

# THE GEOMETRY AND STRUCTURE OF ISOTROPY IRREDUCIBLE HOMOGENEOUS SPACES

BY

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To Albert M. Wolf on his sixtieth birthday

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### Introduction

As a first step toward understanding the geometry of a riemannian homogeneous space  $M = G/K$  it is natural to consider the case where  $K$  acts irreducibly on the tangent space. This approach has been very useful in the study of riemannian symmetric spaces; these spaces are now well understood, and their understanding is based on Cartan's classification and structure theory for the irreducible case. Our Chapter I gives a structure theory and classification for nonsymmetric coset spaces  $G/K$  where  $K$  acts irreducibly on the tangent space. For  $K$  compact, this is done in §§ 1–10, then summarized and put into global form in § 11. For  $K$  noncompact, we reduce to the compact case by means of Cartan involutions (§ 12). The results are surprising, for there are a large number of nonsymmetric "isotropy irreducible" coset spaces  $G/K$ , and only a few examples had been known before. One of the more interesting classes is  $\mathbf{SO}(\dim K)/\mathbf{ad}(K)$  for an arbitrary compact simple Lie group  $K$ .

Chapter II concentrates on the study of complex and almost complex structures on isotropy irreducible coset spaces, and § 13 is a definitive treatment of this matter. More general structures are introduced and studied in § 14; the quaternionic structures are needed in § 16, and I believe that the notion of commuting structure will become important in riemannian geometry.

Chapter III is the goal of this paper—the riemannian geometry of isotropy irreducible coset spaces. The riemannian metric is unique up to a constant scalar factor; it is an Einstein metric with sectional curvature of one sign. We determine the holonomy group, the full group of isometries, and (in the almost complex case) the full group of almost hermitian isometries. The chapter ends with an examination of riemannian manifolds in which the local isometry group at a point is irreducible on the tangent space.

The de Rham decomposition shows that a riemannian manifold has parallel Ricci tensor if and only if it is locally a product of Einstein manifolds. Our isotropy irreducible riemannian manifolds have parallel Ricci tensor. Thus the classification results of Chapter I provide new examples of Einstein manifolds, and those examples are neither symmetric nor kaehlerian. I have hopes that those examples, especially the  $\mathbf{SO}(\dim K)/\mathbf{ad}(K)$  which show a clear pattern, will contribute toward an understanding of Einstein manifolds.

I wish to thank Lois B. Wolf for checking some of my calculations on  $\mathbf{E}_7$  and  $\mathbf{E}_8$ .

### Chapter I. The structure and classification of nonsymmetric isotropy irreducible coset spaces $G/K$

In this chapter we study and classify the coset spaces  $M = G/K$  which satisfy the conditions

- (i)  $G$  is a connected Lie group and  $K$  is a closed subgroup,
- (ii)  $M = G/K$  is a reductive<sup>(1)</sup> coset space on which  $G$  acts effectively<sup>(2)</sup>, and
- (iii) the linear isotropy action (on the tangent space of  $M$ ) of the identity component  $K_0$  of  $K$  is a representation which is irreducible over the real number field.

Conditions (ii) and (iii) together are equivalent to

[(ii)  $\cup$  (iii)] Let  $\chi$  be the linear isotropy representation of  $K$  on the tangent space of  $M$ . Then  $\chi$  is a faithful representation of  $K$  and  $\chi|_{K_0}$  is irreducible over the real number field.

The general case can be transformed or reduced to the case where  $K$  is compact, by means of Cartan involutions. This is done in § 12. For the next eleven sections, however, we avoid technical difficulties by generally making the working hypothesis

- (iv)  $K$  is compact.

The euclidean spaces and the irreducible riemannian symmetric spaces are the best known spaces which satisfy (i)–(iv). So we avoid duplication of standard material with the working hypothesis

- (v)  $(G, K)$  is not a symmetric pair, i.e.  $K_0$  is not the identity component of the fixed point set of an involutive automorphism of  $G$ .

In this chapter we need a certain amount of notation.  $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$ , and  $E_8$  refer to the Cartan classification types of simple Lie groups and algebras. We use boldface to denote the compact simply connected groups. Thus

$\mathbf{A}_n = \mathbf{SU}(n+1)$ , special unitary group,

$\mathbf{B}_n = \mathbf{Spin}(2n+1)$ , two sheeted covering of the rotation group  $\mathbf{SO}(2n+1)$ ;

$\mathbf{C}_n = \mathbf{Sp}(n)$ , unitary symplectic group;

$\mathbf{D}_n = \mathbf{Spin}(2n)$ , double covering of  $\mathbf{SO}(2n)$ ;

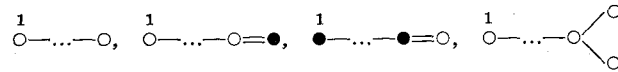
$\mathbf{G}_2$  is the automorphism group of the Cayley algebra; and so on. German letters denote Lie algebras; thus  $\mathfrak{A}_n, \mathfrak{Sp}(n)$  and  $\mathfrak{G}$  are the Lie algebras of Lie groups  $\mathbf{A}_n, \mathbf{Sp}(n)$  and  $G$ . If  $K$  is a Lie subgroup of  $G$ , then  $\mathfrak{K}$  denotes the corresponding subalgebra of  $\mathfrak{G}$ . If  $g \in G$ , then  $\text{ad}(g)$  denotes both the inner automorphism  $x \rightarrow gxg^{-1}$  of  $G$ , and the corresponding automorphism of  $\mathfrak{G}$ ; the latter is a representation which we usually denote  $\text{ad}_G$ .

Let  $\mathfrak{K}$  be a semisimple Lie algebra. Given a Cartan subalgebra  $\mathfrak{H}$  and an ordering of the roots, we have a system  $\{\alpha_1, \dots, \alpha_l\}$  of simple roots. If  $\pi$  is a linear representation of

<sup>(1)</sup> This means that the Lie algebra  $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$  where  $\mathfrak{M}$  is a vector space complement of  $\mathfrak{K}$  such that  $\text{ad}_G(K)\mathfrak{M} = \mathfrak{M}$ .

<sup>(2)</sup> In other words the identity element  $1 \in G$  is the only element which acts on  $M$  as the identity transformation.

$\mathfrak{g}$  on a complex vector space, then every weight  $\lambda$  satisfies the condition that  $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$  are integers; here  $\langle, \rangle$  denotes the inner product dual to the Killing form. If  $\pi$  is absolutely irreducible and  $\lambda$  is the highest weight, we generally denote  $\pi$  by  $\pi_\lambda$  because  $\lambda$  specifies  $\pi$  up to equivalence. In turn  $\lambda$  is specified by the nonnegative integers  $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$ . Our notation for  $\lambda$  and  $\pi_\lambda$  is the following: if the integer  $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle \neq 0$ , then we write it next to the vertex of the Dynkin diagram of  $\mathfrak{g}^{\mathbb{C}}$  which specifies  $\alpha_i$ . For example



denote the usual (“vector”) representations of  $A_{n-1}$  as  $SU(n)$ ,  $B_n$  as  $SO(2n+1)$ ,  $C_n$  as  $Sp(n)$  and  $D_n$  as  $SO(2n)$ , respectively. And the adjoint representations are given by

$A_1$	$\begin{matrix} 2 \\ \circ \end{matrix}$	$G_2$	$\begin{matrix} 1 \\ \bullet \equiv \circ \end{matrix}$
$A_n, n > 1$	$\begin{matrix} 1 & & 1 \\ \circ - \dots - \circ \end{matrix}$	$F_4$	$\begin{matrix} & & & 1 \\ & & \bullet - \bullet \equiv \circ - \circ & \\ \circ - \circ - \circ - \circ - \circ & & & \end{matrix}$
$B_n, n > 2$	$\begin{matrix} 1 \\ \circ - \circ - \dots - \circ = \bullet \end{matrix}$	$E_6$	$\begin{matrix} & & & & & \\ & & & & & \circ 1 \\ \circ - \circ - \circ - \circ - \circ - \circ & & & & &   \\ & & & & & \circ \end{matrix}$
$C_n, n > 1$	$\begin{matrix} 2 \\ \bullet - \bullet - \dots - \bullet = \circ \end{matrix}$	$E_7$	$\begin{matrix} 1 \\ \circ - \circ - \circ - \circ - \circ - \circ - \circ \\   \\ \circ \end{matrix}$
$D_n, n > 3$	$\begin{matrix} 1 \\ \circ - \circ - \dots - \circ \begin{matrix} \diagup \circ \\ \diagdown \circ \end{matrix} \end{matrix}$	$E_8$	$\begin{matrix} \circ - \circ - \circ - \circ - \circ - \circ - \circ - \circ \\   \\ \circ \end{matrix}$ $\begin{matrix} & & & & & & & 1 \end{matrix}$

Note that we are using the dot convention: if there are two lengths of roots, then the short roots are black in the Dynkin diagram.

**1.  $G$  is a compact simple Lie group**

We will prove:

1.1. THEOREM. *Let  $M = G/K$  satisfy conditions (i) through (v) above. Then  $G$  is a compact simple Lie group.*

The proof is divided into several steps, some of which are stated as fairly general lemmas for purposes of reference when we come to the case of noncompact  $K$ .

1.2. LEMMA. *Let  $M = G/K$  be a reductive coset space of connected real Lie groups such that  $G$  acts effectively on  $M$  and the linear isotropy representation  $\pi$  of  $K$  is  $\mathbb{R}$ -irreducible. Suppose that  $G$  is not semisimple. Then either  $G$  is a circle group and  $K = \{1\}$ , so  $(G, K)$  is a*

symmetric pair with symmetry  $g \rightarrow g^{-1}$ ; or  $G$  is the semidirect product  $K \times_{\pi} \mathbf{R}^n$  with  $n = \dim M$  and  $(G, K)$  is a symmetric pair with symmetry  $(k, v) \rightarrow (k, v^{-1})$  [ $k \in K$  and  $v \in \mathbf{R}^n$ ].

*Proof.* Let  $\mathfrak{G}$  and  $\mathfrak{K}$  denote the Lie algebras of  $G$  and  $K$ , and let  $\mathfrak{S}$  be the radical of  $\mathfrak{G}$ . Let  $\mathfrak{A}$  denote the last nonzero term in the derived series of  $\mathfrak{S}$ . Then  $\mathfrak{A}$  is an ideal in  $\mathfrak{G}$ . We cannot have  $\mathfrak{A} \subset \mathfrak{K}$  because  $G$  acts effectively on  $M$ , so  $\mathfrak{K} \not\subseteq \mathfrak{K} + \mathfrak{A} \subset \mathfrak{G}$ . Now  $\mathbf{R}$ -irreducibility of  $\pi$  says  $\mathfrak{G} = \mathfrak{K} + \mathfrak{A}$ . As  $M = G/K$  is reductive we have an  $\text{ad}_G(K)$ -stable decomposition  $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$  with  $\mathfrak{M} \subset \mathfrak{A}$ , and  $\mathfrak{M}$  is an abelian Lie algebra because  $\mathfrak{A}$  is abelian. Now  $\mathfrak{G} = \mathfrak{K} + {}_{\pi}\mathfrak{R}^n$  semidirect sum where  $\mathfrak{R}^n = \mathfrak{M}$  is the Lie algebra of the real vector group of dimension  $n = \dim M$ .

Let  $V$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{M} = \mathfrak{R}^n$ . Then  $V = \mathbf{R}^n/D$ , quotient of a vector group by a discrete additive subgroup.  $\pi(K)$  acts on  $\mathbf{R}^n$  *qua*  $\mathfrak{R}^n$  and preserves  $D$ , and this linear action is irreducible. If  $n > 1$  it follows that  $D = \{1\}$ , so  $V$  is the vector group  $\mathbf{R}^n$ ; then  $K \cap V = \{1\}$  and  $G$  is the semidirect product  $K \times_{\pi} \mathbf{R}^n$  as asserted; in that case  $(k, v) \rightarrow (k, v^{-1})$  is an involutive automorphism of  $G$  with fixed point set  $K$  so  $(G, K)$  is a symmetric pair. If  $n = 1$  there is also the possibility that  $D \neq \{1\}$ . Then  $G = V$  is a circle group and  $K = \{1\}$ , and it is immediate that  $(G, K)$  is a symmetric pair under the involutive automorphism  $g \rightarrow g^{-1}$  of  $G$ , q.e.d.

The relevant special case of Lemma 1.2 is

(1.3) *Under conditions (i) through (v),  $G$  is semisimple.*

1.4. LEMMA. *Let  $M = G/K$  be a reductive coset space of connected real Lie groups such that  $G$  acts effectively on  $M$  and the linear isotropy representation of  $K$  is  $\mathbf{R}$ -irreducible. Suppose that  $G$  is semisimple but not simple. Then  $K$  is simple,  $G$  is locally isomorphic to  $K \times K$  with  $K$  embedded diagonally, and  $(G, K)$  is a symmetric pair with symmetry  $(k_1, k_2) \rightarrow (k_2, k_1)$  [ $k_i \in K$ ].*

*Proof.* We may divide out the center of  $G$ , assuming  $G = G_1 \times \dots \times G_r$  with  $G_i$  centerless and simple. Let  $\beta_i: G \rightarrow G_i$  denote the projection.

If  $\beta_i(K) \neq G_i$  for some index  $i$  then  $K \subset \beta_i^{-1}\beta_i K \subsetneq G$ , so  $K = \beta_i^{-1}\beta_i K$  by irreducibility of the linear isotropy representation.  $r > 1$  because  $G$  is not simple, so there is an index  $j \neq i$ ; then  $G_j \subset K$  so  $G$  is not effective on  $M$ . That contradiction shows that  $\beta_i(K) = G_i$  for every index  $i$ .

The Lie group  $K$  is reductive because the linear isotropy representation is faithful and fully reducible. Let  $K''$  be the kernel of  $\beta_1|_K$ ; now  $K = K' \cdot K''$  local direct product, and  $\beta_1: K' \cong G_1$ . In particular  $K'$  is simple. If we have an index  $i$  with  $\beta_i(K') \neq G_i$  then  $\beta_i(K') = \{1\}$  so  $G_i$  is in the centralizer of  $K'$ . But  $K''$  is the centralizer of  $K'$  and  $G_i \not\subset K$ .

Thus  $\beta_i: K' \cong G_i$  for every index  $i$ . That shows that each  $\beta_i(K'') = \{1\}$ , so  $K = K'$ ,  $K$  is simple and each  $\beta_i: K \cong G_i$ .

Identify  $G_i$  with  $K$  by  $\beta_i$ . Then  $G = K \times \dots \times K$  ( $r$  times) with  $K$  embedded diagonally.  $K = \{(g_1, \dots, g_r) \in G: g_1 = \dots = g_r\}$ . If  $r > 2$  then  $K \subsetneq KG_r \subsetneq G$ , contradicting irreducibility of the linear isotropy representation. As  $r > 1$  now  $G = K \times K$  with  $K$  embedded diagonally. Then  $(k_1, k_2) \rightarrow (k_2, k_1)$  is an involutive automorphism of  $G$  with fixed point set  $K$  and our assertions are proved, q.e.d.

In view of (1.3), the relevant special case of Lemma 1.4 is

(1.5) Under conditions (i) through (v),  $G$  is simple.

*Proof of Theorem 1.1.* (1.5) says that  $G$  is a simple Lie group. If  $G$  is noncompact we choose a maximal compactly embedded subalgebra  $\mathfrak{Q} \subsetneq \mathfrak{G}$  such that  $\mathfrak{K} \subset \mathfrak{Q}$ , and then  $\mathfrak{K} = \mathfrak{Q}$  by irreducibility of the linear isotropy representation; it follows that  $(G, K)$  is a symmetric pair. Thus  $G$  is compact, q.e.d.

The analysis of coset spaces  $M = G/K$  satisfying conditions (i) through (v) is now reduced to a specific problem on compact simple Lie groups.

**2. The case of equal ranks**

If  $\text{rank } G = \text{rank } K$  the result is

2.1. THEOREM. Let  $M = G/K$  be a coset space of compact connected Lie groups with  $G$  acting effectively and  $\text{rank } G = \text{rank } K$ . Let  $\chi$  be the linear isotropy representation of  $K$  on the tangent space of  $M$ . Then  $\chi$  is  $\mathbf{R}$ -irreducible if and only if, either  $M = G/K$  is an irreducible symmetric coset space, or the center of  $K$  is the cyclic group of order 3. In the latter case there are just six possibilities, as follows.

	$G$	$K$	$\chi$
1	$G_2$	$SU(3)$	$\begin{matrix} 1 & & 1 \\ \circ - \circ \oplus \circ - \circ \end{matrix}$
2	$F_4$	$SU(3) \cdot SU(3)$	$\begin{matrix} 1 & & 2 & & 1 & & 2 \\ (\circ - \circ \otimes \circ - \circ) \oplus (\circ - \circ \otimes \circ - \circ) \end{matrix}$
3	$E_6/Z_3$	$SU(3) \cdot SU(3) \cdot SU(3)$	$\begin{matrix} 1 & & 1 & & 1 & & 1 & & 1 \\ (\circ - \circ \otimes \circ - \circ \otimes \circ - \circ) \oplus (\circ - \circ \otimes \circ - \circ \otimes \circ - \circ) \end{matrix}$
4	$E_7/Z_2$	$[SU(3) \times SU(6)]/Z_6$	$\begin{matrix} 1 & & 1 & & 1 & & 1 \\ (\circ - \circ \otimes \circ - \circ - \circ - \circ - \circ) \oplus (\circ - \circ \otimes \circ - \circ - \circ - \circ - \circ) \end{matrix}$
5	$E_8$	$SU(9)/Z_3$	$\begin{matrix} 1 & & & & & & & & 1 \\ (\circ - \circ - \circ - \circ - \circ - \circ - \circ - \circ) \oplus (\circ - \circ - \circ - \circ - \circ - \circ - \circ - \circ) \end{matrix}$
6	$E_8$	$[SU(3) \times E_6]/Z_3$	$\begin{matrix} 1 & & 1 & & \circ & & 1 & & \circ & & 1 \\ (\circ - \circ \otimes \circ - \circ - \circ - \circ - \circ) \oplus (\circ - \circ \otimes \circ - \circ - \circ - \circ - \circ) \end{matrix}$

Theorem 2.1 can be checked directly by computing the linear isotropy representation for each of the nonsymmetric pairs  $(G, K)$  listed by Borel and de Siebenthal [4]. That calculation is extremely unpleasant without *a priori* knowledge of irreducibility of  $\chi$ , so we avoid the unpleasantness by using B. Kostant's result ([20], Theorem 8.13.3, p. 296):

2.2. THEOREM. *Let  $G$  be a connected reductive Lie group, let  $T$  be a Cartan subgroup, and let  $A$  be a subgroup of  $T$ . Let  $K$  be the identity component of the centralizer of  $A$  in  $G$ , let  $Z$  be the center of  $K$ , and suppose  $Z_0 \subset A$ . Let  $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$  be the orthogonal decomposition under the Killing form. Decompose  $\mathfrak{M}^{\mathbb{C}} = \sum \mathfrak{M}_i$  where  $\text{ad}_G(A)$  acts on  $\mathfrak{M}_i$  by a multiple of an irreducible complex representation  $\alpha_i$  and where the  $\alpha_i$  are distinct characters on  $A$ . Then  $\text{ad}_G(K)$  preserves  $\mathfrak{M}_i$ , acting there by an irreducible complex representation  $\pi_i$ , and the  $\pi_i$  are mutually inequivalent.*

*Proof.*  $\text{ad}(K)$  preserves  $\mathfrak{M}_i$  because  $K$  centralizes  $A$ . An equivalence  $\pi_i \sim \pi_j$  would restrict to an equivalence  $\alpha_i \sim \alpha_j$ , and imply  $\alpha_i = \alpha_j$ ; thus the  $\pi_i$  are mutually inequivalent. Now we need only prove that each  $\pi_i$  is irreducible.

Suppose  $\pi_i$  reducible and decompose  $\pi_i = \sum \beta_a$  with  $\beta_a$  irreducible. Then  $\mathfrak{M}_i = \sum \mathfrak{M}_{(a)}$  where  $\mathfrak{M}_{(a)}$  is the representation space of  $\beta_a$ . Order the  $\mathfrak{T}^{\mathbb{C}}$ -roots of  $\mathfrak{K}^{\mathbb{C}}$  and let  $\lambda_a$  denote the highest weight of  $\beta_a$ . Let  $a \neq b$  be indices. Then  $\lambda_a$  and  $\lambda_b$  coincide on  $\mathfrak{Z}^{\mathbb{C}}$  because  $Z_0 \subset A$ , so they differ only on the intersection of  $\mathfrak{T}^{\mathbb{C}}$  with the semisimple part of the reductive Lie algebra  $\mathfrak{K}^{\mathbb{C}}$ . There all highest weights are in the closure of the positive Weyl chamber, so  $\langle \lambda_a, \lambda_b \rangle \geq 0$ .

Suppose  $\langle \lambda_a, \lambda_b \rangle > 0$ . As  $\lambda_a$  and  $\lambda_b$  are  $\mathfrak{T}^{\mathbb{C}}$ -roots of  $\mathfrak{G}^{\mathbb{C}}$  it follows that  $\nu = \lambda_a - \lambda_b$  is a root. Choose nonzero root vectors  $E_\nu \in \mathfrak{G}_\nu$ ,  $E_a \in \mathfrak{G}_{\lambda_a}$  and  $E_b \in \mathfrak{G}_{\lambda_b}$ . Then  $[E_\nu, E_b] = cE_a$  for some  $c \neq 0$ . If  $g \in A$  then  $c \cdot \alpha_i(g) E_a = c \cdot \text{ad}(g) E_a = \text{ad}(g)[E_\nu, E_b] = [\text{ad}(g) E_\nu, \text{ad}(g) E_b] = [\text{ad}(g) E_\nu, \alpha_i(g) E_b]$ , so  $\text{ad}(g) E_\nu = E_\nu$ . Thus  $E_\nu \in \mathfrak{K}^{\mathbb{C}}$ . Now  $E_a \in \mathfrak{M}_{(a)}$  and  $E_a = c^{-1}[E_\nu, E_b] \in [\mathfrak{K}^{\mathbb{C}}, \mathfrak{M}_{(b)}] \subset \mathfrak{M}_{(b)}$  so  $a = b$ . In other words  $a \neq b$  implies  $\lambda_a \perp \lambda_b$ .

Decompose  $\mathfrak{K}^{\mathbb{C}} = \sum \mathfrak{K}_r$ , direct sum of its center  $\mathfrak{Z}^{\mathbb{C}}$  and its simple ideals. Then each  $\beta_a = \otimes \beta_{a,r}$  with  $\beta_{a,r}$  an irreducible complex representation of  $\mathfrak{K}_r$ , and each  $\lambda_a = \sum \lambda_{a,r}$  where  $\lambda_{a,r} \in \mathfrak{T}^{\mathbb{C}} \cap \mathfrak{K}_r$  is the highest weight of  $\beta_{a,r}$ . If  $a \neq b$  then  $\lambda_a \perp \lambda_b$  says, for each index  $r$ , that at most one of the  $\lambda_{a,r}$  can be nonzero. Now  $\mathfrak{K}^{\mathbb{C}} = \mathfrak{Q}_1^{\mathbb{C}} \oplus \mathfrak{Q}_2^{\mathbb{C}}$  and  $\mathfrak{K} = \mathfrak{Q}_1 \oplus \mathfrak{Q}_2$  where  $\mathfrak{Q}_1^{\mathbb{C}}$  is the sum of all  $\mathfrak{K}_r$  for which  $\lambda_{a,r} \neq 0$  and  $\mathfrak{Q}_2^{\mathbb{C}}$  is the sum of the remaining  $\mathfrak{K}_r$ . This decomposes  $\pi_i = (\tau_a \otimes 1) \oplus (1 \otimes \tau'_a)$  where  $\tau_a$  represents  $\mathfrak{Q}_1^{\mathbb{C}}$  and  $\tau'_a$  represents  $\mathfrak{Q}_2^{\mathbb{C}}$ . Thus  $\tau_a \otimes 1 = \beta_a$  and  $1 \otimes \tau'_a = \sum_{b \neq a} \beta_b$ . Let  $L_1$  and  $L_2$  be the analytic subgroups of  $K$  with respective Lie algebras  $\mathfrak{Q}_1$  and  $\mathfrak{Q}_2$ . Let  $Z_1$  and  $Z_2$  be their centers so  $Z = Z_1 \cdot Z_2$ ; let  $A_1$  and  $A_2$  denote the projections of  $A$  on  $Z_1$  and  $Z_2$  so  $A \subset A_1 \cdot A_2$ . Then  $1 \otimes \tau'_a$  annihilates  $A_1$  because it annihilates  $L_1$ . This forces  $\tau_a \otimes 1$  to annihilate  $A_1$  because it and  $1 \otimes \tau'_a$  both represent on  $\mathfrak{M}_i$ . Thus  $\pi_i$  annihilates

$A_1$ . Similarly  $\tau_a \otimes 1$ , thus also  $1 \otimes \tau'_a$ , thus  $\pi_i$ , annihilates  $A_2$ . Now  $\pi_i$  annihilates  $A \subset A_1 \cdot A_2$ , so  $\alpha_i = 1$ , which is absurd. This contradiction shows that we cannot have two distinct summands  $\beta_a$  and  $\beta_b$  of  $\pi_i$ . In other words,  $\pi_i$  is irreducible, q.e.d.

*Proof of Theorem 2.1.* Let  $Z$  be the center of  $K$  and let  $T$  be a maximal torus of  $G$  such that  $Z \subset T \subset K$ . The rank condition says ([4], Theoreme 5; or [20], Theorem 8.10.2, p. 276) that  $K$  is the identity component of the centralizer of  $Z$  in  $G$ . Assume  $\chi$  to be  $\mathbf{R}$ -irreducible. Then Schur's Lemma says that  $Z$  is a circle group or a cyclic group of some finite order  $m$ . If  $Z$  has an element of order 2 then  $M = G/K$  is an irreducible symmetric coset space. If not,  $Z$  is cyclic of odd finite order  $m > 1$ . Then we apply Theorem 2.2 with  $A = Z$  to obtain the decomposition  $\mathfrak{M}^c = \sum \mathfrak{M}_\varepsilon$  where  $\varepsilon$  runs through a set of  $m$ -th roots of 1 and where we have chosen a generator  $z$  of  $Z$  such that  $\alpha_\varepsilon(z) = \varepsilon$ . As  $z$  has order  $m$  we have a primitive  $m$ -th root  $\eta$  of 1 such that  $\mathfrak{M}_\eta \neq 0$ . Then  $\mathfrak{M}^c = \mathfrak{M}_\eta + \mathfrak{M}_{\bar{\eta}}$  by  $\mathbf{R}$ -irreducibility of  $\chi$ . If  $m > 3$  then  $[\mathfrak{M}_\eta, \mathfrak{M}_{\bar{\eta}}] \subset \mathfrak{R}^c$ ,  $[\mathfrak{M}_\eta, \mathfrak{M}_{\eta^2}] \subset \mathfrak{M}_{\eta^2} = 0$  and  $[\mathfrak{M}_{\bar{\eta}}, \mathfrak{M}_{\bar{\eta}^2}] \subset \mathfrak{M}_{\bar{\eta}^2} = 0$ ; that implies  $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{R}$  so that  $M = G/K$  is an irreducible symmetric coset space. In other words, if  $M = G/K$  is not an irreducible symmetric coset space then  $Z$  has order 3. Conversely if  $Z$  has order 3 then Theorem 2.2 shows that  $\chi$  is  $\mathbf{R}$ -irreducible.

Consider the case where the center  $Z$  of  $K$  has order 3. The classification of all such pairs  $(\mathfrak{G}, \mathfrak{R})$  is given by Borel and de Siebenthal [4] (or see [20], Theorem 8.10.9, p. 280).  $G$  is centerless, thus of the listed global form.  $\chi = \beta \oplus \bar{\beta}$  for some irreducible complex representation  $\beta$  of  $K$  such that  $\beta(K)$  has center of order 3 and  $\beta$  has degree  $\deg \beta = \frac{1}{2} \dim M = \frac{1}{2} [\dim G - \dim K]$ . In these low degrees there is no choice;  $\beta$  and  $\chi$  are as listed because there are no other possibilities. Now  $K$  has the listed global form because  $\beta$  is faithful, q.e.d.

### 3. The case where $G$ is exceptional and $\text{rank } G > \text{rank } K$

Here the classification is given by

**3.1 THEOREM** (E. B. Dynkin<sup>(1)</sup>). *The following is a complete list of the coset spaces  $G/K$  of compact connected Lie groups where (a)  $G$  acts effectively, (b)  $\text{rank } G > \text{rank } K$ , (c)  $G$  is an exceptional group and (d)  $K$  acts irreducibly on the tangent space.*

$E_6/A_2$  is the only one for which the isotropy representation is not absolutely irreducible.  $E_6/C_4$  and  $E_6/F_4$  are the only ones which are symmetric. In  $G_2/A_1$ , the  $A_1$  is the principal three dimensional subgroup.

The result follows from Theorem 14.1 of E. B. Dynkin's paper [7]. Dynkin writes  $\tilde{G}$  for our  $\mathfrak{R}^c$ ,  $G$  for our  $\mathfrak{G}^c$ ,  $\chi_{\tilde{G}}$  for the representation of  $\mathfrak{R}^c$  on the complexification of the tangent

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<sup>(1)</sup> As will be seen from the proof, the result is essentially due to Dynkin.



	$G$	$K$	Isotropy representation of $K$ on tangent space
1	$G_2$	$A_1$	10 ○
2	$F_4$	$A_1 \cdot G_2$	4 1 ○ ⊗ ● ≡ ○
3	$E_6$	$A_2$	1 4 4 1 ○ — ○ ⊕ ○ — ○
4	$E_6$	$G_2$	1 1 ● ≡ ○
5	$E_6$	$A_2 \cdot G_2$	1 1 1 ○ — ○ ⊗ ● ≡ ○
6	$E_6$	$C_4$	1 ● — ● — ● = ○
7	$E_6$	$F_4$	1 ● — ● = ○ — ○
8	$E_7$	$A_2$	4 4 ○ — ○
9	$E_7$	$G_2 \cdot C_3$	1 1 ● ≡ ○ ⊗ ● — ● = ○
10	$E_7$	$A_1 \cdot F_4$	2 1 ○ ⊗ ● — ● = ○ — ○
11	$E_8$	$G_2 \cdot F_4$	1 1 ● ≡ ○ ⊗ ● — ● = ○ — ○

space of  $G/K$ . Thus we are looking for Dynkin's classification of pairs  $(G, \tilde{G})$  consisting of a complex exceptional simple Lie algebra and a complex subalgebra  $\tilde{G}$  such that (a)  $\chi_{\tilde{G}}$  is absolutely irreducible or (b)  $\chi_{\tilde{G}} = \beta \oplus \tilde{\beta}$  where  $\tilde{\beta}$  is absolutely irreducible and has no nonzero symmetric bilinear invariant.  $\tilde{G} = \mathfrak{K}^C$  will be a semisimple  $S$ -subalgebra in Dynkin's terminology because it is a maximal subalgebra which has lower rank. Following ([7], Theorem 14.1) now, the pair  $(G, K)$  is listed in our theorem under the number

- 1 if rank  $K = 1$ ;
- 3, 4, 6, 7, 8 if rank  $K > 1$  and  $K$  is simple;
- 2, 5, 9, 10, 11 if  $K$  is not simple.

This completes the proof that  $G/K$  is one of the spaces that we have listed. On the other hand, all the listed pairs  $(\mathfrak{G}^C, \mathfrak{K}^C)$  exist, and given such a pair one can find a Cartan involution of  $\mathfrak{G}^C$  which preserves  $\mathfrak{K}^C$ ; then the pair  $(\mathfrak{G}, \mathfrak{K})$  consists of the respective fixed point sets, so  $G/K$  exists, q.e.d.

#### 4. The case where $G$ is classical and $K$ is not simple

The result is:

4.1. THEOREM. *The only nonsymmetric coset spaces  $G/K$  of compact connected Lie groups, where (a)  $G$  acts effectively, (b)  $\text{rank } G > \text{rank } K$ , (c)  $G$  is a classical group, (d)  $K$  is not simple, and (e)  $K$  acts  $\mathbf{R}$ -irreducibly on the tangent space, are the*

$$\mathbf{SU}(pq)/\mathbf{SU}(p) \times \mathbf{SU}(q), \quad p > 1, \quad q > 1, \quad pq > 4,$$

with the action of  $\mathbf{SU}(pq)$  rendered effective.

Here the inclusion is the tensor product of the usual linear representations of  $\mathbf{SU}(p)$  and  $\mathbf{SU}(q)$ , and the isotropy representation is the tensor product of the adjoint representations of  $\mathbf{SU}(p)$  and  $\mathbf{SU}(q)$ . Let  $m$  be the least common multiple of  $p$  and  $q$ . Then globally

$$G = \mathbf{SU}(pq)/\mathbf{Z}_m \quad \text{and} \quad K = \{\mathbf{SU}(p)/\mathbf{Z}_p\} \times \{\mathbf{SU}(q)/\mathbf{Z}_q\}.$$

For the proof we first need some remarks on linear groups. Here  $*$  denotes dual representation,  $\text{ad}_L$  denotes the adjoint representation of a Lie group  $L$ , and  $\mathbf{1}_L$  denotes the trivial representation of degree 1.

(4.2) *Let  $\delta: \mathbf{SL}(n, \mathbf{C}) \rightarrow \mathbf{GL}(n, \mathbf{C})$  denote the usual matrix representation of the complex special (determinant 1) linear group. Then  $\delta \otimes \delta^* = \mathbf{1}_{\mathbf{SL}(n, \mathbf{C})} \oplus \text{ad}_{\mathbf{SL}(n, \mathbf{C})}$ .*

For the Lie algebra  $\mathfrak{GL}(n, \mathbf{C})$  consists of all  $n \times n$  complex matrices, so  $\mathbf{SL}(n, \mathbf{C})$  acts on it by conjugation via  $\delta \otimes \delta^*$ . This action decomposes into the trivial action  $\mathbf{1}_{\mathbf{SL}(n, \mathbf{C})}$  on scalar matrices and the adjoint representation on matrices of trace zero.

(4.3) *Let  $\delta: \mathbf{Sp}(n, \mathbf{C}) \rightarrow \mathbf{GL}(2n, \mathbf{C})$  denote the usual matrix representation of the complex symplectic group. Then  $\text{ad}_{\mathbf{Sp}(n, \mathbf{C})} = S^2(\delta)$ , second symmetrization, which is the action on polynomials of degree 2.*

For  $\text{ad}_{\mathbf{Sp}(n, \mathbf{C})}$  is an irreducible summand of degree  $\dim \mathbf{Sp}(n, \mathbf{C}) = 2n^2 + n$  in the representation  $\delta \otimes \delta^* = \delta \otimes \delta$  on  $\mathfrak{GL}(2n, \mathbf{C})$ , hence contained in the representation on symmetric matrices or the representation on skew matrices. The latter has degree  $2n^2 - n$ , which excludes it. The former is  $S^2(\delta)$  and has degree  $2n^2 + n$ , which yields our assertion.

(4.4) *Let  $\delta: \mathbf{SO}(n, \mathbf{C}) \rightarrow \mathbf{GL}(n, \mathbf{C})$  denote the usual matrix representation of the complex special orthogonal group. Then  $\text{ad}_{\mathbf{SO}(n, \mathbf{C})} = \Lambda^2(\delta)$ , second alternation, which is the action on differential forms of degree 2.*

For  $\delta$  maps  $\mathfrak{SO}(n, \mathbf{C})$  onto the set of all antisymmetric  $n \times n$  complex matrices, and  $\otimes \delta \delta^* = \delta \otimes \delta$ .

(4.5) Let  $K_i$  be groups, let  $F$  be a field of characteristic  $\neq 2$ , and let  $\alpha_i: K_i \rightarrow \mathbf{GL}(n_i, F)$  be linear representations. Then

$$\begin{aligned}\Lambda^2(\alpha_1 \otimes \alpha_2) &= \{S^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes S^2(\alpha_2)\} \quad \text{on } K_1 \times K_2; \\ S^2(\alpha_1 \otimes \alpha_2) &= \{S^2(\alpha_1) \otimes S^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \quad \text{on } K_1 \times K_2.\end{aligned}$$

For let  $f$  and  $g$  be bilinear forms. Then so is  $f \otimes g$ . If  $f$  and  $g$  are both symmetric or both antisymmetric, one checks that  $f \otimes g$  is symmetric. If one of  $\{f, g\}$  is symmetric and the other is antisymmetric, one checks that  $f \otimes g$  is antisymmetric. Now the assertion follows from the decomposition  $\Lambda^2(\alpha_1 \otimes \alpha_2) + S^2(\alpha_1 \otimes \alpha_2) = (\alpha_1 \otimes \alpha_2) \otimes (\alpha_1 \otimes \alpha_2) = (\alpha_1 \otimes \alpha_1) \otimes (\alpha_2 \otimes \alpha_2) = \{\Lambda^2(\alpha_1) \oplus S^2(\alpha_1)\} \otimes \{\Lambda^2(\alpha_2) \oplus S^2(\alpha_2)\}$ .

*Proof of Theorem 4.1.* According to Dynkin ([6], Theorems 1.3 and 1.4),  $K^C \subset G^C$  is one of the inclusions

- (1)  $\mathbf{SL}(p_1, \mathbf{C}) \otimes \mathbf{SL}(p_2, \mathbf{C}) \subset \mathbf{SL}(p_1 p_2, \mathbf{C})$ ,
- (2)  $\mathbf{Sp}(p_1, \mathbf{C}) \otimes \mathbf{SO}(p_2, \mathbf{C}) \subset \mathbf{Sp}(p_1 p_2, \mathbf{C})$ ,
- (3)  $\mathbf{Sp}(p_1, \mathbf{C}) \otimes \mathbf{Sp}(p_2, \mathbf{C}) \subset \mathbf{SO}(4p_1 p_2, \mathbf{C})$ ,
- (4)  $\mathbf{SO}(p_1, \mathbf{C}) \otimes \mathbf{SO}(p_2, \mathbf{C}) \subset \mathbf{SO}(p_1 p_2, \mathbf{C})$ .

Here  $K = K_1 \cdot K_2$  local direct product,  $K_i^C$  is a complex simple classical group with usual linear representation  $\alpha_i: K_i^C \rightarrow \mathbf{GL}(n_i, \mathbf{C})$  and  $K_1^C \otimes K_2^C$  just denotes  $(\alpha_1 \otimes \alpha_2)(K_1^C \otimes K_2^C)$ . The cases are (1)  $n_i = p_i$ , (2)  $n_1 = 2p_1$  and  $n_2 = p_2$ , (3)  $n_i = 2p_i$ , (4)  $n_i = p_i$ .

Let  $\pi$  be the representation of  $K$  on the tangent space of  $G/K$ . Then the representation  $\psi$  of  $K$  on  $\mathfrak{G}$  decomposes as  $\psi = \text{ad}_K \oplus \pi = \{\text{ad}_{K_1} \otimes \mathbf{1}_{K_2}\} \oplus \{\mathbf{1}_{K_1} \otimes \text{ad}_{K_2}\} \oplus \pi$ . Now we check the four cases.

Case (1). Using (4.2) we have  $\mathbf{1} \oplus \psi = (\alpha_1 \otimes \alpha_2) \otimes (\alpha_1 \otimes \alpha_2)^* = (\alpha_1 \otimes \alpha_1^*) \otimes (\alpha_2 \otimes \alpha_2^*) = (\mathbf{1}_{K_1} \oplus \text{ad}_{K_1}) \otimes (\mathbf{1}_{K_2} \oplus \text{ad}_{K_2}) = \mathbf{1}_{K_1 \times K_2} \oplus \{\text{ad}_{K_1} \otimes \mathbf{1}_{K_2}\} \oplus \{\mathbf{1}_{K_1} \otimes \text{ad}_{K_2}\} \oplus \{\text{ad}_{K_1} \otimes \text{ad}_{K_2}\}$ . Thus  $\pi = \text{ad}_{K_1} \otimes \text{ad}_{K_2}$ , absolutely irreducible. This is the case of the theorem.

Case (2). Using (4.3), (4.4) and (4.5), we have  $\psi = S^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes S^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} = \{\text{ad}_{K_1} \otimes [\mathbf{1}_{K_2} \oplus \eta_2]\} \oplus \{[\mathbf{1}_{K_1} \oplus \eta_1] \otimes \text{ad}_{K_2}\} = \{\text{ad}_{K_1} \otimes \mathbf{1}_{K_2}\} \oplus \{\mathbf{1}_{K_1} \otimes \text{ad}_{K_2}\} \oplus \{\text{ad}_{K_1} \otimes \eta_2\} \oplus \{\eta_1 \otimes \text{ad}_{K_2}\}$  for some representations  $\eta_i$  of  $K_i$ . Thus  $\pi = \sigma \oplus \tau$  where  $\sigma = \text{ad}_{K_1} \otimes \eta_2$  and  $\tau = \eta_1 \otimes \text{ad}_{K_2}$ . As  $\pi$  is irreducible over  $\mathbf{R}$  we must have that (a)  $\tau = \sigma^*$  and (b)  $\sigma$  has no symmetric bilinear invariant. But (a) says  $\eta_2 = \text{ad}_{K_2}$ , which violates (b). Thus our case (2) is excluded.

Case (3). Using (4.3), (4.4) and (4.5), we have  $\psi = \Lambda^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes S^2(\alpha_2)\} = \{\text{ad}_{K_1} \otimes [\mathbf{1}_{K_2} \oplus \eta_2]\} \oplus \{[\mathbf{1}_{K_1} \oplus \eta_1] \otimes \text{ad}_{K_2}\}$ . As in case (2), this violates irreducibility of  $\pi$  over  $\mathbf{R}$ ; thus case (3) is excluded.

Case (4) is also excluded by the argument used for case (2).

Finally, back in the admissible case (1), we have  $p_i > 1$  so that  $K_i^{\mathbb{C}} = \mathbf{SL}(p_i, \mathbf{C})$  is semi-simple, and we have  $p_1 p_2 > 4$  because  $\mathbf{SU}(4)/\mathbf{SU}(2) \otimes \mathbf{SU}(2) = \mathbf{SU}(4)/\mathbf{SO}(4)$ , which is symmetric, q.e.d.

### 5. A problem in representation theory

Our classification problem for coset spaces  $G/K$  satisfying conditions (i)–(v), is now reduced to the case where  $G$  is a compact simple classical group and  $K$  is simple. On the Lie algebra level,  $\mathfrak{G}$  is  $\mathfrak{SU}(N)$ ,  $\mathfrak{Sp}(N)$  or  $\mathfrak{SO}(N)$  for some integer  $N$ , and we view the inclusion  $\mathfrak{K} \rightarrow \mathfrak{G}$  as a linear representation  $\pi$ . If  $\pi$  is not absolutely irreducible it has image in a direct sum  $\mathfrak{Q} = \mathfrak{Q}_1 \oplus \mathfrak{Q}_2$  of Lie algebras of classical groups, and our simplicity conditions give  $\mathfrak{K} \subsetneq \mathfrak{Q} \subsetneq \mathfrak{G}$ , contradicting irreducibility of  $K$  on the tangent space. Now  $\pi$  is absolutely irreducible, so it has a highest weight  $\lambda$ ; thus  $\pi = \pi_\lambda$ . Let  $\chi$  denote the representation of  $\mathfrak{K}$  on the tangent space of  $G/K$ , so  $\text{ad}_G \circ \pi_\lambda = \text{ad}_K \oplus \chi$ . We must express  $\mathbf{R}$ -irreducibility of  $\chi$  in terms of  $\lambda$ .

Let  $l$  be the rank of  $K$ . The choice of highest weight  $\lambda$  implied a choice of maximal torus  $T \subset K$  and the choice of a system  $\{\alpha_1, \dots, \alpha_l\}$  of simple  $\mathfrak{Q}^{\mathbb{C}}$ -roots of  $\mathfrak{K}^{\mathbb{C}}$ . Let  $\xi_r$  denote the linear form on  $\mathfrak{Q}^{\mathbb{C}}$  specified by the conditions

$$\frac{2\langle \xi_r, \alpha_r \rangle}{\langle \alpha_r, \alpha_r \rangle} = 1, \quad \frac{2\langle \xi_r, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 0 \quad \text{for } i \neq r.$$

Then the highest weights of absolutely irreducible representations of  $\mathfrak{K}$  are just the linear forms  $\eta = \sum n_i \xi_i$ ,  $n_i$  integers,  $n_i \geq 0$ . The representation of highest weight  $\eta$  is denoted  $\pi_\eta$ . The weights and representations  $\xi_i$  and  $\pi_{\xi_i}$  are called *basic*. The representation dual to  $\pi_\eta$ , which we denote  $\pi_\eta^*$ , has highest weight which we denote  $\eta^*$ . Note that  $(\sum n_i \xi_i)^* = \sum n_i \xi_i^*$ .

**5.1. PROPOSITION.** *If  $G = \mathbf{SU}(N)$ , then*

- (1)  $\lambda = k\xi_r$ , for some integer  $k \geq 1$  and some basic weight  $\xi_r \neq \xi_r^*$ ,
- (2)  $\chi = \pi_{\lambda+\lambda^*}$ , absolutely irreducible, and
- (3)  $N = \deg \pi_\lambda$  satisfies  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim K + 1$

*Proof.* If  $\lambda = \lambda^*$  then  $\pi_\lambda$  maps  $K$  into a subgroup  $L = \mathbf{SO}(N)$  or  $\mathbf{Sp}(\frac{1}{2}N)$  of  $G$ .  $K$  does not map onto the subgroup because  $G/K$  is not symmetric, so  $K \subsetneq L \subsetneq G$ . That violates irreducibility. Thus  $\lambda \neq \lambda^*$ .

