7 [2, Theorem 4].

4.7. THEOREM. *Let (M, ds^2) be a connected riemannian manifold that has a nilpotent transitive group of isometries. Then the following conditions are equivalent.*

(i) *(M, ds^2) has mean curvature $\geqq 0$ everywhere.*

(ii) *(M, ds^2) has mean curvature $=0$ everywhere.*

(iii) *(M, ds^2) has mean curvature $\leqq 0$ everywhere.*

(iv) *(M, ds^2) has every sectional curvature $\geqq 0$.*

(v) *(M, ds^2) has every sectional curvature $=0$.*

(vi) *(M, ds^2) has every sectional curvature $\leqq 0$.*

(vii) *(M, ds^2) is isometric to the product of an euclidean space and a flat riemannian torus.*

Proof. Let N denote the identity component of the closure of a nilpotent transitive group of isometries. Then N is a connected nilpotent transitive Lie group of isometries of (M, ds^2). Its isotropy subgroups are central by Lemma 2.5, hence trivial; thus N is simply transitive on (M, ds^2). Now we may view ds^2 as a left invariant riemannian metric on N.

Lemma 4.3 says that (i) implies (v); so (ii) implies (v).

We use Lemma 4.6 to prove that (iii) implies (v). Let $\mathfrak{Z}$ be the last nonzero term of the lower central series of $\mathfrak{N}$. Then $\mathfrak{Z}$ is central in $\mathfrak{N}$. Let $0\neq X\in\mathfrak{Z}$. Assume (iii), so $\boldsymbol{k}(X)\leqq 0$. Lemma 4.6 says $\boldsymbol{k}(X)\geqq 0$. Thus $\boldsymbol{k}(X)=0$ and Lemma 4.6 says $\langle [\mathfrak{N}, \mathfrak{N}], X\rangle=0$. If $\mathfrak{N}$ is noncommutative then $X\in\mathfrak{Z}\subset[\mathfrak{N}, \mathfrak{N}]$ and so $\langle [\mathfrak{N}, \mathfrak{N}], X\rangle$

$\neq 0$. That proves N commutative, so every sectional curvature of (N, ds^2) is zero. Thus (iii) implies (v).

Now (i), (ii) and (iii) each implies (v). It follows that (iv) and (vi) each implies (v). But (v) implies (vii) by [6, Théorème 4], and (vii) clearly implies each of (i), (ii), (iii), (iv), (v) and (vi). Q.E.D.

4.8. Corollary. *Let (M, ds^2) be a connected riemannian Einstein manifold that has a nilpotent transitive group of isometries. Then (M, ds^2) is isometric to the product of an euclidean space and a flat riemannian torus.*

Proof. We have the hypothesis of Theorem 4.7 as well as condition (i), (ii) or (iii); thus we have condition (vii) of Theorem 4.7. Q.E.D.

Our last application is a refinement of [8, Corollary 5.8].

4.9. Theorem. *Let (M, ds^2) be a compact connected locally homogeneous riemannian manifold with mean curvature $\geqq 0$ everywhere. Suppose that the fundamental group $\pi_1(M)$ has a solvable subgroup of finite index.*

Let $\pi: \tilde{M} \to M$ be the universal riemannian covering, G the largest connected group of isometries of $(\tilde{M}, \pi^* ds^2)$, K an isotropy subgroup of G, $\mathfrak{L}$ an $\mathrm{ad}_G(K)$-invariant Levi factor of $\mathfrak{G}$ and L its analytic subgroup of G, and $\mathfrak{R}$ and $\mathfrak{N}$ the solvable and nilpotent radicals of $\mathfrak{G}$ and R and N their analytic subgroups of G. Let Γ denote the group $\cong \pi_1(M)$ of deck transformations of $\tilde{M} \to M$. Suppose that $\pi^* ds^2$ is consistent with $\mathfrak{G} = \mathfrak{R} + \mathfrak{L}$.

1. *L is compact.*

2. *$R = N \cdot (K \cap R)_0$ semidirect product, where $(K \cap R)_0$ is a torus group whose Lie algebra $\mathfrak{K} \cap \mathfrak{R}$ acts effectively on $\mathfrak{N}$ in the adjoint representation.*

3. *Γ has a torsion free normal nilpotent subgroup Δ of finite index, $\Delta \subset N \cdot Z_L(N)_0$ where $Z_L(N)$ is the centralizer of N in L, and Δ projects isomorphically to a discrete subgroup with compact quotient in N.*

Proof. Compactness of L is part of [8, Corollary 5.8], and the decomposition $R = N \cdot (K \cap R)_0$ follows from Theorem 4.1. In the proof of [8, Corollary 5.8] it is shown that Γ has a nilpotent subgroup Δ of finite index, and that the identity component of the closure of $R\Delta$ in G has form

$$F = R \cdot U \text{ for some torus } U \subset L.$$

Enlarge U to a maximal torus T of F. Then $T = T_N \cdot T_{R/N} \cdot U$ local direct product, where T_N is a maximal torus of N and $T_{R/N}$ is an $\mathrm{ad}_G(R)$-conjugate of $(K \cap R)_0$. These constructions are not changed if Δ is cut down to a subgroup of finite index. So we first cut Δ down to $\Delta \cap F$, then [8, (4.5)] to a torsion free group, and finally to a normal subgroup of Γ.

Let V be the kernel of the action of T on N, i.e. the centralizer $Z_T(N)$. Then $T_N \subset V_0 \subset T_N \cdot U$. Let $F^* = F/V$; then $F^* = N^* \cdot T^*$ where $N^* = N/T_N$ simply connected nilpotent group and

$$T^* = (T_{R/N} \cdot U)/\{(T_{R/N} \cdot U) \cap V\}.$$

Δ^* is the projection of Δ to F^*. As Δ is a discrete subgroup with compact quotient in F, the same is true of Δ^* in F^*. By construction of F^* and the fact that T_N is central in F, conjugation represents T^* faithfully as a group of automorphisms of N^*, so L. Auslander's result [1] says that $\Delta^* \cap N^*$ has finite index in Δ^*. Again cutting Δ down, we may assume $\Delta^* \subset N^*$, i.e. that $\Delta \subset N \cdot Z_U(N)_0 \subset N \cdot Z_L(N)_0$. Q.E.D.

References

1. L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Ann. of Math. **71** (1960), 579–590.

2. G. Jensen, *Homogeneous Einstein spaces*, Thesis, University of California, Berkeley, 1968.

3. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200–221.

4. K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.

5. E. J. Taft, *Orthogonal conjugacies in associative and Lie algebras*, Trans. Amer. Math. Soc. **113** (1964), 18–29.

6. J. A. Wolf, *Sur la classification des variétés riemanniennes homogènes à courbure constante*, C. R. Acad. Sci. Paris **250** (1960), 3443–3445.

7. ———, *Curvature in nilpotent Lie groups*, Proc. Amer. Math. Soc. **15** (1964), 271–274.

8. ———, *Growth of finitely generated solvable groups and curvature of riemannian manifolds*, J. Differential Geometry, (to appear).

University of California,
Berkeley, California