(4.2) says  $\pi_\lambda \otimes \pi_{\lambda^*} = \mathbf{1}_K \oplus \text{ad}_K \oplus \chi$ .  $\pi_{\lambda+\lambda^*}$  is a summand of  $\pi_\lambda \otimes \pi_{\lambda^*}$ , hence of  $\text{ad}_K$  or of  $\chi$ . In the former case  $\pi_{\lambda+\lambda^*} = \text{ad}_K$ . Let  $\mu$  be the highest root so that  $\text{ad}_K = \pi_\mu$ . Now  $\mu = \lambda + \lambda^*$ . As  $\lambda \neq \lambda^*$ , this says  $\mu = \sum n_i \xi_i$  with at least two of the  $n_i$  nonzero. The only case is where

$\lambda + \lambda^* = \mu$ :  $\overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{1}{\circ}$ . Then  $K = \mathrm{SU}(l+1)$ ,  $\lambda$  is  $\xi_1$  or  $\xi_i$ , and  $\pi_\lambda(K) = G$ . But  $K \neq G$ .

Thus  $\pi_{\lambda+\lambda^*}$  is a summand of  $\chi$ . As  $\chi$  is  $\mathbf{R}$ -irreducible and  $\pi_{\lambda+\lambda^*}$  is real, now  $\chi = \pi_{\lambda+\lambda^*}$ , absolutely irreducible. Thus (2) is proved. (3) follows by taking degrees.

Decompose  $\lambda = \sum n_i \xi_i$  and let  $m$  be the number of indices  $i$  with  $n_i > 0$ . Then  $m$  is the multiplicity of  $\mathrm{ad}_K$  in  $\pi_\lambda \otimes \pi_{\lambda^*}$ . We have just seen that  $\mathrm{ad}_K$  has multiplicity zero in  $\chi$ . Thus  $m = 1$ , so  $\lambda$  has form  $k\xi_r$ , q.e.d.

The orthogonal case is more delicate:

5.2 PROPOSITION. *If  $G = \mathbf{SO}(N)$ , then there are three cases:*

(a)  $\lambda = k\xi_r$  for some basic weight  $\xi_r = \xi_r^*$ , and  $\chi = \pi_{2\lambda - \alpha_r}$ , absolutely irreducible;

(b)  $\lambda = k(\xi_r + \xi_r^*)$  for some basic weight  $\xi_r \neq \xi_r^*$ , and  $\chi = \pi_{2\lambda - \alpha_r} \oplus \pi_{2\lambda - \alpha_r^*}$ , not absolutely irreducible;

(c)  $K = \mathbf{G}_2$  and  $G = \mathbf{SO}(7)$ ,  $\pi_\lambda = \chi = \pi_{\xi_1}$ ,  $\overset{1}{\bullet} \equiv \overset{1}{\circ}$ , and  $\pi_{2\lambda - \alpha_1} = \mathrm{ad}_K = \pi_{\xi_2}$ ,  $\overset{1}{\bullet} \equiv \overset{1}{\circ}$ .

In all cases,  $\lambda = \lambda^*$  with  $\pi_\lambda$  real, and  $N = \deg \pi_\lambda$  satisfies  $\frac{1}{2}(\deg \pi_\lambda)^2 = \deg \chi + \frac{1}{2} \deg \pi_\lambda + \dim K$ .

*Proof.* By (4.4),  $\Lambda^2(\pi_\lambda) = \mathrm{ad}_K \oplus \chi$ . As  $\pi_\lambda$  is orthogonal, this proves the last statement. Now  $\chi = \beta_1 \oplus \dots \oplus \beta_p$  with  $\beta_i$  absolutely irreducible. As  $\chi$  is  $\mathbf{R}$ -irreducible, either  $p = 1$ , or  $p = 2$  with  $\beta_1 \neq \beta_2 = \beta_1^*$ , or  $p = 2$  with  $\beta_1 = \beta_1^* = \beta_2$  symplectic. Let  $\alpha_i$  be a simple root not orthogonal to  $\lambda$ . Let  $V$  be the representation space of  $\pi_\lambda$  and choose weight vectors  $v_\lambda, v_{\lambda - \alpha_i}$ . Then  $v_{\lambda - \alpha_i} \wedge v_\lambda \in \Lambda^2(V)$  is a weight vector of weight  $2\lambda - \alpha_i$  for  $\Lambda^2(\pi_\lambda)$  which is annihilated by every positive root space of  $\mathfrak{K}^C$ , so  $\pi_{2\lambda - \alpha_i}$  is a summand of  $\Lambda^2(\pi_\lambda)$ .

If  $\pi_{2\lambda - \alpha_i} = \mathrm{ad}_K$  then  $\alpha_i$  is a terminal vertex on the Dynkin diagram of  $\mathfrak{K}^C$ . For otherwise we have two different simple roots  $\alpha', \alpha''$  not orthogonal to  $\alpha_i$ , so the highest root  $\mu = 2\lambda - \alpha_i$  is not orthogonal to  $\alpha'$  nor to  $\alpha''$ . That implies  $\mathfrak{K}^C$  of type  $A_l$ , and then  $\alpha'$  and  $\alpha''$  are terminal so  $l = 3$ . Thus  $\lambda$ :  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$  so  $\pi_\lambda$  maps  $\mathfrak{K} = \mathfrak{SU}(4)$  isomorphically onto  $\mathfrak{G} = \mathfrak{SO}(6)$ . As  $\mathfrak{K} \neq \mathfrak{G}$  this is impossible. Thus  $\alpha_i$  is terminal.

Let  $\pi_{2\lambda - \alpha_i} = \mathrm{ad}_K$ . Now  $\alpha_i$  is terminal; let  $\alpha'$  be the unique simple root not orthogonal to it. Then we have

$$\lambda: \begin{array}{c} n_i \qquad n' \\ \circ \text{---} \circ \text{---} \circ \\ \alpha_i \qquad \alpha' \end{array} \quad \text{and} \quad \mu = 2\lambda - \alpha_i: \begin{array}{c} 2n_i - 2 \quad n' + e \\ \circ \text{---} \circ \text{---} \circ \\ \alpha_i \qquad \alpha' \end{array}$$

where  $e$  is 1, 2 or 3. If  $\alpha'$  is not terminal, then  $\mu: \overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\left(-\overset{1}{\circ}=\bullet \text{ or } -\overset{1}{\circ}\begin{array}{c} \circ \\ \circ \end{array}\right)$ ,

so  $K$  is an orthogonal group and  $\lambda: \overset{1}{\circ}-\overset{1}{\circ}-\dots$ . But then  $\pi_\lambda(K)=G$ , which is excluded. Thus  $\alpha'$  is terminal. In other words,  $K$  has rank 2 with simple roots  $\alpha_i$  and  $\alpha'$ . The possibilities are

$$(a) \quad \begin{array}{c} \circ-\circ \\ \alpha_i \quad \alpha' \end{array} \quad (b) \quad \begin{array}{c} \circ=\bullet \\ \alpha_i \quad \alpha' \\ \bullet=\circ \end{array} \quad (c) \quad \begin{array}{c} \circ\equiv\bullet \\ \alpha_i \quad \alpha' \\ \bullet\equiv\circ \end{array}$$

In case (a),  $\mu: \overset{1}{\circ}-\overset{1}{\circ}$ , so  $2n_i-2$  is odd. In case (b),  $\mu: \overset{1}{\circ}=\bullet$ , so  $\lambda: \overset{1}{\circ}=\bullet$  and  $K=\mathbf{SO}(5)=G$ . In case (c),  $\mu: \overset{1}{\circ}\equiv\bullet$ , so  $\lambda: \overset{1}{\circ}\equiv\bullet$ ; then we are in alternative (c) of the proposition.

Now we may assume  $\pi_{2\lambda-\alpha_i} \neq \text{ad}_K$  for every simple root not orthogonal to  $\lambda$ . Let  $\lambda=k\xi_r$  for some  $r$ . Then  $\lambda=\lambda^*$  says  $\xi_r=\xi_r^*$ . Now  $\pi_{2\lambda-\alpha_r}$  is a summand of  $\chi$ . If they are not equal then  $\chi=\pi_{2\lambda-\alpha_r} \oplus (\pi_{2\lambda-\alpha_r})^* = \pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_r}$ , so  $2\lambda-\alpha_r$  has multiplicity  $\geq 2$  in  $\Lambda^2(\pi_\lambda)$ . That being impossible, now  $\chi=\pi_{2\lambda-\alpha_r}$ . Now suppose  $\lambda$  not of the form  $k\xi_r$ . Then we have  $\lambda=k\xi_r+t\xi_s$  and  $\chi=\pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_s}$ . The summands of  $\chi$  must be dual, so  $\xi_s=\xi_r^*$ . They must be distinct because  $2\lambda-\alpha_r$  has multiplicity 1 in  $\Lambda^2(\pi_\lambda)$ ; so  $\xi_r \neq \xi_r^*$ . Now  $\lambda=\lambda^*$  says  $\lambda=k(\xi_r+\xi_r^*)$ , and we have  $\chi=\pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_r^*}$ . q.e.d.

The symplectic case is more delicate:

**5.3 PROPOSITION.** *If  $G=\mathbf{Sp}(N)$ , then*

- (1)  $\lambda=k\xi_r$  for some basic weight  $\xi_r=\xi_r^*$ , and  $\pi_\lambda$  is not real on  $K$ ;
- (2)  $\chi=\pi_{2\lambda}$ , absolutely irreducible; and
- (3)  $2N=\text{deg } \pi_\lambda$  satisfies  $\frac{1}{2}\{(\text{deg } \pi_\lambda)^2+\text{deg } \pi_\lambda\}=\text{deg } \pi_{2\lambda}+\dim K$ .

*Proof.* (4.3) says  $S^2(\pi_\lambda)=\text{ad}_K \oplus \chi$ , so  $\pi_{2\lambda}$  is a summand of  $\text{ad}_K$  or of  $\chi$ . If  $\pi_{2\lambda}=\text{ad}_K$  then the highest root  $\mu=2\lambda$ , so there is a simple root  $\alpha_i$  with  $\mu: \overset{n_i}{\sim}\circ\sim$  and  $n_i \geq 2$ . That occurs only for  $K=\mathbf{Sp}(l)$ , and then  $\mu: \overset{2}{\bullet}-\dots-\bullet=\circ$ , so  $\lambda: \overset{1}{\bullet}-\dots-\bullet=\circ$  and  $\pi_\lambda(K)=G$ . That is excluded. Now  $\pi_{2\lambda}$  is a summand of  $\chi$ . As  $\pi_{2\lambda}$  is real and  $\chi$  is  $\mathbf{R}$ -irreducible, this shows  $\chi=\pi_{2\lambda}$  absolutely irreducible.

Suppose that  $\lambda$  is not a multiple of a basic weight. Then we have distinct simple roots  $\alpha'$  and  $\alpha''$  not orthogonal to  $\lambda$ . Let  $V$  be the representation space of  $\pi_\lambda$ ; choose nonzero weight vectors  $u \in V_\lambda$ ,  $v \in V_{\lambda-\alpha'}$  and  $w \in V_{\lambda-\alpha''}$ ; let  $\{x_1, \dots, x_t\}$  be a basis of  $V_{\lambda-\alpha'-\alpha''}$ . Let  $Y$  denote the weight space of weight  $2\lambda-\alpha'-\alpha''$  for  $\pi_\lambda \otimes \pi_\lambda$  on  $V \otimes V$ ; now  $Y \cap \Lambda^2(V)$  has basis  $\{v \wedge w; u \wedge x_1, \dots, u \wedge x_t\}$ , so it has dimension  $t+1$ . B. Kostant's method for decomposing a tensor product shows that  $\pi_{2\lambda-\alpha'-\alpha''}$  is a summand of multiplicity  $t$  in  $\pi_\lambda \otimes \pi_\lambda$ .

Its multiplicity in  $\Lambda^2(\pi_\lambda)$  is at most  $l-1$  because each of the subrepresentations  $\pi_{2\lambda-\alpha'}$  and  $\pi_{2\lambda-\alpha''}$  of  $\Lambda^2(\pi_\lambda)$  has  $2\lambda-\alpha'-\alpha''$  for a weight. Now  $\pi_{2\lambda-\alpha'-\alpha''}$  is a subrepresentation of  $S^2(\pi_\lambda) = \pi_{2\lambda} + \text{ad}_K$ . This shows that  $\text{ad}_K = \pi_{2\lambda-\alpha'-\alpha''}$ .

Now the highest root  $\mu = 2\lambda - \alpha' - \alpha''$ . If  $\alpha$  is a simple root adjacent to  $\alpha'$  or  $\alpha''$  in the Dynkin diagram of  $\mathfrak{K}^C$ , it follows that  $\mu$  is not orthogonal to  $\alpha$ , so  $-\mu$  is joined to  $\alpha$  in the extended Dynkin diagram. If  $K$  is not of type  $A_l$ , then there is a unique simple root  $\alpha_0$  joined to  $-\mu$  in the extended diagram.  $\alpha_0$  is the only simple root adjacent to  $\alpha'$  or  $\alpha''$ ; now it is adjacent to both, so it is interior to the diagram and satisfies  $2\langle\mu, \alpha_0\rangle/\langle\alpha_0, \alpha_0\rangle \geq 2$ .

Those two properties contradict each other; thus  $K$  is of type  $A_l$  and  $\mu: \overset{1}{\circ} - \dots - \overset{1}{\circ}$ . As  $\mu = 2\lambda - \alpha' - \alpha''$ , now  $l=2$  and  $\mu: \overset{1}{\circ} - \overset{1}{\circ}$ . Thus  $2\lambda = \mu + \alpha' + \alpha'': \overset{2}{\circ} - \overset{2}{\circ}$ . Now  $\lambda: \overset{1}{\circ} - \overset{1}{\circ}$  so  $\pi_\lambda$  is orthogonal. That is absurd. We have proved that  $\lambda$  is a multiple  $k\xi_r$  of a basic weight.

$\xi_r = \xi_r^*$  because  $\lambda = \lambda^*$ , and (3) comes from  $S^2(\pi_\lambda) = \pi_{2\lambda} \oplus \text{ad}_K$  by taking degrees, q.e.d.

Propositions 5.1, 5.2 and 5.3 do several things. They identify  $\chi$  in terms of  $\lambda$ , giving a formula for  $\deg \pi_\lambda$ . And they limit the possibilities for  $\lambda$ .

Recall the H. Weyl degree formula:

$$\deg \pi_\nu = \prod_{\alpha > 0} \frac{\langle \nu + g, \alpha \rangle}{\langle g, \alpha \rangle}, \text{ where } g = \frac{1}{2} \sum_{\alpha > 0} \alpha. \quad (5.4)$$

We need a modification involving some new notation. We have the system of simple roots  $\{\alpha_1, \dots, \alpha_l\}$ . Given any positive root  $\alpha$ , there is a unique expression  $\alpha = \sum a_i \alpha_i$  where  $a_i \geq 0$  are integers. Recall the level  $l(\alpha) = \sum a_i$ . Now define

$$\hat{a}_i = a_i \|\alpha_i\|^2; \quad \hat{l}(\alpha) = \sum \hat{a}_i, \text{ modified level.} \quad (5.5)$$

We calculate for  $\nu = \sum n_i \xi_i$ :

$$\begin{aligned} 2\langle \nu, \alpha \rangle &= \sum (2\langle \nu, \alpha_i \rangle) a_i = \sum \frac{2\langle \nu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \cdot a_i \|\alpha_i\|^2 = \sum n_i \hat{a}_i, \\ 2\langle g, \alpha \rangle &= \sum (2\langle g, \alpha_i \rangle) a_i = \sum \frac{2\langle g, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \cdot a_i \|\alpha_i\|^2 = \sum \hat{a}_i = \hat{l}(\alpha), \\ \frac{\langle \nu + g, \alpha \rangle}{\langle g, \alpha \rangle} &= \frac{\langle \nu, \alpha \rangle}{\langle g, \alpha \rangle} + 1 = \frac{\sum n_i \hat{a}_i}{\hat{l}(\alpha)} + 1 = \frac{\hat{l}(\alpha) + \sum n_i \hat{a}_i}{\hat{l}(\alpha)}. \end{aligned}$$

Substituting back into (5.4) we now have

$$\deg \pi_\nu = \prod_{\alpha > 0} \frac{\hat{l}(\alpha) + \sum n_i \hat{a}_i}{\hat{l}(\alpha)} \text{ where } \nu = \sum n_i \xi_i. \tag{5.6}$$

The next five sections consist of applying (5.6) to Propositions 5.1, 5.2 and 5.3, obtaining the classification of pairs  $(G, K)$  with  $G$  classical and  $K$  simple.

**6. The case where  $G$  is unitary and  $K$  is simple**

The result is:

6.1. THEOREM. Let  $G$  be a special unitary group and let  $K$  be a compact connected non-symmetric subgroup. Let  $\chi$  denote the representation of  $K$  on the tangent space of  $G/K$ . Then  $\chi$  is irreducible over the real number field, if and only if  $(G, K)$  is one of the following

$G$	$K$	$\pi_\lambda$	$\chi$
$SU\left(\frac{n(n-1)}{2}\right)$	$SU(n)$ ( $n \geq 5$ )	$\overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{1}{\circ}$	$\overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{1}{\circ} - \overset{1}{\circ}$
$SU\left(\frac{n(n+1)}{2}\right)$	$SU(n)$ ( $n \geq 3$ )	$\overset{2}{\circ} - \overset{2}{\circ} - \dots - \overset{2}{\circ}$	$\overset{2}{\circ} - \overset{2}{\circ} - \dots - \overset{2}{\circ} - \overset{2}{\circ}$
$SU(27)$	$E_6$	$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$   $\overset{1}{\circ}$	$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$   $\overset{1}{\circ}$
$SU(16)$	$Spin(10)$	$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{matrix} \nearrow \overset{1}{\circ} \\ \searrow \overset{1}{\circ} \end{matrix}$	$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{matrix} \nearrow \overset{1}{\circ} \\ \searrow \overset{1}{\circ} \end{matrix}$

where the inclusion  $K \rightarrow G$  is the absolutely irreducible representation  $\pi_\lambda$  of highest weight  $\lambda$ . In each case  $\chi = \pi_{\lambda + \lambda^*}$ , absolutely irreducible.

In each of the cases listed,  $\chi$  is irreducible.

Now assume  $\chi$  irreducible. Proposition 5.1 says that the inclusion  $K \rightarrow G$  is an absolutely irreducible representation  $\pi_{k\xi_r}$  for some basic weight  $\xi_r \neq \xi_r^*$ , and that

$$(\deg \pi_{k\xi_r})^2 = \deg \pi_{k(\xi_r + \xi_r^*)} + \dim K + 1. \tag{6.2}$$

To compute these degree we denote sets of positive roots by

$$P_r = \{\alpha = \sum a_i \alpha_i > 0 : a_r \neq 0 = a_{r^*}\} \text{ and } S_r = \{\alpha = \sum a_i \alpha_i > 0 : a_r \neq 0 \neq a_{r^*}\},$$

where  $r^*$  is the integer  $t$  such that  $\alpha_r^* = \alpha_t$ . Now define

$$p_{r,k} = \prod_{P_r} \frac{\hat{l}(\alpha) + k\hat{a}_r}{\hat{l}(\alpha)}, \quad s_{r,k} = \prod_{S_r} \frac{\hat{l}(\alpha) + k\hat{a}_r}{\hat{l}(\alpha)}, \quad t_{r,k} = \prod_{S_r} \frac{\hat{l}(\alpha) + k\hat{a}_r + k\hat{a}_{r^*}}{\hat{l}(\alpha)}. \tag{6.3}$$



We compute from (5.6)

$$\deg \pi_{k\xi_r} = \prod_{\alpha>0} \frac{\hat{l}(\alpha) + k\hat{a}_r}{\hat{l}(\alpha)} = p_{r,k} \cdot s_{r,k},$$

$$\deg \pi_{k(\xi_r + \xi_r^*)} = \prod_{\alpha>0} \frac{\hat{l}(\alpha) + k\hat{a}_r + k\hat{a}_{r^*}}{\hat{l}(\alpha)} = p_{r,k}^2 \cdot t_{r,k}.$$

Now (6.2) becomes 
$$p_{r,k}^2 \{s_{r,k}^2 - t_{r,k}\} = 1 + \dim K. \quad (6.4)$$

To limit  $k$ , we need a growth estimate:

6.5. LEMMA. *If  $1 \leq h < k$ , then  $p_{r,h}^2 \{s_{r,h}^2 - t_{r,h}\} < p_{r,k}^2 \{s_{r,k}^2 - t_{r,k}\}$ .*

*Proof.* As  $p_{r,h}^2 < p_{r,k}^2$  visibly, it suffices to show that  $0 \leq s_{r,h}^2 - t_{r,h} \leq s_{r,k}^2 - t_{r,k}$ . From a glance at (6.3) we see that  $s_{r,x}$  and  $t_{r,x}$  are smooth functions of  $x$  for  $x > 0$ . Thus we need only prove  $d/dx(s_{r,x}^2 - t_{r,x}) \geq 0$  for  $x \geq 1$ . For every root  $\beta = \sum b_j \alpha_j \in S_r$  we define

$$s_\beta(x) = 2 \hat{b}_r \left\{ \frac{x\hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)^2} \right\} \cdot \prod_{S_r - \beta} \frac{x^2 \hat{a}_r^2 + 2x\hat{a}_r \hat{l}(\alpha) + \hat{l}(\alpha)^2}{\hat{l}(\alpha)^2}$$

$$t_\beta(x) = \frac{\hat{b}_r + \hat{b}_{r^*}}{\hat{l}(\beta)} \cdot \prod_{S_r - \beta} \frac{\hat{l}(\alpha) + x\hat{a}_r + x\hat{a}_{r^*}}{\hat{l}(\alpha)}$$

so that  $d/dx(s_{r,x}^2) = \sum_{S_r} s_\beta(x)$  and  $d/dx(t_{r,x}) = \sum_{S_r} t_\beta(x)$ . Now we must prove  $\sum_{S_r} s_\beta(x) \geq \sum_{S_r} t_\beta(x)$ . For this it suffices to prove:

- (A) if  $\beta = \beta^*$ , then  $s_\beta(x) \geq t_\beta(x)$  for  $x \geq 1$ ;
- (B) if  $\beta \neq \beta^*$ , then  $s_\beta(x) + s_{\beta^*}(x) \geq t_\beta(x) + t_{\beta^*}(x)$  for  $x \geq 1$ .

To prove (A) and (B) we first observe that

$$\frac{x^2 \hat{a}_r^2 + 2x\hat{a}_r \hat{l}(\alpha) + \hat{l}(\alpha)^2}{\hat{l}(\alpha)^2} \geq \frac{x\hat{a}_r + x\hat{a}_{r^*} + \hat{l}(\alpha)}{\hat{l}(\alpha)} \quad \text{if } \alpha = \alpha^*, \quad (6.6)$$

$$\frac{x^2 \hat{a}_r^2 + 2x\hat{a}_r \hat{l}(\alpha) + \hat{l}(\alpha)^2}{\hat{l}(\alpha)^2} \cdot \frac{x^2 \hat{a}_{r^*}^2 + 2x\hat{a}_{r^*} \hat{l}(\alpha^*) + \hat{l}(\alpha^*)^2}{\hat{l}(\alpha^*)^2}$$

$$\geq \frac{x\hat{a}_r + x\hat{a}_{r^*} + \hat{l}(\alpha)}{\hat{l}(\alpha)} \cdot \frac{x\hat{a}_{r^*} + x\hat{a}_r + \hat{l}(\alpha^*)}{\hat{l}(\alpha^*)} \quad \text{if } \alpha \neq \alpha^*. \quad (6.7)$$

Inequality (6.6) is clear. For (6.7), observe that  $\hat{l}(\alpha) = \hat{l}(\alpha^*)$  and expand. If  $\beta = \beta^*$  we also have

$$2 \hat{\delta}_r \left\{ \frac{x \hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)^2} \right\} \geq \frac{2 \hat{\delta}_r \hat{l}(\beta)}{\hat{l}(\beta)^2} = \frac{\hat{\delta}_r + \hat{\delta}_{r^*}}{\hat{l}(\beta)};$$

with (6.6), this proves (A). Using (6.6) and (6.7), the proof of (B) reduces to checking that

$$\begin{aligned} 2 \hat{\delta}_r \left\{ \frac{x \hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)^2} \right\} \left\{ \frac{x^2 \hat{b}_{r^*}^2 + 2 x \hat{b}_{r^*} \hat{l}(\beta^*) + \hat{l}(\beta^*)^2}{\hat{l}(\beta^*)^2} \right\} &+ 2 \hat{\delta}_{r^*} \left\{ \frac{x \hat{b}_{r^*} + \hat{l}(\beta^*)}{\hat{l}(\beta^*)^2} \right\} \left\{ \frac{x^2 \hat{b}_r^2 + 2 x \hat{b}_r \hat{l}(\beta) + \hat{l}(\beta)^2}{\hat{l}(\beta)^2} \right\} \\ &\geq \left\{ \frac{\hat{\delta}_r + \hat{\delta}_{r^*}}{\hat{l}(\beta)} \right\} \left\{ \frac{\hat{l}(\beta^*) + x \hat{b}_r + x \hat{b}_{r^*}}{\hat{l}(\beta^*)} \right\} + \left\{ \frac{\hat{\delta}_{r^*} + \hat{\delta}_r}{\hat{l}(\beta^*)} \right\} \left\{ \frac{\hat{l}(\beta) + x \hat{b}_{r^*} + x \hat{b}_r}{\hat{l}(\beta)} \right\}. \end{aligned}$$

That inequality is checked by expanding out and using  $\hat{l}(\beta) = \hat{l}(\beta^*)$ , q.e.d.

We reformulate Lemma 6.5 as follows.

6.8. LEMMA.  $\pi_\lambda$  is among the following representations;

$$K = \text{SU}(n+1), n \geq 2, \text{ and } \lambda: \begin{array}{cccc} & 2 & & \\ \circ & - & \circ & \dots & - & \circ \\ \alpha_1 & & \alpha_2 & & & \alpha_n \end{array}$$

$$K = \text{SU}(n+1), 1 < r < \frac{n+1}{2}, \text{ and } \lambda: \begin{array}{ccccccc} & & & & 1 & & \\ \circ & - & \dots & - & \circ & - & \dots & - & \circ \\ \alpha_1 & & & & \alpha_r & & & & \alpha_n \end{array}$$

$$K = \text{Spin}(2n), n = 2m+1 \geq 5, \text{ and } \lambda: \begin{array}{ccccccc} & & & & & & 1 \\ \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ & & & & & & & & \circ \end{array}$$

$$K = \mathbf{E}_6, \text{ and } \lambda: \begin{array}{ccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ | & & | & & | & & | & & | \\ \circ & & \circ & & \circ & & \circ & & \circ \end{array} \text{ or } \lambda: \begin{array}{ccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ | & & | & & | & & | & & | \\ \circ & & \circ & & \circ & & \circ & & \circ \end{array}$$

*Proof.* As  $\lambda \neq \lambda^*$ ,  $K$  must be of type  $A_n$  ( $n \geq 2$ ),  $D_n$  ( $n = 2m+1 \geq 5$ ) or  $E_6$ . If  $\pi_{\xi_r} \otimes \pi_{\xi_r^*} = 1_K \oplus \text{ad}_K$ , then  $K$  is of type  $A_n$  and  $\xi_r: \begin{array}{cccc} & 1 & & \\ \circ & - & \circ & - & \dots & - & \circ \end{array}$ . Then Lemma 6.5 says  $p_{r,2} \{s_{r,2}^2 + t_{r,2}\} \geq 1 + \dim K$ , so  $k=2$  by (6.4) and Lemma 6.5. This is the first possibility listed in Lemma 6.8. Now suppose  $\pi_{\xi_r} \otimes \pi_{\xi_r^*} \neq 1_K \oplus \text{ad}_K$ . Then the latter is a proper summand of  $\pi_{\xi_r} \otimes \pi_{\xi_r^*}$  and we have  $p_{r,1} \{s_{r,1}^2 - t_{r,1}\} \geq 1 + \dim K$ . Then (6.4) and Lemma 6.5 say that  $\lambda = \xi_r$ , basic weight which is not self dual. These are the remaining possibilities listed in Lemma 6.8, q.e.d.

We now run through the cases of Lemma 6.8.

6.9. LEMMA. The representation  $\pi_\lambda$  given by  $\lambda: \begin{array}{cccc} & 2 & & \\ \circ & - & \circ & - & \dots & - & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_n \end{array}$ ,  $n \geq 2$  maps  $\text{SU}(n+1)$

into  $\text{SU}\left(\frac{(n+1)(n+2)}{2}\right)$  and satisfies  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \text{SU}(n+1) + 1$ .

*Proof.* The roots of  $\mathrm{SU}(n+1)$  are the roots  $\pm(\alpha_i + \alpha_{i+1} + \dots + \alpha_j)$ ,  $1 \leq i \leq j \leq n$ , where  $\{\alpha_1, \dots, \alpha_n\}$  are the simple roots. We have  $r=1$  and observe that

$$P_1 = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \alpha_{n-1}\} \text{ and } S_1 = \{\alpha_1 + \dots + \alpha_n\}.$$

All roots have the same length, which we normalize to be 1, so  $\hat{l}(\alpha_1 + \dots + \alpha_n) = q$  and  $\hat{a}_i = a_i$ . Now

$$p_{1,2} = \frac{1+2}{1} \cdot \frac{2+2}{2} \cdot \frac{3+2}{3} \cdots \frac{n-1+2}{n-1} = \frac{n(n+1)}{2},$$

$$s_{1,2} = \frac{n+2}{2} \quad \text{and} \quad t_{1,2} = \frac{n+4}{2}.$$

Thus

$$\begin{aligned} p_{1,2}^2 (s_{1,2}^2 - t_{1,2}) &= \frac{1}{4} n^2 (n+1)^2 \left\{ \frac{(n+2)^2}{n^2} - \frac{(n+4)}{n} \right\} \\ &= \frac{1}{4} n^2 (n+1)^2 \cdot \frac{4}{n^2} = (n+1)^2 = \dim \mathrm{SU}(n+1) + 1, \text{ q.e.d.} \end{aligned}$$

6.10. LEMMA. *The representation  $\pi_\lambda$  of  $\mathrm{SU}(n+1)$  given by  $\lambda: \overset{1}{\circ} \cdots \cdots \overset{1}{\circ} \cdots \cdots \overset{1}{\circ}$ ,  
 $\alpha_1 \qquad \qquad \qquad \alpha_r \qquad \qquad \qquad \alpha_n$*

$1 < r < n/2$ , satisfies  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathrm{SU}(n+1) + 1$  if and only if  $r=2$ .

*Proof.* We go by induction on  $r$ . First let  $r=2$ . Then  $P_r$  consists of

root	$\alpha_2$	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2 + \alpha_3$	$\dots$	$\alpha_1 + \dots + \alpha_{n-3}$	$\alpha_1 + \dots + \alpha_{n-2}$
		$\alpha_2 + \alpha_3$	$\alpha_2 + \alpha_3 + \alpha_4$	$\dots$	$\alpha_2 + \dots + \alpha_{n-2}$	
level	1	2	3	$\dots$	$n-3$	$n-2$

so

$$p_{2,1} = \frac{2}{1} \cdot \left( \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n-2}{n-3} \right) \frac{n-1}{n-2} = 2 \left( \frac{n-2}{2} \right) \frac{n-1}{n-2} = \frac{(n-2)(n-1)}{2}.$$

And  $S_r = \{\alpha_2 + \dots + \alpha_{n-1}, \alpha_1 + \dots + \alpha_{n-1}, \alpha_2 + \dots + \alpha_n, \alpha_1 + \dots + \alpha_n\}$  so that

$$s_{2,1} = \frac{n-1}{n-2} \left( \frac{n}{n-1} \right)^2 \frac{n+1}{n} = \frac{n(n+1)}{(n-2)(n-1)} \quad \text{and} \quad t_{2,1} = \frac{n}{n-2} \left( \frac{n+1}{n-1} \right)^2 \frac{n+2}{n} = \frac{n+2}{n-2} \left( \frac{n+1}{n-1} \right)^2.$$

Thus

$$p_{2,1}^2 (s_{2,1}^2 - t_{2,1})$$

$$\begin{aligned} &= \frac{1}{4} (n-2)^2 (n-1)^2 \left\{ \frac{n^2 (n+1)^2}{(n-2)^2 (n-1)^2} - \frac{(n+2)(n+1)^2}{(n-2)(n-1)^2} \right\} \\ &= \frac{1}{4} (n-2)^2 (n-1)^2 \left\{ \frac{n^2 (n+1)^2 - (n-2)(n+2)(n+1)^2}{(n-2)^2 (n-1)^2} \right\} \\ &= \frac{1}{4} \{4n^2 + 8n + 4\} = (n+1)^2 = \dim \mathrm{SU}(n+1). \end{aligned}$$

This proves the assertion for  $r=2$ .

Suppose  $r \geq 3$  and suppose the lemma known for  $r-1$  and all  $n > 2r-2$ . We decompose  $P_r$  into the subset  $P'_r$  given by  $a_1=0$  and the complementary subset  $P''_r = \{\alpha_1 + \dots + \alpha_r, \alpha_1 + \dots + \alpha_{r+1}, \dots, \alpha_1 + \dots + \alpha_{n-r}\}$ . Then we have the factorization

$$p_{r,1} = p'_{r,1} \cdot p''_{r,1} = p'_{r,1} \cdot \frac{r+1}{r} \cdot \frac{r+2}{r+1} \cdots \frac{n-r+1}{n-r} = p'_{r,1} \cdot \frac{n-r+1}{r}.$$

Similarly  $S_r$  consists of the set  $S'_r$  given by  $a_1 = a_n = 0$  and the complementary set

$$S''_r = \{\alpha_1 + \dots + \alpha_{n-r+1}, \alpha_r + \dots + \alpha_n; \dots; \alpha_1 + \dots + \alpha_{n-1}, \alpha_2 + \dots + \alpha_n; \alpha_1 + \dots + \alpha_n\}.$$

Thus

$$\begin{aligned} s_{r,1} &= s'_{r,1} s''_{r,1} = s'_{r,1} \cdot \left\{ \frac{n-r+2}{n-r+1} \cdot \frac{n-r+3}{n-r+2} \cdots \frac{n}{n-1} \right\}^2 \frac{n+1}{n} \\ &= s'_{r,1} \cdot \frac{n(n+1)}{(n-r+1)^2} \end{aligned}$$

and

$$t_{r,1} = t'_{r,1} t''_{r,1} = t'_{r,1} \cdot \left\{ \frac{n-r+3}{n-r+1} \cdots \frac{n+1}{n-1} \right\}^2 \frac{n+2}{n} = t'_{r,1} \cdot \frac{n(n+1)^2 (n+2)}{(n-r+1)^2 (n-r+2)^2}.$$

Now  $q = n-r+1 < n$  satisfies  $n(q+1)^2 > (n+2)q^2$ , so

$$\frac{n^2(n+1)^2}{(n-r+1)^4} > \frac{n(n+1)^2(n+2)}{(n-r+1)^2(n-r+2)^2}.$$

This shows

$$(s_{r,1}^2 - t_{r,1}) > (s'_{r,1}{}^2 - t'_{r,1}) \frac{n^2(n+1)^2}{(n-r+1)^4}.$$

The induction hypothesis, applied to the  $\mathrm{SU}(n-1)$  with simple roots  $\{\alpha_2, \dots, \alpha_{n-1}\}$ , says that  $p'_{r,1}{}^2 (s'_{r,1}{}^2 - t'_{r,1}) \geq (n-1)^2$ . Now we have

$$\begin{aligned} p_{r,1} (s_{r,1}^2 - t_{r,1}) &> (n-1)^2 \left( \frac{n-r+1}{r} \right)^2 \frac{n^2(n+1)^2}{(n-r+1)^4} = \frac{n^2(n-1)^2}{r^2(n-r+1)^2} (n+1)^2 \\ &> (n+1)^2 = \dim \mathrm{SU}(n+1) + 1, \text{ q.e.d.} \end{aligned}$$

6.11. LEMMA. *The representation  $\pi_\lambda$  of  $\mathrm{Spin}(2n)$ ,  $n = 2m+1 \geq 5$ , given by  $\lambda$ :*

$\circ - \circ - \dots - \circ \begin{cases} \circ 1 \\ \circ \end{cases}$ , *satisfies  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathrm{Spin}(2n) + 1$  if and only if  $n = 5$ .*

*Proof.* We label the simple roots  $\circ - \circ - \dots - \circ \begin{cases} \circ \alpha_1 \\ \circ \alpha_2 \end{cases}$ , then  $\lambda = \xi_1$  and  $\pi_\lambda$  is the half

spin representation,  $\deg \pi_\lambda = 2^{n-1}$ . The usual representation  $\pi_{\xi_n}: \mathrm{Spin}(2n) \rightarrow \mathrm{SO}(2n)$  satisfies

$$\Lambda^{n-1}(\pi_{\xi_n}) = \pi_{\xi_1 + \xi_2} = \pi_{\lambda + \lambda^*}, \quad \text{degree} \binom{2n}{n-1}.$$

Thus our degree equation is  $2^{2n-2} = \binom{2n}{n-1} + (2n^2 - n) + 1$ , i.e.,

$$2^{2n-2} - \frac{(n+2)(n+3)\dots(2n)}{(n-1)!} = 2n^2 - n + 1,$$

where  $n \geq 5$  is an odd integer. 5 is a solution, and one checks that the left side grows more than the right side when  $n+2$  replaces  $n$ . Thus 5 is the only solution, q.e.d.

6.12. LEMMA. The representation  $\pi_\lambda: \mathbf{E}_6 \rightarrow \mathbf{SU}(27)$  given by  $\lambda: \begin{array}{c} \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array}$  satisfies  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathbf{E}_6 + 1$ . The representation  $\pi_\lambda: \mathbf{E}_6 \rightarrow \mathbf{SU}(351)$  given by  $\lambda: \begin{array}{c} \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array}$  does not satisfy  $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathbf{E}_6 + 1$ .

Proof. We label the simple roots  $\begin{array}{c} \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \\ \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \\ \alpha_6 \end{array}$ . Then  $\xi_5 = \xi_1^*$  and a calculation shows that  $\deg \pi_{\xi_1} = 27$  and  $\deg \pi_{\xi_1+\xi_1^*} = 650$ , so that  $(\deg \pi_{\xi_1})^2 = 27^2 = 729 = 650 + 78 + 1 = \deg \pi_{\xi_1+\xi_1^*} + \dim \mathbf{E}_6 + 1$ . Also,  $\xi_4 = \xi_2^*$  and a calculation shows that  $\deg \pi_{\xi_2} = 351$  and  $\deg \pi_{\xi_2+\xi_2^*} = 70\,070$ , so that  $(\deg \pi_{\xi_2})^2 = 123\,201 > 70\,070 + 78 + 1 = \deg \pi_{\xi_2+\xi_2^*} + \dim \mathbf{E}_6 + 1$ , q.e.d.

Theorem 6.1 now follows from Lemmas 6.8, 6.9, 6.10, and 6.11.

### 7. The case where $G$ is symplectic and $K$ is simple

Here we have the classification:

7.1. THEOREM. Let  $G$  be a unitary symplectic group and let  $K$  be a compact connected simple subgroup. Then the representation  $\chi$  of  $K$  on the tangent space of  $G/K$  is irreducible over the real number field, if and only if  $(G, K)$  is one of the following.

$G$	$K$	$\pi_\lambda$	$\chi$
$\mathbf{Sp}(2)$	$\mathbf{SU}(2)$	$\begin{array}{c} 3 \\ \circ \end{array}$	$\begin{array}{c} 6 \\ \circ \end{array}$
$\mathbf{Sp}(7)$	$\mathbf{Sp}(3)$	$\begin{array}{c} 1 \\ \bullet - \bullet = \circ \end{array}$	$\begin{array}{c} 2 \\ \bullet - \bullet = \circ \end{array}$
$\mathbf{Sp}(10)$	$\mathbf{SU}(6)$	$\begin{array}{c} 1 \\ \circ - \circ - \circ - \circ - \circ \end{array}$	$\begin{array}{c} 2 \\ \circ - \circ - \circ - \circ - \circ \end{array}$
$\mathbf{Sp}(16)$	$\mathbf{Spin}(12)$	$\begin{array}{c} 1 \\ \circ - \circ - \circ - \circ \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array}$	$\begin{array}{c} 2 \\ \circ - \circ - \circ - \circ \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array}$
$\mathbf{Sp}(28)$	$\mathbf{E}_7$	$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ \\   \\ \circ \end{array}$	$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ \\   \\ \circ \end{array}$

where the inclusion  $K \rightarrow G = \mathrm{Sp}(n) \subset \mathrm{GL}(2n, \mathbb{C})$  is the absolutely irreducible representation  $\pi_\lambda$  of  $K$ , of highest weight  $\lambda$  and degree  $2n$ . In each case  $\chi = \pi_{2\lambda}$ , absolutely irreducible.

In each of the cases listed,  $\chi$  is irreducible.

Now assume  $\chi$  irreducible. Proposition 5.3 shows that the inclusion  $K \rightarrow G = \mathrm{Sp}(N)$  is an absolutely irreducible representation  $\pi_{k\xi_r}$  for some basic weight  $\xi_r = \xi_r^*$ , that  $2N = \deg \pi_{k\xi_r}$ , that  $\chi = \pi_{2k\xi_r}$ , and that

$$\frac{1}{2} (\deg \pi_{k\xi_r})^2 + \frac{1}{2} \deg \pi_{k\xi_r} = \deg \pi_{2k\xi_r} + \dim K. \quad (7.2)$$

The fact that  $\pi_{k\xi_r}$  is symplectic can be reformulated as follows using results of A. I. Mal'cev ([11], § 6); the details are carried out by E. B. Dynkin in Table 12 of [6].

7.3 LEMMA. *The positive integer  $k$  is odd and the basic weight  $\xi_r$  is one of the following.*

Type of $K$	$\xi_r$	Conditions
$A_n$	$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_r \text{---} \dots \text{---} \alpha_n$ 1 above $\alpha_r$	$n = 4s + 1, \quad r = 2s + 1$
$B_n$	$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_n = \bullet$ 1 above $\alpha_n$	$n = 4s + 1$ or $4s + 2$
$C_n$	$\alpha_1 = \bullet \text{---} \alpha_2 = \bullet \text{---} \dots \text{---} \alpha_r = \bullet \text{---} \dots \text{---} \alpha_{n-1} = \bullet \text{---} \alpha_n = \circ$ 1 above $\alpha_r$	$r \geq 1, \quad r$ odd
$D_n$	$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_s \text{---} \alpha_{s+1} \text{---} \alpha_{s+2} \text{---} \dots \text{---} \alpha_n$ 1 above $\alpha_{s+1}$	$n = 4s + 2, \quad s \geq 1$
$E_7$	$\alpha_1 \text{---} \alpha_2 \text{---} \alpha_3 \text{---} \alpha_4 \text{---} \alpha_5 \text{---} \alpha_6 \text{---} \alpha_7$ 1 below $\alpha_1$ or 1 below $\alpha_5$ or 1 below $\alpha_7$	

This lemma is used with a precise growth estimate:

7.4. LEMMA. *Either  $k = 1$  or  $k = 3$ . If  $k = 3$  and  $\mathrm{rank} K > 1$ , then  $\dim K \geq \frac{22}{9} (\deg \pi_{\xi_r})^2 + \frac{5}{3} \deg \pi_{\xi_r}$ .*

*Proof.* Define  $f(x) = \prod_{\alpha > 0} \frac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}$  where the product runs over the positive roots  $\alpha = \sum a_i \alpha_i$  of  $K$ . Then (5.6) and (7.2) say  $\frac{1}{2} f(k)^2 + \frac{1}{2} f(k) = f(2k) + \dim K$ .

Let  $x \geq 2$ ; we will prove that  $F(x) = \frac{1}{2} f(x)^2 + \frac{1}{2} f(x) - f(2x)$  is a strictly increasing function of  $x$ . As the second term is increasing, it suffices to show that  $d/dx \{ \frac{1}{2} f(x)^2 - f(2x) \} \geq 0$ , i.e., that

$$\sum_{\beta > 0} \left\{ \prod_{\substack{\alpha > 0 \\ \alpha \neq \beta}} \left( \frac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 \right\} \left\{ \frac{\hat{b}_r}{\hat{l}(\beta)} \cdot \frac{x\hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)} \right\} \geq \sum_{\beta > 0} \left\{ \prod_{\substack{\alpha > 0 \\ \alpha \neq \beta}} \frac{2x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right\} \left\{ \frac{2\hat{b}_r}{\hat{l}(\beta)} \right\}.$$

We will prove this inequality term by term. Dividing the  $\beta$ -term by  $\hat{b}_r/\hat{l}(\beta)$ , that amounts to showing

$$\left\{ \prod_{\substack{\alpha > 0 \\ \alpha \neq \beta}} \left( \frac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 \right\} \left\{ \frac{x\hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)} \right\} \geq 2 \prod_{\substack{\alpha > 0 \\ \alpha \neq \beta}} \left( \frac{2x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right). \quad (7.5)$$

for every root  $\beta > 0$ .

Let  $\beta$  be a fixed positive root and let  $S$  be any set of positive roots which does not contain  $\beta$ . We define

$$A_S = \prod_S \left( \frac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2, \quad B_S = \frac{x\hat{b}_r + \hat{l}(\beta)}{\hat{l}(\beta)}, \quad \text{and } C_S = 2 \prod_S \frac{2x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}.$$

If  $\alpha \notin S \cup \{\beta\}$  is a positive root, then

$$\left( \frac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 \geq \frac{2x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}.$$

Thus, in order to prove (7.5) it suffices to find a set  $S$  such that  $A_S B_S \geq C_S$ .

If  $\beta = \alpha_r$  we take  $S$  empty. Then  $A_S = 1$ ,  $B_S = x + 1 \geq 2$  and  $C_S = 2$ , so  $A_S B_S \geq C_S$ . Now suppose  $\beta \neq \alpha_r$ . Then  $\text{rank } K > 1$ . Let  $\alpha_s$  be a simple root adjacent to  $\alpha_r$  in the Dynkin diagram of  $K$ , and consider  $\gamma = \sum c_i \alpha_i$  defined as follows:

- (1)  $\|\alpha_r\| = \|\alpha_s\|$ ,  $\gamma = \alpha_r + \alpha_s$ ,  $\frac{x\hat{c}_r + \hat{l}(\gamma)}{\hat{l}(\gamma)} = \frac{x+2}{2}$ ;
- (2)  $\|\alpha_r\|^2 = 2\|\alpha_s\|^2$ ,  $\gamma = \alpha_r + 2\alpha_s$ ,  $\frac{x\hat{c}_r + \hat{l}(\gamma)}{\hat{l}(\gamma)} = \frac{x+2}{2}$ ;
- (3)  $2\|\alpha_r\|^2 = \|\alpha_s\|^2$ ,  $\gamma = 2\alpha_r + \alpha_s$ ,  $\frac{x\hat{c}_r + \hat{l}(\gamma)}{\hat{l}(\gamma)} = \frac{x+2}{2}$ .

If  $\beta = \gamma$  we take  $S = \{\alpha_r\}$ ; then  $A_S = (x+1)^2$ ,  $B_S = \frac{1}{2}(x+2)$  and  $C_S = 4x+2$ ;  $x \geq 2$  says  $A_S \geq 9$  so  $A_S B_S \geq \frac{9}{2}x + 9 > 4x + 2 = C_S$ . Now we may assume  $\alpha_r \neq \beta \neq \gamma$  and take  $S = \{\alpha_r, \gamma\}$ ; then  $A_S = \frac{1}{4}(x+1)^2(x+2)^2$ ,  $B_S \geq 1$  and  $C_S = (2x+1)(2x+2)$ , so  $A_S B_S \geq A_S \geq \frac{1}{4}(x+1)^2 4^2 = 4x^2 + 8x + 4 > 4x^2 + 6x + 2 = C_S$ . This completes the proof that  $dF(x)/dx > 0$  for  $x \geq 2$ .

$\pi_{3\xi_r}$  is symplectic because  $\pi_{k\xi_r}$  is symplectic, and  $\pi_{3\xi_r}(K) \subseteq \mathbb{S}p(\frac{1}{2} \deg \pi_{3\xi_r})$ . Thus  $F(3) = \deg S^2(\pi_{3\xi_r}) - \deg \pi_{6\xi_r} \geq \dim K$ . As  $F(x)$  is increasing for  $x \geq 2$ , as  $F(k) = \dim K$ , and as  $k$  is an odd positive integer, it follows that  $k$  is 1 or 3. This proves the first statement.

Suppose  $\text{rank } K > 1$ . Then we have  $\alpha_s$  and  $\gamma$  as defined above. Let  $V$  be the set of all positive roots except for  $\alpha_s$  and  $\gamma$ . Now

$$\begin{aligned} (\deg \pi_{k\xi_r})^2 &= (k+1)^2 \left(\frac{k+2}{2}\right)^2 \prod_V \left(\frac{k\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2 \\ \deg \pi_{2k\xi_r} &= (2k+1)(k+1) \prod_V \frac{2k\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \end{aligned}$$

and we have termwise inequalities for the factors corresponding to roots  $\alpha \in V$ . Thus

$$4(2k+1)(k+1)(\deg \pi_{k\xi_r})^2 \geq (k+1)^2(k+2)^2 \deg \pi_{2k\xi_r} \quad (7.6)$$

Suppose  $k \neq 1$ , i.e.,  $k = 3$ . Then (7.6) says  $\frac{7}{25}(\deg \pi_{3\xi_r})^2 \geq \deg \pi_{6\xi_r}$ , so the identity  $F(3) = \dim K$  gives us

$$\dim K = \frac{1}{2}(\deg \pi_{3\xi_r})^2 + \frac{1}{2} \deg \pi_{3\xi_r} - \deg \pi_{6\xi_r} \geq \frac{11}{50}(\deg \pi_{3\xi_r})^2 + \frac{1}{2} \deg \pi_{3\xi_r}.$$

Suppose further that  $\text{rank } K > 1$ . Looking at the  $\alpha_r$ -term and the  $\gamma$ -term in the degree formula, we notice

$$\deg \pi_{3\xi_r} \geq \frac{3+1}{1+1} \cdot \frac{3+2}{1+2} \deg \pi_{\xi_r} = \frac{10}{3} \deg \pi_{\xi_r}.$$

Thus  $\frac{11}{50}(\deg \pi_{3\xi_r})^2 + \frac{1}{2} \deg \pi_{3\xi_r} \geq \frac{22}{9}(\deg \pi_{\xi_r})^2 + \frac{5}{3} \deg \pi_{\xi_r}$ .

This proves the second statement, q.e.d.

Now we can run through cases.

7.7. LEMMA. *The representation  $\pi_\lambda$  of  $SU(n+1)$  given by  $\lambda$ :  $\overset{k}{\circ} - \overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \dots - \overset{\alpha_r}{\circ} - \dots - \overset{\alpha_n}{\circ}$ ,  $n = 4s + 1$ ,  $r = 2s + 1$ ,  $k = 1$  or  $3$ , satisfies  $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2} \deg \pi_\lambda = \deg \pi_{2\lambda} + \dim SU(n+1)$ , if and only if  $\lambda$ :  $\overset{3}{\circ}$  or  $\lambda$ :  $\overset{1}{\circ} - \circ - \circ - \circ - \circ$ .*

*Proof.* First suppose  $k = 3$ . If  $s = 0$ , then  $n = 1$  and  $\deg \pi_{m\xi_1} = m + 1$ . Thus  $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2} \deg \pi_\lambda = \frac{1}{2}4^2 + \frac{1}{2}4 = 10 = 7 + 3 = \deg \pi_{2\lambda} + \dim SU(2)$ , which is our case  $\lambda$ :  $\overset{3}{\circ}$ . Now we prove by induction on  $s$  that  $\frac{1}{2}(\deg \pi_{3\xi_r})^2 + \frac{1}{2} \deg \pi_{3\xi_r} > \deg \pi_{6\xi_r} + \dim SU(n+1)$  for  $s \geq 1$ .



By Lemma 7.4 it suffices to prove  $\dim \text{SU}(n+1) \leq \frac{22}{9} (\deg \pi_{\xi_r})^2$  for  $s \geq 1$ . For  $s=1$  this says  $35 \leq \frac{22}{9} \cdot 20^2$ , which is clear. The induction hypothesis is

$$(4s+2)^2 \leq \frac{22}{9} \binom{4s+2}{2s+1}^2, \text{ i.e., } 36r^2 \leq 22 \binom{2r}{r}^2.$$

We write this in the form  $(1 \cdot 2 \dots \cdot r)^2 \cdot 36r^2 \leq 22 \{(r+1)(r+2) \dots (2r)\}^2$ . If we raise  $s$  to  $s+1$ ,  $r$  goes to  $r+2$ , and we multiply the left side by  $\frac{(r+2)^4 (r+1)^2}{r^2}$ , and the right side by  $\frac{(2r+1)^2 (2r+2)^2 (2r+3)^2 (2r+4)^2}{(r+1)^2 (r+2)^2}$ . The factor for the right side is larger, so the inequality persists.

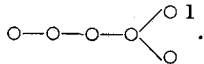
We may now assume  $k=1$ , so  $\lambda = \xi_r$ , and  $s \geq 1$ . If  $s=1$  then  $\frac{1}{2} (\deg \pi_{\lambda})^2 + \frac{1}{2} \deg \pi_{\lambda} = \frac{1}{2} \cdot 20^2 + \frac{1}{2} \cdot 20 = 210 = 175 + 35 = \deg \pi_{2\lambda} + \dim \text{SU}(n+1)$ . That is our case  $\lambda: \circ - \circ - \overset{1}{\circ} - \circ - \circ$ . Now we prove by induction on  $s$  that  $\frac{1}{2} (\deg \pi_{\xi_r})^2 + \frac{1}{2} \deg \pi_{\xi_r} > \deg \pi_{2\xi_r} + \dim \text{SU}(2r)$  for  $s \geq 2$ . It suffices to show that  $\frac{1}{2} (\deg \pi_{\xi_r})^2 \geq \deg \pi_{2\xi_r} + \dim \text{SU}(2r)$ . For  $s=2$ ,  $\deg \pi_{\xi_r} = 252$ ,  $\deg \pi_{2\xi_r} = 19404$ , and  $\dim \text{SU}(2r) = 99$ , so the inequality is clear. Let  $s > 2$ , let  $S$  be the set of all roots  $\alpha > 0$  where  $\alpha_r \neq 0$ , and divide  $S$  into the set  $T = \{\alpha \in S: \alpha_1 = 0 = \alpha_n\}$  and its complement  $U = \{\alpha_1 + \dots + \alpha_r, \alpha_r + \dots + \alpha_n; \dots; \alpha_1 + \dots + \alpha_{n-1}, \alpha_2 + \dots + \alpha_n; \alpha_1 + \dots + \alpha_n\}$ . Let  $L$  be the subgroup  $\text{SU}(n-1)$  of  $\text{SU}(n+1)$  with simple root system  $\{\alpha_2, \dots, \alpha_{n-1}\}$ . By induction on  $r$ ,  $\frac{1}{2} (\deg \tau_{\xi_r})^2 \geq \deg \tau_{2\xi_r} + (2r-2)^2 - 1$  where  $\tau_v$  is the representation of  $L$  with highest weight  $v$ . Now  $\deg \pi_{\xi_r} = u \cdot \deg \tau_{\xi_r}$  and  $\deg \pi_{2\xi_r} = v \cdot \deg \tau_{\xi_r}$  where

$$u = \prod_U \frac{1+l(\alpha)}{l(\alpha)} = \left\{ \frac{r+1}{r} \dots \frac{2r-1}{2r-2} \right\}^2 \cdot \frac{2r}{2r-1} = \frac{4r-2}{r}$$

and 
$$v = \prod_U \frac{2+l(\alpha)}{l(\alpha)} = \left\{ \frac{r+2}{r} \dots \frac{2r}{2r-2} \right\}^2 \cdot \frac{2r+1}{2r-1} = 4 \frac{(2r-1)(2r+1)}{(r+1)^2}.$$

Now  $u^2 > v$  shows that  $\frac{1}{2} (\deg \pi_{\xi_r})^2$  grows more than  $\deg \pi_{2\xi_r}$  when  $r$ , hence when  $s$ , is raised. Also  $u^2 > \frac{(2r)^2 - 1}{(2r-2)^2 - 1}$  so  $\frac{1}{2} (\deg \pi_{\xi_r})^2$  grows faster than  $\dim \text{SU}(2r)$  when  $r$ , hence when  $s$ , is raised. Thus our inequalities persist when  $s$  is raised, q.e.d.

7.8 LEMMA. The group  $K$  is of type  $B_n$  or  $D_n$ , if and only if  $\lambda$  is given by  $\overset{3}{\bullet}$  or by



*Proof.* Let  $K$  be of type  $B_n$ . Then Lemma 7.3 says  $\lambda: \underset{\alpha_n}{\circ} - \underset{\alpha_{n-1}}{\circ} - \dots - \underset{\alpha_2}{\circ} = \overset{k}{\bullet} \underset{\alpha_1}{\circ}$  and  $n = 4s + 1$  or  $4s + 2$ , and Lemma 7.4 says that  $k = 1$  or  $3$ . Suppose  $k = 3$  and  $n > 1$ . Then Lemma 7.4 says that  $2n^2 + n = \dim K > \frac{22}{9} (\deg \pi_{\xi_1})^2 + \frac{5}{3} \deg \pi_{\xi_1} = \frac{22}{9} \cdot 2^{2n} + \frac{5}{3} \cdot 2^n$ . That inequality has no integral solution  $n > 1$ . Thus  $k = 3$  implies  $n = 1$ ; that is the case  $\lambda: \overset{3}{\bullet}$  which occurs as  $\lambda: \overset{3}{\circ}$  in Lemma 7.7. We may now assume  $k = 1$ , so  $\lambda = \xi_1$  and  $n > 1$ .  $\pi_{2\xi_1}$  can be obtained by composition of the inclusion  $\mathbf{B}_n = \mathbf{SO}(2n+1) \subset \mathbf{SU}(2n+1) = \mathbf{A}_{2n}$  with the representation  $\pi_{\xi_n}$  of  $\mathbf{A}_{2n}$ ; thus  $\deg \pi_{2\xi_1} = \binom{2n+1}{n}$ . As  $\deg \pi_{\xi_1} = 2^n$ , our equation is  $2^{2n-1} + 2^{n-1} = \binom{2n+1}{n} + 2n^2 + n$ . There are no integral solutions  $n > 1$ .

Let  $K$  be of type  $D_n$ . Then Lemma 7.3 says that  $\lambda: \underset{\alpha_n}{\circ} - \underset{\alpha_{n-1}}{\circ} - \dots - \underset{\alpha_3}{\circ} \begin{matrix} \circ \alpha_1 \\ \circ \alpha_2 \end{matrix} \overset{k}{\circ}$   $n = 4s + 2, s \geq 1$ ; and Lemma 7.4 says that  $k$  is 1 or 3. If  $k = 3$ , then Lemma 7.4 and  $\deg \pi_{\xi_1} = 2^{n-1}$  say that  $2n^2 - n = \dim K \geq \frac{22}{9} \cdot 2^{2n-2} + \frac{5}{3} \cdot 2^{n-1}$ . There are no integral solutions  $n \geq 6$ . Thus  $k = 1$  and our equation is  $2^{2n-3} + 2^{n-2} = \deg \pi_{2\xi_1} + 2n^2 - n$ . If  $n = 6$  then  $\deg \pi_{2\xi_1} = 462 = 2^9 + 2^4 - 72 + 6$ , so we have the solution  $\lambda: \underset{\alpha_n}{\circ} - \underset{\alpha_{n-1}}{\circ} - \dots - \underset{\alpha_3}{\circ} \begin{matrix} \circ 1 \\ \circ \end{matrix}$ . Now we will prove by induction on  $n$  that, for  $n > 6$ , the representation  $\pi_{2\xi_1}$  of  $\mathbf{D}_n$  satisfies

$$\deg \pi_{2\xi_1} < 2^{2n-3} + 2^{n-2} - 2n^2 + n,$$

i.e., that

$$2^{2n-3} > \deg \pi_{2\xi_1} - 2^{n-2} + 2n^2 - n.$$

For when  $n$  is raised to  $n + 1$ , the new roots  $\alpha > 0$  with  $a_1 \neq 0$  are  $\{\alpha_1 + \alpha_3 + \alpha_4 + \dots + \alpha_{n+1}; \alpha_1 + \alpha_2 + \dots + \alpha_{n+1}; \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \dots + \alpha_{n+1}; \dots; \alpha_1 + \alpha_2 + 2\alpha_3 + \dots + 2\alpha_n + \alpha_{n+1}\}$ , so  $\deg \pi_{2\xi_1}$  is multiplied by  $\frac{n+2}{n} \cdot \frac{n+3}{n+1} \cdot \dots \cdot \frac{2n+1}{2n-1} = 2 \frac{2n+1}{n+1} < 4$ . Similarly  $2^{n-2}$  is doubled and  $2n^2 - n$  is multiplied by a factor less than 4. But  $2^{2n-3}$  is multiplied by 4. Thus our equality for  $n = 6$  becomes strict inequality for  $n > 6$ , q.e.d.

**7.9. LEMMA.** *The representation  $\pi_\lambda$  of  $\mathbf{Sp}(n)$  given by  $\bullet - \bullet - \dots - \overset{k}{\bullet} - \dots - \bullet = \circ$ ,  $n - r + 1$  and  $k$  odd,  $1 \leq r \leq n$ , satisfies  $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2} \deg \pi_\lambda = \deg \pi_{2\lambda} + \dim \mathbf{Sp}(n)$  if and only if  $\lambda: \bullet - \bullet = \overset{1}{\circ}$ .*

*Proof.* As before,  $k$  is 1 or 3. Let  $s = n - r + 1$ ; then  $\deg \pi_{\xi_r} = \binom{2n}{s} - \binom{2n}{s-2}$  with the

usual convention that  $\binom{m}{0} = 1$  and  $\binom{m}{-1} = 0$ . Thus  $\deg \pi_{\xi_r} \geq 2n$ . If  $k=3$  then Lemma 7.4 says  $2n^2 + n = \dim \mathbf{Sp}(n) > \frac{22}{9} (\deg \pi_{\xi_r})^2 \geq \frac{22}{9} \cdot 4n^2 > 8n^2 > 2n^2 + n$ . Thus  $k=1$ .

Given an integer  $b$  with  $1 \leq b \leq n$  we define

$$P(n, b) = \frac{1}{2} (\deg \pi_b)^2 - \deg \pi_{2b} \quad \text{and} \quad Q(n, b) = \frac{1}{2} \deg \pi_b - \dim \mathbf{Sp}(n).$$

Then our degree equation is  $P(n, r) + Q(n, r) = 0$ . Suppose  $b < n$ . Let  $\tau_r$  denote the representation of highest weight  $\nu$  for the subgroup  $\mathbf{Sp}(n-1)$  with simple root system  $\{\alpha_1, \dots, \alpha_{n-1}\}$ . Then  $\deg \pi_{\xi_b} = u \cdot \deg \tau_{\xi_b}$  and  $\deg \pi_{2\xi_b} = v \cdot \deg \tau_{2\xi_b}$  where

$$u^2 = \prod_{\substack{\alpha > 0 \\ a_n \neq 0}} \left( \frac{\hat{a}_b + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 > \prod_{\substack{\alpha > 0 \\ a_n \neq 0}} \frac{2\hat{a}_b + \hat{l}(\alpha)}{\hat{l}(\alpha)} = v.$$

Thus: (7.10)  
if  $b < n$  then  $P(n, b) > P(n-1, b)$ .

We observe  $\dim \mathbf{Sp}(n) = w \cdot \dim \mathbf{Sp}(n-1)$  where  $w = \frac{2n^2 + n}{2n^2 - 3n + 1}$ . We compute  $u = \frac{2n(2n+1)}{(n-b+1)(n+b+5)}$ . Thus the condition for  $u > w$  is  $3n^2 - 12n - 3 + 4b + b^2 > 0$ . If  $b=1$  this says  $n \geq 4$ ; if  $b=2$  it says  $n \geq 4$ ; if  $b \geq 3$  it is automatic.

Thus: (7.11)  
if  $b < n$ , and if  $n \geq 4$  or  $b \geq 3$ , then  $Q(n, b) > Q(n-1, b)$ .

$\pi_{\xi_n}$  maps  $\mathbf{Sp}(n)$  onto  $\mathbf{Sp}(\frac{1}{2} \deg \pi_{\xi_n})$ ; thus  $r < n$ . Let  $L$  denote the subgroup  $\mathbf{Sp}(r+2)$  of  $\mathbf{Sp}(n)$  with simple root system  $\{\alpha_1, \dots, \alpha_{r+2}\}$ , and let  $\tau$  denote its representation of highest weight  $\xi_r$ . Then  $\tau$  maps  $L$  onto a proper subgroup of  $\mathbf{Sp}(\frac{1}{2} \deg \tau)$ ; thus  $P(r+2, r) + Q(r+2, r) \geq 0$ . If  $r+2 < n$  then (7.10) and (7.11) say that  $P(n, r) + Q(n, r) > 0$ . Thus  $r = n-2$ .

Suppose  $n \geq 5$ . Then  $Q(n, n-2) \geq 0$ . Define  $U = \{\alpha_{n-2}; \alpha_{n-1} + \alpha_{n-2}, \alpha_{n-2} + \alpha_{n-3}; \alpha_n + \alpha_{n-1} + \alpha_{n-2}, \alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3}; \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3}\}$  and let  $V$  be the complementary set of positive roots. As  $V$  contains the highest root, we define (recall  $r = n-2$ )

$$v_1 = \prod_V \left( \frac{\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 \quad \text{and} \quad v_2 = \prod_V \frac{2\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}$$

and have  $v_1 > v_2$ . We also define

$$u_1 = \frac{1}{2} \prod_U \left( \frac{\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)} \right)^2 \quad \text{and} \quad u_2 = \prod_U \frac{2\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}$$

so that  $P(n, n-2) = u_1 v_1 - u_2 v_2$ . But

$$u_1 = \frac{1}{2} \cdot \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{5}{4} = 50 \quad \text{and}$$

$$u_2 = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{4}{2} \cdot \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{6}{4} = 50 = u_1.$$

Thus  $P(n, n-2) > 0$ . As  $Q(n, n-2) \geq 0$ , that contradicts  $P(n, n-2) + Q(n, n-2) = 0$ . Thus  $n < 5$ . As  $n \geq 3$  now  $n$  must be 3 or 4.

Let  $n = 4$ . Then  $\lambda$ :  $\bullet - \bullet - \overset{1}{\bullet} = \circ$ , so  $\deg \pi_\lambda = 48$  and  $\deg \pi_{2\lambda} = 825$ , so  $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2}(\deg \pi_{2\lambda}) = 1176 > 825 + 36 = \deg \pi_{2\lambda} + \dim \text{Sp}(4)$ . Thus  $n \neq 4$ .

Now  $n = 3$  so  $\lambda$ :  $\bullet - \bullet = \overset{1}{\circ}$ . Here  $\deg \pi_\lambda = 14$  and  $\deg \pi_{2\lambda} = 84$ , so  $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2} \deg \pi_{2\lambda} = 105 = 84 + 21 = \deg \pi_{2\lambda} + \dim \text{Sp}(3)$ , q.e.d.

7.12 LEMMA. If  $K = \mathbb{E}_7$  then  $\lambda$  is given by  $\overset{1}{\circ} - \circ - \circ - \circ - \circ - \circ - \circ$ , degree 56.

*Proof.* We number the simple roots of  $\mathbb{E}_7$  by  $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ} - \overset{\alpha_4}{\circ} - \overset{\alpha_5}{\circ} - \overset{\alpha_6}{\circ} - \overset{\alpha_7}{\circ}$ . Then Lemmas

7.3 and 7.4 say that  $\lambda = k\xi_r$ ;  $r$  is 1, 3 or 7; and  $k$  is 1 or 3. We compute  $\deg \pi_{\xi_1} = 56$ ;  $\deg \pi_{\xi_3} = 27\,664$ ;  $\deg \pi_{\xi_7} = 912$ ;  $\deg \pi_{2\xi_1} = 1\,463$ ;  $\deg \pi_{2\xi_3} = 109\,120\,648$ ;  $\deg \pi_{2\xi_7} = 84\,645$ . Now  $\dim \mathbb{E}_7 = 133$  and we have

$$\frac{22}{9} (27664)^2 > \frac{22}{9} (912)^2 > \frac{22}{9} (56)^2 > 133.$$

Thus Lemma 7.4 says  $k = 1$ . Finally we compute

$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_1})^2 + \frac{1}{2}(\deg \pi_{\xi_1}) &= 1\,596 = \deg \pi_{2\xi_1} + \dim \mathbb{E}_7; \\ \frac{1}{2}(\deg \pi_{\xi_3})^2 + \frac{1}{2}(\deg \pi_{\xi_3}) &= 382\,662\,280 > \deg \pi_{2\xi_3} + \dim \mathbb{E}_7; \\ \frac{1}{2}(\deg \pi_{\xi_7})^2 + \frac{1}{2}(\deg \pi_{\xi_7}) &= 416\,328 > \deg \pi_{2\xi_7} + \dim \mathbb{E}_7. \end{aligned}$$

Thus  $\lambda = \xi_1$ , q.e.d.

Theorem 7.1 now follows from Lemmas 7.3, 7.7, 7.8, 7.9 and 7.12.

### 8. The case where $G$ is orthogonal and $\chi$ reduces

As the first step in the classification for  $G$  orthogonal and  $K$  simple, we prove:

8.1. THEOREM. Let  $G$  be a simple<sup>(1)</sup> special orthogonal group and let  $K$  be a proper compact connected simple subgroup. Let  $\chi$  be the representation of  $K$  on the tangent space of  $G/K$ . Then  $\chi$  is irreducible over the real number field but not absolutely irreducible, if and only

<sup>(1)</sup> This means  $G = \mathbf{SO}(n)$ ,  $n > 2$ ,  $n \neq 4$ .

if the inclusion  $K \rightarrow G$  is given by  $\overset{1}{\circ} \dots \overset{1}{\circ}$ , adjoint representation of  $\mathrm{SU}(n+1)$ ,  $n \geq 2$ .

In that case  $\chi$  is given by  $\overset{3}{\circ} \text{---} \overset{3}{\circ} \oplus \text{---} \overset{3}{\circ}$  if  $n=2$ , by  $\overset{1}{\circ} \text{---} \overset{2}{\circ} \oplus \overset{2}{\circ} \text{---} \dots \text{---} \overset{1}{\circ}$  if  $n > 2$ .

Note that the adjoint representation of  $\mathrm{SU}(2)$  simply maps  $\mathrm{SU}(2)$  onto  $\mathrm{SO}(3)$ , and thus is not interesting in the context of the theorem.

Proposition 5.2 says that the inclusion  $K \rightarrow G = \mathrm{SO}(N)$  is an absolutely irreducible representation  $\pi_{k(\xi_r + \xi_r^*)}$ , for some basic weight  $\xi_r \neq \xi_r^*$ , that  $N = \deg \pi_{k(\xi_r + \xi_r^*)}$ , that  $\chi = \pi_{2k(\xi_r + \xi_r^*) - \alpha_r} \oplus \pi_{2k(\xi_r + \xi_r^*) - \alpha_r^*}$ , and that  $\frac{1}{2} (\deg \pi_{k(\xi_r + \xi_r^*)})^2 - \frac{1}{2} \deg \pi_{k(\xi_r + \xi_r^*)} = \deg \chi + \dim K$ . We write the latter in the form

$$\frac{1}{2} (\deg \pi_{k(\xi_r + \xi_r^*)})^2 = \frac{1}{2} \deg \pi_{k(\xi_r + \xi_r^*)} + 2 \deg \pi_{2k(\xi_r + \xi_r^*) - \alpha_r} + \dim K. \quad (8.2)$$

We now make a growth estimate on  $k$ , proving:

8.3 LEMMA. *The integer  $k$  is equal to 1.*

*Proof.* For each integer  $m > 0$  we define

$$U_m = \frac{1}{2} (\deg \pi_{m(\xi_r + \xi_r^*)})^2, \quad V_m = \frac{1}{2} \deg \pi_{m(\xi_r + \xi_r^*)} \quad \text{and} \quad W_m = 2 \deg \pi_{2m(\xi_r + \xi_r^*) - \alpha_r}.$$

We also define multipliers by

$$U_{m+1} = u_m U_m, \quad V_{m+1} = v_m V_m \quad \text{and} \quad W_{m+1} = w_m W_m.$$

$\xi_r \neq \xi_r^*$  shows that  $K$  is of type  $A_n$ ,  $D_{2n+1}$  or  $E_6$ . Thus all simple roots have the same norm.

Now a glance at (5.6) shows that

$$u_m = \prod_{\alpha > 0} u_m(\alpha), \quad v_m = \prod_{\alpha > 0} v_m(\alpha) \quad \text{and} \quad w_m = \prod_{\alpha > 0} w_m(\alpha),$$

where  $u_m(\alpha) = v_m(\alpha)^2$ ,  $v_m(\alpha) = \frac{(m+1)(a_r + a_{r^*}) + l(\alpha)}{m(a_r + a_{r^*}) + l(\alpha)}$  and

$$w_m(\alpha) = \frac{2(m+1)a_{r^*} + 2ma_r + \sum a_i + l(\alpha)}{2ma_{r^*} + 2(m-1)a_r + \sum a_i + l(\alpha)}$$

and the summation  $\sum a_i$  is extended over all simple roots adjacent to  $\alpha_r$  in the Dynkin diagram of  $K$ .

Observe  $v_m(\alpha_r) = \frac{m+2}{m+1} = v_m(\alpha_{r^*})$ . Also  $w_m(\alpha_r) = \frac{2m+1}{2m-1}$ ,  $w_m(\alpha_{r^*}) = \frac{2m+3}{2m+1}$  if  $\alpha_{r^*} \perp \alpha_r$ ,

and  $w_m(\alpha_{r^*}) = \frac{m+2}{m+1}$  if  $\alpha_{r^*}$  and  $\alpha_r$  are adjacent. Now  $(m+2)^4(2m-1)(2m+1) = 4m^6 +$

$32m^5 + 95m^4 + 120m^3 + 40m^2 - 32m - 16 > 4m^6 + 24m^5 + 59m^4 + 76m^3 + 54m^2 + 20m + 3 = (m+1)^4(2m+1)(2m+3)$ . Thus  $\left(\frac{m+2}{m+1}\right)^4 > \frac{2m+1}{2m-1} \cdot \frac{2m+3}{2m+1} > \frac{2m+1}{2m-1} \cdot \frac{m+2}{m+1}$ . This proves

$$u_m(\alpha_r) \cdot u_m(\alpha_{r*}) > w_m(\alpha_r) \cdot w_m(\alpha_{r*}). \quad (8.4)$$

Let  $\alpha$  be a positive root,  $\alpha_r \neq \alpha \neq \alpha_{r*}$ . Denote  $a = a_r + a_{r*}$ ,  $s = \sum a_i$  and  $l = (\alpha)$ . Then

$$u_m(\alpha) = 1 + \frac{(2m+1)a^2 + 2al}{(ma+l)^2} \quad \text{and} \quad w_m(\alpha) = 1 + \frac{2a}{2(m-1)a + 2a_{r*} + s + l}.$$

We will prove that  $u_m(\alpha) \geq w_m(\alpha)$ , i.e., that

$$(*) \quad \{(2m+1)a^2 + 2al\} \{2(m-1)a + 2a_{r*} + s + l\} \geq 2a(ma+l)^2.$$

For  $m=1$  this inequality is  $(3a^2 + 2al)(2a_{r*} + s + l) \geq 2a^3 + 4a^2l + 2al^2$ , i.e.,  $6a^2a_{r*} + 3a^2s + 4laa_{r*} + 2las \geq la^2 + 2a^3$ . If  $a_r=0$ , then  $a_{r*}=a$  and the inequality follows; if  $a_r > 1$  then  $\xi_r$ :  $\begin{array}{ccccccc} \circ & - & \circ & - & \circ & - & \circ \\ & & & & & & \circ \\ & & & & & & | \\ & & & & & & \circ \end{array}$  so  $s \geq a_r$  and the inequality follows; now suppose  $a_r=1$ . If  $a_{r*} > 0$

then  $2a_{r*} \geq a$  and the inequality follows, so suppose  $a_{r*}=0$ . Then the inequality says  $3s + 2ls \geq l + 2$ ; as  $\alpha \neq \alpha_r$ , we have  $s \geq 1$  and  $l \geq 2$  so this is clear. Now (\*) is proved for  $m=1$ .

To prove (\*) for  $m > 1$  we let  $m$  range as a real variable and we differentiate. Thus we must prove  $2a^2\{2(m-1)a + 2a_{r*} + s + l\} + \{(2m+1)a^2 + 2al\}2a \geq 4ma^3 + 2a^2l$  which is clear by inspection. This completes the proof of

$$u_m(\alpha) \geq w_m(\alpha) \quad \text{for} \quad \alpha > 0, \quad \alpha_r \neq \alpha \neq \alpha_{r*}. \quad (8.5)$$

Combining (8.4) and (8.5) we have  $u_m > w_m$ . And  $v_m > 1$  shows  $u_m > v_m$ . This says

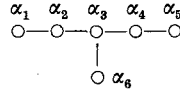
$$U_{m+1} - \{V_{m+1} + W_{m+1} + \dim K\} > U_m - \{V_m + W_m + \dim K\}. \quad (8.6)$$

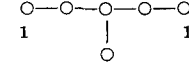
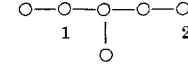
Let  $\nu = \xi_r + \xi_r^*$ . Then  $\pi_\nu$  is orthogonal and  $\pi_\nu(K) \not\subseteq \mathbf{S0}$  ( $\deg \pi_\nu$ ). As  $\pi_{2\nu - \alpha_r}$  and  $\pi_{2\nu - \alpha_{r*}}$  are summands of  $\Lambda^2(\pi_\nu)$ , this shows that  $U_1 \geq V_1 + W_1 + \dim K$ . If  $k > 1$  then repetition of (8.6) says  $U_k > V_k + W_k + \dim K$ . But (8.2) says  $U_k = V_k + W_k + \dim K$ . This proves  $k=1$ , q.e.d.

8.7. LEMMA.  $K$  is of type  $A_n$ ,  $1 \leq r \leq n/2$ .

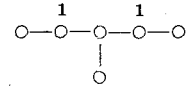
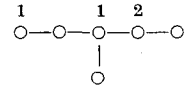
*Proof.* Suppose that  $K$  is not of type  $A_n$ ; then  $\xi_r \neq \xi_r^*$  implies that  $\xi_r$  is given by

$$\begin{array}{c} \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array} \quad \text{or} \quad \begin{array}{c} 1 \\ \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array} \quad \text{or} \quad \begin{array}{c} 1 \\ \circ \dots - \circ \begin{array}{l} \circ \\ \circ \end{array} \end{array} \quad (n \text{ odd}, n \geq 5).$$

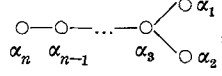
Suppose  $K = E_6$ . We number the simple roots  as before. If  $r = 1$ ,

then  $\lambda = \xi_1 + \xi_5$ :  and  $2\lambda - \alpha_1 = \xi_2 + 2\xi_5$ : .

Then we calculate  $\deg \pi_\lambda = 650$  and  $\deg \pi_{2\lambda - \alpha_1} = 78\,975$ . Thus  $\frac{1}{2}(\deg \pi_\lambda)^2 - \frac{1}{2} \deg \pi_\lambda = 210\,925 > 158\,028 = 2 \deg \pi_{2\lambda - \alpha_1} + \dim K$ . Now  $r \neq 1$ . If  $r = 2$ , then

$\lambda = \xi_2 + \xi_4$ :  and  $2\lambda - \alpha_2 = \xi_1 + \xi_3 + 2\xi_4$ : .

We then calculate  $\deg \pi_\lambda = 70\,070$  and  $\deg \pi_{2\lambda - \alpha_2} = 252\,808\,452$ . Thus  $\frac{1}{2}(\deg \pi_\lambda)^2 - \frac{1}{2} \deg \pi_\lambda = 2\,454\,867\,415 > 505\,616\,982 = 2 \deg \pi_{2\lambda - \alpha_2} + \dim K$ . We conclude  $K \neq E_6$ .

Now suppose  $K = D_n$ ,  $n \geq 5$ . We number the simple roots , and will prove by induction on  $n$  that

$$(*) \quad \frac{1}{2}(\deg \pi_{\xi_1 + \xi_2})^2 > \frac{1}{2} \deg \pi_{\xi_1 + \xi_2} + 2 \deg \pi_{2\xi_1 + 2\xi_2 - \alpha_1} + \dim D_n.$$

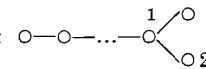
For  $n = 5$  we have  $\deg \pi_{\xi_1 + \xi_2} = 210$  and  $\deg \pi_{2\xi_1 + 2\xi_2 - \alpha_1} = 6\,930$ ; thus  $\frac{1}{2}(\deg \pi_{\xi_1 + \xi_2})^2 = 22\,050 > 14\,010 = \frac{1}{2} \deg \pi_{\xi_1 + \xi_2} + 2 \deg \pi_{2\xi_1 + 2\xi_2 - \alpha_1} + \dim D_n$ . Now suppose  $n > 5$ . Let  $\tau_\nu$  denote the representation of highest weight  $\nu$  for the subgroup  $D_{n-1}$  with simple root system  $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ . Then we have multipliers defined by

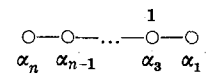
$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_1 + \xi_2})^2 &= t \cdot \frac{1}{2}(\deg \tau_{\xi_1 + \xi_2})^2, & \deg \pi_{\xi_1 + \xi_2} &= u \cdot \deg \tau_{\xi_1 + \xi_2}, \\ 2 \deg \pi_{2\xi_1 + 2\xi_2 - \alpha_1} &= v \cdot 2 \deg \tau_{2\xi_1 + 2\xi_2 - \alpha_1} \end{aligned}$$

and  $\dim D_n = w \cdot \dim D_{n-1}$ . From  $\pi_{\xi_1 + \xi_2} = \Lambda^{n-1}(\pi_{\xi_n})$ ,  $\tau_{\xi_1 + \xi_2} = \Lambda^{n-2}(\tau_{\xi_{n-1}})$  and  $\dim D_q = 2q^2 - q$ , we have

$$t = u^2, \quad u = \frac{2n(2n-1)}{(n+1)(n-1)} \quad \text{and} \quad w = \frac{2n^2 - n}{2n^2 - 5n + 3}.$$

The positive roots of  $D_n$  which contribute to  $v$  are (a) those with  $a_n > 0, a_3 > 0$  and  $a_2 = 0$ ,

and (b) those with  $a_n > 0$  and  $a_2 > 0$ . As  $2\xi_1 + 2\xi_2 - \alpha_1$ : , those satisfy-

ing (a) form a system  and contribute a factor of  $\binom{n}{2} / \binom{n-1}{2} = \frac{n}{n-2}$

to  $v$ . The roots  $\alpha > 0$  which satisfy (b) are  $\{\alpha_2 + \alpha_3 + \dots + \alpha_n, \alpha_1 + \alpha_2 + \dots + \alpha_n, \alpha_1 + \alpha_2 + \dots + \alpha_n + \alpha_2 + \dots + \alpha_n\}$ .

$2\alpha_3 + \alpha_4 + \dots + \alpha_n, \dots, \alpha_1 + \alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-1} + \alpha_n\}$ . They contribute to  $v$  a factor of

$$\frac{n+2}{n-1} \cdot \frac{n+3}{n} \left\{ \frac{n+5}{n+1} \cdots \frac{2n+1}{2n-3} \right\} = \frac{4(2n-1)(2n+1)}{(n+1)(n+4)}.$$

Thus 
$$v = \frac{4n(4n^2-1)}{(n-2)(n+1)(n+4)}.$$

Notice  $n(2n-1) = 2n^2 - n > 2n^2 - n - 1 = (2n+1)(n-1)$ , and  $(n-2)(n+4) = n^2 + 2n - 8 > n^2 - 1$  as  $n > 5$ ; now this shows  $t > v$ . And  $t > u$  and  $t > w$  are clear. Thus, using the induction hypothesis on  $\mathbf{D}_{n-1}$ , we have  $\frac{1}{2}(\deg \pi_{\xi_1 + \xi_2})^2 = t \cdot \frac{1}{2}(\deg \tau_{\xi_1 + \xi_2})^2 > t \cdot \frac{1}{2} \deg \tau_{\xi_1 + \xi_2} + t \cdot 2 \deg \tau_{2\xi_1 + 2\xi_2 - \alpha_1} + t \cdot \dim \mathbf{D}_{n-1} > \frac{1}{2} \deg \pi_{\xi_1 + \xi_2} + 2 \deg \tau_{2\xi_1 + 2\xi_2 - \alpha_1} + \dim \mathbf{D}_n$ . Now (\*) is proved. This shows  $K \neq \mathbf{D}_n$ , completing the proof of the lemma, q.e.d.

8.8. LEMMA. A representation  $\pi_\lambda$  of  $\mathrm{SU}(n+1)$ ,  $\lambda = \xi_r + \xi_r^*$  with  $\xi_r \neq \xi_r^*$ , satisfies

$$\frac{1}{2}(\deg \pi_\lambda)^2 = \frac{1}{2} \deg \pi_\lambda + 2 \deg \pi_{2\lambda - \alpha_r} + \dim \mathrm{SU}(n+1) \text{ if and only if } \lambda: \overset{1}{\circ} - \dots - \overset{1}{\circ}.$$

*Proof.* We label the simple roots  $\overset{\circ}{\alpha_1} - \overset{\circ}{\alpha_2} - \dots - \overset{\circ}{\alpha_n}$ . Now  $\xi_r^* = \xi_{n+1-r}$ , so we may assume

$$1 \leq r \leq \frac{1}{2}n.$$

First suppose  $r=1$ . If  $n=2$  then  $\deg \pi_\lambda = 8$  and  $\deg \pi_{2\lambda - \alpha_1} = 10$ ; thus  $\frac{1}{2}(\deg \pi_\lambda)^2 = 32 = 4 + 2 \cdot 10 + 8 = \frac{1}{2} \deg \pi_\lambda + 2 \deg \pi_{2\lambda - \alpha_1} + \dim \mathbf{A}_2$ . Now let  $n > 2$ .  $\pi_\lambda$  is the adjoint representation,

$$\deg \pi_\lambda = n^2 + 2n = \dim \mathrm{SU}(n+1).$$

$2\lambda - \alpha_1 = \xi_2 + 2\xi_n$ ; we compute  $\deg \pi_{2\lambda - \alpha_1} = \frac{1}{4}(n-1)n(n+2)(n+3)$ . Thus  $\frac{1}{2}(\deg \pi_\lambda)^2 = \frac{1}{2}n^2(n+2)^2 = \frac{1}{2}n(n+2)\{1 + (n-1)(n+3) + 2\} = \frac{1}{2} \deg \pi_\lambda + 2 \deg \pi_{2\lambda - \alpha_1} + \dim \mathrm{SU}(n+1)$ . This proves the equality for  $r=1$ .

Suppose  $r > 1$  and let  $\mathrm{SU}(n-1)$  denote the subgroup of that type with simple root system  $\{\alpha_2, \dots, \alpha_{n-1}\}$ . Let  $\tau_\nu$  denote the representation of  $\mathrm{SU}(n-1)$  with highest weight  $\nu$ . Finally define multipliers by

$$\deg \pi_\lambda = x \cdot \deg \tau_\lambda, \quad \deg \pi_{2\lambda - \alpha_r} = y \cdot \deg \tau_{2\lambda - \alpha_r}, \quad \text{and} \quad \dim \mathrm{SU}(n+1) = z \cdot \dim \mathrm{SU}(n-1). \quad (8.9)$$

Let  $r = \frac{1}{2}n$ . If  $r=2$  then  $\lambda: \overset{1}{\circ} - \overset{1}{\circ} - \overset{\circ}{\circ} - \overset{\circ}{\circ}$  and  $2\lambda - \alpha_r: \overset{1}{\circ} - \overset{3}{\circ} - \overset{\circ}{\circ} - \overset{\circ}{\circ}$ . For that case we calculate  $\frac{1}{2}(\deg \pi_\lambda)^2 = \frac{1}{2}(75)^2 > \frac{1}{2}(75) + 2(700) + 24 = \frac{1}{2} \deg \pi_\lambda + 2 \deg \pi_{2\lambda - \alpha_r} + \dim \mathrm{SU}(n+1)$ . Now suppose  $r > 2$ . The roots of  $\mathrm{SU}(n+1)$  which are not roots of  $\mathrm{SU}(n-1)$  are the  $\alpha_1 + \dots + \alpha_m$  and the  $\alpha_i + \dots + \alpha_n$ . Thus



$$\begin{aligned}
 x &= \left\{ \frac{r+1}{r} \cdot \frac{r+3}{r+1} \cdot \frac{r+4}{r+2} \cdots \frac{2r+1}{2r-1} \right\}^2 \cdot \frac{2r+2}{2r} = 4 \frac{(r+1)(2r+1)^2}{r(r+2)^2} \text{ and} \\
 y &= \left\{ \frac{r}{r-1} \cdot \frac{r+1}{r} \cdot \frac{r+5}{r+1} \right\} \left\{ \frac{r+3}{r} \cdot \frac{r+4}{r+1} \right\} \left\{ \frac{r+6}{r+2} \cdot \frac{r+7}{r+3} \cdots \frac{2r+3}{2r-1} \right\}^2 \frac{2r+4}{2r} \\
 &= 16 \frac{r+1}{r-1} \cdot \frac{(2r+1)^2 (2r+3)^2}{(r+2)(r+3)(r+4)(r+5)}.
 \end{aligned}$$

By expanding we check  $(r-1)(r+3)(r+4) \geq r(r+2)^2$  and  $(2r+1)^2(r+5)(r+1) > r(r+2)(2r+3)^2$ ; it follows that  $x^2 > y$ . That  $x^2 > x$  and  $x^2 > z$  are clear.

Now suppose  $1 < r < n/2$ . Then we compute

$$x = \left\{ \frac{r+1}{r} \cdot \frac{r+2}{r+1} \cdots \frac{n-r+1}{n-r} \right\}^2 \left\{ \frac{n-r+3}{n-r+1} \cdot \frac{n-r+4}{n-r+2} \cdots \frac{n+1}{n-1} \right\}^2 \cdot \frac{n+2}{n} = \frac{n(n+1)^2(n+2)}{r^2(n-r+2)^2}$$

and

$$\begin{aligned}
 y &= \left\{ \frac{r}{r-1} \cdot \frac{r+1}{r} \cdot \frac{r+3}{r+1} \cdot \frac{r+4}{r+2} \cdots \frac{n-r+2}{n-r} \cdot \frac{n-r+5}{n-r+1} \right\} \left\{ \frac{r+2}{r} \cdot \frac{r+3}{r+1} \cdots \frac{n-r+1}{n-r-1} \cdot \frac{n-r+3}{n-r} \right. \\
 &\quad \left. \cdot \frac{n-r+4}{n-r+1} \right\} \cdot \left\{ \frac{n-r+6}{n-r+2} \cdot \frac{n-r+7}{n-r+3} \cdots \frac{n+3}{n-1} \right\}^2 \cdot \frac{n+4}{n} \\
 &= \frac{n(n+1)^2(n+2)^2(n+3)^2(n+4)}{(r-1)r(r+1)(r+2)(n-r+2)(n-r+3)(n-r+4)(n-r+5)}.
 \end{aligned}$$

An extremely unpleasant expansion shows  $x^2 > y$ . Again  $x^2 > x$  and  $x^2 > z$  are clear. We have proved:

$$x^2 > y, \quad x^2 > x \quad \text{and} \quad x^2 > z \quad \text{for} \quad 2 \leq r \leq \frac{1}{2}n. \quad (8.10)$$

We have proved  $\frac{1}{2}(\deg \tau_\lambda)^2 = \frac{1}{2} \deg \tau_\lambda + 2 \deg \tau_{2\lambda - \alpha_r} + \dim \text{SU}(n-1)$  for  $r=2$ . By induction, we have  $\frac{1}{2}(\deg \tau_\lambda)^2 > \frac{1}{2} \deg \tau_\lambda + 2 \deg \tau_{2\lambda - \alpha_r} + \dim \text{SU}(n-1)$  for  $r > 2$ . Now (8.9) and (8.10) give us  $\frac{1}{2}(\deg \pi_\lambda)^2 > \frac{1}{2} \deg \pi_\lambda + 2 \deg \pi_{2\lambda - \alpha_r} + \dim \text{SU}(n+1)$  for  $r > 1$ , q.e.d.

Theorem 8.1 is immediate from Lemmas 8.3, 8.7 and 8.8.

## 9. The estimate for the case where $G$ is orthogonal with $\chi$ absolutely irreducible

The estimate is:

**9.1 PROPOSITION.** *Let  $\xi_r$  be a basic weight of a compact connected simple Lie group  $K$ . For every integer  $m \geq 1$ , define*

$$U_m = \frac{1}{2} (\deg \pi_{m\xi_r})^2, \quad V_m = \frac{1}{2} \deg \pi_{m\xi_r} \quad \text{and} \quad W_m = \deg \pi_{2m\xi_r - \alpha_r}.$$

If  $m \geq 2$  then  $U_{m+1} - V_{m+1} - W_{m+1} > U_m - V_m - W_m$ . If  $\pi_{\xi_r}(K) \not\cong \mathbf{SO}(\deg \pi_{\xi_r})$  in case  $\alpha_r$  is a terminal vertex<sup>(1)</sup> on the Dynkin diagram of  $K$ , then  $U_2 - V_2 - W_2 > U_1 - V_1 - W_1$ .

*Proof.* We define multipliers by

$$U_{m+1} = u_m U_m, \quad V_{m+1} = v_m V_m \quad \text{and} \quad W_{m+1} = w_m W_m.$$

Given a root  $\alpha > 0$  we define  $a = \hat{a}_r$  and  $l = \hat{l}(\alpha)$ . Let  $S$  be the set of all simple roots adjacent to  $\alpha_r$  in the Dynkin diagram of  $K$ , and let  $\alpha_i \in S$ . Then  $n_i = \frac{2 \langle \alpha_i, 2m\xi_r - \alpha_r \rangle}{\langle \alpha_i, \alpha_i \rangle} = -\frac{2 \langle \alpha_i, \alpha_r \rangle}{\langle \alpha_i, \alpha_i \rangle}$  is 1 if  $\|\alpha_i\|^2 \geq \|\alpha_r\|^2$ , and is  $\frac{\|\alpha_r\|^2}{\|\alpha_i\|^2}$  otherwise. Observe  $k\xi_r - \alpha_r = (k-2)\xi_r + \sum_S n_i \xi_i$ . For our given  $\alpha$  we define  $s = \sum_S n_i \hat{a}_i$ . Now a glance at (5.6) shows

$$\begin{aligned} w_m &= \prod_{\alpha > 0} w_m(\alpha), & w_m(\alpha) &= \frac{2ma + s + l}{2(m-1)a + s + l} = 1 + \frac{2a}{2(m-1)a + s + l}, \\ v_m &= \prod_{\alpha > 0} v_m(\alpha) & v_m(\alpha) &= \frac{(m+1)a + l}{ma + l}, \quad \text{and} \\ u_m &= \prod_{\alpha > 0} u_m(\alpha), & u_m(\alpha) &= v_m(\alpha)^2 = 1 + \frac{(2m+1)a^2 + 2la}{(ma + l)^2}. \end{aligned}$$

If  $m \geq 2$  then  $\{(2m+1)a^2 + 2la\} \{2(m-1)a + s + l\} \geq 2a(ma + l)^2$ , with strict inequality when  $a > 0$ . Thus  $u_m(\alpha) \geq w_m(\alpha)$ , and  $u_m(\alpha) > w_m(\alpha)$  in case  $a > 0$ . Now

$$\text{if } m \geq 2 \text{ then } u_m > w_m. \quad (9.2)$$

Now let  $m = 1$ . We compute

$$u_1(\alpha) = 1 + \frac{3a^2 + 2la}{(a+l)^2} \quad \text{and} \quad w_1(\alpha) = 1 + \frac{2a}{s+l}.$$

Suppose  $a > 0$ . If  $s \geq \frac{2}{3}a$ , then  $3as + 2ls > al + 2a^2$ , so  $(3a^2 + 2la)(s+l) > 2a(a+l)^2$ . Thus

$$\text{if } s \geq \frac{2}{3}a > 0 \text{ then } u_1(\alpha) > w_1(\alpha). \quad (9.3)$$

Suppose  $s < \frac{2}{3}a$ . If  $a_r \geq 3$  then  $\alpha$  must be one of a few roots of exceptional groups, and one easily checks that

$$\alpha = 3\alpha_1 + \alpha_2 \quad \text{for } K = \mathbf{G}_2 \quad \begin{array}{c} \alpha_1 \\ \bullet \equiv \circ \\ \alpha_2 \end{array} \quad \text{and } r = 1 \quad (9.4)$$

is the only possibility. Now suppose  $a_r = 2$ . If  $\alpha_r$  is a terminal vertex of the Dynkin diagram

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<sup>(1)</sup> In other words, there is no condition if  $\alpha_r$  is interior to the Dynkin diagram. But if  $\alpha_r$  is not interior, then  $\pi_{\xi_r}$  must be orthogonal with image  $\neq \mathbf{SO}(\deg \pi_{\xi_r})$ .

and  $\alpha_0$  denotes the unique adjacent root, then  $s < \frac{2}{3}a$  says  $n_0 a_0 \|\alpha_0\|^2 < \frac{4}{3} \|\alpha_r\|^2$ . If  $\|\alpha_r\|^2 \geq \|\alpha_0\|^2$  then  $n_0$  is the quotient so  $a_0 < \frac{4}{3}$ . As  $a_0 \neq 0$  because  $2\alpha_r$  is not a root, now  $a_0 = 1$  and  $\alpha_0 + 2\alpha_r$  is a root, which contradicts  $\|\alpha_r\|^2 \geq \|\alpha_0\|^2$ . Now  $\|\alpha_r\|^2 < \|\alpha_0\|^2$ , so  $n_0 = 1$  and  $a_0 < \frac{4}{3} \|\alpha_r\|^2 / \|\alpha_0\|^2 \leq \frac{2}{3}$ . That says  $a_0 = 0$ , contradicting the fact that  $2\alpha_r$  is not a root. Thus  $\alpha_r$  must be interior to the Dynkin diagram of  $K$ , and by the argument for terminal vertices we have  $\alpha_1$  and  $\alpha_2$  in  $S$  with  $a_1 > 0$  and  $a_2 > 0$ . If  $\|\alpha_r\|^2 \geq \|\alpha_1\|^2$  then  $a_1 \|\alpha_r\|^2 = n_1 \hat{a}_1 < s < \frac{4}{3} \|\alpha_r\|^2$  so  $a_1 = 1$ . If also  $\|\alpha_r\|^2 \geq \|\alpha_2\|^2$  then  $a_2 = 1$  and  $s \geq 2 \|\alpha_r\|^2 > \frac{2}{3}a$ ; thus  $\|\alpha_r\|^2 < \|\alpha_2\|^2$ , and now  $\|\alpha_2\|^2 = 2 \|\alpha_r\|^2$  because  $\text{rank } K \geq 3$ . We calculate  $\|\alpha_r\|^2 + 2a_2 \|\alpha_r\|^2 = n_2 \hat{a}_2 \leq s < \frac{4}{3} \|\alpha_r\|^2$ ; that is impossible. Thus  $\|\alpha_r\|^2 < \|\alpha_1\|^2$ . Similarly  $\|\alpha_r\|^2 < \|\alpha_2\|^2$ . But  $\begin{smallmatrix} \alpha_1 & \alpha_r & \alpha_2 \\ \circ = & \bullet = & \circ \end{smallmatrix}$  cannot be contained in a Dynkin diagram. We have proved  $a_r \neq 2$ . Finally suppose  $a_r = 1$ . Then  $s < \frac{2}{3}a$  says  $s < \frac{2}{3} \|\alpha_r\|^2$ . Let  $\alpha_0 \in S$ . If  $\|\alpha_r\|^2 \geq \|\alpha_0\|^2$  then  $n_0 \hat{a}_0 = a_0 \|\alpha_r\|^2 \leq s < \frac{2}{3} \|\alpha_r\|^2$ ; thus  $a_0 = 0$ . If  $\|\alpha_r\|^2 < \|\alpha_0\|^2$  then  $n_0 \hat{a}_0 > a_0 \|\alpha_r\|^2$  so again  $a_0 = 0$ . We have proved:

$$\text{if } s < \frac{2}{3}a, \text{ then either } \alpha = \alpha_r \text{ or } \alpha \text{ is given by (9.4).} \quad (9.5)$$

We eliminate the odd case (9.4). There  $\deg \pi_{\xi_1} = 7$  and  $\deg \pi_{2\xi_1} = 27$ , so  $u_1 = 729/49$ . Also  $2\xi_1 - \alpha_1 = \xi_2$  and  $\deg \pi_{\xi_2} = 14$ , and  $4\xi_1 - \alpha_1 = 2\xi_1 + \xi_2$  and  $\pi_{2\xi_1 + \xi_2}$  has degree 189, so  $w_1 = 189/14$ . As  $729 \cdot 14 = 10\,206 > 9\,261 = 49 \cdot 189$ , this shows:

$$\text{if } K = \mathbf{G}_2 \quad \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \bullet = & \circ \end{smallmatrix} \text{ and } r = 1, \text{ then } u_1 > w_1. \quad (9.6)$$

Now we need roots to overcome  $\alpha_r$ . We look for a set  $\Gamma$  of positive roots such that

$$\alpha_r \in \Gamma \quad \text{and} \quad \prod_{\Gamma} u_1(\alpha) > \prod_{\Gamma} w_1(\alpha). \quad (9.7)$$

If  $\text{rank } K \geq 4$  and  $\alpha_r$  is not a terminal vertex on the Dynkin diagram of  $K$ , then we choose a subdiagram  $\Delta$  of rank 4 with  $\alpha_r$  interior to  $\Delta$ . We run through the possibilities for  $\Delta$ .

(1)  $\Delta$ :  $\begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_r & \alpha_4 \\ \circ - & \circ - & \circ - & \circ \end{smallmatrix}$ . Here we define  $\Gamma = \{\alpha_r, \alpha_2 + \alpha_r, \alpha_r + \alpha_4, \alpha_2 + \alpha_r + \alpha_4, \alpha_1 + \alpha_2 + \alpha_r + \alpha_4\}$  and calculate  $\prod_{\Gamma} u_1(\alpha) = 16 > 140/9 = \prod_{\Gamma} w_1(\alpha)$ .

(2)  $\Delta$ : contains  $\begin{smallmatrix} \alpha_1 & \alpha_r & \alpha_2 \\ \circ - & \circ = & \bullet \end{smallmatrix}$ . Define  $\Gamma = \{\alpha_r, \alpha_r + \alpha_2, \alpha_r + 2\alpha_2, \alpha_1 + \alpha_r, \alpha_1 + \alpha_r + \alpha_2, \alpha_1 + \alpha_r + 2\alpha_2, \alpha_1 + 2\alpha_r + 2\alpha_2\}$ . Then  $\prod_{\Gamma} u_1(\alpha) = 49 > 286/7 = \prod_{\Gamma} w_1(\alpha)$ .

(3)  $\Delta$ : contains  $\begin{smallmatrix} \alpha_1 & \alpha_r & \alpha_2 \\ \bullet - & \bullet = & \circ \end{smallmatrix}$ . Define  $\Gamma = \{\alpha_r, \alpha_1 + \alpha_r, \alpha_r + \alpha_2, 2\alpha_r + \alpha_2, \alpha_1 + \alpha_r + \alpha_2, \alpha_1 + 2\alpha_r + \alpha_2, 2\alpha_1 + 2\alpha_r + \alpha_2\}$ . Then  $\prod_{\Gamma} u_1(\alpha) = 2025/49 > 286/7 = \prod_{\Gamma} w_1(\alpha)$ .

(4)  $\Delta$ :  $\begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_r & \alpha_4 \\ \bullet = & \circ - & \circ - & \circ \end{smallmatrix}$ . Here we imitate case (1), defining  $\Gamma = \{\alpha_r, \alpha_2 + \alpha_r, \alpha_r + \alpha_4, \alpha_2 + \alpha_r + \alpha_4, 2\alpha_1 + \alpha_2 + \alpha_r + \alpha_4\}$  and calculating  $\prod_{\Gamma} u_1(\alpha) = 16 > 140/9 = \prod_{\Gamma} w_1(\alpha)$

(5)  $\Delta$ :  $\overset{\alpha_1}{\circ} = \overset{\alpha_2}{\bullet} - \overset{\alpha_r}{\bullet} - \overset{\alpha_4}{\bullet}$ . Again we imitate case (1), defining  $\Gamma = \{\alpha_r, \alpha_2 + \alpha_r, \alpha_r + \alpha_4, \alpha_2 + \alpha_r + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_r + 2\alpha_4\}$  and calculating  $\prod_{\Gamma} u_1(\alpha) = 16 > 140/9 = \prod_{\Gamma} w_1(\alpha)$ .

(6)  $\Delta$ :  $\overset{\alpha_1}{\circ} - \overset{\alpha_r}{\circ} \begin{cases} \overset{\alpha_2}{\circ} \\ \overset{\alpha_3}{\circ} \end{cases}$ . Define  $\Gamma = \{\alpha_r; \alpha_1 + \alpha_r, \alpha_2 + \alpha_r, \alpha_3 + \alpha_r; \alpha_1 + \alpha_2 + \alpha_r, \alpha_2 + \alpha_3 + \alpha_r, \alpha_3 + \alpha_1 + \alpha_r; \alpha_1 + \alpha_2 + \alpha_3 + \alpha_r\}$ . Then  $\prod_{\Gamma} u_1(\alpha) = 625/9 > 49 = \prod_{\Gamma} w_1(\alpha)$ .

This covers all possibilities for  $\Delta$ . We have proved: if  $\text{rank } K \geq 4$  and  $\alpha_r$  is interior to the Dynkin diagram of  $K$ , then there exists a set  $\Gamma$  satisfying (9.7). In fact, in considering cases (2) and (3), we also proved this for  $K$  of type  $B_3$  or  $C_3$ . As  $A_3$  is the only other type of rank 3, and as the rank must be at least 3 if there is to be a root interior to the diagram we summarize as follows.

*If  $K \neq A_3$  and  $\alpha_r$  is interior to the diagram, then a set  $\Gamma$  exists satisfying (9.7).* (9.8)

Now let  $\alpha_r$  be a terminal vertex on the Dynkin diagram of  $K$ . We examine some possibilities  $\Psi$  for a subdiagram containing  $\alpha_r$ ; the numbering is continued from the cases for  $\Delta$  listed above.

(7)  $\Psi$ :  $\overset{\alpha_r}{\circ} \begin{cases} \overset{\alpha_3}{\circ} \\ \overset{\alpha_2}{\circ} \end{cases} - \overset{\alpha_4}{\circ} - \overset{\alpha_5}{\circ}$ . Here we define  $\Gamma = \{\alpha_r, \alpha_r + \alpha_3, \alpha_r + \alpha_2 + \alpha_3, \alpha_r + \alpha_2 + \alpha_3 + \alpha_4, \alpha_r + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_r + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_r + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \alpha_r + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$ . Then  $\prod_{\Gamma} u_1(\alpha) = 441/16 > 55/2 = \prod_{\Gamma} w_1(\alpha)$ .

(8)  $\Psi$ :  $\overset{\alpha_r}{\circ} - \overset{\circ}{\circ} - \overset{\circ}{\circ} - \overset{\circ}{\circ} - \overset{\circ}{\circ}$ . In this case we define  $\Gamma$  to be all positive roots on the system  $\Psi$  which involve  $\alpha_r$ . Then  $\prod_{\Gamma} u_1(\alpha) = 169 > 154 = \prod_{\Gamma} w_1(\alpha)$ .

(9)  $\Psi$ :  $\overset{\alpha_r}{\bullet} = \overset{\alpha_2}{\circ} - \overset{\alpha_1}{\circ}$ . We define  $\Gamma = \{\alpha_r, \alpha_r + \alpha_2, 2\alpha_r + \alpha_2, \alpha_r + \alpha_2 + \alpha_1, 2\alpha_r + \alpha_2 + \alpha_1, 2\alpha_r + 2\alpha_2 + \alpha_1\}$  and calculate  $\prod_{\Gamma} u_1(\alpha) = 1225/64 > 18 = \prod_{\Gamma} w_1(\alpha)$ .

(10)  $\Psi$ :  $\overset{\alpha_r}{\circ} = \overset{\alpha_2}{\bullet} - \overset{\alpha_1}{\bullet}$ . We define  $\Gamma = \{\alpha_r, \alpha_r + \alpha_2, \alpha_r + \alpha_2 + \alpha_1, \alpha_r + 2\alpha_2, \alpha_r + 2\alpha_2 + \alpha_1, \alpha_r + 2\alpha_2 + 2\alpha_1\}$  and calculate  $\prod_{\Gamma} u_1(\alpha) = 36 > 273/10 = \prod_{\Gamma} w_1(\alpha)$ .

(11)  $\Psi$ :  $\overset{\alpha_r}{\bullet} - \overset{\bullet}{\bullet} = \overset{\circ}{\circ} - \overset{\circ}{\circ}$ . Then  $u_1 = \frac{9^3 \cdot 11^4}{2^4 \cdot 5^2 \cdot 13^2} > \frac{1287}{10} = w_1$ .

(12)  $\Psi$ :  $\overset{\bullet}{\bullet} - \overset{\bullet}{\bullet} = \overset{\circ}{\circ} - \overset{\alpha_r}{\circ}$ . Then  $u_1 = \frac{6561}{16} > \frac{26163}{98} = w_1$ .

(13)  $\Psi$ :  $\overset{\bullet}{\bullet} \equiv \overset{\circ}{\circ} \alpha_r$ . Then  $u_1 = \frac{121}{4} > \frac{221}{11} = w_1$ .

$$(14) \Psi: \begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array} \alpha_r. \text{ Then } u_1 = 15^6 > \frac{3^5 \cdot 5^2 \cdot 13 \cdot 23}{2^2 \cdot 7^2} = w_1.$$

If the terminal vertex  $\alpha_r$  is not contained as shown in one of the configurations  $\Psi$  just considered, then the position of  $\alpha_r$  in the Dynkin diagram of  $K$  must (by classification) be one of the following

$$(u) \begin{array}{c} \alpha_r \\ \circ - \dots - \circ \end{array}, \text{ rank } K > 1$$

$$(f_1) \begin{array}{c} \alpha_r \\ \circ - \dots - \circ = \bullet \end{array}, \text{ rank } K > 1$$

$$(f_2) \begin{array}{c} \alpha_r \\ \circ - \dots - \circ \begin{array}{l} / \circ \\ \backslash \circ \end{array} \end{array}, \text{ rank } K > 3$$

$$(s_1) \begin{array}{c} \alpha_r \\ \bullet - \dots - \bullet = \circ \end{array}, \text{ rank } K > 1$$

$$(s_2) \begin{array}{c} \alpha_r \\ \circ \end{array}, \text{ rank } K = 1$$

$$(s_3) \begin{array}{c} \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array} \alpha_r$$

except for the case which is settled by (9.6). In case (u),  $\pi_{\xi_r}$  is unitary, not self dual, hence not orthogonal. In cases (f<sub>i</sub>),  $\pi_{\xi_r}(K)$  is the full  $\mathbf{SO}$  (deg  $\pi_{\xi_r}$ ). In cases (s<sub>i</sub>),  $\pi_{\xi_r}$  is symplectic, hence not orthogonal. Thus those cases are excluded by hypothesis in considering the inequality of Proposition 9.1 for  $m=1$ . We summarize as follows.

(9.9) *If  $\alpha_r$  is a terminal vertex with  $\pi_{\xi_r}(K) \not\subseteq \mathbf{SO}$  (deg  $\pi_{\xi_r}$ ), then a set  $\Gamma$  exists satisfying (9.7).*

Combine (9.3), (9.5), (9.8) and (9.9), using the fact that  $u_1(\alpha) = 1 = w_1(\alpha)$  whenever  $a=0$ . This gives:

(9.10) *If  $\xi_r \neq \circ - \overset{1}{\circ} - \circ$ , and if  $\pi_{\xi_r}(K) \subseteq \mathbf{SO}$  (deg  $\pi_{\xi_r}$ ) in case  $\alpha_r$  is terminal, then  $u_1 > w_1$ .*

We also notice

(9.11) *If  $\xi_r = \circ - \overset{1}{\circ} - \circ$ , then  $U_2 - V_2 - W_2 = 15 > 0 = U_1 - V_1 - W_1$ .*

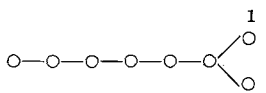
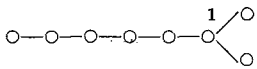




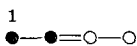
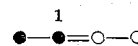
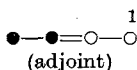
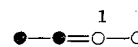
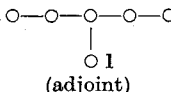
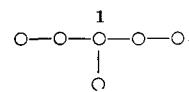
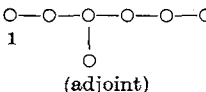
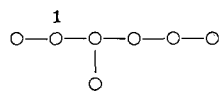
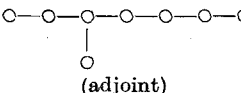
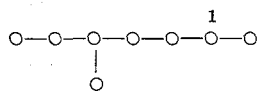
We complete the proof of Proposition 9.1. Let  $m \geq 1$  be an integer. If  $m=1$ , suppose  $\xi_r \neq \circ - \overset{1}{\circ} - \circ$ , and further assume  $\pi_{\xi_r}(K) \subseteq \mathbf{SO}$  (deg  $\pi_{\xi_r}$ ) if  $\alpha_r$  is a terminal vertex. Then  $u_m > w_m$  by (9.2) and (9.10). Notice also  $u_m > v_m > 1$ . Now  $U_{m+1} - V_{m+1} - W_{m+1} = u_m U_m - v_m V_m - w_m W_m > u_m (U_m - V_m - W_m) > U_m - V_m - W_m$ . With (9.11), this proves our assertions, q.e.d.

**10. The classification for the case where  $G$  is orthogonal with  $\chi$  absolutely irreducible**

The classification is:

**10.1 THEOREM.** *Let  $G$  be a simple special orthogonal group and let  $K$  be a proper compact connected simple subgroup. Then the representation  $\chi$  of  $K$  on the tangent space of  $G/K$  is absolutely irreducible, if and only if  $(G, K)$  is one of the following*

$G$	$K$	$\pi_\lambda$	$\chi$
$SO(5)$	$SU(2)/Z_2$	$\overset{4}{\circ}$	$\overset{6}{\circ}$
$SO(20)$	$SU(4)/Z_4$	$\overset{2}{\circ}-\overset{2}{\circ}-\overset{2}{\circ}$	$\overset{1}{\circ}-\overset{2}{\circ}-\overset{1}{\circ}$
$SO(70)$	$SU(8)/Z_4$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$
$SO(2n^2+n)$	$SO(2n+1)$	$\overset{2}{\circ}=\bullet$ if $n=2$ $\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\bullet$ if $n>2$ (adjoint)	$\overset{1}{\circ}=\bullet$ if $n=2$ $\overset{1}{\circ}-\overset{2}{\circ}=\bullet$ if $n=3$ $\overset{2}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\bullet$ if $n>3$
$SO(2n^2+3n)$	$SO(2n+1)$	$\overset{2}{\circ}-\dots-\overset{2}{\circ}=\bullet$ $n \geq 2$	$\overset{2}{\circ}=\bullet$ if $n=2$ $\overset{2}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\bullet$ if $n>2$
$SO(16)$	$Spin(9)$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}=\bullet$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}=\bullet$
$SO(2n^2-n-1)$	$Sp(n)/Z_2$	$\bullet-\overset{1}{\circ}-\dots-\bullet=\overset{1}{\circ}$ $n \geq 3$	$\bullet-\bullet-\overset{1}{\circ}=\overset{1}{\circ}$ if $n=3$ $\bullet-\bullet-\bullet-\dots-\bullet=\overset{1}{\circ}$ if $n>3$
$SO(2n^2+n)$	$Sp(n)/Z_2$	$\overset{2}{\bullet}-\dots-\bullet=\overset{2}{\circ}$ $n \geq 3$ (adjoint)	$\overset{2}{\bullet}-\bullet-\dots-\bullet=\overset{2}{\circ}$ $n \geq 3$
$SO(42)$	$Sp(4)/Z_2$	$\bullet-\bullet-\bullet=\overset{1}{\circ}$	$\bullet-\bullet-\bullet=\overset{2}{\circ}$
$SO(2n^2-n)$	$SO(2n)/Z_2$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\overset{1}{\circ}$ $n \geq 4$ (adjoint)	$\overset{1}{\circ}-\overset{1}{\circ}=\overset{1}{\circ}$ if $n=4$ $\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\overset{1}{\circ}$ if $n>4$
$SO(2n^2+n-1)$	$SO(2n)/Z_2$	$\overset{2}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\overset{1}{\circ}$ $n \geq 4$	$\overset{2}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}=\overset{1}{\circ}$ $n \geq 4$

$G$	$K$	$\pi_\lambda$	$\chi$
SO (128)	Spin (16)/Z <sub>2</sub>		
SO (7)	G <sub>2</sub>		
SO (14)	G <sub>2</sub>		
SO (26)	F <sub>4</sub>		
SO (52)	F <sub>4</sub>		
SO (78)	E <sub>6</sub> /Z <sub>3</sub>		
SO (133)	E <sub>7</sub> /Z <sub>2</sub>		
SO (248)	E <sub>8</sub>		

where the inclusion  $K \rightarrow G = \mathbf{SO}(N) \subset \mathbf{GL}(N, \mathbf{C})$  is the absolutely irreducible representation  $\pi_\lambda$  of  $K$  with highest weight  $\lambda$  indicated in the chart.

Remark.  $\mathbf{SO}(5)/\{\mathbf{SU}(2)/\mathbf{Z}_2\} = \{\mathbf{Sp}(2)/\mathbf{Z}_2\}/\{\mathbf{SU}(2)/\mathbf{Z}_2\}$ .

If  $K$  is not of type  $A_n$  then we notice that the adjoint representation of  $K$  is one of the possibilities for  $\pi_\lambda$ . Combining this with Theorem 8.1 and with the fact that  $\dim K \neq 4$  for  $K$  simple, we have the following.

10.2 COROLLARY. Let  $K$  be a compact connected simple Lie group,  $n = \dim K$ . Let  $\pi: K \rightarrow \mathbf{SO}(n)$  denote the adjoint representation and let  $\chi$  denote the representation of  $K$  on the tangent space of  $\mathbf{SO}(n)/\pi(K)$ . Then  $\chi$  is irreducible over the real number field, and  $\chi$  is absolutely irreducible if and only if  $K$  is not of type  $A_l$  for  $l \geq 2$ .

We go on to prove Theorem 10.1.

Let  $\chi$  be absolutely irreducible. We will eliminate all possibilities for  $\pi_\lambda$  except those listed in the theorem, and in the process we will check that  $\chi$  is absolutely irreducible for the listed  $\pi_\lambda$ .

For the moment we set aside the case where  $K = \mathbf{G}_2$  and  $G = \mathbf{SO}(7)$ . Then Proposition 5.2 says that the inclusion  $K \rightarrow G$  is an absolutely irreducible representation  $\pi_{k\xi_r}$  for some basic weight  $\xi_r = \xi_r^*$  of  $K$ , and that

$$\frac{1}{2} (\deg \pi_{k\xi_r})^2 = \frac{1}{2} \deg \pi_{k\xi_r} + \deg \pi_{2k\xi_r - \alpha_r} + \dim K. \quad (10.3)$$

If  $\pi_{\xi_r}(K) \not\subseteq \mathbf{SO}(\deg \pi_{\xi_r})$ , then  $\frac{1}{2} (\deg \pi_{\xi_r})^2 + \frac{1}{2} \deg \pi_{\xi_r} \geq \deg \pi_{2\xi_r - \alpha_r} + \dim K$ , so Proposition 9.1 says  $k=1$ . If either  $\pi_{\xi_r}(K) = \mathbf{SO}(\deg \pi_{\xi_r})$  or  $\pi_{\xi_r}(K) \not\subseteq \mathbf{SO}(\deg \pi_{\xi_r})$ , then we still have  $\pi_{2\xi_r}(K) \subseteq \mathbf{SO}(\deg \pi_{2\xi_r})$  because  $\xi_r = \xi_r^*$ , so Proposition 9.1 says  $k=2$ . We now run through cases.  $\lambda$  denotes  $k\xi_r$ .

10.4 LEMMA. *If  $K$  is of type  $A_n$ , then (1)  $n=1$  with  $\lambda: \overset{4}{\circ}$ , or (2)  $n=3$  with  $\lambda: \overset{2}{\circ}-\overset{2}{\circ}-\circ$  or (3)  $n=7$  with  $\lambda: \circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ-\circ$ .*

*Remark.* In case (1),  $G/K = \mathbf{SO}(5)/(\mathbf{SU}(2)/\{\pm I\}) = \mathbf{Sp}(2)/\mathbf{SU}(2)$ ; in the latter,  $\mathbf{A}_1 \rightarrow \mathbf{C}_2$  is given by  $\overset{3}{\circ}$ .

*Proof.*  $\xi_r = \xi_r^*$  says that  $n=2r-1$  and  $K$  has diagram  $\overset{\alpha_1}{\circ}-\overset{\alpha_2}{\circ}-\dots-\overset{\alpha_r}{\circ}-\dots-\overset{\alpha_n}{\circ}$ . Now orthogonality of  $\pi_{k\xi_r}$  is equivalent to  $kr \equiv 0 \pmod{2}$ .

Let  $r=1$  so  $n=1$ . The representation with highest weight  $\overset{m}{\circ}$  has degree  $m+1$ . Now (10.3) becomes  $\frac{1}{2}(k+1)^2 = \frac{1}{2}(k+1) + (2k-1) + 3$ , which has solutions 4 and  $-1$ . Thus  $k=4$ ,  $\lambda: \overset{4}{\circ}$ .

Let  $r=2$  so  $n=3$ . Then  $\xi_r: \overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} = \overset{1}{\circ} \begin{matrix} \circ \\ \circ \end{matrix}$  so  $\pi_{\xi_r}(K) = \mathbf{SO}(6)$ . Thus  $k=2$ , and we check for  $\lambda: \overset{2}{\circ}-\overset{2}{\circ}-\circ$  that  $\frac{1}{2} (\deg \pi_\lambda)^2 = 200 = 10 + 175 + 15 = \frac{1}{2} (\deg \pi_\lambda) + \deg \pi_{2\lambda - \alpha_3} + \dim \mathbf{A}_3$ .

If  $r=3$  then  $n=5$  and  $kr \equiv 0 \pmod{2}$  says  $k=2$ . But for  $\lambda: \overset{2}{\circ}-\overset{2}{\circ}-\overset{2}{\circ}-\circ-\circ$  we compute  $\frac{1}{2} (\deg \pi_\lambda)^2 = \frac{1}{2}(175)^2 > \frac{1}{2}(175) + 11\,340 + 35 = \frac{1}{2} \deg \pi_\lambda + \deg \pi_{2\lambda - \alpha_5} + \dim \mathbf{A}_5$ .

If  $r=4$  then  $n=7$  and  $k=1$ . We check for  $\lambda: \overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\circ-\circ$  that  $\frac{1}{2} (\deg \pi_\lambda)^2 - \frac{1}{2} \deg \pi_\lambda = 2415 = 2352 + 63 = \deg \pi_{2\lambda - \alpha_4} + \dim \mathbf{A}_7$ .

Let  $r > 4$ ; we will prove  $\frac{1}{2} (\deg \pi_{\xi_r})^2 > \frac{1}{2} (\deg \pi_{\xi_r}) + \deg \pi_{2\xi_r - \alpha_r} + \dim \mathbf{A}_{2r-1}$ . Let  $\tau_\nu$  denote the representation of highest weight  $\nu$  for the subgroup  $\mathbf{A}_{2r-3}$  with simple root system



$\{\alpha_2, \alpha_3, \dots, \alpha_{2r-2}\}$ . By induction if  $r > 5$ , and as just seen if  $r = 5$ , we have  $\frac{1}{2} (\deg \tau_{\xi_r})^2 \geq \frac{1}{2} \deg \tau_{\xi_r} + \deg \tau_{2\xi_r - \alpha_r} + \dim \mathbf{A}_{2r-3}$ . Now define multipliers by

$$\deg \pi_{\xi_r} = x \cdot \deg \tau_{\xi_r}, \quad \deg \pi_{2\xi_r - \alpha_r} = y \cdot \deg \tau_{2\xi_r - \alpha_r} \quad \text{and} \quad \dim \mathbf{A}_{2r-1} = z \cdot \dim \mathbf{A}_{2r-3}.$$

We compute 
$$x = \left\{ \prod_{a_1 > 0 = a_n} \frac{a_r + l(\alpha)}{l(\alpha)} \right\} \cdot \left\{ \prod_{a_n > 0 = a_1} \frac{a_r + l(\alpha)}{l(\alpha)} \right\} \frac{n+1}{n} = \frac{4r-2}{r}.$$

To calculate  $y$  we note  $2\xi_r - \alpha_r = \xi_{r-1} + \xi_{r+1}$ . The roots involving  $\alpha_1$  and just one of  $\{\alpha_{r-1}, \alpha_{r+1}\}$  are  $\alpha_1 + \dots + \alpha_{r-1}$  and  $\alpha_1 + \dots + \alpha_r$ ; those involving  $\alpha_n$  and just one of  $\{\alpha_{r-1}, \alpha_{r+1}\}$  are  $\alpha_r + \dots + \alpha_n$  and  $\alpha_{r+1} + \dots + \alpha_n$ .

$$y = \left\{ \frac{r}{r-1} \cdot \frac{r+1}{r} \right\}^2 \cdot \left\{ \prod_{l=r+1}^{2r-2} \frac{l+2}{l} \right\}^2 \cdot \frac{2r+1}{2r-1} = \frac{4r^2(2r-1)(2r+1)}{(r-1)^2(r+2)^2}.$$

Observe (check for  $r = 5$  and differentiate; iterate three times) that  $(2r-1)(r-1)^2(r+2)^2 - r^4(2r+1) = 2r^4 - 8r^3 - 5r^2 + 12r - 4$  is positive for  $r \geq 5$ . It follows that  $x^2 > y$ . And  $x > 1$  gives  $x^2 > x > 1$ . Finally notice that  $z = (4r^2 - 1)/(4r^2 - 8r + 3) < x^2$ . Now  $\frac{1}{2} (\deg \pi_{\xi_r})^2 - \frac{1}{2} \deg \pi_{\xi_r} - \deg \pi_{2\xi_r - \alpha_r} - \dim \mathbf{A}_{2r-1} > x^2 \left\{ \frac{1}{2} (\deg \tau_{\xi_r})^2 - \frac{1}{2} \deg \tau_{\xi_r} - \deg \tau_{2\xi_r - \alpha_r} - \dim \mathbf{A}_{2r-3} \right\} \geq 0$ , which proves our assertion. Proposition 9.1 shows that we may replace  $\xi_r$  by any multiple and retain the inequality. Thus we cannot have  $r > 4$ , q.e.d.

**10.5. LEMMA.** *If  $K$  is of type  $B_n$ ,  $n \geq 2$ , then*

- (1)  $\pi_\lambda$  is the adjoint representation, given by  $\lambda: \circ = \overset{2}{\bullet}$  for  $n = 2$  and by  $\lambda: \circ - \overset{1}{\circ} - \dots - \circ = \bullet$  for  $n > 2$ , or
- (2)  $\pi_\lambda$  has degree  $2n^2 + 3n$ , given by  $\lambda: \overset{2}{\circ} - \circ - \dots - \circ = \bullet$ , or
- (3)  $n = 4$  and  $\pi_\lambda$  is the spin representation,  $\lambda: \circ - \circ - \circ = \overset{1}{\bullet}$ .

*Proof.* We first examine  $\mathbf{B}_2: \overset{\alpha_1}{\bullet} = \overset{\alpha_2}{\circ}$ . If  $\lambda = k\xi_1$  then  $k = 2$  because  $\pi_{\xi_1}$  is symplectic; we check  $\frac{1}{2} (\deg \pi_{2\xi_1})^2 = 50 = 5 + 35 + 10 = \frac{1}{2} (\deg \pi_{2\xi_1}) + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{B}_2$ . If  $\lambda = k\xi_2$  then  $k = 2$  because  $\pi_{\xi_2}(\mathbf{B}_2) = \mathbf{SO}(5)$ ; we check  $\frac{1}{2} (\deg \pi_{2\xi_2})^2 = 98 = 7 + 81 + 10 = \frac{1}{2} \deg \pi_{2\xi_2} + \deg \pi_{4\xi_2 - \alpha_2} + \dim \mathbf{B}_2$ . Our assertions are proved for  $n = 2$ .

Now assume  $n > 2$ . We number the simple roots by  $\overset{\alpha_1}{\bullet} = \overset{\alpha_2}{\circ} - \dots - \overset{\alpha_n}{\circ}$ . Suppose  $\lambda = \xi_1$ . Then  $\deg \pi_\lambda = 2^n$  and  $\deg \pi_{2\lambda - \alpha_1} = \deg \pi_{\xi_2} = \binom{2n+1}{n-1}$ , so (10.3) says  $\frac{1}{2} \{2^{2n} - 2^n\} - \left\{ \binom{2n+1}{n-1} + 2n^2 + n \right\} = 0$ . The only solution is  $n = 4$ , giving case (3) of the lemma. Now suppose  $\lambda = 2\xi_1$ . Let  $\tau_\nu$  denote the representation of highest weight  $\nu$  for the subgroup  $\mathbf{B}_{n-1}$  with simple root system  $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ . Define multipliers by

$$\deg \pi_{2\xi_1} = x \cdot \deg \tau_{2\xi_1}, \quad \deg \pi_{4\xi_1 - \alpha_1} = y \cdot \deg \tau_{4\xi_1 - \alpha_1} \quad \text{and} \quad \dim \mathbf{B}_n = z \cdot \dim \mathbf{B}_{n-1}.$$

The roots involving  $\alpha_1$  and  $\alpha_n$  are  $\{\alpha_1 + \dots + \alpha_n, 2\alpha_1 + \alpha_2 + \dots + \alpha_n, \dots, 2\alpha_1 + \dots + 2\alpha_{n-1} + \alpha_n\}$ . And  $\alpha_1 + \dots + \alpha_{n-1}$  is the only root involving  $\alpha_1$  and  $\alpha_{n-1}$  but not  $\alpha_n$ . Thus

$$x = \frac{2n+1}{2n-1} \left\{ \frac{2n+4}{2n} \cdot \frac{2n+6}{2n+2} \cdots \frac{4n}{4n-4} \right\} = 2 \frac{2n+1}{n+1};$$

$$y = \frac{2n}{2n-2} \cdot \frac{2n+3}{2n-1} \cdot \frac{2n+6}{2n} \left\{ \frac{2n+10}{2n+2} \cdot \frac{2n+12}{2n+4} \cdots \frac{4n+4}{4n-4} \right\} = 4 \frac{n(2n+1)(2n+3)}{(n-1)(n+2)(n+4)}$$

if  $n > 5$ ;

$$y = \frac{715}{63} \text{ if } n=5, \quad y=11 \text{ if } n=4, \quad y=\frac{54}{5} \text{ if } n=3.$$

If  $n > 5$  then  $(n-1)(n+4) > (n+1)^2$  and  $(n+2)(2n+1) > n(2n+3)$ , implying  $x^2 > y$ . If  $n=5$  then  $x^2 = 121/9 > 715/63 = y$ ; if  $n=4$  then  $x^2 = 324/25 > 11 = y$ ; if  $n=3$  then  $x^2 = 49/4 > 54/5 = y$ . Thus we have  $x^2 > y$ . Note that  $x > 1$  so  $x^2 > x > 1$  and that  $z = (2n^2 + n)/(2n^2 - 3n + 1) < 3 < x < x^2$ . By induction if  $n > 3$ , and as we proved if  $n=2$ ,  $\frac{1}{2}(\deg \tau_{2\xi_1})^2 \geq \frac{1}{2} \deg \tau_{2\xi_1} + \deg \tau_{4\xi_1 - \alpha_1} + \dim \mathbf{B}_{n-1}$ . Now  $\frac{1}{2}(\deg \pi_{2\xi_1})^2 - \{\frac{1}{2} \deg \pi_{2\xi_1} + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{B}_n\} > x^2 \{\frac{1}{2}(\deg \tau_{2\xi_1})^2 - \frac{1}{2} \deg \tau_{2\xi_1} - \deg \tau_{4\xi_1 - \alpha_1} - \dim \mathbf{B}_{n-1}\} \geq 0$ , violating (10.3). Thus  $\lambda \neq 2\xi_1$  for  $n > 2$ .

Suppose  $\lambda = k\xi_n$ . Then  $k=2$  because  $\pi_{\xi_n}(\mathbf{B}_n) = \mathbf{SO}(2n+1)$ . Now we compute  $\deg \pi_{2\xi_n} = 2n^2 + 3n$  and  $\deg \pi_{4\xi_n - \alpha_n} = \frac{1}{2}n(2n-1)(n+1)(2n+5)$ . Thus  $\frac{1}{2}(\deg \pi_{2\xi_n})^2 = \frac{1}{2}(4n^4 + 12n^3 + 9n^2) = \frac{1}{2}(2n^2 + 3n) + \frac{1}{2}(4n^4 + 12n^3 + 3n^2 - 5n) + \frac{1}{2}(4n^2 + 2n) = \frac{1}{2} \deg \pi_{2\xi_n} + \deg \pi_{4\xi_n - \alpha_n} + \dim \mathbf{B}_n$ .

Suppose  $\lambda = k\xi_{n-1}$ . Then  $k=1$  because  $\pi_{\xi_{n-1}}(\mathbf{B}_n) \subsetneq \mathbf{SO}(2n^2 + n)$ , so  $\pi_\lambda$  is the adjoint representation  $\pi_{\xi_{n-1}}$ . Calculating separately for  $n=3$  ( $2\xi_{n-1} - \alpha_{n-1}$ :  $\overset{1}{\circ} - \overset{2}{\circ} = \overset{2}{\bullet}$ ) and  $n > 3$  ( $2\xi_{n-1} - \alpha_{n-1}$ :  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{1}{\circ} = \overset{2}{\bullet}$ ),  $\deg \pi_{2\xi_{n-1} - \alpha_{n-1}} = \frac{1}{2}(n-1)(2n+3)n(2n+1)$ . Thus  $\frac{1}{2}(\deg \pi_{\xi_{n-1}})^2 = \frac{1}{2}(4n^4 + 4n^3 + n^2) = \frac{1}{2}(2n^2 + n) + \frac{1}{2}(4n^4 + 4n^3 - 5n^2 - 3n) + \frac{1}{2}(4n^2 + 2n) = \frac{1}{2} \deg \pi_{\xi_{n-1}} + \deg \pi_{2\xi_{n-1} - \alpha_{n-1}} + \dim \mathbf{B}_n$ .

We summarize the last four paragraphs as follows: Lemma 10.5 gives precisely those cases for which  $\lambda = k\xi_r$ , with  $r=1, n-1$ , or  $n$ . Thus we need only show that  $2 \leq r \leq n-2$  violates (10.3).

Suppose  $r=2 \leq n-2$ . Then  $\lambda = \xi_2$ . We retain the notation that  $\tau_\nu$  denotes the representation of highest weight  $\nu$  for the  $\mathbf{B}_{n-1}$  with simple root system  $\{\alpha_1, \dots, \alpha_{n-1}\}$ . If  $n=4$  we check  $\frac{1}{2}(\deg \pi_{\xi_2})^2 = 3528 > 42 + 2772 + 36 = \frac{1}{2}(\deg \tau_{\xi_2}) + \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{B}_4$ . Now assume  $n > 4$ ; by induction we have  $\frac{1}{2}(\deg \tau_{\xi_2})^2 - \{\frac{1}{2} \deg \tau_{\xi_2} + \deg \tau_{2\xi_2 - \alpha_2} + \dim \mathbf{B}_{n-1}\} > 0$ . Define multipliers by

$$\deg \pi_{\xi_2} = u \cdot \deg \tau_{\xi_2} \quad \text{and} \quad \deg \pi_{2\xi_2 - \alpha_2} = v \cdot \deg \tau_{2\xi_2 - \alpha_2}.$$

Then we calculate

$$u = \frac{2n(2n+1)}{(n+2)(n-1)} \quad \text{and} \quad v = \frac{4n(2n+1)(2n+3)}{(n-2)(n+2)(n+5)} < u^2.$$

As  $u^2 > u > (\dim \mathbf{B}_n)/(\dim \mathbf{B}_{n-1}) > 0$ , we now have

$$\begin{aligned} & \frac{1}{2} (\deg \pi_{\xi_2})^2 - \left\{ \frac{1}{2} \deg \pi_{\xi_2} + \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{B}_n \right\} \\ & > u^2 \left[ \frac{1}{2} (\deg \tau_{\xi_2})^2 - \left\{ \frac{1}{2} \deg \tau_{\xi_2} + \deg \tau_{2\xi_2 - \alpha_2} + \dim \mathbf{B}_{n-1} \right\} \right] > 0, \end{aligned}$$

violating (10.3). Thus  $r \neq 2$  for  $n \geq 4$ . Now we need only show that  $3 \leq r \leq n-2$  violates (10.3).

Suppose  $3 \leq r \leq n-2$ . Then  $n \geq 5$  and  $\lambda = \xi_r$ . As before we define multipliers by

$$\deg \pi_{\xi_r} = s \cdot \deg \tau_{\xi_r} \quad \text{and} \quad \deg \pi_{2\xi_r - \alpha_r} = t \cdot \deg \tau_{2\xi_r - \alpha_r}.$$

Now we compute

$$s = \frac{\binom{2n+1}{n-r+1}}{\binom{2n-1}{n-r}} = \frac{(n+r+1) \dots (2n+1)}{(n-r+1)!} \cdot \frac{(n-r)!}{(n+r) \dots (2n-1)} = \frac{2n(2n+1)}{(n+r)(n-r+1)} \quad \text{and}$$

$$t = \frac{2n(2n+1)(2n+2)(2n+3)}{(n-r)(n+r)(n-r+3)(n+r+3)} < s^2.$$

If  $r < n-2$  we use induction, and if  $r = n-2$  we use our verification for the adjoint representation of  $\mathbf{B}_{n-1}$ , concluding that  $\frac{1}{2} (\deg \pi_{\xi_r})^2 - \left\{ \frac{1}{2} \deg \pi_{\xi_r} + \deg \pi_{2\xi_r - \alpha_r} + \dim \mathbf{B}_n \right\} > s^2 \left[ \frac{1}{2} (\deg \tau_{\xi_r})^2 - \left\{ \frac{1}{2} \deg \tau_{\xi_r} + \deg \tau_{2\xi_r - \alpha_r} + \dim \mathbf{B}_{n-1} \right\} \right] > 0$ . This violates (10.3). Thus we cannot have  $3 \leq r \leq n-2$ , q.e.d.

10.6 LEMMA. *If  $K$  is of type  $C_n$ ,  $n \geq 3$ , then*

- (1)  $\pi_\lambda$  is the adjoint representation, given by  $\lambda: \overset{2}{\bullet} - \bullet - \dots - \bullet = \circ$ , or
- (2)  $\pi_\lambda$  has degree  $2n^2 - n - 1$ , given by  $\lambda: \bullet - \overset{1}{\bullet} - \dots - \bullet = \circ$ , or
- (3)  $n=4$  and  $\lambda: \bullet - \bullet - \bullet - \overset{1}{\bullet} = \circ$ .

*Remark.* Here the similarity between type  $B_n$  and  $C_n$  is striking. It suggests the possibility of a cohomological treatment of our results on representations.

*Proof.* We number the simple roots  $\overset{\alpha_1}{\circ} = \overset{\alpha_2}{\bullet} - \dots - \overset{\alpha_{n-1}}{\bullet} - \overset{\alpha_n}{\bullet}$ . Suppose  $\lambda = k\xi_n$ . Then  $k=2$  because  $\pi_{\xi_n}$  is symplectic, so  $\lambda = 2\xi_n$  and  $\pi_\lambda$  is the adjoint representation. We check  $\frac{1}{2} (\deg \pi_{2\xi_n})^2 = \frac{1}{2} (4n^4 + 4n^3 + n^2) = \frac{1}{2} (2n^2 + n) + \frac{1}{2} (4n^4 + 4n^3 - 5n^2 - 3n) + \frac{1}{2} (4n^2 + 2n) = \frac{1}{2} \deg \pi_{2\xi_n} + \deg \pi_{4\xi_n - \alpha_n} + \dim \mathbf{C}_n$ .

Suppose  $\lambda = k\xi_{n-1}$ . Then  $k=1$  because  $\pi_{\xi_{n-1}}(\mathbf{C}_n) \stackrel{\subset}{=} \mathbf{SO}(2n^2 - n - 1) = \mathbf{SO}(\deg \pi_{\xi_{n-1}})$ . Now we check  $\frac{1}{2}(\deg \pi_{\xi_{n-1}})^2 = \frac{1}{2}(4n^4 - 4n^3 - 3n^2 + 2n + 1) = \frac{1}{2}(2n^2 - n - 1) + \frac{1}{2}(4n^4 - 4n^3 - 9n^2 + n + 2) + \frac{1}{2}(4n^2 + 2n) = \frac{1}{2}\deg \pi_{\xi_{n-1}} + \deg \pi_{2\xi_{n-1} - \alpha_{n-1}} + \dim \mathbf{C}_n$ .

Let  $\tau_\nu$  denote the representation of highest weight  $\nu$  for the subgroup  $\mathbf{C}_{n-1}$  with simple root system  $\{\alpha_1, \dots, \alpha_{n-1}\}$ . If  $n=4$  we compute  $\frac{1}{2}(\deg \pi_{\xi_1})^2 = 882 = 21 + 825 + 36 = \frac{1}{2}\deg \pi_{\xi_1} + \deg \pi_{2\xi_1 - \alpha_1} + \dim \mathbf{C}_4$ ; this gives case (3) of the lemma. Now let  $n > 4$ . As just seen for  $n=5$ , and by induction for  $n > 5$ , we have  $\frac{1}{2}(\deg \tau_{\xi_1})^2 \geq \frac{1}{2}\deg \tau_{\xi_1} + \deg \tau_{2\xi_1 - \alpha_1} + \dim \mathbf{C}_{n-1}$ . Define multipliers by

$$\deg \pi_{\xi_1} = a \cdot \deg \tau_{\xi_1} \quad \text{and} \quad \deg \pi_{2\xi_1 - \alpha_1} = b \cdot \deg \tau_{2\xi_1 - \alpha_1}.$$

We compute  $a = 2 \frac{2n+1}{n+2}$  and  $b = 4 \frac{(n+1)(2n+1)(2n+3)}{(n-1)(n+4)(n+5)}$ . One checks  $a^2 > b$  for  $n \geq 5$ . As before, it follows that  $\frac{1}{2}(\deg \pi_{\xi_1})^2 > \frac{1}{2}\deg \pi_{\xi_1} + \deg \pi_{2\xi_1 - \alpha_1} + \dim \mathbf{C}_n$ . If  $\lambda = k\xi_1$  then this shows  $k=1$  by case (10) for  $\Psi$  in the proof of Proposition 9.1, and the latter violates (10.3). Thus  $\lambda \neq k\xi_1$  for  $n \geq 5$ .

Let  $\lambda = k\xi_1$ . As just seen, this implies  $n \leq 4$ , so  $n$  is 3 or 4. If  $n=3$  then  $k=2$  because  $\overset{1}{\circ} = \bullet - \bullet$  is symplectic, and we calculate  $\frac{1}{2}(\deg \pi_{2\xi_1})^2 = 3528 > 42 + 1638 + 21 = \frac{1}{2}(\deg \pi_{2\xi_1}) + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{C}_3$ ; thus  $n=4$ . If  $n=4$  then  $k=1$  because  $\overset{1}{\circ} = \bullet - \bullet - \bullet$  is orthogonal, and we have already checked (10.3) for  $\lambda = \xi_1$  with  $n=4$ .

The proof of Lemma 10.6 is now reduced to showing that we cannot have  $2 \leq r \leq n-2$ .

Let  $r = 2 \leq n-2$  and define multipliers by

$$\deg \pi_{\xi_2} = c \cdot \deg \tau_{\xi_2} \quad \text{and} \quad \deg \pi_{2\xi_2 - \alpha_2} = d \cdot \deg \tau_{2\xi_2 - \alpha_2}.$$

We compute  $c = \frac{2n(2n+1)}{(n-1)(n+3)}$ . If  $n \geq 6$  we calculate  $d = 4 \frac{n(2n+1)(2n+3)}{(n-2)(n+3)(n+6)} < c^2$ ; if  $n=5$  then  $d = 195/18 < 3025/256 = c^2$ ; if  $n=4$  then  $d = 396/35 < 576/49 = c^2$ ; now  $c^2 > d$  for  $n \geq 4$ . And trivially  $c^2 > c > (\dim \mathbf{C}_n)/(\dim \mathbf{C}_{n-1}) > 0$ . As has been checked when  $n=4$ , and by induction if  $n > 4$ ,  $\frac{1}{2}(\deg \tau_{\xi_2})^2 \geq \frac{1}{2}\deg \tau_{\xi_2} + \deg \tau_{2\xi_2 - \alpha_2} + \dim \mathbf{C}_{n-1}$ ; it follows that  $\frac{1}{2}(\deg \pi_{\xi_2})^2 > \frac{1}{2}\deg \pi_{\xi_2} + \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{C}_n$ . Now Proposition 9.1 shows that  $\lambda = k\xi_2$ ,  $n \geq 4$ , would contradict (10.3). The proof of Lemma 10.6 is thus reduced to showing that we cannot have  $3 \leq r \leq n-2$ . As before, we define multipliers by

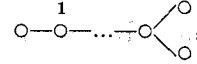
$$\deg \pi_{\xi_r} = u \cdot \deg \tau_{\xi_r} \quad \text{and} \quad \deg \pi_{2\xi_r - \alpha_r} = v \cdot \deg \tau_{2\xi_r - \alpha_r};$$

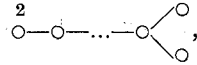
we compute

$$u = \frac{2n(2n+1)}{(n-r+1)(n+r+2)} \quad \text{and} \quad v = \frac{2n(2n+1)(2n+2)(2n+3)}{(n-r)(n-r+3)(n+r+2)(n+r+5)}$$

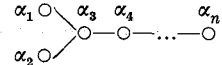
and we check  $u^2 > v$  and  $u^2 > u > (\dim \mathbf{C}_n)/(\dim \mathbf{C}_{n-1}) > 1$ . As we saw for  $r = n - 2$ , and by induction on  $n - r$  if  $r < n - 2$ , we have  $\frac{1}{2}(\deg \tau_{\xi_r})^2 \geq \frac{1}{2} \deg \tau_{\xi_r} + \deg \tau_{2\xi_r - \alpha_r} + \dim \mathbf{C}_{n-1}$ . It follows that  $\lambda = \xi_r$  violates (10.3). Now Proposition 9.1 says  $\lambda = \xi_r$  if  $\lambda = k\xi_r$ . Thus we cannot have  $3 \leq r \leq n - 2$ , q.e.d.

10.7 LEMMA. *If  $K$  is of type  $D_n$ ,  $n \geq 4$ , then*

(1)  $\pi_\lambda$  is the adjoint representation, given by  $\lambda$ : , or

(2)  $\pi_\lambda$  has degree  $2n^2 + n + 1$ , given by  $\lambda$ : , or

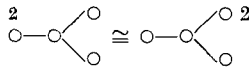
(3)  $n = 8$  and  $\pi_\lambda$  is the half spin representation given by  $\lambda$ : 

*Proof.* We label the simple roots by  and let  $\tau_\nu$  denote the representation of highest weight  $\nu$  for the subgroup  $\mathbf{D}_{n-1}$  with simple root system  $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ .

If  $\lambda = k\xi_n$  then  $k = 2$  because  $\pi_{\xi_n}(\mathbf{D}_n) = \mathbf{SO}(2n)$ . And we check  $\frac{1}{2}(\deg \pi_{2\xi_n})^2 = \frac{1}{2}(4n^4 + 4n^3 - 3n^2 - 2n + 1) = \frac{1}{2}(2n^2 + n - 1) + \frac{1}{2}(4n^4 + 4n^3 - 9n^2 - n + 2) + \frac{1}{2}(4n^2 - 2n) = \frac{1}{2} \deg \pi_{2\xi_n} + \deg \pi_{4\xi_n - \alpha_n} + \dim \mathbf{D}_n$ .

If  $\lambda = k\xi_{n-1}$  then  $k = 1$  because  $\pi_{\xi_{n-1}}(\mathbf{D}_n) \subsetneq \mathbf{SO}(2n^2 - n)$ . Then  $\pi_\lambda = \pi_{\xi_{n-1}}$ , adjoint representation, and we check  $\frac{1}{2}(\deg \pi_\lambda)^2 = \frac{1}{2}(4n^4 - 4n^3 + n^2) = \frac{1}{2}(2n^2 - n) + \frac{1}{2}(4n^4 - 4n^3 - 5n^2 + 3n) + \frac{1}{2}(4n^2 - 2n) = \frac{1}{2} \deg \pi_\lambda + \deg \pi_{2\lambda - \alpha_{n-1}} + \dim \mathbf{D}_n$ .

If  $\lambda = \xi_1$  then (10.3) says  $2^{2n-3} = 2^{n-2} + \binom{2n}{n-2} + (2n^2 - n)$ ;  $n = 8$  is the only solution, and  $\pi_{\xi_1}$  is orthogonal for  $n = 8$ ; this gives case (3) of the lemma.

Define multipliers by  $\deg \pi_{2\xi_1} = p \cdot \deg \tau_{2\xi_1}$  and  $\deg \pi_{4\xi_1 - \alpha_1} = q \cdot \deg \tau_{4\xi_1 - \alpha_1}$ . Then  $p = 2 \frac{2n-1}{n}$ , and  $q = 4 \frac{n(2n-1)(2n+1)}{(n-2)(n+1)(n+4)}$  for  $n \geq 5$ . So  $p^2 > q$  for  $n \geq 5$ . As  $p^2 > p > (\dim \mathbf{D}_n)/(\dim \mathbf{D}_{n-1}) > 1$ , we have  $\frac{1}{2}(\deg \pi_{2\xi_1})^2 - \{\frac{1}{2} \deg \pi_{2\xi_1} + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{D}_n\} > p^2[\frac{1}{2}(\deg \tau_{2\xi_1})^2 - \{\frac{1}{2} \deg \tau_{2\xi_1} + \deg \tau_{4\xi_1 - \alpha_1} + \dim \mathbf{D}_{n-1}\}] \geq 0$ , the last inequality having checked on  for  $n = 5$ , and being the induction hypothesis of  $n > 5$ . Now (10.3) shows  $\lambda \neq 2\xi_1$  if  $n > 4$ .

We have just seen that  $\lambda = k\xi_1$  implies  $n = 8$  and  $k = 1$ , or  $n = 4$  and  $k = 2$ ; the latter is included in case (2) of the lemma. If  $\lambda = k\xi_2$  we change notation, coming back to the case  $\lambda = k\xi_1$ . If  $\lambda = k\xi_r$ ,  $3 \leq r \leq n - 2$ , then  $k = 1$  because  $\pi_{\xi_r} = \Lambda^{n-r+1}(\pi_{\xi_n})$  maps  $\mathbf{D}_n$  onto a proper subgroup of  $\mathbf{SO}(\deg \pi_{\xi_r})$ . Now we need only check that  $\lambda \neq \xi_r$  whenever  $3 \leq r \leq n - 2$ .

Let  $\lambda = \xi_r$ ,  $3 \leq r \leq n-2$ . Define multipliers by  $\deg \pi_{\xi_r} = u \cdot \deg \tau_{\xi_r}$  and  $\deg \pi_{2\xi_r - \alpha_r} = v \cdot \deg \tau_{2\xi_r - \alpha_r}$ . Then, calculating separately for  $r=3$  and  $r>3$ ,

$$u = \frac{2n(2n-1)}{(n+r-1)(n-r+1)} \quad \text{and} \quad v = \frac{4n(n+1)(2n-1)(2n+1)}{(n-r)(n-r+3)(n+r-1)(n+r+2)}.$$

It follows that  $u^2 > v$ . Now  $\frac{1}{2}(\deg \pi_{\xi_r})^2 - \{\frac{1}{2} \deg \pi_{\xi_r} + \deg \pi_{2\xi_r - \alpha_r} + \dim \mathbf{D}_n\} > u^2 [\frac{1}{2}(\deg \tau_{\xi_r})^2 - \{\frac{1}{2} \deg \tau_{\xi_r} + \deg \tau_{2\xi_r - \alpha_r} + \dim \mathbf{D}_{n-1}\}] \geq 0$ , where the last inequality was checked

$\left( \begin{array}{c} 1 \\ \circ - \circ - \dots - \circ \end{array} \right)$  for  $n-r=2$  and is the induction hypothesis if  $n-r>2$ . This con-

tradicts (10.3). Thus  $\lambda \neq \xi_r$  for  $3 \leq r \leq n-2$ , q.e.d.

We finally come to the easy case.

10.8. LEMMA. *If  $K$  is an exceptional group, then*

- (1)  $K = \mathbf{G}_2$  and  $\lambda: \bullet \equiv \overset{1}{\circ}$  or  $\lambda: \overset{1}{\bullet} \equiv \circ$ ; or
- (2)  $K = \mathbf{F}_4$  and  $\lambda: \bullet - \bullet - \overset{1}{\circ} - \circ$  or  $\lambda: \overset{1}{\bullet} - \bullet - \circ - \circ$ ; or
- (3)  $K = \mathbf{E}_6$  and  $\lambda: \begin{array}{c} \circ - \circ - \overset{1}{\circ} - \circ - \circ \\ | \\ \circ \end{array}$ ; or
- (4)  $K = \mathbf{E}_7$  and  $\lambda: \begin{array}{c} \circ - \circ - \overset{1}{\circ} - \circ - \circ - \circ \\ | \\ \circ \end{array}$ ; or
- (5)  $K = \mathbf{E}_8$  and  $\lambda: \begin{array}{c} \circ - \circ - \overset{1}{\circ} - \circ - \circ - \circ - \circ \\ | \\ \circ \end{array}$ .

In each case, the first-mentioned possibility for  $\lambda$  is the case where  $\pi_\lambda$  is the adjoint representation.

*Proof.* Let  $K = \mathbf{G}_2: \bullet \equiv \overset{\alpha_1}{\circ} \overset{\alpha_2}{\circ}$ . We compute  $\frac{1}{2}(\deg \pi_{\xi_2})^2 = 98 = 7 + 77 + 14 = \frac{1}{2} \deg \pi_{\xi_2} + \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{G}_2$ . Thus  $\lambda = \xi_2$  is admissible.  $\lambda = \xi_1$  is case (c) of Proposition 5.2, and

$$\frac{1}{2}(\deg \pi_{2\xi_1})^2 = \frac{1}{2}(729) > \frac{1}{2}(27) + 189 + 14 = \frac{1}{2} \deg \pi_{2\xi_1} + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{G}_2.$$

Now  $\lambda$  is  $\xi_1$  or  $\xi_2$  by Proposition 9.1.

Let  $K = \mathbf{F}_4: \bullet - \bullet - \overset{\alpha_4}{\circ} \overset{\alpha_3}{\circ} \overset{\alpha_2}{\circ} \overset{\alpha_1}{\circ}$ . We compute

$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_1})^2 &= 1352 = 26 + 1274 + 52 = \frac{1}{2} \deg \pi_{\xi_1} + \deg \pi_{2\xi_1 - \alpha_1} + \dim \mathbf{F}_4, \\ \frac{1}{2}(\deg \pi_{\xi_2})^2 &= 831 \cdot 538 > 637 + 420 \cdot 147 + 52 = \frac{1}{2} \deg \pi_{\xi_2} + \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{F}_4, \\ \frac{1}{2}(\deg \pi_{\xi_3})^2 &= \frac{1}{2}(74 \cdot 529) > \frac{1}{2}(273) + 19 \cdot 278 + 52 = \frac{1}{2} \deg \pi_{\xi_3} + \deg \pi_{2\xi_3 - \alpha_3} + \dim \mathbf{F}_4, \\ \frac{1}{2}(\deg \pi_{\xi_4})^2 &= 338 = 13 + 273 + 52 = \frac{1}{2} \deg \pi_{\xi_4} + \deg \pi_{2\xi_4 - \alpha_4} + \dim \mathbf{F}_4. \end{aligned}$$

Now Proposition 9.1 says that  $\lambda$  is  $\xi_1$  or  $\xi_4$ .

Let  $K = \mathbf{E}_6$ :  $\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_6 \end{array}$ . Then  $\lambda = k\xi_r$ ,  $\xi_r = \xi_r^*$ , says  $r=3$  or  $r=6$ . We calculate

$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_3})^2 - \frac{1}{2} \deg \pi_{\xi_3} &= 4\,276\,350 > 2\,453\,814 + 78 = \deg \pi_{2\xi_3 - \alpha_3} + \dim \mathbf{E}_6 \\ \frac{1}{2}(\deg \pi_{\xi_6})^2 - \frac{1}{2} \deg \pi_{\xi_6} &= 3003 = 2925 + 78 = \deg \pi_{2\xi_6 - \alpha_6} + \dim \mathbf{E}_6. \end{aligned}$$

Now Proposition 9.1 says  $\lambda = \xi_6$ .

Let  $K = \mathbf{E}_7$ :  $\begin{array}{ccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_7 \end{array}$ . Then  $\pi_{\xi_1}$ ,  $\pi_{\xi_3}$  and  $\pi_{\xi_5}$  are symplectic while  $\pi_{\xi_2}$ ,

$\pi_{\xi_4}$ ,  $\pi_{\xi_6}$  and  $\pi_{\xi_7}$  are orthogonal. We compute

$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_1})^2 - \frac{1}{2} \deg \pi_{\xi_1} &= 1\,069\,453 > 915\,705 + 133 = \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_2})^2 - \frac{1}{2} \deg \pi_{\xi_2} &= 1\,183\,491 > 980\,343 + 133 = \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_3})^2 - \frac{1}{2} \deg \pi_{\xi_3} &= 382\,634\,616 > 209\,868\,813 + 133 = \deg \pi_{2\xi_3 - \alpha_3} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_4})^2 - \frac{1}{2} \deg \pi_{\xi_4} &= 66\,886\,348\,375 > 19\,903\,763\,880 + 133 = \deg \pi_{2\xi_4 - \alpha_4} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_5})^2 - \frac{1}{2} \deg \pi_{\xi_5} &= 37\,363\,690 > 24\,386\,670 + 133 = \deg \pi_{2\xi_5 - \alpha_5} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_6})^2 - \frac{1}{2} \deg \pi_{\xi_6} &= 8778 = 8645 + 133 = \deg \pi_{2\xi_6 - \alpha_6} + \dim \mathbf{E}_7, \\ \frac{1}{2}(\deg \pi_{\xi_7})^2 - \frac{1}{2} \deg \pi_{\xi_7} &= 415\,416 > 365\,750 + 133 = \deg \pi_{2\xi_7 - \alpha_7} + \dim \mathbf{E}_7. \end{aligned}$$

Now Proposition 9.1, together with case (7) of  $\Psi$  in its proof, shows that  $\lambda = \xi_6$ .

Let  $K = \mathbf{E}_8$ :  $\begin{array}{cccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_8 \end{array}$ . Then we calculate<sup>(1)</sup>

$$\begin{aligned} \deg \pi_{\xi_1} &= 248; \quad \deg \pi_{\xi_3} = 30\,380; \quad \deg \pi_{\xi_5} = 2\,450\,240; \quad \deg \pi_{\xi_4} = 146\,325\,270; \\ \deg \pi_{\xi_6} &= 6\,899\,079\,264; \quad \deg \pi_{\xi_8} = 6\,696\,000; \quad \deg \pi_{\xi_7} = 3875; \quad \deg \pi_{\xi_2} = 147\,250. \end{aligned}$$

Now we compute

$$\begin{aligned} \frac{1}{2}(\deg \pi_{\xi_1})^2 - \frac{1}{2} \deg \pi_{\xi_1} &= 30\,628 = 30\,380 + 248 = \deg \pi_{2\xi_1 - \alpha_1} + \dim \mathbf{E}_8, \\ \frac{1}{2}(\deg \pi_{\xi_2})^2 - \frac{1}{2} \deg \pi_{\xi_2} &= 461\,457\,010 > 344\,452\,500 + 248 = \deg \pi_{2\xi_2 - \alpha_2} + \dim \mathbf{E}_8 \\ \frac{1}{2}(\deg \pi_{\xi_3})^2 - \frac{1}{2} \deg \pi_{\xi_3} &= 3\,001\,836\,803\,680 \\ &> 1\,283\,242\,632\,840 + 248 = \deg \pi_{2\xi_3 - \alpha_3} + \dim \mathbf{E}_8, \\ \frac{1}{2}(\deg \pi_{\xi_4})^2 - \frac{1}{2} \deg \pi_{\xi_4} &= 10\,705\,542\,247\,123\,815 \\ &> 2\,118\,568\,836\,696\,000 + 248 = \deg \pi_{2\xi_4 - \alpha_4} + \dim \mathbf{E}_8, \end{aligned}$$

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<sup>(1)</sup> Note that this does not agree with Dynkin's table 30 in [6], which is incorrect for  $\mathbf{E}_8$ .

$$\begin{aligned} \frac{1}{2} (\deg \pi_{\xi_6})^2 - \frac{1}{2} \deg \pi_{\xi_6} &= 23\,798\,647\,342\,027\,851\,216 \\ &> 1\,704\,723\,757\,359\,480\,000 + 248 = \deg \pi_{2\xi_6 - \alpha_6} + \dim \mathbf{E}_8, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} (\deg \pi_{\xi_6})^2 - \frac{1}{2} \deg \pi_{\xi_6} &= 22\,418\,204\,652\,000 \\ &> 7\,723\,951\,192\,125 + 248 = \deg \pi_{2\xi_6 - \alpha_6} + \dim \mathbf{E}_8, \end{aligned}$$

$$\frac{1}{2} (\deg \pi_{\xi_7})^2 - \frac{1}{2} \deg \pi_{\xi_7} = 7\,505\,875 > 6\,696\,000 + 248 = \deg \pi_{2\xi_7 - \alpha_7} + \dim \mathbf{E}_8,$$

$$\frac{1}{2} (\deg \pi_{\xi_8})^2 - \frac{1}{2} \deg \pi_{\xi_8} = 10\,841\,207\,625 > 6\,899\,079\,264 + 248 = \deg \pi_{2\xi_8 - \alpha_8} + \dim \mathbf{E}_8.$$

Now Proposition 9.1 says  $\lambda = \xi_1$ , q.e.d.

Theorem 10.1 now follows from Lemmas 10.4, 10.5, 10.6, 10.7 and 10.8.

## 11. Summary and global formulation

We summarize and reformulate the results of Chapter I as follows.

**11.1 THEOREM.** *The table on pages 107–110 gives a complete list of the nonsymmetric simply connected coset spaces  $M = G/K$ , where (1)  $G$  is a connected Lie group acting effectively on  $M$  and (2)  $K$  is a compact subgroup whose linear isotropy action  $\chi$  on the tangent space of  $M$  is irreducible over the real number field. (Explanation of table:  $\pi$  denotes the inclusion  $K \rightarrow G$ ; if  $G$  is locally isomorphic to a classical linear group, then  $\pi$  is listed as a linear representation of  $\mathfrak{K}$ ; if  $G$  is exceptional, then  $\pi$  is given as a linear representation  $\alpha$  of  $\mathfrak{G}$  and a linear representation  $\beta$  of  $\mathfrak{K}$  such that  $\beta(\mathfrak{K}) \subset \alpha(\mathfrak{G})$ . The center of  $G$  is denoted  $Z$ , and  $N_G(K)$  is the normalizer of  $K$  in  $G$ .)*

*Let  $M' = G'/K'$  be an effective coset space of a connected Lie group by a compact subgroup, where the identity component  $K'_0$  acts irreducibly on the tangent space of  $M'$ . Then there is an entry  $M = G/K$  in the table, a central subgroup  $Q \subset Z$  of  $G$ , and a subgroup  $K'' \subset N_G(K)$  with  $K = K'_0 = N_G(K)_0$ , such that  $G' = G/Q$  and  $K' = (Q \cdot K'')/Q$ .*

*Proof.* Theorems 1.1, 2.1, 3.1, 4.1, 6.1, 7.1 and 8.1 give us all the information in the chart except for (a) the global isomorphism classes of  $G$  and  $K$  within their respective local isomorphism classes (b)  $Z$  and (c)  $N_G(K)/Z$ . As  $Z$  is specified (and so listed) by  $G$ , we need only check items (a) and (c).

Let  $\text{rank } G = \text{rank } K$ . Then  $G$  is centerless and  $K$  has center of order 3. This specifies the listed forms of  $G$  and  $K$ .

Let  $\text{rank } G > \text{rank } K$ . Then  $K$  is centerless. Let  $\tilde{G}$  denote the simply connected group with Lie algebra  $\mathfrak{G}$ , let  $\tilde{K}$  be the subgroup generated by  $\mathfrak{K}$ , and let  $Q$  denote the center of  $\tilde{K}$ . Then  $G = \tilde{G}/Q$ ,  $\tilde{K} = K/Q$  and  $Z = \tilde{Z}/Q$  where  $\tilde{Z}$  is the center of  $\tilde{G}$ .



$G$	$K$	$Z$	$N_G(K)/ZK$	$\pi$	$\kappa$	Conditions
$SU(pq)/Z_m$	$\{SU(p)/Z_p\} \times \{SU(q)/Z_q\}$	$Z_{pq}/m$	$\{1\}$			$p \geq q \geq 2, pq > 4$ $m = \text{l.c.m.}\{p, q\}$
$SU(16)/Z_4$	$SO(10)/Z_2$	$Z_4$	$\{1\}$			—
$SU(27)/Z_3$	$E_6/Z_3$	$Z_9$	$\{1\}$			—
$SU\left(\frac{n(n-1)}{2}\right)/Z_m$	$SU(n)/Z_n$	$Z_{n(n-1)/2m}$	$\{1\}$			$n \geq 5$ $n$ even: $m = n/2$ $n$ odd: $m = n$
$SU\left(\frac{n(n+1)}{2}\right)/Z_m$	$SU(n)/Z_n$	$Z_{n(n+1)/2m}$	$\{1\}$			$n \geq 3, m$ as above
$Sp(2)/Z_2$	$SO(3)$	$\{1\}$	$\{1\}$			—
$Sp(7)/Z_2$	$Sp(3)/Z_2$	$\{1\}$	$\{1\}$			—
$Sp(10)/Z_2$	$SU(6)/Z_6$	$\{1\}$	$Z_2$			—
$Sp(16)/Z_2$	$SO(12)/Z_2$	$\{1\}$	$\{1\}$			—
$Sp(28)/Z_2$	$E_7/Z_2$	$\{1\}$	$\{1\}$			—
$SO(20)$	$SU(4)/Z_4$	$Z_2$	$Z_2$			—
$SO(70)/Z_2$	$SU(8)/Z_8$	$\{1\}$	$Z_2$			—
$Spin(n^2-1), n$ odd	$Z_2 \times Z_2, n$ odd	$Z_2$	$\{1\}, n \equiv 1 \text{ or } n \equiv 2$			$n \geq 3$
$SO(n^2-1), n$ even	$SU(n)/Z_n$	$\{1\}, n$ even	$Z_2, n \equiv 3 \text{ or } n \equiv 0 \pmod{4}$			$n \geq 3$

$\mathcal{G}$	$K$	$Z$	$N_G(K)/ZK$	$\pi$	$\chi$	Conditions
$\text{Spin}(16)/Z_2 (\neq \text{SO}(16))$	$\text{SO}(9)$	$Z_2$	$\{1\}$	 into $\text{SO}(16)$		—
$\text{SO}(2n^2+n)$	$\text{SO}(2n+1)$	$\{1\}, n \text{ odd}$ $Z_2, n \text{ even}$	$\{1\}$	 if $n=2$ if $n=3$ if $n>2$	 if $n=2$ if $n=3$ if $n>3$	$n \geq 2$
$\text{SO}(2n^2+3n)$	$\text{SO}(2n+1)$	$\{1\}, n \text{ odd}$ $Z_2, n \text{ even}$	$\{1\}$			$n \geq 2$
$\text{Spin}(42)$	$\text{Sp}(4)/Z_2$	$Z_4$	$\{1\}$	 into $\text{SO}(42)$		—
$\text{Spin}(2n^2-n-1), n \equiv 0$ or $n \equiv 1 \pmod{4}$ $\text{SO}(2n^2-n-1), n \equiv 2$ or $n \equiv 3 \pmod{4}$	$\text{Sp}(n)/Z_2$	$Z_2 \times Z_2, n \equiv 0$ $Z_2 \times Z_2, n \equiv 1$ $\{1\}, n \equiv 2$ $Z_2, n \equiv 3$	$\{1\}$	 into $\text{SO}(2n^2-n-1)$	 if $n=3$ if $n>3$	$n \geq 3$
$\text{Spin}(2n^2+n), n \equiv 0$ or $n \equiv 3 \pmod{4}$ $\text{SO}(2n^2+n), n \equiv 1$ or $n \equiv 2 \pmod{4}$	$\text{Sp}(n)/Z_2$	$Z_2 \times Z_2, n \equiv 0$ $\{1\}, n \equiv 1;$ $Z_2, n \equiv 2$	$\{1\}$	 into $\text{SO}(2n^2+n)$		$n \geq 3$
$\text{Spin}(128)/Z_2 (\neq \text{SO}(128))$	$\text{SO}(16)/Z_2$	$Z_2$	$\{1\}$	 into $\text{SO}(128)$		—
$\text{Spin}(2n^2-n), n \text{ even}$ $\text{SO}(2n^2-n), n \text{ odd}$	$\text{SO}(2n)/Z_2$	$Z_2 \times Z_2, n \equiv 0$ $Z_4, n \equiv 2$ $\{1\}, n \text{ odd}$	$\{1\}, n \text{ even}$ $Z_2, n \text{ odd}$	 into $\text{SO}(2n^2-n)$	 if $n=4$ if $n>4$	$n \geq 4$
$\text{Spin}(2n^2+n-1), n \text{ even}$ $\text{SO}(2n^2+n-1), n \text{ odd}$	$\text{SO}(2n)/Z_2$	$Z_2$	$\{1\}$	 into $\text{SO}(2n^2+n-1)$		$n \geq 4$
$\text{Spin}(7)$	$G_2$	$Z_2$	$\{1\}$	 into $\text{SO}(7)$		—

$G$	$K$	$Z$	$N_G(K)/ZK$	$\pi$	$\chi$	Conditions
$\text{Spin}(14)$	$G_2$	$Z_4$	$\{1\}$	$\bullet \equiv \bullet \rightarrow \text{into } \text{SO}(14)$	$\begin{matrix} 3 \\ \bullet \equiv \bullet \end{matrix}$	—
$\text{Spin}(26)$	$F_4$	$Z_4$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(26)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$	—
$\text{Spin}(52)$	$F_4$	$Z_2 \times Z_2$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(52)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$	—
$\text{Spin}(78)$	$E_6/Z_3$	$Z_4$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(78)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	—
$\text{SO}(133)$	$E_7/Z_2$	$\{1\}$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(133)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	—
$\text{Spin}(248)$	$E_8$	$Z_2 \times Z_2$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(248)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	—
$G_2$	$\text{SO}(3)$	$\{1\}$	$\{1\}$	$\begin{matrix} 6 \\ \bullet \end{matrix}$	$\begin{matrix} 10 \\ \bullet \end{matrix}$	$G_2 \subset \text{SO}(7)$ by $\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet$
$G_2$	$\text{SU}(3)$	$\{1\}$	$Z_2$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SU}(3)$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$	$F_4 \subset \text{SO}(26)$ by $\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$
$F_4$	$\text{SO}(3) \times G_2$	$\{1\}$	$\{1\}$	$\begin{matrix} 2 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SO}(3) \times G_2$	$\begin{matrix} 4 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$	$E_6 \subset \text{SU}(27)$
$F_4$	$\{\text{SU}(3) \times \text{SU}(3)\}/Z_3$	$\{1\}$	$Z_2$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \{\text{SU}(3) \times \text{SU}(3)\}/Z_3$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	by $\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$
$E_6$	$\text{SU}(3)/Z_3$	$Z_3$	$\{1\}$	$\begin{matrix} 2 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \text{SU}(3)/Z_3$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	
$E_6$	$G_2$	$Z_3$	$\{1\}$	$\begin{matrix} 2 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } G_2$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet$	
$E_6/Z_3$	$\{\text{SU}(3)/Z_3\} \times G_2$	$\{1\}$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \{\text{SU}(3)/Z_3\} \times G_2$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	
$E_6/Z_3$	$\frac{\text{SU}(3) \times \text{SU}(3) \times \text{SU}(3)}{Z_2 \times Z_3}$	$\{1\}$	$\{1\}$	$\begin{matrix} \bullet \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \text{into } \frac{\text{SU}(3) \times \text{SU}(3) \times \text{SU}(3)}{Z_2 \times Z_3}$	$\begin{matrix} 1 \\ \bullet \end{matrix} \bullet \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$	

$G$	$K$	$Z$	$N_G(K)/ZK$	$\pi$	$\mathcal{K}$	Conditions
$E_7$	$SU(3)/Z_3$	$Z_2$	$Z_2$			$E_7 \subset Sp(28) \subset SU(56)$
$E_7/Z_2$	$\{Sp(3)/Z_2\} \times G_2$	$\{1\}$	$\{1\}$			by
$E_7/Z_2$	$SO(3) \times F_4$	$\{1\}$	$\{1\}$			by
$E_7/Z_2$	$\{SU(3) \times SU(6)\}/Z_6$	$\{1\}$	$Z_2$			by
$E_8$	$G_2 \times F_4$	$\{1\}$	$\{1\}$			$E_8 \subset SO(248)$
$E_8$	$SU(9)/Z_3$	$\{1\}$	$Z_2$			by
$E_8$	$\{SU(3) \times E_6\}/Z_3$	$\{1\}$	$Z_2$			by

Suppose that  $G$  is of type  $A_l, C_l, G_2, F_4, E_6, E_7$  or  $E_8$ . Then we have  $\tilde{G}$  realized as a simply connected linear group and the inclusion  $\pi: K \rightarrow \tilde{G}$  is given as a linear representation  $\pi_\lambda$ . Now  $Q$  is cyclic of some order  $q$ . Let  $L_{rt}$  be the root lattice of  $\mathfrak{K}^C$  and let  $L_{wt}$  be the weight lattice. Then the class  $[\lambda]$  of  $\lambda$  in  $\Lambda = L_{wt}/L_{rt}$  has order  $q$  in the finite abelian group  $\Lambda$  which is isomorphic to the center of the simply connected version of  $K$ . This specifies the listed forms of  $G$  and  $K$ .

Suppose that  $G$  is of type  $D_l$  or  $B_l$ , locally isomorphic to  $\mathbf{SO}(m)$  with  $m$  being  $2l$  or  $2l+1$ . In order to repeat the trick used above, we must replace  $\mathbf{SO}(m)$  by its two-sheeted covering  $\tilde{G} = \mathbf{Spin}(m)$ . As a Lie algebra inclusion,  $\pi$  is given as an absolutely irreducible representation  $\pi_\lambda: \mathfrak{K} \rightarrow \mathfrak{SD}(m)$ . Now we compose  $\pi_\lambda$  with the spin representation  $\sigma$  of  $\mathfrak{D}_l$  or  $\mathfrak{B}_l$ , looking at the highest summand  $\pi_\nu$  of  $\sigma \cdot \pi_\lambda$ . If  $\delta_1 > \dots > \delta_m$  are the weights of the usual representation of  $\mathfrak{SD}(m)$ , then  $\frac{1}{2} \sum_{i=1}^l \delta_i$  is the highest weight of  $\sigma$ . Now let  $\lambda = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_m$  be the weights of  $\pi_\lambda$ ; it follows that

$$\nu = \frac{1}{2}(\lambda_1 + \lambda_2 + \dots + \lambda_l).$$

If  $\gamma$  is any weight of  $\sigma \cdot \pi_\lambda$ , then  $\gamma = \frac{1}{2}(\pm \lambda_1 \pm \dots \pm \lambda_l)$  for some choice of signs, so  $\gamma \equiv \nu$  modulo the lattice generated by the  $\lambda_i$ . If  $\pi_\lambda$  has a zero weight, i.e., if  $\pi_\lambda(K)$  is centerless, then it follows that  $[\gamma] = [\nu]$  in  $\Lambda$ , so the projection  $\sigma \cdot \pi_\lambda(\mathfrak{K}) \rightarrow \pi_\nu(\mathfrak{K})$  is an isomorphism on the group level. In that case we will usually use  $\pi_\nu(\mathfrak{K})$  rather than  $\sigma \cdot \pi_\lambda(\mathfrak{K})$  to find the order  $q$  of the center  $Q$  of  $K$ . Note there that  $Q$  is cyclic.

Let  $K'$  denote the subgroup of  $\mathbf{SO}(m)$  generated by  $\pi_\lambda(\mathfrak{K})$  and let  $Q'$  be its center. The orders  $q$  and  $q'$  of  $Q$  and  $Q'$  are related by  $q = q'$  if  $[\nu] = [\lambda]$  or by  $q = 2q'$  if  $[2\nu] = [\lambda]$ , and we can read off  $q'$  from the diagram of  $\lambda$ . Thus we need only check which of the two situations apply.

If  $K$  is of type  $A_l$  ( $l+1$  odd),  $G_2, F_4, E_6$  or  $E_8$ , then the simply connected group with Lie algebra  $\mathfrak{K}$  has center of odd order. Thus  $q$  and  $q'$  are odd, so  $q = q'$  because  $q = 2q'$  is impossible. Now we need only check the cases where  $K$  is of type  $A_{2r-1}, B_l, C_l, D_l$  or  $E_7$ .

Suppose that  $\pi_\lambda$  is the adjoint representation. Then  $q' = 1$ ,  $\lambda$  is the highest root and  $\nu$  is half the sum of the positive roots. As noted in § 5, this implies  $\frac{2\langle \nu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1$  for every simple root  $\alpha_i$ . Thus  $\nu = \sum \xi_i$ , sum of the basic weights.

Type  $A_{2r-1}$ :  $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \dots - \overset{\alpha_l}{\circ}$ . Then  $\Lambda = \{z\} \cong \mathbf{Z}_2$ , and  $[\xi_k] = z^k$ . Thus  $[\nu] = [\xi_r] = z^r$  so  $q = 2$ .

Type  $B_l$ :  $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \dots - \overset{\alpha_l}{\bullet}$ . Then  $\Lambda = \{z\} \cong \mathbf{Z}_2$ ,  $[\xi_k] = 1$  for  $k < l$  and  $[\xi_l] = z$ . Thus  $[\nu] = [\xi_l] = z$  so  $q = 2$ .

Type  $C_l$ :  $\overset{\alpha_1}{\bullet} - \overset{\alpha_2}{\bullet} - \dots - \overset{\alpha_{l-1}}{\bullet} = \overset{\alpha_l}{\circ}$ . Then  $\Lambda = \{z\} \cong \mathbf{Z}_2$  and  $[\xi_k] = z^k$ . If  $l = 4r$  or  $l = 4r + 3$  then  $[\nu] = 1$  so  $q = 1$ ; if  $l = 4r + 1$  or  $l = 4r + 2$  then  $[\nu] = z$  so  $q = 2$ .

Type  $D_{2r}$ :  $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \dots - \overset{\alpha_{2r-1}}{\circ} \begin{matrix} \circ \\ \circ \end{matrix}$ . Then  $\Lambda = \{z_1\} \times \{z_2\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ ,  $[\xi_k] = z_1^k$  for  $k < 2r-1$ ,

$[\xi_{2r-1}] = z_1 z_2$  and  $[\xi_r] = z_2$ . If  $r = 2t$  then  $[\nu] = 1$  so  $q = 1$ ; if  $r = 2t + 1$  then  $[\nu] = z_1$  so  $q = 2$ .

Type  $E_7$ :  $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ} - \overset{\alpha_4}{\circ} - \overset{\alpha_5}{\circ} - \overset{\alpha_6}{\circ} - \overset{\alpha_7}{\circ}$ . We compute  $2\nu = 27\alpha_1 + \sum_2^7 n_i \alpha_i$ ; thus  $[\nu] \neq 1$  in  $\Lambda \cong \mathbf{Z}_2$

so  $q = 2$ .

We extend the method:

11.2 LEMMA. If  $\alpha_i$  is a simple root of  $K$  let  $h_i$  be the non-negative integer defined by:  $\{0, \alpha_i, \dots, h_i \alpha_i\}$  are weights of  $\pi_\lambda$  but  $(h_i + 1)\alpha_i$  is not. Then  $\nu = \sum \{\frac{1}{2} h_i (h_i + 1)\} \xi_i$ .

*Proof of lemma.* Let  $\{\psi - p\alpha_i, \dots, \psi + q\alpha_i\}$  be a maximal  $\alpha_i$ -string of weights of  $\pi_\lambda$ . Let  $\mathfrak{G}_i$  be the simple three dimensional subalgebra of  $\mathfrak{R}$  with positive root  $\alpha_i$ .  $\pi_\lambda(\mathfrak{G}_i)$  has trace 0 on the sum of the weight spaces of the string; thus  $\langle \sum_{j=-p}^q \psi + j\alpha_i, \alpha_i \rangle = 0$ . If  $\psi > 0$ ,  $\psi$  not a multiple of  $\alpha_i$ , then each  $\psi + j\alpha_i > 0$ . Now the positive weight system decomposes into  $S_0 \cup \bigcup_{j=1}^r S_j$ , where  $S_0 = \{0, \alpha_i, \dots, h_i \alpha_i\}$  and the  $S_j$  are strings. Thus

$$\frac{2\langle \nu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \sum_{j=0}^{h_i} \frac{\langle j\alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{1}{2} h_i (h_i + 1). \quad \text{Q.e.d.}$$

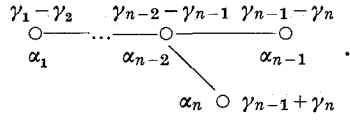
If  $\pi_\lambda$  is given by  $\overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ}$  or by  $\bullet - \bullet - \bullet - \overset{1}{\circ}$  then we calculate that  $h_i = 1$  for each  $i$ , so  $\nu = \sum \xi_i$ . Thus

$$\text{if } \pi_\lambda: \overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ} \text{ then } q = 2 = 2q'; \quad \text{if } \pi_\lambda: \bullet - \bullet - \bullet - \overset{1}{\circ} \text{ then } q = 1 = q'.$$

Let  $\pi_\lambda: \overset{2}{\circ} - \dots - \overset{2}{\circ} = \bullet$ . The usual matrix representation  $\pi_\gamma: \overset{1}{\circ} - \dots - \overset{1}{\circ} = \bullet$  of  $\mathfrak{B}_n$  as  $\mathfrak{C}\mathfrak{D}(2n+1)$  has weights  $\gamma_1 > \gamma_2 > \dots > \gamma_{2n+1}$ ,  $\gamma_{n+1} = 0$ ,  $\gamma_j + \gamma_{2n+2-j} = 0$ ,  $\{\gamma_1, \dots, \gamma_n\}$  linearly independent.  $\Lambda^2(\pi_\gamma)$  is the adjoint representation, so  $\{\pm \gamma_i, \pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n; i \neq j}$  are the roots, and the simple roots are given by  $\frac{\gamma_1 - \gamma_2}{\alpha_1} \overset{\alpha_2}{\circ} - \frac{\gamma_2 - \gamma_3}{\alpha_2} \overset{\alpha_3}{\circ} - \dots - \frac{\gamma_{n-1} - \gamma_n}{\alpha_{n-1}} \overset{\alpha_n}{\circ} = \bullet$ .  $S^2(\pi_\gamma) = \pi_\lambda \oplus 1$ , so  $\{\pm \gamma_1, \pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n}$  are the weights of  $\pi_\lambda$ . Thus  $\{\alpha_1, \dots, \alpha_n, 2\alpha_n\}$  are weights of  $\pi_\lambda$  while  $\{2\alpha_1, \dots, 2\alpha_{n-1}\}$  are not. Now  $\nu = \xi_1 + \dots + \xi_{n-1} + 3\xi_n$ . Thus  $[\nu] \neq 1 \in \Lambda$ , so  $q = 2q' = 2$ .

Let  $\pi_\lambda: \overset{2}{\circ} - \dots - \overset{2}{\circ} \begin{matrix} \circ \\ \circ \end{matrix}$ . The usual matrix representation  $\pi_\gamma: \overset{1}{\circ} - \dots - \overset{1}{\circ} \begin{matrix} \circ \\ \circ \end{matrix}$  has weights

$\gamma_1 > \dots > \gamma_{2n}$ ,  $\gamma_j + \gamma_{2n+1-j} = 0$ ,  $\{\gamma_1, \dots, \gamma_n\}$  linearly independent.  $\Lambda^2(\pi_\gamma)$  is the adjoint representation, so  $\{\pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n; i \neq j}$  are the roots and the simple roots are given by



$S^2(\pi_\gamma) = \pi_\gamma \oplus \mathbf{1}$ , so  $\{\pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n}$  are the weights of  $\pi_\lambda$ . Thus  $\{\alpha_1, \dots, \alpha_n\}$  are weights of  $\pi_\lambda$  while  $\{2\alpha_1, \dots, 2\alpha_n\}$  are not. Now  $\nu = \sum \xi_i$ . Thus  $[\nu] = 1$  and  $q = 1 = q'$  if  $n$  is even,  $[\nu] \neq 1$  and  $q = 2 = 2q'$  if  $n$  is odd.

Let  $\pi_\lambda$ :  $\overset{1}{\bullet} - \bullet - \dots - \bullet = \circ$ . The usual matrix representation  $\pi_\gamma$ :  $\overset{1}{\bullet} - \bullet - \dots - \bullet = \circ$  of  $\mathfrak{C}_n$  has weights  $\gamma_1 > \dots > \gamma_{2n}$ ,  $\gamma_j + \gamma_{2n+1-j} = 0$ ,  $\{\gamma_1, \dots, \gamma_n\}$  linearly independent.  $S^2(\pi_\gamma)$  is the adjoint representation, so  $\{\pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n}$  are the roots together with  $\pm \gamma_i \mp \gamma_i = 0$ , and the simple roots are given by  $\overset{\gamma_1 - \gamma_2}{\bullet} - \overset{\gamma_2 - \gamma_3}{\bullet} - \dots - \overset{\gamma_{n-1} - \gamma_n}{\bullet} = \overset{2\gamma_n}{\circ}$ .  $\Lambda^2(\pi_\gamma) = \pi_\lambda \oplus \mathbf{1}$ , so  $\{\pm \gamma_i \pm \gamma_j\}_{1 \leq i, j \leq n, i \neq j}$  are the nonzero weights of  $\pi_\lambda$ . Thus  $\{\alpha_1, \dots, \alpha_{n-1}\}$  are weights of  $\pi_\lambda$  while  $\{2\alpha_1, \dots, 2\alpha_{n-1}, \alpha_n\}$  are not. Now  $\nu = \sum_{i=1}^{n-1} \xi_i$ . If  $n$  has form  $4r$  or  $4r+1$  then  $[\nu] = 1$  and  $q = 1 = q'$ ; if  $n$  has form  $4r+2$  or  $4r+3$ , then  $[\nu] \neq 1$  so  $q = 2 = 2q'$ .

There remain only the three cases in which  $\pi_\lambda$  has no zero weight, i.e., in which  $K'$  has nontrivial center.

Let  $\pi_\lambda$ :  $\circ - \circ - \circ = \overset{1}{\bullet}$ . Then  $K'$  is simply connected, so  $q > q'$  is impossible. Thus  $q = q'$ .

Let  $\pi_\lambda$ :  $\circ - \circ - \circ - \circ = \overset{1}{\circ}$ . Then  $K' = \text{SU}(8)/\mathbf{Z}_4$  in  $G' = \text{SO}(70)$ . Let  $g$  generate the center of  $\tilde{G} = \text{Spin}(70)$ . Then  $z$  has order 4 and  $z^2 \in \tilde{K}$ . If  $z \notin K$ , then  $Q = \{1, z^2\}$ , so  $G = \tilde{G}/Q = \text{SO}(70) \supset K = \tilde{K}/Q = \text{SU}(8)/\mathbf{Z}_8$ . That inclusion contradicts the original setup. Thus  $z \in \tilde{K}$  and  $q = 2q' = 4$ .

Finally let  $\pi_\lambda$ :  $\circ - \circ - \circ - \circ - \circ - \circ \begin{matrix} \circ \\ \diagup \\ \diagdown \\ \circ \end{matrix} \overset{1}{\circ}$ , half spin representation of  $\mathfrak{D}_8$ . Let  $\pi_\gamma$ :  $\overset{1}{\circ} - \circ - \circ - \circ - \circ - \circ \begin{matrix} \circ \\ \diagup \\ \diagdown \\ \circ \end{matrix}$  denote the usual representation as  $\mathfrak{S}\mathfrak{D}(16)$  and let  $\gamma_1 > \dots > \gamma_{16}$

denote its weights. Then  $\lambda = \frac{1}{2}(\gamma_1 + \dots + \gamma_8)$ . Now let  $\lambda = \lambda_1 \geq \dots \geq \lambda_{64}$  denote the *positive* weights of  $\pi_\lambda$ ; they are just the  $\frac{1}{2}(\gamma_1 \pm \gamma_2 \pm \dots \pm \gamma_8)$ , where the number of minus signs is even. Let  $S_{\epsilon, j}$  consist of those of the form  $\frac{1}{2}(\gamma_1 + \epsilon\gamma_2 + \epsilon_3\gamma_3 + \dots + \epsilon_8\gamma_8)$ ,  $\epsilon = \pm 1$ ,  $\epsilon_k = \pm 1$ , in which just  $j$  of the signs  $\epsilon_k$  are  $-1$ ; let  $\sum_{\epsilon, j}$  be the sum of the elements of  $S_{\epsilon, j}$ ; note that  $S_{+, j}$  is empty for  $j$  odd and  $S_{-, j}$  is empty for  $j$  even. Now  $\nu = \sum \lambda_i$  is given by

$$\nu = (\sum_{+, 0} + \sum_{+, 8}) + (\sum_{+, 2} + \sum_{+, 4}) + \sum_{-, 1} + \sum_{-, 3} + \sum_{-, 5}.$$

Given  $\lambda_i = \frac{1}{2}(\gamma_1 + \gamma_2 + \sum_{j=3}^8 \epsilon_j \gamma_j) \in S_{+, k}$  we have  $'\lambda_i = \frac{1}{2}(\gamma_1 + \gamma_2 - \sum_{j=3}^8 \epsilon_j \gamma_j) \in S_{+, 6-k}$ , and  $'(' \lambda_i) = \lambda_i$  and  $\lambda_i + '\lambda_i = \gamma_1 + \gamma_2$ . As  $S_{+, 0}$  has just one element  $\lambda$  and as  $S_{+, 2}$  has  $\binom{6}{2} = 15$  elements, this shows

$$(\sum_{+,0} + \sum_{+,6}) + (\sum_{+,2} + \sum_{+,4}) = 16\gamma_1 + 16\gamma_2.$$

$S_{-,j}$  has  $\binom{6}{j}$  elements. The coefficient of  $\gamma_i$  is  $-1$  for  $\binom{5}{j-1}$  of them,  $+1$  for the others. Thus

$$\begin{aligned}\sum_{-,1} &= 3\gamma_1 - 3\gamma_2 + 2(\gamma_3 + \dots + \gamma_8), \quad \sum_{-,3} = 10\gamma_1 - 10\gamma_2 \quad \text{and} \\ \sum_{-,5} &= 3\gamma_1 - 3\gamma_2 - 2(\gamma_3 + \dots + \gamma_8).\end{aligned}$$

This shows  $\nu = 32\gamma_1$ . Let  $L$  be the lattice spanned by the  $\lambda_i$ .  $\sigma$  is the spin representation of  $\mathfrak{S}\mathfrak{D}(128 = \deg \pi_\lambda)$ ; the weights of  $\sigma \cdot \pi_\lambda$  are the  $\frac{1}{2}(\pm \lambda_1 \pm \dots \pm \lambda_{64})$ , so any two of them differ by an element of  $L$ . Let  $L_{rt}$  be the root lattice.  $2\gamma_1$  is an integral linear combination of roots; thus  $\nu = 32\gamma_1 \in L_{rt} \subset L$ , so  $L$  contains every weight of  $\sigma \cdot \pi_\lambda$ . This shows that  $K$  and  $K'$  are isomorphic, so  $q = q'$ .

This completes the verification of the material in the first three columns of the chart.

The description of normalizers is based on a simple remark:

**11.3 LEMMA.** *Let  $A$  be the group of all automorphisms of  $K$  which extend to inner automorphisms of  $G$ . Then  $A_0$  consists of the inner automorphisms of  $K$ , and  $g \rightarrow \text{ad}(g)|_K$  maps  $N_G(K)/ZK$  isomorphically onto  $A/A_0$ . If  $\text{rank } K < \text{rank } G$ , then  $Z$  is the centralizer of  $K$  in  $G$ , and  $g \rightarrow \text{ad}(g)|_K$  maps  $N_G(K)/Z$  isomorphically onto  $A$ .*

*Proof.* Let  $\beta$  denote the map  $g \rightarrow \text{ad}(g)|_K$ . Then  $\beta$  is a homomorphism of  $N_G(K)$  onto  $A$ , and the kernel of  $\beta$  is the centralizer of  $K$  in  $G$ . If  $g$  is in that centralizer but  $g \notin Z$ , then  $K$  is the connected centralizer of  $g$  in  $G$ , so  $g \in K$ . Thus  $\ker \beta \subset ZK$ . As  $\beta(ZK) = A_0$ , this shows  $\beta: N_G(K)/ZK \cong A/A_0$ . Now if  $\text{rank } K < \text{rank } G$  then  $K$  is not a connected centralizer, so  $Z = \ker \beta$  and  $\beta: N_G(K)/Z \cong A$ , q.e.d.

Now suppose  $\text{rank } K < \text{rank } G$  until we state the contrary.

If  $K$  has no outer automorphism, then Lemma 11.3 says  $N_G(K)/ZK = \{1\}$ .

Let  $\tilde{K}$  have a central element  $z$  of order  $m > 2$ . Let  $\alpha$  be an automorphism of  $G$  which is outer on  $K$ . Then  $\alpha$  lifts to  $\tilde{G}$  and  $\alpha(z) = z^{-1} \neq z$ . As  $z$  is central in  $\tilde{G}$ , now  $\alpha$  is outer on  $\tilde{G}$ . Thus  $\alpha$  is outer on  $G$  and  $N_G(K)/ZK = \{1\}$ .

Let  $K$  be of type  $D_n$ ,  $n > 4$ , where  $\pi$  is the half spin representation. Let  $\alpha$  be an automorphism of  $G$  which is outer on  $K$ . Then  $\alpha$  interchanges the two half spin representations, so it is not defined on  $\pi(K)$ . If  $\alpha$  were inner on  $G$  it would be defined on  $\pi(K)$ . Thus  $N_G(K)/ZK = \{1\}$ .

Let  $\pi: K \rightarrow G$  come from the adjoint representation of  $K$ ,  $\text{ad}: K \rightarrow \mathfrak{SO}(p)$ , where  $p = \dim K$ . Let  $\alpha \in \mathfrak{O}(p)$  be any outer automorphism of  $K$ . Then  $\alpha \in A$  if and only if  $\det \alpha = 1$ ,



Let  $q$  be the number of positive roots of  $K$  and let  $l = \text{rank } K$ , so  $p = l + 2q$ . If  $\alpha$  is not the triality automorphism of  $\mathbf{D}_4$ , we may multiply it by an inner automorphism and assume that  $\alpha$  is  $-I$  on the Cartan subalgebra and simple interchange  $E_\varphi \leftrightarrow E_{-\varphi}$  of root vectors. Then  $\det \alpha = (-1)^{l+q}$ , so  $N_G(K)/ZK$  is  $\{1\}$  if  $l+q$  is even,  $\mathbf{Z}_2$  if  $l+q$  is odd. Now we notice

$$A_{n-1}: l = n - 1, \quad p = n^2 - 1, \quad q = \frac{n(n-1)}{2}, \quad l + q = \frac{(n+2)(n-1)}{2};$$

$$D_n: l = n, \quad p = 2n^2 - n, \quad q = n^2 - n, \quad l + q = n^2;$$

$$E_6: l = 6, \quad p = 78, \quad q = 36, \quad l + q = 42.$$

Finally, for  $\mathbf{D}_4$ , triality now has determinant 1 because its cube has determinant 1.

Let  $\pi: \overset{1}{\circ} - \dots - \overset{\alpha_q}{\circ} - \dots - \overset{\alpha_{2q-1}}{\circ}$ . Then  $\pi(\text{SU}(2q))$  preserves the bilinear form on  $\Lambda^q(\mathbf{C}^{2q})$  given by  $(v_1 \wedge \dots \wedge v_q, w_1 \wedge \dots \wedge w_q) = v_1 \wedge \dots \wedge v_q \wedge w_1 \wedge \dots \wedge w_q$  where  $\Lambda^{2q}(\mathbf{C}^{2q})$  is identified with  $\mathbf{C}$ , and the linear version of  $G$  is in the symplectic or special orthogonal group of that form. Let  $\underline{g}$  denote complex conjugation of  $\mathbf{C}^{2q}$  and  $\mathbf{C}$ ; it extends to the outer automorphism  $\alpha$  of  $K$ . Note that  $\underline{g}$  preserves the form. In the symplectic case ( $q$  odd), this puts  $\underline{g}$  in the linear version of  $G$ , so  $N_G(K)/ZK \cong \mathbf{Z}_2$ . In the orthogonal case ( $q$  even),  $\underline{g}$  acts on the real form of  $\mathbf{C}^q$  with determinant  $(-1)^{q/2}$ . In the case  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$  we have  $q=4$ , so  $N_G(K)/ZK = \mathbf{Z}_2$ . Now look at  $\overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ}$ . There  $q=2$  so we have  $\underline{g} \in \text{SO}(6)$ , and  $\underline{g}$  persists from  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$  to  $\overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ}$ , so again  $N_G(K)/ZK \cong \mathbf{Z}_2$ .

In the usual representation  $\overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{1}{\circ} \begin{matrix} \circ \\ \diagup \diagdown \\ \circ \end{matrix}$  of  $\text{SO}(2n)$ , the outer automorphism is conjugation  $\alpha$  by  $\underline{a} = \text{diag} \{-1; 1, \dots, 1\} \in \mathbf{O}(2n)$ , using a basis  $\{v_1, \dots, v_{2n}\}$  of  $\mathbf{R}^{2n}$ . In  $S^2(\mathbf{R}^{2n})$  the  $(-1)$ -eigenvectors of  $S^2(\underline{a})$  are  $\{v_1 v_2, v_1 v_3, \dots, v_1 v_{2n}\}$ ; there are  $2n-1$  of them, so  $S^2(\underline{a}) \notin \text{SO}(2n^2 - n)$ . Thus  $N_G(K)/ZK = \{1\}$  for  $\pi: \overset{2}{\circ} - \overset{2}{\circ} - \dots - \overset{2}{\circ} \begin{matrix} \circ \\ \diagup \diagdown \\ \circ \end{matrix}$ .

Let  $G = \mathbf{E}_6$  and  $K = \text{SU}(3)/\mathbf{Z}_3$ , and let  $\alpha = \text{ad}(g)$  be an inner automorphism of  $G$  which is outer on  $K$ . An outer automorphism  $\beta$  of  $G$  is induced by a complex-antilinear map of  $\mathbf{C}^{27}$ , sending  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{matrix} \circ \\ | \\ \circ \end{matrix}$  to  $\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{matrix} \circ \\ | \\ \circ \end{matrix}$ ; it preserves  $\overset{2}{\circ} - \overset{2}{\circ}$  so we may assume  $\beta(K) = K$ . If  $\beta$  is inner on  $K$ , then we may assume  $\beta|_K = 1$ , so  $K$  is in the fixed point set  $F_\beta$  of  $\beta$ . Then  $K = F_\beta$ . But the fixed point set of an outer automorphism of  $\mathbf{E}_6$  has rank 4. Thus  $\beta$  is outer on  $K$ . Similarly  $\beta^{-1}\alpha$  is outer on  $K$ . Now  $\alpha = \beta \cdot \beta^{-1}\alpha$  is inner on  $K$ . That is absurd, so  $\alpha$  cannot exist.

Let  $G = \mathbf{E}_7$  and  $K = \text{SU}(3)/\mathbf{Z}_3$ . Let  $\nu$  denote the complex conjugation automorphism

of  $\mathrm{SU}(56)$  with fixed point set  $\mathrm{SO}(56)$ . We may assume embeddings chosen so that  $\nu$  preserves each of  $\mathrm{Sp}(28) \supset G \supset K$ .  $\nu$  is necessarily inner on  $G$ , but it interchanges the summands  $\overset{6}{\circ}-\circ$  and  $\circ-\overset{6}{\circ}$  of  $K \rightarrow \mathrm{SU}(56)$ , so it is outer on  $K$ . Thus  $N_G(K)/ZK \cong \mathbf{Z}_2$ .

This completes the determination of  $N_G(K)/ZK$  for the cases  $\mathrm{rank} K < \mathrm{rank} G$ .

Let  $G = \mathbf{E}_6/\mathbf{Z}_3$  and  $K = (\mathrm{SU}(3))^3/(\mathbf{Z}_3)^2$ . Then  $Z = \{1, z, z^{-1}\}$  is the center of  $K$ . Let  $\alpha$  be an automorphism, inner on  $G$  and outer on  $K$ . Then  $\alpha(z) = z^{-1}$ . Choose a maximal torus  $T$  containing  $z$  and let  $\beta$  be the outer automorphism of  $G$  which is given on  $T$  by  $x \rightarrow x^{-1}$ . Then  $\beta(K) = K$  and  $\beta$  is outer on  $K$ . Now  $(\alpha\beta)(z) = z$ , so  $\alpha\beta$  is inner on  $K$ , whence  $\alpha\beta$  is trivial on a maximal torus  $S \subset K$ . Thus  $\alpha\beta$  is inner on  $G$ . It follows that  $\beta$  is inner on  $G$ . Now  $N_G(K)/ZK = \{1\}$ .

Finally let  $\mathrm{rank} K = \mathrm{rank} G$  with  $G \neq \mathbf{E}_6/\mathbf{Z}_3$ . Then  $-I$  is in the Weyl group of  $G$ , so every element is conjugate to its inverse. Let  $\{1, z, z^{-1}\} = Z$ ; there exists  $g \in G$  with  $gzg^{-1} = z^{-1}$ , and so  $\mathrm{ad}(g)$  is outer on  $Z$ . Thus  $N_G(K)/ZK \cong \mathbf{Z}_2$ .

This completes the proof of Theorem 11.1, q.e.d.

## 12. Extension to noncompact isotropy subgroup

The extension *per se* is

**12.1 THEOREM.** *Let  $M_u = G_u/H_u$  be an effective reductive coset space where  $H_u$  is a compact connected group with  $\mathbf{R}$ -irreducible linear isotropy representation  $\chi_u$ . Let  $\sigma$  be an involutive automorphism of  $\mathfrak{G}_u$  which preserves  $\mathfrak{H}_u$ . Decompose  $\mathfrak{G}_u = \mathfrak{F}_u + \mathfrak{P}_u$  into  $(\pm 1)$ -eigenspaces of  $\sigma$ , and define  $\mathfrak{G} = \mathfrak{F}_u + \sqrt{-1}\mathfrak{P}_u$  and  $\mathfrak{H} = (\mathfrak{H}_u \cap \mathfrak{F}_u) + \sqrt{-1}(\mathfrak{H}_u \cap \mathfrak{P}_u)$ . Let  $H \subset G$  and  $H_u^C \subset G_u^C$  denote the connected Lie groups with Lie algebras  $\mathfrak{H} \subset \mathfrak{G}$  and  $\mathfrak{H}_u^C \subset \mathfrak{G}_u^C$ , respectively, such that  $M = G/H$  and  $M_u^C = G_u^C/H_u^C$  are simply connected effective coset spaces.<sup>(1)</sup> Let  $\chi$  and  $\chi_u^C$  denote their respective linear isotropy representations. Then there are only three possibilities, as follows.*

1.  $\chi_u$  and  $\chi$  are absolutely irreducible while  $\chi_u^C$  is  $\mathbf{R}$ -irreducible but not absolutely irreducible.
2.  $\chi_u = \beta \oplus \beta$  with  $\beta \not\sim \beta$ ,  $\chi_u^C$  is not  $\mathbf{R}$ -irreducible, and  $\chi$  is  $\mathbf{R}$ -irreducible if and only if  $\beta \cdot \sigma \sim \beta$ .
3.  $\chi_u = \beta \oplus \beta$ ,  $G_u$  is not semisimple,  $\chi$  is  $\mathbf{R}$ -irreducible, and  $\chi_u^C$  is not  $\mathbf{R}$ -irreducible.

*Proof.* First suppose  $\chi_u$  absolutely irreducible. As  $\chi$  has the same extension to a complex representation of  $\mathfrak{H}_u^C$ , it too is absolutely irreducible. If  $\chi_u^C$  reduces over  $\mathbf{R}$  then the complex extension of  $\chi_u$  reduces over  $\mathbf{C}$ ; thus  $\chi_u^C$  is  $\mathbf{R}$ -irreducible.

<sup>(1)</sup> For example  $G = G'/Z$  and  $H = H'/Z$ , where  $G'$  is the connected simply connected Lie group with Lie algebra  $\mathfrak{G}$ ,  $H'$  is the analytic subgroup with Lie algebra  $\mathfrak{H}$ , and  $Z$  is the kernel of the action of  $G'$  on  $M = G'/H' = G/H$ .

Now suppose  $\chi_u$  not absolutely irreducible. Then it has form  $\beta \oplus \bar{\beta}$ . Decompose  $\mathfrak{G}_u = \mathfrak{H}_u + \mathfrak{M}_u$  and  $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ ;  $\mathfrak{M}_u^c = \mathfrak{M}^c = \mathfrak{M}' + \mathfrak{M}''$ , where  $\beta$  has representation space  $\mathfrak{M}'$  and  $\bar{\beta}$  has representation space  $\mathfrak{M}''$ . Let  $\tau$  and  $\tau_u$  be the respective conjugations of  $\mathfrak{G}^c = \mathfrak{G}_u^c$  over  $\mathfrak{G}$  and  $\mathfrak{G}_u$ . Now  $\tau_u$  interchanges  $\mathfrak{M}'$  and  $\mathfrak{M}''$  and  $\tau = \tau_u \sigma$ . If  $\beta \neq \bar{\beta}$  then  $\sigma \mathfrak{M}'$  is  $\mathfrak{M}'$  or  $\mathfrak{M}''$ . In the first case  $\tau$  interchanges  $\mathfrak{M}'$  and  $\mathfrak{M}''$  so  $\chi$  is  $\mathbf{R}$ -irreducible. In the second case  $\tau$  preserves both  $\mathfrak{M}'$  and  $\mathfrak{M}''$  so  $\chi$  reduces over  $\mathbf{R}$ . If  $\beta \sim \bar{\beta}$  then we can choose  $\mathfrak{M}'$  and  $\mathfrak{M}''$  to be  $\sigma$ -stable, so  $\tau$  interchanges them and  $\chi$  is  $\mathbf{R}$ -irreducible. This proves the statements on  $\chi$ . On the other hand reduction of  $\chi_u$  over  $\mathbf{C}$  amounts to reduction of  $\chi_u^c$  over  $\mathbf{R}$ , so the statements on  $\chi_u^c$  are immediate, q.e.d.

In order to use the extension to reduce enumeration problems to the compact case we need its converse.

**12.2 THEOREM.** *Let  $M = G/H$  be an effective reductive coset space of connected Lie groups, where  $H$  has  $\mathbf{R}$ -irreducible linear isotropy representation  $\chi$ .*

*If  $G$  is not semisimple then it is the semidirect product  $H \times_{\omega} V$ , where  $\omega$  is a faithful  $\mathbf{R}$ -irreducible linear representation of  $H$  on a vector group  $V$  or  $H = \{1\}$  and  $G$  is a circle group.*

*If  $G$  is semisimple then it has an involutive automorphism  $\sigma$ , unique up to  $\text{ad}_G(H)$ -conjugacy in the automorphism group of  $G$ , such that*

- (i) *the fixed point set  $F$  of  $\sigma$  is a maximal compactly embedded subgroup<sup>(1)</sup> of  $G$ ,*
- (ii)  *$\sigma(H) = H$ , and*
- (iii) *the fixed point set  $F \cap H$  of  $\sigma|_H$  is a maximal compact subgroup of  $H$ .*

*Decompose  $\mathfrak{G} = \mathfrak{H} + \mathfrak{P}$  into  $(\pm 1)$ -eigenspaces of  $\sigma$ . Define  $\mathfrak{G}_u = \mathfrak{H} + \sqrt{-1}\mathfrak{P}$  and  $\mathfrak{H}_u = (\mathfrak{H} \cap \mathfrak{H}) + \sqrt{-1}(\mathfrak{H} \cap \mathfrak{P})$ . Let  $H_u \subset G_u$  denote the connected Lie groups with Lie algebras  $\mathfrak{H}_u \subset \mathfrak{G}_u$  such that the "compact version"  $M_u = G_u/H_u$  of  $M$  is a simply connected effective coset space. Then  $G_u, H_u$  and  $M_u$  are compact and there are only two possibilities, as follows.*

1. *The linear isotropy representation  $\chi_u$  of  $H_u$  is  $\mathbf{R}$ -irreducible.*
2. *There is a simply connected effective coset space  $A/B$  of compact connected Lie groups such that  $B$  has absolutely irreducible linear isotropy representation,  $G_u = A \times A$ ,  $H_u = B \times B$ ,  $M_u = (A/B) \times (A/B)$ ,  $\mathfrak{G} = \mathfrak{A}^c$  and  $\mathfrak{H} = \mathfrak{B}^c$ .*

*Proof.* If  $G$  is not semisimple then the assertion is contained in Lemma 1.2.

We now assume  $G$  semisimple.  $H$  is a reductive subgroup of  $G$  because its linear isotropy representation is faithful and fully reducible. Now a result of Mostow on Cartan involutions [12] gives the existence of  $\sigma$  satisfying (i), (ii) and (iii). Compactness of  $G_u$ , and thus of  $H_u$  and  $M_u$ , is immediate.

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<sup>(1)</sup> This means  $F = \text{ad}^{-1}(F')$  for some maximal compact subgroup  $F'$  of  $\text{ad}(G)$ .

Assume  $\mathbf{R}$ -reducibility of  $\chi_u$ ; we must check the statements of (2). Lemma 1.4 shows that  $G$  is simple. If  $G_u$  is not simple then  $G$  must be a complex Lie group *qua* real Lie group. In that case the inclusion  $\mathfrak{H} \rightarrow \mathfrak{G}$  defines a homomorphism over  $\mathbf{C}$ ,  $\phi: \mathfrak{H}^{\mathbf{C}} \rightarrow \mathfrak{G}$ . Let  $\mathfrak{L} = \phi(\mathfrak{H}^{\mathbf{C}})$ ;  $\mathbf{R}$ -irreducibility of  $\chi$  says that either  $\mathfrak{H} = \mathfrak{L}$  or  $\mathfrak{L} = \mathfrak{G}$ . If  $\mathfrak{L} = \mathfrak{G}$  it follows that  $\mathfrak{H}$  is a real form of  $\mathfrak{G}$  and that  $\chi_u$  is absolutely irreducible; thus  $\mathfrak{H} = \mathfrak{L}$ . Now  $\mathfrak{H} \subset \mathfrak{G}$  is an inclusion of complex Lie algebras,  $\mathfrak{G}_u = \mathfrak{A} \oplus \mathfrak{A}$ , where  $\mathfrak{A}$  is the compact real form of  $\mathfrak{G}$ , and  $\mathfrak{H}_u = \mathfrak{B} \oplus \mathfrak{B}$ , where  $\mathfrak{B}$  is a subalgebra of  $\mathfrak{A}$ . The assertions of (2) follow from simple connectivity of  $M_u$ . Now the proof is reduced to the demonstration that  $G_u$  cannot be simple when  $\chi_u$  is  $\mathbf{R}$ -reducible.

If  $G_u$  was simple with  $\chi_u$  reducible over  $\mathbf{R}$ , and if we decomposed  $\chi_u = \beta \oplus \gamma$ , the real representations  $\beta$  and  $\gamma$  would be equivalent. On the other hand,  $H_u$  would be a maximal subgroup of  $G_u$  because of  $\mathbf{R}$ -irreducibility of  $\chi$ , so we could not have  $G_u/H_u = \mathbf{Spin}(8)/\mathbf{G}_2$ . Thus the following lemma would give a contradiction, q.e.d. modulo lemma.

**12.3 LEMMA.** *Let  $M_u = G_u/H_u$  be a simply connected effective coset space of connected compact Lie groups. Suppose that  $H_u$  has linear isotropy representation  $\chi_u = \beta_1 \oplus \beta_2$  with  $\beta_i$  absolutely irreducible and  $\beta_1 \sim \beta_2$ . Then  $G_u = \mathbf{Spin}(8)$ ,  $H_u = \mathbf{G}_2$ ,  $M_u$  is the product  $\mathbf{S}^7 \times \mathbf{S}^7$  of spheres, and  $\chi_u: \overset{1}{\bullet} \equiv \overset{1}{\circ} \oplus \overset{1}{\bullet} \equiv \overset{1}{\circ}$ ; or  $G_u = M_u = \mathbf{T}^2$  and  $H_u = \{1\}$ .*

*Proof.* If  $\text{rank } G_u = \text{rank } H_u$  then  $H_u$  is the connected centralizer of its center, and Theorem 2.2 says  $\beta_1 \not\sim \beta_2$ . Now  $\text{rank } H_u < \text{rank } G_u$ . If  $H_u$  has a central element  $z \neq 1$  then  $\beta_1(z) = \beta_2(z) \neq I$  so  $H_u$  is the connected centralizer of  $z$ ; that violates the rank condition. Now  $H_u$  is centerless.

Suppose that  $\mathfrak{H}_u$  is a maximal subalgebra of  $\mathfrak{G}_u$ . The rank condition says that  $\mathfrak{G}_u$  is semisimple. Now Lemma 1.4 says that  $\mathfrak{G}_u$  is simple. As  $\mathfrak{H}_u \subset \mathfrak{G}_u$  does not appear on Dynkin's list ([7], page 231), now  $\mathfrak{G}_u$  is classical simple, so (a)  $\mathfrak{G}_u = \mathfrak{SU}(N)$ , (b)  $\mathfrak{G}_u = \mathfrak{SO}(N)$ , or (c)  $\mathfrak{G}_u = \mathfrak{Sp}(N)$ . We view the inclusion  $\mathfrak{H}_u \rightarrow \mathfrak{G}_u$  as a linear representation  $\pi$ . If it is not absolutely irreducible then maximality of  $\mathfrak{H}_u$  shows  $M_u$  irreducible symmetric with  $\chi_u$  absolutely irreducible. Now  $\pi = \pi_\lambda$  for some highest weight  $\lambda$  of  $\mathfrak{H}_u$ . Let  $\psi$  denote  $\text{ad}_{\mathfrak{G}_u} \cdot \pi$ , so  $\psi = \text{ad}_{H_u} \oplus \beta_1 \oplus \beta_2$ . We go by cases.

(a)  $\mathfrak{G}_u = \mathfrak{SU}(N)$ . Then  $\psi \oplus I_{H_u} = \pi_\lambda \otimes \pi_{\lambda^*}$ , so  $\pi_{\lambda+\lambda^*}$  is a summand of  $\psi$ .  $\lambda \neq \lambda^*$  because  $\mathfrak{SO}(N) \neq \mathfrak{G}_u \neq \mathfrak{Sp}(N/2)$ . If  $\pi_{\lambda+\lambda^*}$  is one of the  $\beta_i$ , say  $\beta_1$ , then  $\beta_1 \sim \beta_2$  says that  $\lambda + \lambda^*$  is a weight of multiplicity  $\geq 2$  in  $\pi_\lambda \otimes \pi_{\lambda^*}$ . Now  $\pi_{\lambda+\lambda^*}$  is a summand of  $\text{ad}_{H_u}$ , so  $\mathfrak{H}_u = \mathfrak{G}_u$ . That is impossible.

(b)  $\mathfrak{G}_u = \mathfrak{SO}(N)$ . Then  $\psi = \Lambda^2(\pi_\lambda)$ , so  $\pi_{2\lambda-\alpha_i}$  is a summand of  $\psi$  for each simple root  $\alpha_i$  not orthogonal to  $\lambda$ . Every weight of the form  $2\lambda - \alpha_i$  has multiplicity 1 in  $\Lambda^2(\pi_\lambda)$ , so

$\pi_{2\lambda-\alpha_i} \neq \beta_j$ ; thus each  $\pi_{2\lambda-\alpha_i}$  is a summand of  $\text{ad}_{H_u}$ . If  $\mathfrak{H}_u$  is not simple it has form  $\mathfrak{Sp}(n_1) \oplus \mathfrak{Sp}(n_2)$ ,  $4n_1n_2 = N$ , or  $\mathfrak{SO}(n_1) \oplus \mathfrak{SO}(n_2)$ ,  $n_1n_2 = N$ , as in the proof of Theorem 4.1. Then  $\pi_\lambda$  is the tensor product of usual vector representations of the summands and each  $2\lambda - \alpha_i$  has nonzero values on both summands. That prevents  $\pi_{2\lambda-\alpha_i}$  from being a summand of  $\text{ad}_{H_u}$ . Now  $\mathfrak{H}_u$  is simple,  $\lambda$  is a multiple  $k\xi_r$  of a basic weight, and  $\text{ad}_{H_u} = \pi_{2k\xi_r-\alpha_r}$ . Thus  $\mathfrak{H}_u$  has highest root  $2k\xi_r - \alpha_r$ , and the second and third paragraphs of the proof of Proposition 5.2 show  $G_u/H_u = \text{Spin}(7)/\mathbf{G}_2$ . That implies absolute irreducibility of  $\chi_u$ , which is impossible.

(c)  $\mathfrak{G}_u = \mathfrak{Sp}(N)$ . Then  $\psi = S^2(\pi_\lambda)$ , which has  $\pi_{2\lambda}$  as a summand. As  $2\lambda$  is a weight of multiplicity 1 in  $S^2(\pi_\lambda)$ ,  $\pi_{2\lambda}$  cannot be one of the  $\beta_i$ , so  $\pi_{2\lambda}$  is a summand of  $\text{ad}_{H_u}$ . As  $\pi_\lambda$  is faithful this shows that  $H_u$  is simple and  $\text{ad}_{H_u} = \pi_{2\lambda}$ . Now  $2\lambda$  is the highest root of  $H_u$  and it follows from classification that either  $2\lambda: \overset{2}{\circ}$  or  $2\lambda: \bullet-\bullet-\dots-\bullet=\circ$ . That says that either  $\lambda: \overset{1}{\circ}$  or  $\lambda: \bullet-\bullet-\dots-\bullet=\circ$ . Now  $\mathfrak{H}_u = \mathfrak{G}_u$ . That is impossible.

For purposes of Theorem 12.2 we could stop at this point. But we continue because the lemma is relevant to the results of § 14.

We have proved that  $\mathfrak{H}_u$  is not a maximal subalgebra of  $\mathfrak{G}_u$ . Thus we have  $\mathfrak{G}_u = \mathfrak{H}_u + \mathfrak{M}_1 + \mathfrak{M}_2$ , where  $\mathfrak{M}_i$  is the representation space of  $\beta_i$  and  $\mathfrak{L} = \mathfrak{H}_u + \mathfrak{M}_1$  is an algebra. Let  $\gamma$  denote the representation of  $\mathfrak{L}$  on  $\mathfrak{M}_2$ , so  $\beta_2 = \gamma|_{\mathfrak{H}_u}$ ; thus  $\gamma$  is absolutely irreducible and Theorem 2.2 shows  $\text{rank } \mathfrak{L} < \text{rank } \mathfrak{G}$ . Similarly Theorem 2.2 and absolute irreducibility of  $\beta_1$  shows  $\text{rank } \mathfrak{H}_u < \text{rank } \mathfrak{L}$ . If  $\mathfrak{H}_u$  contains a nonzero ideal of  $\mathfrak{L}$  then that ideal is killed by  $\beta_1$ , hence also by  $\beta_2$ , contradicting effectiveness of  $G_u$  on  $M_u$ ; now  $L/H_u$  is effective and isotropy irreducible.

If  $L$  is not simple then Lemma 1.4 says that  $\mathfrak{L} \cong \mathfrak{H}_u \oplus \mathfrak{H}_u$  with  $\mathfrak{H}_u$  simple and embedded diagonally, so  $\beta_1$  is the adjoint representation of  $H_u$ . Then  $\beta_2 \sim \text{ad}_{H_u}$  and Corollary 10.2 says that  $\gamma(L)$  is the full  $\mathbf{SO}(\mathfrak{M}_2)$ . As  $\gamma$  is faithful on the diagonal  $H_u$  of  $L$ , it is faithful on  $L$ ; thus  $L \cong \mathbf{SO}(m)$ ,  $m = \dim \mathfrak{M}_2$ , and  $G_u/L$  is effective. That says  $G_u = \mathbf{SO}(m+1)$ . Note that  $m=4$  because  $L$  is semisimple but not simple. Now  $H_u \subset L \subset G_u$  is given by  $\text{SU}(2) \subset \mathbf{SO}(4) \subset \mathbf{SO}(5)$ , so  $m=4$  is equal to  $\dim \text{SU}(2) = 3$ . That is absurd. Thus  $L$  is simple. If  $G_u$  is not simple then  $\mathfrak{G}_u = \mathfrak{L} \oplus \mathfrak{L}$  and  $\dim \mathfrak{M}_2 = \dim \mathfrak{L} > \dim \mathfrak{M}_1$ ; thus  $G_u$  is also simple.

Suppose that  $H_u$  is not simple. Then Theorem 11.1 and the classification of symmetric spaces  $A/B$  ( $A$  simple,  $\text{rank } B < \text{rank } A$ ) show that  $L/H_u$  is one of

- (a)  $\text{SU}(pq)/\text{SU}(p) \cdot \text{SU}(q)$ ,  $p \geq 2$ ,  $q \geq 2$ ;  $N = (p^2 - 1)(q^2 - 1)$ .
- (b)  $\mathbf{F}_4/\mathbf{SO}(3) \times \mathbf{G}_2$ ;  $N = 35$ .
- (c)  $\mathbf{E}_6/\text{SU}(3) \times \mathbf{G}_2$ ;  $N = 46$ .

- (d)  $E_7/Sp(3) \times G_2$ ;  $N = 98$ .
- (e)  $E_7/SU(2) \times F_4$ ;  $N = 78$ .
- (f)  $E_8/G_2 \times F_4$ ;  $N = 182$ .

Here  $N$  is the dimension. Note that  $N = \dim \mathfrak{M}_i = \dim G_u/L$ .  $SU(m)$  ( $m \geq 4$ ),  $E_6$ ,  $E_7$  and  $E_8$  cannot be symmetric subgroups of lower rank in a simple group. If  $G_u/L$  is symmetric, now it must be  $E_6/F_4$ , which has dimension 26 ( $\neq 35$ ). Thus  $G_u/L$  is not symmetric, so it is listed in Theorem 11.1. Now  $L \neq E_8$  because  $\dim SO(248)/E_8 = 30132 > 182$ . And  $L \neq E_7$  because  $\dim SO(133)/E_7 = 8645 > \dim Sp(28)/E_7 = 1463 > 98 > 78$ . Similarly  $L \neq E_6$  because  $\dim SO(78)/E_6 = 2925 > \dim SU(27)/E_6 = 705 > 46$ , and  $L \neq F_4$  because  $\dim SO(52)/F_4 = 1274 > \dim SO(26)/F_4 = 283 > 35$ . Thus  $L = SU(m)$ ,  $m = pq$  not prime, with  $N = (p^2 - 1)(q^2 - 1) < m^2$ . As  $m \neq 3$  and  $\text{rank } L < \text{rank } G_u$  now  $G_u/L$  is  $(a_1) SO(20)/SU(4)$ ,  $(a_2) SO(70)/SU(8)$ ,  $(a_3) SO(m^2 - 1)/SU(m)$ ,  $(a_4) Sp(10)/SU(6)$  or  $(a_5) SU(\frac{1}{2}m(m \pm 1))/SU(m)$ . Each case is eliminated because it has dimension  $> m^2$ . Thus  $H_u$  is simple.

Now  $H_u$ ,  $L$  and  $G_u$  are all simple, and we have  $\beta_2(H_u) \subsetneq \gamma(L) \subset SO(\mathfrak{M}_2)$ . Suppose  $\gamma(L) \neq SO(\mathfrak{M}_2)$ . Then (Dynkin [6], pages 253 and 364)  $\gamma$  is given

$$\begin{aligned} &\text{by } \overset{k}{\bullet} \equiv \circ \text{ with } k \geq 1, \text{ by } \overset{k}{\bullet} = \circ - \dots - \circ \text{ with } n \geq 1 \text{ and } k \geq 1 \text{ and } k(n+1) \text{ odd,} \\ &\text{by } \overset{6}{\circ}, \text{ by } \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}, \text{ by } \overset{1}{\bullet} = \circ - \overset{1}{\circ} - \overset{1}{\circ}, \text{ by } \overset{2}{\circ} = \bullet - \bullet, \\ &\text{by } \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{array}{l} \diagup \circ \\ \diagdown \circ \end{array}, \text{ by } \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{array}{l} | \\ \circ \end{array}, \text{ or by } \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{array}{l} | \\ \circ \end{array}. \end{aligned}$$

If  $G_u/L$  is not symmetric then it is listed in Theorem 11.1 with  $\chi = \gamma$ ; the only cases are  $SO(14)/G_2$  with  $\gamma: \overset{3}{\bullet} \equiv \circ$ ,  $SO(7)/G_2$  with  $\gamma: \overset{1}{\bullet} \equiv \circ$ ,  $Sp(2)/SU(2)$  with  $\gamma: \overset{6}{\circ}$ , and  $SU(15)/SU(6)$  with  $\gamma: \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$ . If  $L$  is of type  $G_2$  then  $L/H_u$  has dimension 11; this eliminates the first two cases. If  $L$  is of type  $A_1$  then  $H_u$  is  $\{1\}$ ; this eliminates the third case. If  $L$  is of type  $A_5$  then  $\dim L/H_u < \dim L = 35 < \deg \gamma$ ; this eliminates the last case. Now  $G_u/L$  is symmetric. A simple symmetric subgroup of lower rank is a simple group is of classical type or  $F_4$ ; this eliminates the possibilities

$$\overset{k}{\bullet} \equiv \circ, \quad \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{array}{l} | \\ \circ \end{array} \text{ and } \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \begin{array}{l} | \\ \circ \end{array}$$

for  $\gamma$ . If  $L$  is of type  $A_n$  then  $G_u/L = SU(3)/SO(3)$ , so  $H_u = \{1\}$ ; that is impossible. If  $L$  is of type  $B_n$  then  $G_u/L$  is  $SU(2n+1)/SO(2n+1)$  or  $SO(2n+2)/SO(2n+1)$ ; the first has isotropy representation  $\overset{2}{\circ} - \circ - \dots - \circ = \bullet$  and the second has  $\overset{1}{\circ} - \circ - \dots - \circ = \bullet$ ; those

are not possibilities for  $\gamma$ . If  $L$  is of type  $C_n (n > 2)$  then  $G_u/L = \text{SU}(2n)/\text{Sp}(n)$  which has isotropy representation  $\bullet \overset{1}{\text{---}} \bullet \text{---} \dots \text{---} \bullet = \circ$  not among the choices of  $\gamma$ . If  $L$  is of type  $D_n$

then  $G_u/L$  and the isotropy representations are  $\text{SU}(2n)/\text{SO}(2n)$  and  $\overset{2}{\circ} \text{---} \circ \text{---} \dots \text{---} \circ \begin{matrix} \diagup \circ \\ \diagdown \circ \end{matrix}$

which is not a possibility for  $\gamma$ . Now  $\gamma$  does not exist. This contradicts the assumption  $\gamma(L) \subseteq_{\neq} \text{SO}(\mathfrak{M}_2)$ .

Now  $H_u, L$  and  $G_u$  are simple with  $\gamma(L) = \text{SO}(\mathfrak{M}_2)$ . Thus  $G_u/L$  is of the form  $\text{SO}(2m)/\text{SO}(2m-1)$  with  $\deg \beta_i = 2m-1$ . As  $\mathbf{B}_{m-1}$  has no outer automorphism,  $L/H_u$  is listed in Theorem 11.1 with  $L = \text{SO}(2m-1)$  or  $L = \text{Spin}(2m-1)$ , and  $\dim H_u = (2m-1)(m-2)$ . The latter says  $\dim H_u = [(m-2)/(m-1)] \dim L$ . That shows  $L/H_u = \text{Spin}(7)/\mathbf{G}_2$  and  $G_u/L = \text{Spin}(8)/\text{Spin}(7)$ , so  $M_u = G_u/H_u = \text{Spin}(8)/\mathbf{G}_2 = \mathbf{S}^7 \times \mathbf{S}^7$ , q.e.d.

12.4 *Remark.* The proof of Lemma 12.3 actually shows: *If  $M_u = G_u/H_u$  is an effective coset space of compact connected Lie groups, and if  $H_u$  is a maximal proper connected subgroup of  $G_u$  and has linear isotropy representation which is a sum of copies of the same irreducible complex representation, then the linear isotropy representation of  $H_u$  is absolutely irreducible.*

The classification of simply connected isotropy irreducible reductive coset spaces  $M = G/H$ ,  $G$  connected and effective on  $M$ , now splits into three parts.

1. The case where  $G$  is not semisimple. Here all spaces  $M$  are constructed as follows. Let  $\beta$  be a faithful irreducible complex representation of a reductive Lie algebra  $\mathfrak{G}$ . If  $\beta$  is equivalent to a real representation, define  $\pi = \beta$ ,  $n = \deg \beta$ , and let  $H$  be the analytic subgroup of  $\mathbf{GL}(n, \mathbf{R})$  with Lie algebra  $\beta(\mathfrak{G})$ . Otherwise, define  $\pi = \beta \oplus \bar{\beta}$ ,  $n = 2 \deg \beta$ , and let  $H$  be the analytic subgroup of  $\mathbf{GL}(n, \mathbf{R})$  with Lie algebra  $(\beta \oplus \bar{\beta})(\mathfrak{G})$ . Then  $H$  is a closed subgroup of  $\mathbf{GL}(n, \mathbf{R})$  and  $G = H \times_{\pi} \mathbf{R}^n$ . The various possibilities for the pair  $(\beta, \mathfrak{G})$  are known from É. Cartan's theory of representations of real semisimple Lie algebras.

2. The case where  $G$  is semisimple and  $\chi$  is absolutely irreducible. These are the spaces  $M = G/H$  constructed as follows.  $M_u = G_u/H_u$  either is an arbitrary nonhermitian compact simply connected irreducible symmetric space, or is any space listed in Theorem 11.1 with linear isotropy representation  $\chi_u$  absolutely irreducible. For each such  $M_u$  one must find all  $\text{ad}(H_u)$ -conjugacy classes of involutive automorphisms  $\sigma$  of  $\mathfrak{G}_u$  which preserve  $\mathfrak{G}_u$ . For each such triple  $(\mathfrak{G}_u, \mathfrak{G}_u, \sigma)$  one has the space  $M = G/H$  constructed in Theorem 12.1. All possible  $M = G/H$  are constructed this way.

3. The case where  $G$  is semisimple and  $\chi$  is not absolutely irreducible. All such spaces  $M = G/H$  are constructed, as in Theorems 12.1 and 12.2, from the compact version  $M_u = G_u/H_u$  and an involutive automorphism  $\sigma$  of  $\mathfrak{G}_u$  which preserves  $\mathfrak{G}_u$ , as follows.

(3a)  $G$  is a complex simple Lie group,  $A$  is a compact real form (hence a maximal compact subgroup),  $M_u = G_u/H_u = (A \times A)/(B \times B) = (A/B) \times (A/B)$ , and  $\sigma$  is interchange of the two compact simple factors of  $G_u = A \times A$ . Here  $A/B$  either is an arbitrary non-hermitian compact simply connected irreducible symmetric space or is any of the spaces listed in Theorem 11.1 with absolutely irreducible linear isotropy representation.

(3b)  $M_u = G_u/H_u$  is an irreducible hermitian symmetric space and  $\sigma$  is any involutive automorphism of  $\mathfrak{G}_u$  which preserves  $\mathfrak{H}_u$  and does not interchange the two inequivalent irreducible summands of  $\chi_u$ .

(3c)  $M_u = G_u/H_u$  is any space listed in Theorem 11.1 with linear isotropy representation  $\chi_u$  which is not absolutely irreducible, and  $\sigma$  is any involutive automorphism of  $G_u$  which preserves  $H_u$  and does not interchange the two summands of  $\chi_u$ .

Problem 1 was settled by É. Cartan, as mentioned above. Problem 2 is straightforward and quite tedious. The techniques relevant to problem 2 are all needed for problem 3, which we will settle in § 13 in the context of invariant almost complex structures.

## Chapter II. Invariant structures on isotropy irreducible coset spaces

In this chapter we study complex and quaternionic structures on isotropy irreducible coset spaces. Complex structures are considered in § 13. There we see that an isotropy irreducible coset space  $G/H$  carries an invariant complex structure if and only if either it is hermitian symmetric or  $G$  and  $H$  are complex Lie groups. We see that  $G/H$  carries an invariant almost complex structure if and only if the linear isotropy representation is not absolutely irreducible, and it then turns out that  $H$  must be connected, except when  $G$  and  $H$  are complex groups. These characterizations lead to an easy classification. Quaternionic structures are considered in § 14, partly because they are needed later in our description of linear holonomy groups, and partly to illustrate the general notions of invariant structure and commuting structure. Invariant quaternionic structures turn out to exist only on those isotropy irreducible coset spaces which are the quaternionic symmetric spaces of [18] and their noncompact versions.

### 13. Complex structures

We first settle the case of compact isotropy subgroup:

**13.1 THEOREM.** *Let  $M = G/K$ , where  $G$  is a connected Lie group acting effectively,  $K$  is compact, and the linear isotropy representation of the identity component  $K_0$  is  $\mathbf{R}$ -irreducible. Then  $M$  has a  $G$ -invariant complex structure if and only if it is a hermitian symmetric space. If  $M$  is not euclidean then the following conditions are equivalent.*



1.  $M$  has a  $G$ -invariant almost complex structure.
2.  $M$  has precisely two  $G$ -invariant almost complex structures.
3.  $K$  is connected and its linear isotropy representation is not absolutely irreducible (so necessarily  $\chi = \beta \oplus \bar{\beta}$ ,  $\beta$  irreducible complex,  $\beta + \bar{\beta}$ ).

13.2 COROLLARY. Let  $M = G/K$  where  $G$  is a connected Lie group acting effectively,  $K$  is compact, and the identity component  $K_0$  has  $\mathbb{R}$ -irreducible linear isotropy representation  $\chi$ . Suppose that  $M$  has a  $G$ -invariant almost complex structure but that  $M$  is not hermitian symmetric. Then  $G = \tilde{G}/E$  and  $\tilde{K} = \tilde{K}E/E$  where  $E$  is an arbitrary central subgroup of  $\tilde{G}$  and all possibilities are given as follows.

$\tilde{G}$	$\tilde{K}$	Center of $G$	$\chi$
$\text{Spin}(n^2 - 1)$ $n$ odd, $n > 2$	$\text{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\overset{3}{\circ} - \overset{3}{\circ} \oplus \overset{3}{\circ} - \overset{3}{\circ}$ if $n = 3$
$\text{SO}(n^2 - 1)$ $n$ even, $> 3$	$\text{SU}(n)/\mathbb{Z}_n$	$\{1\}$	$\overset{1}{\circ} - \overset{1}{\circ} - \dots - \overset{2}{\circ} \oplus \overset{2}{\circ} - \dots - \overset{1}{\circ} - \overset{1}{\circ}$ if $n > 3$
$G_2$	$\text{SU}(3)$	$\{1\}$	$\overset{1}{\circ} - \overset{1}{\circ} \oplus \overset{1}{\circ} - \overset{1}{\circ}$
$F_4$	$\{\text{SU}(3) \times \text{SU}(3)\}/\mathbb{Z}_3$	$\{1\}$	$\overset{1}{\circ} - \overset{2}{\circ} \otimes \overset{2}{\circ} - \overset{1}{\circ} \oplus (\overset{1}{\circ} - \overset{2}{\circ} \otimes \overset{2}{\circ} - \overset{1}{\circ})$
$E_6$	$\text{SU}(3)/\mathbb{Z}_3$	$\mathbb{Z}_3$	$\overset{1}{\circ} - \overset{4}{\circ} \oplus \overset{4}{\circ} - \overset{1}{\circ}$
$E_6/\mathbb{Z}_3$	$\{\text{SU}(3) \times \text{SU}(3) \times \text{SU}(3)\}/\{\mathbb{Z}_3 \times \mathbb{Z}_3\}$	$\{1\}$	$\overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ}$ $\oplus (\overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ})$
$E_7/\mathbb{Z}_2$	$\{\text{SU}(3) \times [\text{SU}(6)/\mathbb{Z}_2]\}/\mathbb{Z}_3$	$\{1\}$	$\overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$ $\oplus (\overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ})$
$E_8$	$\text{SU}(9)/\mathbb{Z}_3$	$\{1\}$	$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$ $\oplus \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ}$
$E_8$	$\{\text{SU}(3) \times E_6\}/\mathbb{Z}_3$	$\{1\}$	$\left( \overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \right)$ $\oplus \left( \overset{1}{\circ} - \overset{1}{\circ} \otimes \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} \right)$

*Proof.* We first check equivalence of the three conditions listed in the theorem. First assume (3). Decomposing  $\chi = \beta \oplus \bar{\beta}$  and glancing through Theorem 11.1 we see that  $\beta \sim \bar{\beta}$ . Thus the commuting algebra of  $\chi$  (the algebra of linear transformations of  $M_x$  which commute with every element of  $\chi(K)$ ) is  $\mathbb{C}$ , which has precisely two elements of square  $-1$ . Thus (3) implies (2).

(2) implies (1) at a glance.

Assume (1). If  $\chi$  were absolutely irreducible its commuting algebra would be  $\mathbb{R}$ , which has no element of square  $-1$ , so (1) would fail; thus  $\chi$  is not absolutely irreducible. Suppose that  $K$  is not connected. Then  $M$  is not simply connected, so  $G/K$  is not hermitian symmetric. If  $\text{rank } G = \text{rank } K$  it follows that the center of  $K_0$  is generated by an element  $z$  of order 3 and, replacing  $z$  by  $z^{-1}$  if necessary, the almost complex structure  $J$  satisfies  $\chi(z) = \cos(2\pi/3)I + \sin(2\pi/3)J$ . If  $k \in K$  then stability of  $J$  under  $k$  implies  $kz = zk$ . Pre-images of  $k$  and  $z$  in the universal covering group of  $G$  still commute; as the pre-image of  $K_0$  there is the full centralizer of any pre-image of  $z$ , and is connected, it follows that  $k \in K_0$ . Thus disconnectedness of  $K$  implies  $\text{rank } G > \text{rank } K$ . If  $k \in K$ ,  $k \notin K_0$ , now  $\text{ad}(k)$  gives an outer automorphism of  $K_0$  because  $K_0$  is a maximal connected subgroup of lower rank.  $K_0$  is simple by Theorem 11.1; it follows that  $\beta \cdot \text{ad}(k) \sim \bar{\beta} \sim \beta$ , so  $K$  has commuting algebra  $\mathbb{R}$ , which is ridiculous. This contradiction proves  $K$  connected, completing the proof that (1) implies (3).

We have proved equivalence of the three conditions of the theorem. If  $M$  had a  $G$ -invariant complex structure it would be a  $C$ -space in the sense of H.-C. Wang [14] and  $K$  would be contained in the centralizer of a toral subgroup of  $G$ . Thus  $K$  could not be semisimple, so  $M$  would be hermitian symmetric. The theorem is proved.

The corollary follows from the theorem and a glance at Theorem 11.1, q.e.d.

In the case of equal ranks, passage to noncompact isotropy is based on

**13.3 THEOREM.** *Let  $M = G/K$  where  $G$  is a compact connected Lie group acting effectively and  $K$  is a closed connected subgroup of maximal rank. Let  $\alpha$  be an automorphism of  $G$  which preserves  $K$ , thus acts on  $M$ , and which preserves some  $G$ -invariant almost complex structure  $J$  on  $M$ .*

1. *The following conditions are equivalent, and each implies that  $\alpha$  preserves every  $G$ -invariant almost complex structure on  $M$ .*

(1a)  *$\alpha$  is an inner automorphism of  $G$ .*

(1b)  *$\alpha|_K$  is an inner automorphism of  $K$ .*

(1c)  *$\alpha$  is conjugation  $\text{ad}_G(k)$  by some element  $k \in K$ ,*

2. If  $G$  is simple and  $\alpha$  is an outer automorphism of  $G$ , then

(2a)  $G = \mathbf{SU}(2n)/\mathbf{Z}_{2n}$ ,  $K = \mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$ ,  $\alpha|_K$  interchanging the two  $\mathbf{U}(n)$ ; or

(2b)  $G = \mathbf{SO}(2n)/\mathbf{Z}_2$ ,  $K = \{\mathbf{U}(n_1) \times \dots \times \mathbf{U}(n_s) \times \mathbf{SO}(2m)\}/\mathbf{Z}_2$ ,  $n_1 + \dots + n_s + m = n$ ,  $m \geq 2$ ,  $\alpha$  conjugation by  $\text{diag}\{P_1, \dots, P_s; Q\}$ ,  $P_i \in \mathbf{U}(n_i)$ ,  $Q \in \mathbf{O}(2m)$ ,  $\det Q = -1$ ; or

(2c)  $G = \mathbf{E}_6/\mathbf{Z}_3$ ,  $K = \{\mathbf{SU}(3) \times \mathbf{SU}(3) \times L_i\}/\{\mathbf{Z}_3 \times \mathbf{Z}_3\}$ ,  $1 \leq i \leq 3$ ,  $\alpha$  exchanging the two  $\mathbf{SU}(3)$ ,  $\alpha(L_i) = L_i$ , where  $L_1 \subset L_2 \subset L_3$  is  $\mathbf{T}^2 \subset \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(2)) \subset \mathbf{SU}(3)$ .

*Proof.* We first prove (1). Let  $\alpha|_K$  be an inner automorphism of  $K$ . Then  $\alpha|_T$  is the identity for some maximal torus  $T$  of  $K$ . As  $T$  is a maximal torus of  $G$ , now  $\alpha$  is an inner automorphism of  $G$ . So  $\alpha = \text{ad}(g)$  for some  $g \in G$  which centralizes  $T$ . Thus  $g \in T \subset K$ . We have just seen that (1b) implies (1a) and (1c). As (1c) visibly implies (1b), we now need only check that (1a) implies (1b).

Let  $Z$  be the center of  $K$ . If  $z \in Z$  has odd order then  $\alpha(J) = J$  and Theorem 2.2 show that  $\alpha(z) = z$ . Let  $Z_0$  be the identity component of  $Z$ , central toral subgroup of  $K$ ; now  $\alpha|_{Z_0} = 1$ . Let  $L$  be the centralizer of  $Z_0$  in  $G$ . As  $\alpha$  is inner now  $\alpha = \text{ad}(g)$  for some  $g \in L$ . Decompose  $\mathfrak{L} = \mathfrak{L}' \oplus \mathfrak{L}''$  and  $\mathfrak{K} = \mathfrak{K}' \oplus \mathfrak{L}''$  into the derived algebras and the centers; the semisimple parts  $\mathfrak{K}' \subset \mathfrak{L}'$ , this reduces the proof that (1a) implies (1b) to the case where  $K$  is semisimple.

Now  $K$  is semisimple and  $Z$  is finite. If  $K$  were not maximal among the connected subgroups of maximal rank in  $G$ , say  $K \subset L \subset G$ , induction on codimension would prove  $\alpha|_L$  inner on  $L$  and then  $\alpha|_K$  inner on  $K$ . Now we may assume  $K$  maximal among the connected subgroups of  $G$ . If  $Z$  had even order,  $K$  would be a nonhermitian symmetric subgroup of  $G$ , contradicting the existence of the invariant almost complex structure. Thus  $Z$  has odd order and  $\alpha|_Z = 1$ . This proves  $g \in K$ , so  $\alpha|_K$  is inner.

Part (1) of the theorem is proved.

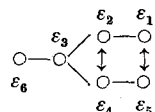
We now assume  $G$  simple and  $\alpha$  outer. Again,  $Z$  is the center of  $K$ ,  $Z_0$  the identity component of  $Z$ , and  $L$  the centralizer of  $Z_0$  in  $G$ . If  $L = G$  then  $Z$  is finite and  $K$  is semisimple; [9] shows that  $G$  is an exceptional group and then  $G$  must be  $\mathbf{E}_6/\mathbf{Z}_3$  because it is centerless simple and admits an outer automorphism. Then it is immediate [9] that  $K = \{\mathbf{SU}(3) \times \mathbf{SU}(3) \times \mathbf{SU}(3)\}/\{\mathbf{Z}_3 \times \mathbf{Z}_3\}$  with  $\alpha$  interchanging the first two factors and preserving the third. If  $K \neq L$ , so  $G$  has a connected subgroup of maximal rank which is not the centralizer of a torus, then [4]  $G$  is exceptional, hence again of type  $\mathbf{E}_6$ , and it follows [9] that  $L = G$ . In the proof of part (2) of the theorem, now, we may assume  $K = L$ , so that  $K$  is the centralizer of the torus  $Z_0$ .

If  $z \in Z$  has odd order then  $\alpha(J) = J$  and Theorem 2.2 show  $\alpha(z) = z$ ; thus  $\alpha|_Z = 1$ . Decompose  $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ ,  $\mathfrak{K} = \mathfrak{L} + \sum \mathfrak{K}_s$ ,  $\mathfrak{M}^c = \sum \mathfrak{M}_i$ , where the  $\mathfrak{K}_s$  are the simple ideals of  $\mathfrak{K}$  and where  $\mathfrak{K}$  acts on  $\mathfrak{M}_i$  by an irreducible representation  $\pi_i$ . Theorem 2.2 and the existence

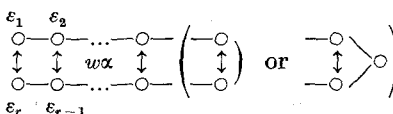
of the almost complex structure show that the  $\pi_i$  are mutually inequivalent and that each  $\pi_i \neq \bar{\pi}_i$ . Triviality of  $\alpha$  on  $\mathfrak{g}$  now shows that  $\pi_i \cdot \alpha \sim \pi_i$  and  $\alpha(\mathfrak{M}_i) = \mathfrak{M}_i$ . Decompose  $\pi_i = \theta_i \otimes (\otimes \eta_{is})$  where  $\theta_i$  represents  $\mathfrak{g}$  and  $\eta_{is}$  represents  $\mathfrak{K}_s$ . If  $\alpha(\mathfrak{K}_s) = \mathfrak{K}_s$  then  $\eta_{is} \cdot \alpha \sim \eta_{is}$ .

Choose a maximal torus  $T$  of  $G$  such that  $Z_0 \subset T \subset K$ ; choose a system  $B = \{\beta_1, \dots, \beta_r\}$  of simple roots of  $G$  such that  $\mathfrak{g}$  has equation  $\beta_1 = \dots = \beta_t = 0$  in  $\mathfrak{X}$ . Then  $B' = \{\beta_1, \dots, \beta_t\}$  is a simple root system for the derived algebra  $\mathfrak{K}' = \sum \mathfrak{K}_s$ . These choices of simple root systems  $B' \subset B$  amount to choices of positive Weyl chambers  $\mathfrak{D}' = \mathfrak{D} \cap \sqrt{-1}\mathfrak{X}' \subset \mathfrak{D}$ ,  $\mathfrak{X}' = \mathfrak{X} \cap \mathfrak{K}'$ . Let  $w$  be an element of the Weyl group of  $\mathfrak{K}'$  which carries  $\alpha(\mathfrak{D}')$  back to  $\mathfrak{D}'$ . Then  $w\alpha$  is the identity on  $\mathfrak{g}$ , permutes  $B'$  and preserves  $\mathfrak{X}'$ . Let  $x \in \mathfrak{D}$ , say  $x = z + x'$  with  $z \in \sqrt{-1}\mathfrak{g}$  and  $x' \in \sqrt{-1}\mathfrak{X}'$ ,  $x$  regular,  $\beta_i(z) > \beta_i(x')$  for  $i > t$ . Then it is immediate that every  $\beta_i(w\alpha x) > 0$ . Thus  $w\alpha(\mathfrak{D}) = \mathfrak{D}$ . As  $\alpha$  is outer, now  $w\alpha$  is a nontrivial automorphism of the Dynkin diagram of  $G$ ; it preserves the diagram of  $K$ , which is obtained from that of  $G$  by deleting the vertices  $\beta_i$  with  $i > t$ , and induces a nontrivial automorphism there. If  $i > t$ , so  $\beta_i$  is not a root of  $K$ , then  $w\alpha$  leaves  $\beta_i$  fixed; for if  $\{x_j\}$  is a dual basis of  $\sqrt{-1}\mathfrak{X}$  relative to the Killing form and the basis  $\{\beta_j\}$ , so that  $\mathfrak{g}$  has basis consisting of the  $\sqrt{-1}x_j$  for  $t < j \leq r$ , then triviality of  $w\alpha$  on  $\mathfrak{g}$  shows  $w\alpha(\sqrt{-1}x_i) = \sqrt{-1}x_i$ . In other words,  $B'$  contains every root of  $B$  which is moved by  $w\alpha$ .

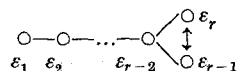
We run through the list of simple groups  $G$  which admit outer automorphisms. If  $G$

is of type  $E_6$    $w\alpha$  then  $B'$  contains  $\epsilon_1, \epsilon_2, \epsilon_4$  and  $\epsilon_5$ , for those are the roots

moved by  $w\alpha$ . If  $B'$  also contains  $\epsilon_3$ , then  $B' \neq B$  shows that  $K' = \text{SU}(6)/\mathbf{Z}_2$  globally, so  $\eta \cdot \alpha \sim \bar{\eta}_i \neq \eta_i$ . Thus  $\epsilon_3 \notin B'$ . Now  $B'$  is  $\{\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5\}$  or  $\{\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5, \epsilon_6\}$ , and both possibilities occur.

Let  $G$  be of type  $A_r$ ,  $r > 1$  . Then  $B' \neq B$  says

that  $r = 2v + 1$  and  $B' = \{\epsilon_1, \dots, \epsilon_v; \epsilon_{v+2}, \dots, \epsilon_r\}$  for the latter are the roots moved by  $w\alpha$ . This case occurs geometrically as orthocomplementation on the grassmannian of  $(v + 1)$ -planes in  $\mathbf{C}^{2v+2}$ .

Let  $G$  be of type  $D_r$ ,  $r > 3$    $w\alpha$ . Then  $B'$  contains  $\epsilon_{r-1}$  and  $\epsilon_r$ .

Thus  $G = \text{SO}(2r)/\mathbf{Z}_2$  and  $K = \{\text{U}(r_1) \times \text{U}(r_2) \times \dots \times \text{U}(r_v) \times \text{SO}(2u)\}/\mathbf{Z}_2$ , where  $r_1 + \dots + r_v + u = r$  and  $u \geq 2$ .  $\alpha$  preserves each of the  $\text{U}(r_s)$ , inducing inner automorphisms on them because  $\eta_{is} \cdot \alpha \sim \eta_{is}$ . Thus  $\alpha$  is conjugation by an orthogonal matrix  $\text{diag}\{P_1, \dots, P_v, Q\}$ , where  $P_s \in \text{U}(r_s)$  is a  $2r_s \times 2r_s$  block and  $Q \in \text{O}(2u)$  has determinant  $-1$ , q.e.d.

Now we can settle the case of noncompact isotropy subgroup.

13.4 THEOREM. Let  $M = G/H$  be an effective reductive coset space in which  $G$  is a connected semisimple Lie group and the linear isotropy representation  $\chi$  of the identity component  $H_0$  is  $\mathbf{R}$ -irreducible.

1.  $M$  has a  $G$ -invariant complex structure if and only if
  - (1a)  $M = G/K$  is a hermitian symmetric coset space, or
  - (1b)  $G$  is a complex Lie group and  $H$  is a complex subgroup.
2. If  $G$  is a complex Lie group, then  $H$  is a complex subgroup,  $\mathfrak{G} = \mathfrak{A}^{\mathbf{C}}$  and  $\mathfrak{H} = \mathfrak{B}^{\mathbf{C}}$ , where  $A/B$  is a coset space of compact connected Lie groups which either is an irreducible nonhermitian symmetric coset space, or is listed in Theorem 11.1 with absolutely irreducible linear isotropy representation  $\beta$ .  $\mathfrak{H}$  has real linear isotropy representation  $\chi = \beta \oplus \bar{\beta}$  (conjugation over  $\mathfrak{G}$ ) with commuting algebra  $\mathbf{C}$ , and  $M$  carries just two  $G$ -invariant almost complex structures, both of which are integrable.
3. If  $G$  is not a complex Lie group then the following conditions are equivalent.
  - (3a)  $M$  has a  $G$ -invariant almost complex structure.
  - (3b)  $M$  has precisely two  $G$ -invariant almost complex structures.
  - (3c)  $H$  is connected and  $\chi$  is not absolutely irreducible (so necessarily  $\chi = \beta \oplus \bar{\beta}$  with  $\beta$  complex irreducible,  $\beta \nabla \bar{\beta}$ ).

(3d)  $H$  is connected, the compact version  $M_u = G_u/H_u$  either is hermitian symmetric or is listed in Corollary 13.2, and  $(\mathfrak{G}, \mathfrak{H})$  is defined from  $(\mathfrak{G}_u, \mathfrak{H}_u)$  [as in Theorem 12.1] by an involutive automorphism  $\sigma$  of  $\mathfrak{G}_u$  which preserves both  $\mathfrak{H}_u$  and the two  $G_u$ -invariant almost complex structures on  $M_u$ .

(3e)  $G/H$  is listed in Table 13.5, 13.6 or 13.7 with the following convention. A second subscript (e.g. the  $D_8$  in  $E_{8, D_8}$ ) denotes Cartan classification type of the maximal compact subgroup, and then (in contrast to the compact case, where it means the simply connected group) boldface means the centerless group if it stands alone as in  $\mathbf{E}_{6, A_1 A_1}$ , or the group with cyclic center of order  $m$  if it occurs in an expression of the type  $[\mathbf{E}_{6, A_1 A_1} \times \mathbf{T}^1]/\mathbf{Z}_m$ ; in that type of expression the  $\mathbf{Z}_m$  is diagonal between the circle group and the center of the simple group.

13.5. Table.  $M_u$  hermitian symmetric.

$M_u$	$G$	$K$	Conditions
$\mathbf{SU}(p+q)/\mathbf{S}[\mathbf{U}(p) \times \mathbf{U}(q)]$ $\mathbf{SU}(2n)/\mathbf{S}[\mathbf{U}(n) \times \mathbf{U}(n)]$	$\mathbf{SU}^{u+v}(p+q)/\mathbf{Z}_{p+q}$ $\mathbf{SL}(n, \mathbf{Q})/\mathbf{Z}_2$ or $\mathbf{SL}(2n, \mathbf{R})/\mathbf{Z}_2$	$\mathbf{S}[\mathbf{U}^u(p) \times \mathbf{U}^v(q)]$ $[\mathbf{SL}(n, \mathbf{C}) \times \mathbf{T}^1]/\mathbf{Z}_n$	$0 \leq 2u \leq p \leq q, 0 \leq 2v \leq q$ $n > 1$
$\mathbf{SO}(2n)/\mathbf{U}(n)$	$\mathbf{SO}^{2r}(2n)/\mathbf{Z}_2$ $\mathbf{SO}^*(2n)/\mathbf{Z}_2$	$\mathbf{U}^r(n)/\mathbf{Z}_2$	$0 \leq 2r \leq n, n \geq 3$

$M_u$	$G$	$K$	Conditions
$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}^r(n)/\mathbb{Z}_2$	$\mathrm{U}^r(n)/\mathbb{Z}_2$	$0 \leq 2r \leq n$
$\mathrm{SO}(2m+2)/\mathrm{SO}(2m) \times \mathrm{SO}(2)$	$\mathrm{SO}^r(2m+2)/\mathbb{Z}_2$	$\{\mathrm{SO}^r(2m) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	$0 \leq r \leq m$
	$\mathrm{SO}^{r+2}(2m+2)/\mathbb{Z}_2$	$\{\mathrm{SO}^r(2m) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	$0 \leq r \leq m$
	$\mathrm{SO}^*(2m+2)/\mathbb{Z}_2$	$\{\mathrm{SO}^*(2m) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
$\mathrm{SO}(2m+3)/\mathrm{SO}(2m+1) \times \mathrm{SO}(2)$	$\mathrm{SO}^r(2m+3)$	$\mathrm{SO}^r(2m+1) \times \mathrm{SO}(2)$	$0 \leq r \leq m$
	$\mathrm{SO}^{r+2}(2m+3)$		
$[\mathrm{E}_6/\mathbb{Z}_3]/[\{\mathrm{SO}(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2]$	$\mathrm{E}_6/\mathbb{Z}_3$	$\{\mathrm{SO}(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
	$\mathrm{E}_{6, A_1 A_1}$	$\{\mathrm{SO}^*(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
		$\{\mathrm{SO}^4(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
	$\mathrm{E}_{6, D_5 T^1}$	$\{\mathrm{SO}(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
		$\{\mathrm{SO}^*(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
		$\{\mathrm{SO}^2(10) \times \mathrm{SO}(2)\}/\mathbb{Z}_2$	—
$[\mathrm{E}_7/\mathbb{Z}_2]/[\{\mathrm{E}_6 \times \mathbb{T}^1\}/\mathbb{Z}_3]$	$\mathrm{E}_7/\mathbb{Z}_2$	$\{\mathrm{E}_6 \times \mathbb{T}^1\}/\mathbb{Z}_3$	—
	$\mathrm{E}_{7, A_7}$	$\{\mathrm{E}_{6, A_1 A_5} \times \mathbb{T}^1\}/\mathbb{Z}_3$	—
	$\mathrm{E}_{7, A_1 D_6}$	$\{\mathrm{E}_{6, A_1 A_5} \times \mathbb{T}^1\}/\mathbb{Z}_3$	—
		$\{\mathrm{E}_{6, D_5 T^1} \times \mathbb{T}^1\}/\mathbb{Z}_3$	—
	$\mathrm{E}_{7, E_8 T^1}$	$\{\mathrm{E}_{6, D_5 T^1} \times \mathbb{T}^1\}/\mathbb{Z}_3$	—
		$\{\mathrm{E}_6 \times \mathbb{T}^1\}/\mathbb{Z}_3$	—

13.6. Table. rank  $H = \text{rank } G$ , but  $M_u$  not hermitian symmetric.

$M_u$	$G$	$H$
$\mathrm{G}_2/\mathrm{SU}(3)$	$\mathrm{G}_2$	$\mathrm{SU}(3)$
	$\mathrm{G}_{2, A_1 A_1}$	$\mathrm{SU}^1(3)$
$\mathrm{F}_4/\mathrm{SU}(3) \cdot \mathrm{SU}(3)$	$\mathrm{F}_4$	$[\mathrm{SU}(3) \times \mathrm{SU}(3)]/\mathbb{Z}_3$
	$\mathrm{F}_{4, B_4}$	$[\mathrm{SU}^1(3) \times \mathrm{SU}(3)]/\mathbb{Z}_3$
	$\mathrm{F}_{4, C_3 C_1}$	$[\mathrm{SU}(3) \times \mathrm{SU}^1(3)]/\mathbb{Z}_3$ and $[\mathrm{SU}^1(3) \times \mathrm{SU}^1(3)]/\mathbb{Z}_3$
$\mathrm{E}_6/\mathrm{SU}(3) \cdot \mathrm{SU}(3) \cdot \mathrm{SU}(3)$	$\mathrm{E}_6/\mathbb{Z}_3$	$[\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)]/[\mathbb{Z}_3 \times \mathbb{Z}_3]$
	$\mathrm{E}_{6, A_1 A_5}$	$[\mathrm{SU}^1(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)]/[\mathbb{Z}_3 \times \mathbb{Z}_3]$ and $[\mathrm{SU}^1(3) \times \mathrm{SU}^1(3) \times \mathrm{SU}^1(3)]/[\mathbb{Z}_3 \times \mathbb{Z}_3]$
	$\mathrm{E}_{6, D_5 T^1}$	$[\mathrm{SU}^1(3) \times \mathrm{SU}^1(3) \times \mathrm{SU}(3)]/[\mathbb{Z}_3 \times \mathbb{Z}_3]$
	$\mathrm{E}_{6, F_4}$	$[\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SU}(3)]/\mathbb{Z}_3$
	$\mathrm{E}_{6, C_4}$	$[\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SU}^1(3)]/\mathbb{Z}_3$
$\mathrm{E}_7/\mathrm{SU}(3) \cdot \mathrm{SU}(6)$	$\mathrm{E}_7/\mathbb{Z}_2$	$[\mathrm{SU}(3) \times \mathrm{SU}(6)]/\mathbb{Z}_6$
	$\mathrm{E}_{7, A_7}$	$[\mathrm{SU}(3) \times \mathrm{SU}^1(6)]/\mathbb{Z}_6$ and $[\mathrm{SU}^1(3) \times \mathrm{SU}^3(6)]/\mathbb{Z}_6$
	$\mathrm{E}_{7, A_1 D_6}$	$[\mathrm{SU}^1(3) \times \mathrm{SU}(6)]/\mathbb{Z}_6$ , $[\mathrm{SU}(3) \times \mathrm{SU}^2(6)]/\mathbb{Z}_6$ , and $[\mathrm{SU}^1(3) \times \mathrm{SU}^2(6)]/\mathbb{Z}_6$
	$\mathrm{E}_{7, E_8 T^1}$	$[\mathrm{SU}^1(3) \times \mathrm{SU}^1(6)]/\mathbb{Z}_6$ and $[\mathrm{SU}(3) \times \mathrm{SU}^3(6)]/\mathbb{Z}_6$
$\mathrm{E}_8/\mathrm{SU}(3) \cdot \mathrm{E}_6$	$\mathrm{E}_8$	$[\mathrm{SU}(3) \times \mathrm{E}_6]/\mathbb{Z}_3$
	$\mathrm{E}_{8, D_8}$	$[\mathrm{SU}(3) \times \mathrm{E}_{6, D_5 T^1}]/\mathbb{Z}_3$ and $[\mathrm{SU}^1(3) \times \mathrm{E}_{6, A_1 A_5}]/\mathbb{Z}_3$
	$\mathrm{E}_{8, A_1 E_7}$	$[\mathrm{SU}^1(3) \times \mathrm{E}_6]/\mathbb{Z}_3$ , $[\mathrm{SU}^1(3) \times \mathrm{E}_{6, D_5 T^1}]/\mathbb{Z}_3$ , $[\mathrm{SU}(3) \times \mathrm{E}_{6, A_1 A_5}]/\mathbb{Z}_3$
$\mathrm{E}_8/[\mathrm{SU}(9)/\mathbb{Z}_3]$	$\mathrm{E}_8$	$\mathrm{SU}(9)/\mathbb{Z}_3$
	$\mathrm{E}_{8, D_8}$	$\mathrm{SU}^1(9)/\mathbb{Z}_3$ and $\mathrm{SU}^4(9)/\mathbb{Z}_3$
	$\mathrm{E}_{8, A_1 E_7}$	$\mathrm{SU}^2(9)/\mathbb{Z}_3$ and $\mathrm{SU}^3(9)/\mathbb{Z}_3$

13.7. Table. rank  $G > \text{rank } H$ . Then  $G = \bar{G}/Z$  and  $H = \bar{H}Z/Z$ , where  $Z$  is an arbitrary central subgroup of  $\bar{G}$  and where  $\bar{G}$  and  $\bar{H}$  are given as follows.

$M_u$	$\bar{G}$	Center of $\bar{G}$	$\bar{H}$	Conditions
$\text{Spin}(n^2 - 1)/\text{ad } \text{SU}(n)$	$\text{Spin}(n^2 - 1)$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\text{SU}(n)/\mathbf{Z}_n$	$n$ odd, $n > 2$
	$\text{SO}^{2r(n-r)}(n^2 - 1)$	$\mathbf{Z}_2$	$\text{SU}^r(n)/\mathbf{Z}_n$	$n$ odd, $n > 2$ , $0 < 2r \leq n$
$\text{SO}(n^2 - 1)/\text{ad } \text{SU}(n)$	$\text{SO}^{2r(n-r)}(n^2 - 1)$	1	$\text{SU}(n)/\mathbf{Z}_n$	$n$ even, $n > 3$ , $0 \leq 2r \leq n$
$\text{E}_6/[\text{SU}(3)/\mathbf{Z}_3]$	simply connected (six-sheeted) covering group of $\text{E}_6, A_5, A_1$	$\mathbf{Z}_6$	$\text{SU}^1(3)/\mathbf{Z}_3$	—
	$\text{E}_6$	$\mathbf{Z}_3$	$\text{SU}(3)/\mathbf{Z}_3$	—

*Proof.* We first dispose of the case where  $G$  is a complex Lie group. There Theorem 12.2 tells us that  $H_0$  is a complex analytic subgroup and that  $\mathfrak{G} = \mathfrak{A}^{\mathbb{C}}$  and  $\mathfrak{H} = \mathfrak{B}^{\mathbb{C}}$  where (i)  $\mathfrak{B} \subset \mathfrak{A}$  are the compact real forms of  $\mathfrak{H} \subset \mathfrak{G}$ , (ii)  $\mathfrak{G}_u = \mathfrak{A} \oplus \mathfrak{A}$  and  $\mathfrak{H}_u = \mathfrak{B} \oplus \mathfrak{B}$ , and (iii)  $\mathfrak{B}$  has absolutely irreducible linear isotropy representation (say  $\beta$ ) in  $A/B$ . Let  $T$  denote the real tangent space of  $M$ , so  $T^{\mathbb{C}} = T' \oplus T''$ , where  $T'$  is the holomorphic tangent space and  $T'' = \bar{T}'$  is the antiholomorphic tangent space. Now  $H_0$  acts on  $T'$  by  $\beta$ , on  $T''$  by  $\bar{\beta}$ . But  $H$  is a complex subgroup of  $G$ , so its linear isotropy action preserves  $T'$  and  $T''$ , and we may view  $\beta$  and  $\bar{\beta}$  as inequivalent (one is holomorphic, the other antiholomorphic) representations of  $H$ . Let  $\mathbf{A}$  denote the commuting algebra of  $H$  on  $T$ , real division algebra by Schur's Lemma,  $\neq \mathbf{R}$  because the complex structure gives it an element of square  $-1$ , and  $\neq \mathbf{Q}$  because  $\beta$  and  $\bar{\beta}$  are inequivalent on  $H$ . Then  $\mathbf{A} = \mathbf{C}$  by elimination of all other possibilities. Thus  $M$  carries precisely two  $G$ -invariant almost complex structures, and these must be the ones defined by the natural complex structure and its conjugate. We have proved all our assertions for the case of a complex group  $G$ .

From now on,  $G$  is not a complex Lie group.

Suppose that  $\chi$  is not absolutely irreducible. Then  $\chi = \beta \oplus \bar{\beta}$  and a glance at Theorem 11.1 shows  $\beta \sim \bar{\beta}$ ; in particular  $\chi$  has commuting algebra  $\mathbf{C}$  so there are precisely two  $G$ -invariant almost complex structures on  $G/H_0$ . On the other hand, if  $M$  carries a  $G$ -invariant almost complex structure than  $\chi$  has commuting algebra  $\neq \mathbf{R}$ , so  $\chi$  cannot be absolutely irreducible. This shows equivalence of (3a), (3b) and (3c), except that it remains to show that (3a) implies connectivity of  $H$ .

Let  $M$  carry a  $G$ -invariant almost complex structure. Then  $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$  and  $\mathfrak{M}^c = \mathfrak{M}' + \mathfrak{M}''$  where,  $\mathfrak{M}'' = \overline{\mathfrak{M}'}$ ,  $H$  acts irreducibly (say by  $\beta$ ) on  $\mathfrak{M}'$  and by  $\bar{\beta}$  on  $\mathfrak{M}''$ , and  $\chi = \beta \oplus \bar{\beta}$  with  $\beta \neq \bar{\beta}$ . Let  $\sigma$  be the involutive automorphism of  $\mathfrak{G}_u$  preserving  $\mathfrak{H}_u$  which defines  $\mathfrak{G}$  and  $\mathfrak{H}$  as in Theorem 12.1. Theorem 12.1(2) shows that  $\sigma$  preserves  $\mathfrak{M}'$  and  $\mathfrak{M}''$ . Let  $h \in H$ . Then  $\text{ad}(h)$  induces an automorphism of  $\mathfrak{G}_u^c$  preserving  $\mathfrak{H}_u^c$ ,  $\mathfrak{M}'$  and  $\mathfrak{M}''$ , the latter two because the almost complex structure of  $M$  is preserved. Glancing through the cases of Corollary 13.2 we see that  $\text{ad}(h)|_{\mathfrak{H}}$  is an inner automorphism because  $\text{ad}(h)$  preserves  $\mathfrak{M}'$  and  $\mathfrak{M}''$ . Thus we may replace  $h$  by an element of  $hH_0$  and assume  $\text{ad}(h)|_{\mathfrak{H}} = 1$ . Now  $\text{ad}(h)$  has eigenvalues 1 on  $\mathfrak{H}$ , and  $\text{ad}(h)|_{\mathfrak{M}'}$  and  $\text{ad}(h)|_{\mathfrak{M}''}$  are in the commuting algebras of  $\beta$  and  $\bar{\beta}$  respectively. Thus there is a complex number  $c \neq 0$  such that  $\text{ad}(h)$  is multiplication by  $c$  on  $\mathfrak{M}'$  and by  $\bar{c}$  on  $\mathfrak{M}''$ . If  $\text{rank } K = \text{rank } G$  then Theorem 2.2 implies  $[\mathfrak{M}', \mathfrak{M}''] = \mathfrak{H}^c$  so  $[\mathfrak{M}', \mathfrak{M}''] \neq 0$  and it follows that  $c\bar{c} = 1$ . If  $\text{rank } K < \text{rank } G$  we check from Corollary 13.2 that  $[\mathfrak{M}', \mathfrak{M}''] \neq 0$  and it follows that  $|c\bar{c}|$  is  $|c|$  or 1. In either case, now,  $|c| = 1$ , so  $h$  is contained in a maximal compact subgroup of  $H$ . Extending  $\sigma$  to  $\mathfrak{G}$  and  $G$  we now have  $\sigma(h) = h$ . On the group level now we have  $h \in G_u$ , viewing both  $G$  and  $G_u$  as  $\mathbf{R}$ -analytic subgroups of a complex group with Lie algebra  $\mathfrak{G}^c$ .  $G_u/(H_u \cup h \cdot H_u)$  has an invariant almost complex structure; now  $h \in H_u$  by Theorem 13.1, and so  $\text{ad}(h)|_{\mathfrak{H}} = 1$  implies that  $h$  is central in  $H_u$ . If  $\text{rank } K < \text{rank } G$  it follows that  $h = 1$ . If  $\text{rank } K = \text{rank } G$  and  $\sigma$  is inner,  $\sigma = \text{ad}(k)$  for some  $k \in H_u$  by Theorem 13.3,  $h$  is contained in any maximal torus  $T$  of  $H_u$  containing  $k$ , so  $h \in T \subset H_0$ . If  $\text{rank } K = \text{rank } G$  and  $\sigma$  is outer then in each of the three cases of Theorem 13.3(2)  $h$  is contained in a toral subgroup  $T$  of  $H_u$  which is fixed by  $\sigma$ , so  $h \in T \subset H_0$ . In any case we have shown that  $H$  is connected. This completes the proof of equivalence of (3a), (3b) and (3c), which are clearly equivalent to (3d) by means of Theorem 12.1.

If  $M$  has a  $G$ -invariant complex structure then [8]  $\mathfrak{G}^c = \mathfrak{Q} + \bar{\mathfrak{Q}}$  where  $\mathfrak{Q}$  is a complex subalgebra with  $\mathfrak{H}^c = \mathfrak{Q} \cap \bar{\mathfrak{Q}}$ . Then we may take  $\mathfrak{Q} = \mathfrak{H}^c + \mathfrak{M}''$  and  $\bar{\mathfrak{Q}} = \mathfrak{H}^c + \mathfrak{M}'$  in the notation above, and we have a map  $f: M_u \rightarrow G^c/L$  given by  $f(gH_u) = gL$ .  $f$  is  $G_u$ -equivariant and maps  $\mathfrak{M}$  isomorphically onto  $\mathfrak{G}^c/\mathfrak{Q}$ ; thus  $f$  is a nonsingular differentiable map with open image. As  $M_u$ , and thus  $f(M_u)$ , is compact,  $f$  must be surjective. Now  $f$  is a complex analytic covering. Theorem 13.1 shows  $M_u$  hermitian symmetric. Thus  $M$  is (indefinite) hermitian symmetric.

Now we need only check the listing in Tables 13.5, 13.6 and 13.7. If  $G$  and  $H$  have the same rank, the same is true (see Theorem 13.3) for their maximal compact subgroups, so  $G$  is centerless and  $G/H$  is simply connected. Now we need only check the equal rank case (Tables 13.5 and 13.6) on the Lie algebra level. We use the notation  $\sigma = \text{ad}(s)$ ,  $s \in H_u$ ,  $s^2 = 1$ , in the case where  $\sigma$  is inner.



$M_u = \mathbf{SU}(p+q)/\mathbf{S}[\mathbf{U}(p) \times \mathbf{U}(q)]$ . If  $\sigma$  is inner then the matrix  $s'$  representing  $s$  has square  $cI$  where  $c^{p+q} = 1$ . Thus  $s'$  is a scalar multiple of  $\text{diag} [-I_u, I_{p-u}, -I_v, I_{p-v}]$  with  $2u \leq p$  and  $2v \leq q$ . Then  $\mathfrak{G} = \mathfrak{S}\mathfrak{U}^{u+v}(p+q)$  and  $\mathfrak{H} = \mathfrak{S}[\mathfrak{U}^u(p) \times \mathfrak{U}^v(q)]$ . If  $\sigma$  is outer then Theorem 13.3 says that  $p$  and  $q$  have a common value  $n$  and that  $\sigma$  interchanges the two local  $\mathbf{U}(n)$  factors of  $H_u$ . Thus  $\mathfrak{H} = \mathfrak{S}\mathfrak{L}(n, \mathbf{C}) \cdot \mathfrak{Z}_0$ , where its connected center  $Z_0$  is a circle group because it is common to  $H_u$ , and  $\mathfrak{G}$  is either  $\mathfrak{S}\mathfrak{L}(n, \mathbf{Q})$  or  $\mathfrak{S}\mathfrak{L}(2n, \mathbf{R})$  according to whether  ${}^t g = -g$  or  ${}^t g = g$ , where  $\sigma = \alpha \cdot \text{ad}(g)$  and  $\alpha$  is complex conjugation; see [21].

$M_u = \mathbf{SO}(2n)/\mathbf{U}(n)$ . Theorem 13.3 says that  $\sigma$  is inner. The matrix  $s'$  representing  $s$  has square  $I$  or  $-I$  and is in  $\mathbf{U}(n)$ , so it is  $\mathbf{U}(n)$ -conjugate to

$$\begin{pmatrix} -I_{2r} & 0 \\ 0 & I_{2n-2r} \end{pmatrix} \text{ or } \begin{pmatrix} -J_r & 0 \\ 0 & J_{n-r} \end{pmatrix}, \quad 2r \leq n,$$

where  $J_t$  is the  $2t \times 2t$  matrix with  $2 \times 2$  blocks  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  down the diagonal and zeros elsewhere. Thus  $\mathfrak{G} = \mathfrak{S}\mathfrak{D}^{2r}(2n)$  [resp.  $\mathfrak{S}\mathfrak{D}^*(2n)$ ] and  $\mathfrak{H} = \mathfrak{U}^r(n)$ .

$M_u = \mathbf{Sp}(n)/\mathbf{U}(n)$ . Theorem 13.3 says that  $\sigma$  is inner. The matrix  $s'$  is diagonalizable over  $\mathbf{Q}$ , so we may take it to be  $\begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$  and then  $\mathfrak{G} = \mathfrak{S}\mathfrak{p}^r(n)$  and  $\mathfrak{H} = \mathfrak{U}^r(n)$ .

$M_u = \mathbf{SO}(n+2)/\mathbf{SO}(n) \times \mathbf{SO}(2)$ ,  $n > 2$ ,  $n \neq 4$ . If  $\sigma$  is inner then  $s'^2$  is  $I$  or  $-I$ , so we may conjugate  $s$  in  $H_u$  and assume that  $s'$  is given by

$$\begin{pmatrix} -I_{2r} & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & \pm I_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & I_m & 0 & 0 \\ -I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then  $\mathfrak{G} = \mathfrak{S}\mathfrak{D}^{m \text{ or } m+2}(n+2)$  and  $\mathfrak{H} = \mathfrak{S}\mathfrak{D}^m(n) \oplus \mathfrak{S}\mathfrak{D}(2)$  with  $2m \leq n$  and  $m = 2r$  or  $m = n - 2r$ ; or  $n = 2m$  with  $\mathfrak{G} = \mathfrak{S}\mathfrak{D}^*(n+2)$  and  $\mathfrak{H} = \mathfrak{S}\mathfrak{D}^*(n) \oplus \mathfrak{S}\mathfrak{D}(2)$ . If  $\sigma$  is outer we are in case (2b) of Theorem 13.3 with  $s = 1$  and  $n_1 = 1$ ; that is the same as the first of the two cases directly above except that  $2r$  is replaced by an odd number.

If  $G$  is exceptional and  $\text{rank } K = \text{rank } G$ , the possibilities for  $\sigma$  are quickly listed up to  $\text{ad}(H_u)$ -conjugacy by means of Theorem 13.3; given such a  $\sigma$ , one looks at its action on a Weyl basis of  $\mathfrak{G}^c$  and calculates the dimension of the fixed point set; the local form of  $G$  is specified by that dimension. These calculations are carried out in a somewhat more general context in [21] and the results are as recorded in Tables 13.5 and 13.6. This completes our checking for the case of equal ranks.

Now suppose  $\text{rank } H < \text{rank } G$ . Then Corollary 13.2 says that  $M_u = G_u/H_u$  is  $\mathbf{Spin}(n^2-1)/\text{ad}\mathbf{SU}(n)$  with  $n > 2$  odd,  $\mathbf{SO}(n^2-1)/\text{ad}\mathbf{SU}(n)$  with  $n > 3$  even, or  $\mathbf{E}_6/[\mathbf{SU}(3)/\mathbf{Z}_3]$ .  $\sigma|_{H_u}$  is

inner because it does not interchange the two summands of the linear isotropy representation. Thus there exists  $s \in H_u$  such that  $\text{ad}(s) \cdot \sigma$  acts trivially on  $H_u$ . If  $G_u = \mathbf{E}_6$  and  $\sigma$  is outer, then the fixed point set  $\mathfrak{F}_u$  of  $\text{ad}(s) \cdot \sigma$  on  $\mathfrak{G}_u$  has rank 4, so it properly contains the maximal subalgebra  $\mathfrak{H}_u$ ; thus  $\sigma$  is inner if  $G = \mathbf{E}_6$ . If  $G_u = \mathbf{SO}(n^2 - 1)$  with  $n$  even,  $n = 2m$ , then  $G_u$  is of type  $B_{2m^2 - 1}$  and so necessarily  $\sigma$  is inner. If  $G_u = \mathbf{Spin}(n^2 - 1)$  with  $n > 2$  odd and if  $\sigma$  is outer, then  $n = 2m + 1$  and the fixed point set  $\mathfrak{F}_u$  of  $\text{ad}(s) \cdot \sigma$  on  $\mathfrak{G}_u$  has rank equal to  $\text{rank } G_u - 1 = \frac{1}{2}(n^2 - 1) - 1 = 2m^2 + 2m - 1$  and contains the maximal subalgebra  $\mathfrak{H}_u$  of rank  $n - 1 = 2m$ . Then  $2m^2 + 2m - 1 = 2m$ , so  $2m^2 = 1$ , which is ridiculous. Thus  $\sigma$  is inner. Now we have verified that  $\sigma$  and  $\sigma|_{H_u}$  are inner. So  $\sigma = \text{ad}(g)$  for some  $g \in G_u$ , and  $\sigma|_{H_u} = \text{ad}(s)|_{H_u}$  for some  $s \in H_u$ . Thus  $gs^{-1}$  centralizes  $H_u$  and consequently is central in  $G_u$ . It follows that  $\sigma = \text{ad}(s)$ ,  $s \in H_u$ ,  $s^2 = 1$ .

$M_u = [\mathbf{Spin}$  or  $\mathbf{SO}](n^2 - 1)/\text{adSU}(n)$ . As we work first on the Lie algebra level we may first assume  $M_u = \mathbf{SO}(n^2 - 1)/\text{adSU}(n)$ ,  $n > 2$  even or odd.  $s$  is  $\text{ad}(H_u)$ -conjugate to an element of  $H_u$  represented by the matrix  $s' = (-1)^r \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} \in \mathbf{SU}(n)$ ,  $2r \leq n$ . Thus  $\mathfrak{H} =$

$\mathfrak{U}^r(n)$ .  $H$  is centerless so it must be  $\mathbf{SU}^r(n)/\mathbf{Z}_n$  globally. The inclusion  $\pi: \mathfrak{H}_u \rightarrow \mathfrak{G}_u$  is the adjoint representation. The  $(+1)$ -eigenspace of  $\pi(s')$  is  $\mathfrak{S}[\mathfrak{U}(r) \times \mathfrak{U}(n-r)]$  which has dimension  $r^2 + (n-r)^2 - 1$ , so the  $(-1)$ -eigenspace of  $\pi(s')$  has dimension  $n^2 - 1 - [r^2 + (n-r)^2 - 1] = 2r(n-r)$ . Thus  $\mathfrak{G} = \mathfrak{S}\mathfrak{D}^{2r(n-r)}(n^2 - 1)$ .

Let  $\theta: \bar{G} \rightarrow G$  be a covering group,  $\bar{H} = \theta^{-1}(H)_0$  and  $\bar{M} = \bar{G}/\bar{H}$ , such that  $\bar{G}$  acts effectively on  $\bar{M}$  and  $\bar{M}$  is simply connected. Then  $G = \bar{G}/Z$  and  $H = (\bar{H}Z)/Z$ , where  $Z$  is an arbitrary central subgroup of  $\bar{G}$  which is characterized up to isomorphism by  $Z \cong \pi_1(\bar{M})$ . To find  $\bar{G}$  we start with the existence of a central  $Z'$  in  $\bar{G}$  such that  $G' = \bar{G}/Z'$  is  $\mathbf{SO}^{2r(n-r)}(n^2 - 1)$  and  $H' = (\bar{H}Z')/Z'$  is the linear group  $\text{ad } \mathbf{SU}^r(n)$ . Then  $Z' = \pi_1(G'/H') = \pi_1(K/L)$ , where  $L \subset K$  are the maximal compact subgroups of  $H' \subset G'$ . Now  $L = \mathbf{S}[\mathbf{U}(r) \times \mathbf{U}(n-r)]/\mathbf{Z}_n$  and  $K = K_1 \times K_2$ , where  $K_1 = \mathbf{SO}(r^2 + (n-r)^2 - 1)$  and  $K_2 = \mathbf{SO}(2r(n-r))$ . As a linear representation, the inclusion  $L \subset K$  is  $\omega_1 \oplus \omega_2$ , where  $\omega_1 = \text{ad}_L$  maps into  $K_1$ , and  $\omega_2 = [\alpha_r \otimes \bar{\alpha}_{n-r}] \oplus [\bar{\alpha}_r \otimes \alpha_{n-r}]$  (where  $\alpha_m$  is the usual vector representation of degree  $m$  of  $\mathbf{U}(m)$ ) maps into  $K_2$ . If  $r = 0$  we know  $G$  and  $K$ ; now assume  $r > 0$  so  $\omega_2$  is faithful.  $\pi_1(K_2) = \mathbf{Z}_2$  because  $n > 2$ , so  $\mathbf{Spin}(2r(n-r)) \rightarrow K_2$  is the universal covering. If  $K_2/\omega_2(L)$  is not simply connected, now  $\omega_2$  lifts to a  $\mathbf{Spin}(2r(n-r))$ -valued representation. Then if  $b_t$  is a 1-parameter subgroup of  $L$ ,  $b_t = 1$  if and only if  $t$  is an integer, the lift of  $\omega_2(b_t)$  to the Clifford algebra is 1 if and only if  $t$  is an integer. We test this with the 1-parameter subgroup  $b_t = \text{diag} [\varepsilon_t, \dots, \varepsilon_t]$ ,  $\varepsilon_t = e^{2\pi\sqrt{-1}t}$ , of the subgroup  $\mathbf{U}(r) \subset L$ . In an orthonormal basis  $\{v_i\}$  of  $\mathbf{R}^{2r(n-r)}$ ,  $\omega_2(b_t)$  has matrix  $\text{diag} [E_t, \dots, E_t]$ ,  $E_t = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$ . Thus the lift to the Clifford algebra on  $\mathbf{R}^{2r(n-r)}$  is

given by  $b_i \rightarrow \prod v_{2j-1} \cdot (\cos \pi t v_{2j} + \sin \pi t v_{2j-1})$  where  $j$  ranges from 1 to  $r(n-r)$  in the product. Thus  $b_1 \rightarrow v_1 \cdot v_2 \cdot \dots \cdot v_{2r(n-r)-1} \cdot v_{2r(n-r)}$ . In other words, the representation  $\omega_2$  does not lift. If  $G'/H'$  is not simply connected, then  $\bar{G} = \mathbf{Spin}^{2r(n-r)}(n^2-1)$  with maximal compact subgroup  $\bar{K} = \bar{K}_1 \cdot \bar{K}_2$ ,  $\bar{K}_i = \mathbf{Spin}(m)$  covering  $K_i = \mathbf{SO}(m)$ , so  $\omega_2$  lifts. That proves  $G'/H'$  simply connected. Thus  $\bar{G} = G' = \mathbf{SO}^{2r(n-r)}(n^2-1)$  for  $r > 1$ .

$M_u = \mathbf{E}_6/[\mathbf{SU}(3)/\mathbf{Z}_3]$ . If  $\sigma \neq 1$ , so  $\sigma = \text{ad}(s)$  with  $s \in H_u$  of order 2, we may conjugate and assume that  $s$  is represented by  $s' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{SU}(3)$ . Then  $H = \mathbf{SU}^1(3)/\mathbf{Z}_3$ . Let  $\alpha$

denote complex conjugation on  $\mathbf{SU}(27)$ ; we view  $G_u = \mathbf{E}_6$  as an  $\alpha$ -stable subgroup of  $\mathbf{SU}(27)$ . The inclusion  $H_u \rightarrow G_u$  is given by the real representation  $\overset{2}{\circ} - \overset{2}{\circ}$  so any element of the image may be conjugated and assumed fixed under  $\alpha$ . Thus we may assume  $\alpha(s) = s$ . Then  $\alpha$  and  $\sigma$  commute, so  $\alpha$  preserves the fixed point set  $K$  of  $\sigma$ , and  $\sigma$  preserves the fixed point set  $F$  of  $\alpha$ . For convenience let  $L = F \cap K$ . Now  $K$  is of type  $D_5 T^1$  or  $A_5 A_1$ ,  $F$  is of type  $F_4$  or  $C_4$ , and  $L$  is a group of rank 4 which is a symmetric subgroup in both  $K$  and  $F$ . Recall the symmetric subgroups of rank 4 in those groups.

$$\begin{array}{ll} C_4: A_3 T^1, C_1 C_3 \text{ and } C_2 C_2. & F_4: B_4 \text{ and } C_3 C_1 \\ A_5 A_1: D_3 A_1, D_3 T^1, C_3 A_1 \text{ and } C_3 T^1. & D_5 T^1: B_1 D_3, B_2 D_2 \text{ and } B_3 D_1. \end{array}$$

Here note  $D_3 = A_3$ ,  $B_2 = C_2$ ,  $D_2 = A_1 A_1$ ,  $D_1 = T^1$  and  $A_1 = B_1 = C_1$ . Despite this,  $D_5 T^1$  is eliminated as a possibility for  $K$ . Thus  $G$  is of type  $E_{6, A_5 A_1}$ .

As before  $\bar{G}/\bar{H} = \bar{M}$  is the simply connected covering group of  $M$  and  $G = \bar{G}/Z$ ,  $H = \bar{H}Z/Z$  where  $Z$  is an arbitrary central subgroup of  $G$ . We start on the matrix level with  $G' = G/Z'$  which has maximal compact subgroup  $K = [\mathbf{SU}(2) \times \mathbf{SU}(6)]/\mathbf{Z}_2$  embedded in  $G' \subset \mathbf{SL}(27, \mathbf{C})$  by the representation  $(\overset{1}{\circ} \otimes \overset{1}{\circ} - \circ - \circ - \circ - \circ) \oplus (\circ \otimes \overset{1}{\circ} - \circ - \circ - \circ)$ . Then the fundamental group  $\pi_1(G') = \pi_1(K) = \mathbf{Z}_2$ . Let  $G'' \rightarrow G'$  be the universal 2-fold covering,  $H''$  the identity component of the inverse image of  $H' = HZ'/Z'$ .  $H'$  is centerless so  $H''$  has center of order 1 or 2. But  $H' = \mathbf{SU}^1(3)/\mathbf{Z}_3$  does not have a covering group with center of order 2. Thus  $G''/H''$  is simply connected and effective. We have now proved that  $G'' = \bar{G}$  is the simply connected group of type  $E_{6, A_5 A_1}$ , that it has center  $\mathbf{Z}_6$ , and that  $H'' = \bar{H} = \mathbf{SU}^1(3)/\mathbf{Z}_3$ , q.e.d.

### 14. Invariant division algebras

Let  $A$  be an associative algebra of linear transformations of a real vector space  $V$ . By an  $A$ -structure on a differentiable manifold  $M$ , we mean a family  $\{A_x\}_{x \in M}$ , where  $A_x$  is an algebra of linear transformations of the tangent space  $M_x$ , and there exist linear iso-

morphisms  $V \rightarrow M_x$  carrying  $A$  to  $A_x$ . If  $F$  is a division algebra  $\mathbf{R}$  (reals),  $\mathbf{C}$  (complex numbers) or  $\mathbf{Q}$  (quaternions), then we view  $F^n$  as a real vector space of dimension  $n \cdot (\dim_{\mathbf{R}} F)$ ; this realizes  $F$  as an associative algebra of linear transformations of a real vector space and allows us to speak of an  $F$ -structure. Note that a  $\mathbf{C}$ -structure is more general than an almost complex structure, but that locally it defines an almost complex structure up to sign.

Let  $G$  be a differentiable transformation group on  $M$ . Then an  $A$ -structure  $\{A_x\}_{x \in M}$  is called  $G$ -invariant if  $x \in M$  and  $g \in G$  imply that  $g$  carries  $A_x$  to  $A_{g(x)}$ .

Suppose that  $G$  is transitive on  $M$ , so  $M = G/H$  where  $H = \{g \in G: g(x_0) = x_0\}$ . If  $\{A_x\}_{x \in M}$  is a  $G$ -invariant  $A$ -structure on  $M$ , then  $A_{x_0}$  satisfies

$$(i) \text{ if } h \in H, \text{ then } h_* A_{x_0} h_*^{-1} = A_{x_0},$$

and  $A_{x_0}$  defines the structure by means of

$$(ii) \text{ } A_{g(x_0)} \text{ is the image of } A_{x_0} \text{ under } g.$$

Conversely, if  $A_{x_0}$  is an algebra of linear transformations of  $M_{x_0}$ , then it defines an  $A_{x_0}$ -structure if and only if it satisfies (i), and in that case a  $G$ -invariant structure is defined by (ii). In particular:

**14.1 LEMMA.** *Let  $M$  be a coset space  $G/H$ , where  $H$  is the isotropy subgroup at a point  $x$ . Then  $M$  has a  $G$ -invariant  $\mathbf{C}$ -structure if and only if  $M_x$  has a complex vector space structure for which every tangent map  $h_{*x}$ ,  $h \in H$ , is either  $\mathbf{C}$ -linear or  $\mathbf{C}$ -antilinear.  $M$  has a  $G$ -invariant  $\mathbf{Q}$ -structure if and only if  $M_x$  has a quaternionic vector space structure for which every  $h_{*x}$ ,  $h \in H$ , is the product of a  $\mathbf{Q}$ -linear map and a  $\mathbf{Q}$ -scalar map.*

Now let  $M = G/H$  be an effective reductive coset space such that the linear isotropy action of  $H_0$  is an  $\mathbf{R}$ -irreducible representation  $\chi$ . If  $\chi$  is not absolutely irreducible Theorem 13.4 shows that its commuting algebra  $A$  is a complex number field;  $A$  is normalized by the non-identity components of  $H$  and thus extends to a  $G$ -invariant  $\mathbf{C}$ -structure on  $M$ . That is the unique  $G$ -invariant  $\mathbf{C}$ -structure on  $M$ . If  $\chi$  is absolutely irreducible, so  $A = \mathbf{R}$ , then Theorem 13.4 shows that there is no  $G$ -invariant  $\mathbf{C}$ -structure on  $M$ . In other words

**14.2 THEOREM.** *Let  $M = G/H$  be an effective reductive coset space, where  $G$  is a connected Lie group,  $H$  is a closed subgroup, and  $H_0$  has  $\mathbf{R}$ -irreducible linear isotropy representation  $\chi$ . Suppose that  $M$  is not euclidean. Then  $M$  has a  $G$ -invariant  $\mathbf{C}$ -structure if and only if  $\chi$  is not absolutely irreducible, and in that case the structure is unique.*

It is known [18] that certain compact riemannian symmetric spaces have invariant  $\mathbf{Q}$ -structures. Essentially they are the ones which are base spaces of 2-sphere fibrations of compact complex homogeneous contact manifolds. We say 'essentially' because there is a mild complication which involves the notion of scalar part, which we now define.

Let  $M = G/H$  be an effective reductive coset space with a  $G$ -invariant  $\mathbf{Q}$ -structure  $\{A_z\}_{z \in M}$ . Then the linear isotropy group  $H$  at  $x$  is a local direct product,  $H_0 = H' \cdot H''$ , where the linear isotropy representation sends  $H'$  into transformations which commute with every element of  $A_x$  and sends  $H''$  into  $A_x$ .  $H'$  is the  $\mathbf{Q}$ -linear part of  $H_0$  and  $H''$  is the  $\mathbf{Q}$ -scalar part of  $H_0$ . These parts are well defined because  $G/H$  is effective and reductive. We use the notation that  $\mathbf{R}^*$ ,  $\mathbf{C}^*$  and  $\mathbf{Q}^*$  are the multiplicative groups of nonzero reals, complex numbers and quaternions, respectively; that  $\mathbf{R}'$ ,  $\mathbf{C}'$  and  $\mathbf{Q}'$  are the respective subgroups consisting of elements of absolute value 1; and that  $\mathbf{R}_+^*$  and  $\mathbf{R}'_+$  are the respective subgroups of  $\mathbf{R}^*$  and  $\mathbf{R}'$  consisting of positive numbers. Now there are three types of possibilities for the analytic subgroup  $H''$  of  $H$ , which we name and list as follows.

(i) The linear isotropy representation maps  $H''$  into  $\mathbf{R}^*$ . Then  $H''$  is isomorphic to  $\mathbf{R}'_+ = \{1\}$  or to  $\mathbf{R}_+^*$ , and we say that  $H$  has *real scalar part*.

(ii) The linear isotropy representation maps  $H''$  into  $\mathbf{C}^*$  but not into a real subfield. Then  $H''$  is isomorphic to  $\mathbf{C}' = \mathbf{T}^1$  or to  $\mathbf{C}^* = \mathbf{C}' \times \mathbf{R}_+^*$ , and we say that  $H$  has *complex scalar part*.

(iii) The linear isotropy representation maps  $H''$  into  $\mathbf{Q}^*$  but not into a complex sub-algebra. Then  $H''$  is isomorphic to  $\mathbf{Q}' = \mathbf{Sp}(1)$  or to  $\mathbf{Q}^* = \mathbf{Q}' \times \mathbf{R}_+^*$ , and we say that  $H$  has *quaternionic scalar part*.

**14.3 THEOREM.** *Let  $M = G/H$  be an effective reductive coset space, where  $G$  is a connected Lie group,  $H$  is a closed subgroup, and  $H_0$  has  $\mathbf{R}$ -irreducible linear isotropy representation  $\chi$ . Suppose that  $M$  is not euclidean.*

1.  $M$  has no  $G$ -invariant  $\mathbf{Q}$ -structure for which  $H$  has real scalar part.

2.  $M$  has a  $G$ -invariant  $\mathbf{Q}$ -structure for which  $H$  has complex scalar part, if and only if the compact version  $M_u = G_u/H_u$  is the complex projective plane.

3.  $M$  has a  $G$ -invariant  $\mathbf{Q}$ -structure for which  $H$  has quaternionic scalar part, if and only if (3a)  $M_u = G_u/H_u$  is one of the compact quaternionic symmetric spaces classified in [18, Theorem 5.4] and the involutive automorphism  $\sigma$  of  $G_u$  which gives the Cartan involution of  $G$  is trivial on the subgroup of  $H_u$  corresponding to the  $\mathbf{Q}$ -scalar part of  $H$ , or (3b)  $\mathfrak{G} = \mathfrak{A}^{\mathbf{C}}$ ,  $\mathfrak{H} = \mathfrak{B}^{\mathbf{C}}$ ,  $G_u = A \times A$ ,  $H_u = B \times B$  and  $M_u = (A/B) \times (A/B)$ , where  $A/B$  is a nonhermitian compact quaternionic symmetric space listed in [18, Theorem 5.4].

*Proof.* We may assume  $H$  connected. (1) is immediate from Theorem 12.1, which shows that  $\chi$  cannot have commuting algebra  $\mathbf{Q}$ .

Let  $M$  have a  $G$ -invariant  $\mathbf{Q}$ -structure with complex scalar part. Then Theorem 13.4

shows that  $H''$  is a circle group  $\mathbf{C}' = \mathbf{T}^1$ , so  $M_u$  is hermitian symmetric, and [18, Theorem 3.7] says that  $M_u$  is the complex projective plane. On the other hand, if  $M_u$  is the complex projective plane, then  $G_u = \mathbf{SU}(3)/\mathbf{Z}_3$  and  $H_u = \mathbf{S}[\mathbf{U}(1) \times \mathbf{U}(2)]/\mathbf{Z}_3 \cong \mathbf{U}(2)$ , and the first line of Table 13.5 shows that either  $G = G_u$  and  $H = H_u$ , or  $G = \mathbf{SU}^1(3)/\mathbf{Z}_3$  and  $H = \mathbf{S}[\mathbf{U}(1) \times \mathbf{U}^1(2)]/\mathbf{Z}_3 \cong \mathbf{U}^1(2)$ ; in both cases  $M$  has a  $G$ -invariant  $\mathbf{Q}$ -structure for which  $H$  has complex scalar part.

Let  $M$  have a  $G$ -invariant  $\mathbf{Q}$ -structure for which  $H$  has quaternionic scalar part. If  $H$  is not semisimple then Theorem 12.1(2) says that the center of  $H$  is a circle group; thus  $H''$  is  $\mathbf{Sp}(1) = \mathbf{Q}'$ , represented by  $\overset{1}{\circ}$  in  $\chi$ . Now suppose that  $G/H$  is not symmetric. Then a glance through the list of Theorem 11.1 eliminates the possibility that  $G$  and  $H$  are complex groups, so  $G_u/H_u$  is listed in Theorem 11.1. Then either  $\chi$  is absolutely irreducible with  $\chi = \chi|_H \otimes \chi|_{H'}$  and  $\chi|_{H'} = \overset{1}{\circ}$ , or  $\chi = \beta \oplus \bar{\beta}$  with  $\beta = \beta|_H \otimes \beta|_{H'}$  and  $\beta|_{H'} = \overset{1}{\circ}$ ; no such spaces are listed in Theorem 11.1. In other words,  $G/H$  is symmetric. Theorems 12.1 and 12.2, with Theorem 5.4 of [18], now show that either  $G_u/H_u$  is one of the symmetric spaces listed in [18, Theorem 5.4] which have quaternionic structure such that  $H_u$  has quaternionic scalar part, or there is a nonhermitian quaternionic symmetric space  $A/B$  listed in [18, Theorem 5.4] such that  $\mathfrak{G} = \mathfrak{A}^{\mathbf{C}}$ ,  $\mathfrak{H} = \mathfrak{B}^{\mathbf{C}}$ ,  $G_u = A \times A$ ,  $H_u = B \times B$  and  $M_u = (A/B) \times (A/B)$ , q.e.d.

The *commuting structure* on a coset space  $G/H$  is the  $G$ -invariant structure  $\{A_x\}_{x \in G/H}$  where  $A_x$  is the commuting algebra of the linear isotropy group at  $x$ . Theorems 14.2 and 14.3 say, for a simply connected noneuclidean reductive isotropy irreducible coset space  $G/H$ , that the commuting structure is an  $\mathbf{R}$ -structure if the linear isotropy representation is absolutely irreducible, is a  $\mathbf{C}$ -structure otherwise, and cannot be a  $\mathbf{Q}$ -structure.

Note that the commuting structure is the structure of the algebra of  $n \times n$  real matrices if and only if  $\chi = \beta_1 \oplus \dots \oplus \beta_n$  with  $\beta_i$  absolutely irreducible real and all the  $\beta_i$  equivalent. In this context see Lemma 12.3 and Remark 12.4.

### Chapter III. Riemannian geometry on isotropy irreducible coset spaces

An isotropy irreducible coset space  $M = G/K$ , with  $K$  compact, has a riemannian metric which is unique up to a constant scalar factor. In § 15 we see that  $M$  is an Einstein manifold and that sectional curvature keeps its sign, and we determine when two such riemannian manifolds are isometric. In § 16 we determine the linear holonomy group of  $M$ . § 17 contains the determination of the full group of isometries; if  $M$  has invariant almost complex structure we also determine the full group of almost hermitian isometries and study the group of almost-analytic diffeomorphisms. Finally, in § 18, we study locally

isotropy irreducible riemannian manifolds and their relations to isotropy irreducible coset spaces.

Most of the results extend immediately to indefinite metric.

### 15. Curvature and equivalence

The elementary properties of isotropy irreducible riemannian homogeneous spaces are given by the following theorem.

**15.1 THEOREM.** *Let  $M$  be an effective coset space  $G/K$  of a connected Lie group by a compact subgroup, where  $K$  is  $\mathbf{R}$ -irreducible on the tangent space.*

1. *If  $ds^2$  and  $d\sigma^2$  are  $G$ -invariant riemannian metrics on  $M$ , then  $ds^2 = c \cdot d\sigma^2$  for some constant  $c > 0$ . In particular  $ds^2$  and  $d\sigma^2$  have the same Levi-Civita connection.*

2. *Choose a  $G$ -invariant riemannian metric  $ds^2$  on  $M$ , let  $\mathbf{r}$  denote the Ricci tensor, and let  $r$  denote the scalar curvature. Then  $(M, ds^2)$  is an Einstein space,  $\mathbf{r} = \frac{r}{n} ds^2$  with  $r$  constant and  $n = \dim M$ .*

3. *If the identity component  $K_0$  is  $\mathbf{R}$ -irreducible on the tangent space, then*

(3a)  *$r < 0$ ,  $(M, ds^2)$  is a riemannian symmetric space of noncompact type, and every sectional curvature satisfies  $\kappa \leq 0$ ; or*

(3b)  *$r = 0$  and  $(M, ds^2)$  is a euclidean space; or*

(3c)  *$r > 0$ ,  $M$  is compact, and  $(M, ds^2)$  has every sectional curvature  $\kappa \geq 0$ .*

*Remark.* Now Theorem 11.1 gives many new examples of Einstein spaces which are neither symmetric nor kählerian.

*Remark.* By uniqueness, the Levi-Civita connection on  $M$  must be the first canonical connection for  $G/K$ .

*Proof.*  $K$  is the isotropy subgroup at some point  $x \in M$ . Let  $\chi$  be the representation of  $K$  on  $M_x$ . As  $\chi$  is  $\mathbf{R}$ -irreducible, any nonzero  $\chi(K)$ -invariant symmetric bilinear form on  $M_x$  is definite and any two are proportional. Thus  $ds_x^2 = c \cdot d\sigma_x^2$  for some  $c > 0$ . If  $z \in M$ ,  $z = g^{-1}(x)$ , then  $ds_z^2 = g^* ds_x^2 = c \cdot g^* d\sigma_x^2 = c \cdot d\sigma_z^2$ . This proves (1). Similarly  $\mathbf{r} = f \cdot ds^2$  for some constant  $f$ , and  $f = r/n$  by definition of  $r$ ; this proves (2).

Let  $\chi(K_0)$  be  $\mathbf{R}$ -irreducible on  $M_x$ . If  $G$  is not semisimple then Lemma 1.2 shows  $(M, ds^2)$  isometric to euclidean space; in particular  $r = 0$ . If  $G$  is noncompact and semisimple then  $K$  is a maximal compact subgroup, so  $(M, ds^2)$  is a riemannian symmetric space of noncompact type; in particular  $r < 0$  and every sectional curvature  $\kappa \leq 0$ . If  $G$  is compact and semisimple then  $M$  is compact. By uniqueness,  $ds_x^2$  is the restriction of a negative multiple of the Killing form of  $\mathfrak{G}$  to the orthocomplement of  $\mathfrak{K}$ ; in particular every sectional curvature  $\kappa \geq 0$  and some  $\kappa > 0$ ; it follows that  $r > 0$ , q.e.d.

Distinct coset spaces may give isometric manifolds. For example, we have euclidean  $(2n)$ -space given as  $\mathbf{SO}(2n) \cdot \mathbf{R}^{2n} / \mathbf{SO}(2n)$ ,  $\mathbf{SU}(n) \cdot \mathbf{R}^{2n} / \mathbf{SU}(n)$  and  $\mathbf{U}(n) \cdot \mathbf{R}^{2n} / \mathbf{U}(n)$ . This is not a phenomenon restricted to euclidean spaces, for  $\mathbf{Spin}(7)/\mathbf{G}_2$  is isometric to the 7-sphere  $\mathbf{S}^7$ . Thus we need the uniqueness theorem:

**15.2 THEOREM.** *Let  $G/K$  and  $A/B$  be simply connected effective coset spaces of connected Lie groups by compact subgroups with  $\mathbf{R}$ -irreducible linear isotropy representations. Let each carry an invariant riemannian metric and suppose that they are isometric. Then*

- (i) *there is an isomorphism of  $G$  onto  $A$  which carries  $K$  onto  $B$ ; or*
- (ii)  *$G/K$  and  $A/B$  are euclidean spaces of the same dimension; or*
- (iii)  *$G/K$  and  $A/B$  are the two presentations  $\mathbf{Spin}(7)/\mathbf{G}_2$  and  $\mathbf{SO}(8)/\mathbf{SO}(7)$  of the sphere  $\mathbf{S}^7$ ; or*
- (iv)  *$G/K$  and  $A/B$  are the two presentations  $\mathbf{G}_2/\mathbf{SU}(3)$  and  $\mathbf{SO}(7)/\mathbf{SO}(6)$  of the sphere  $\mathbf{S}^6$ .*

*Proof.* If one (thus both) of  $G/K$  and  $A/B$  is a euclidean space then we are in case (ii). If  $G/K$  and  $A/B$  are both noneuclidean symmetric coset spaces then we are in case (i). Now we may assume that  $A/B$  is a nonsymmetric coset space, so it is listed in Theorem 11.1, and that  $G/K$  either is a compact irreducible symmetric coset space or is listed in Theorem 11.1.

Let  $M$  be the common riemannian manifold of  $G/K$  and  $A/B$  and write  $M = U/V$ , where  $U$  is the largest connected group of isometries. Then  $G \subset U$  and  $A \subset U$ , and we may assume  $K \subset V$  and  $B \subset V$ . It suffices to prove our assertion in the case where  $U$  is  $G$  or  $A$ ; for then  $G \neq U \neq A$  implies that either  $M = \mathbf{S}^7$  with  $G$  and  $A$  as conjugates of  $\mathbf{Spin}(7)$  in  $\mathbf{SO}(8)$ , or  $M = \mathbf{S}^6$  with  $G$  and  $A$  as conjugates of  $\mathbf{G}_2$  in  $\mathbf{SO}(7)$ , and we are in case (i). Thus we are reduced to considering the case  $A \subsetneq G$  with  $B = A \cap K$ . These situations are classified by A. L. Oniščik ([22], Table 7, p. 219 [p. 29 in the translation], except that Oniščik writes:  $G'$  for  $K$  and  $G''$  for  $A$ , or  $G'$  for  $A$  and  $G''$  for  $K$ ),  $U$  for  $B$ , and  $Sp(2n)$  for  $\mathbf{Sp}(n)$ ,  $A$  and  $G$  are simple and  $B$  is semisimple. Thus  $\pi_2(A/B) = \pi_2(G/K)$  is finite, so  $K$  is semisimple. Now we need only run through the entries on Oniščik's list which have  $G''$  and  $U$  semisimple, checking for isotropy irreducibility. Doing that, we find that we are in case (iii) or case (iv) of the theorem, q.e.d.

## 16. Holonomy

Let  $M$  be a riemannian manifold. If  $x \in M$ , then  $\mathbf{O}(M_x)$  denotes the orthogonal group of the tangent space and  $\mathbf{SO}(M_x)$  is the subgroup consisting of proper rotations. If  $\sigma$  is a sectionally smooth curve in  $M$  with both endpoints at  $x$ , then the parallel translation about  $\sigma$  is an element  $\tau_\sigma^M \in \mathbf{O}(M_x)$ . All such transformations  $\tau_\sigma^M$  compose the *linear holonomy*



group  $H(M, x)$  at  $x$ .  $H(M, x)$  carries the subspace topology from its inclusion in  $\mathbf{O}(M_x)$ ; the arc component of  $I$  is a closed Lie subgroup  $H_0(M, x) \subset \mathbf{SO}(M_x)$  which is called the *restricted linear holonomy group* of  $M$  at  $x$ .  $H_0(M, x)$  consists of all  $\tau_\sigma^M$  for which  $\sigma$  is homotopic (with fixed endpoints) to the trivial curve at  $x$ . If  $\varphi: M' \rightarrow M$  is a riemannian covering,  $\varphi(x') = x$ , then  $\tau_{\varphi\sigma}^{M'} \rightarrow \tau_\sigma^M$  defines an injection  $\varphi_*: H(M', x') \rightarrow H(M, x)$  which is equivariant with the tangent map  $\varphi_*: M'_{x'} \rightarrow M_x$ . Thus restricted linear holonomy is invariant under riemannian coverings.

The holonomy of symmetric spaces is well known, although I cannot find the global statement in the literature:

**16.1 PROPOSITION.** *Let  $M = G/K$  be an effective symmetric coset space with a  $G$ -invariant riemannian metric, where  $G$  is a connected Lie group and  $K$  is a compact subgroup. Let  $\chi$  be the linear isotropy action of  $K$  on  $M_x$ . Decompose  $M \sim M^0 \times M'$  locally as the product of a euclidean space  $M^0$  and a product  $M'$  of irreducible spaces, so  $x = (x^0, x')$  and  $M_x = M_{x^0}^0 \oplus M'_x$ . Then  $K = K^0 \times K'$  and  $\chi = \chi^0 \oplus \chi'$  where  $\chi^0(K^0)$  acts on  $M_{x^0}^0$ ,  $\chi'(K')$  acts on  $M'_x$ , and  $H(M, x) = \chi'(K')$ .*

The proof is immediate, by means of the universal riemannian covering, from ([17], § 7) and ([16], § 3).

By way of contrast, the holonomy of isotropy irreducible nonsymmetric coset spaces is much less complicated:

i

**16.2 THEOREM.** *Let  $M = G/K$  be a nonsymmetric effective coset space with a  $G$ -invariant riemannian metric, where  $G$  is a connected Lie group and  $K$  is a compact subgroup. Suppose that the linear isotropy action  $\chi$  of  $K_0$  on  $M_x$  is  $\mathbf{R}$ -irreducible. Then*

$$\begin{aligned} H(M, x) &= \mathbf{SO}(M_x) \text{ if } M \text{ is orientable,} \\ H(M, x) &= \mathbf{O}(M_x) \text{ if } M \text{ is not orientable.} \end{aligned}$$

An immediate consequence is:

**16.3 COROLLARY.** *If  $\Psi$  is a nonzero parallel differential form on  $M$ , then either  $\Psi$  is a scalar constant, or  $M$  is orientable and  $\Psi$  is a constant multiple of the volume element.*

*Proof of theorem.*  $M$  is orientable if and only if  $H(M, x) \subset \mathbf{SO}(M_x)$ . Thus we need only prove  $H_0(M, x) = \mathbf{SO}(M_x)$ , and for this we may assume  $M$  simply connected. The de Rham decomposition of  $M$  as a product of a euclidean space and some irreducible riemannian manifolds, decomposes the largest connected group of isometries; thus  $\mathbf{R}$ -irreducibility of  $\chi$  implies that  $M$  is an irreducible riemannian manifold. Now ([1] or [13]) either  $M$  is iso-

metric to an irreducible riemannian symmetric space, or  $H(M, x)$  is transitive on the unit sphere in  $M_x$ .

Let  $M$  be isometric to an irreducible riemannian symmetric space. Then Theorem 15.2 says that  $M$  is isometric to a sphere  $S^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ , and Proposition 16.1 now implies that  $H(M, x) = \mathbf{SO}(M_x)$ .

Let  $H(M, x)$  be transitive on the unit sphere  $S^{n-1}$  in  $M_x$ . Then  $H(M, x)$  must be

- (i)  $\mathbf{SO}(n) = \mathbf{SO}(M_x)$ , (ii)  $\mathbf{SU}\left(\frac{n}{2}\right)$ , (iii)  $\mathbf{U}\left(\frac{n}{2}\right)$ , (iv)  $\mathbf{Sp}\left(\frac{n}{4}\right)$ , (v)  $\mathbf{Sp}\left(\frac{n}{4}\right) \cdot \mathbf{T}^1$ , (vi)  $\mathbf{Sp}\left(\frac{n}{4}\right) \cdot \mathbf{Sp}(1)$ ,  
 (vii)  $\mathbf{G}_2$  with  $n = 7$ , (viii)  $\mathbf{Spin}(7)$  with  $n = 8$ ; or (ix)  $\mathbf{Spin}(9)$  with  $n = 16$ .

If the commuting algebra of  $H(M, x)$  on  $M_x$  has an element  $J$  of square  $-I$ , then  $J$  defines a kaehlerian structure on  $M$ . By compactness, the cohomology  $\mathbf{H}^2(M; \mathbf{R}) \neq 0$ , so  $\pi_2(M)$  is infinite by the Hurewicz Theorem. The exact homotopy sequence of  $G \rightarrow G/K = M$  then shows  $\pi_1(K)$  infinite, contradicting semisimplicity of  $K$ . This excludes the possibilities (ii), (iii), (iv) and (v) for  $H(M, x)$ .

In the possibilities remaining,  $H(M, x)$  is its own connected normalizer in  $\mathbf{SO}(M_x)$ . Thus  $\chi(K) \subset H(M, x)$ .

If  $M_x$  has an  $H(M, x)$ -stable structure as a quaternionic vector space, then we have a  $G$ -invariant  $\mathbf{Q}$ -structure on  $M$ , contradicting Theorem 14.3. This excludes the possibility (vi) for  $H(M, x)$ .

If  $M$  has dimension 7 and  $K$  is isomorphic to a subgroup of  $\mathbf{G}_2$  then Theorem 11.1 says  $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$ , so  $H(M, x) = \mathbf{SO}(M_x)$ . This excludes possibility (vii) for  $H(M, x)$ .

Following Dynkin ([6], Table 5), a connected semisimple subgroup of  $\mathbf{Spin}(2m+1)$  absolutely irreducible on  $\mathbf{R}^{2m}$  must be all of  $\mathbf{Spin}(2m+1)$ . As  $\circ - \circ = \overset{1}{\bullet}$  and  $\circ - \circ - \circ = \overset{1}{\bullet}$  are not among the possibilities for  $\chi$  listed in Theorem 11.1, this excludes possibilities (viii) and (ix) for  $H(M, x)$  in the case where  $\chi$  is absolutely irreducible. But if  $\chi$  is not absolutely irreducible, Theorem 11.1 shows that  $M$  cannot have dimension 8 or 16. This excludes the possibilities (viii) and (ix) for  $H(M, x)$ , q.e.d.

## 17. Isometries

Let  $M$  be a riemannian manifold. Then  $\mathbf{I}(M)$  denotes the group of all isometries of  $M$  onto itself. As is now standard, we let  $\mathbf{I}(M)$  carry the compact-open topology, and then  $\mathbf{I}(M)$  is a Lie transformation group on  $M$ . The identity component is denoted  $\mathbf{I}_0(M)$ .

Given an effective coset space  $M = G/K$  with a  $G$ -invariant riemannian metric, one knows  $G \subset \mathbf{I}(M)$ . But in general one does not know how to determine  $\mathbf{I}(M)$ , or even  $\mathbf{I}_0(M)$ , and this can be troublesome.

The determination of  $\mathbf{I}(M)$  is reduced to the simply connected case as follows. Let  $\varphi: M' \rightarrow M$  be the universal riemannian covering. Then  $M = M'/\Gamma$ , where  $\Gamma \subset \mathbf{I}(M')$  is the group of deck transformations of the covering. Every  $g \in \mathbf{I}(M)$  lifts to a  $\varphi$ -fibre preserving map  $g' \in \mathbf{I}(M')$ , and the  $\varphi$ -fibre maps in  $\mathbf{I}(M')$  induce isometries of  $M$ . An element  $g' \in \mathbf{I}(M')$  maps a  $\varphi$ -fibre  $\Gamma(x')$  to another (necessarily  $\Gamma(g'x')$ ) if and only if  $(g'\Gamma)(x') = (\Gamma g')(x')$ . Let  $N_\Gamma$  be the normalizer of  $\Gamma$  in  $\mathbf{I}(M')$ , so its identity component  $N_\Gamma^0$  is the centralizer of  $\Gamma$ . It follows that

$$\mathbf{I}(M) = N_\Gamma/\Gamma \quad \text{and} \quad \mathbf{I}_0(M) = (\Gamma \cdot N_\Gamma^0)/\Gamma.$$

If  $M = G/K$  is a simply connected symmetric coset space, then the determination of  $\mathbf{I}_0(M)$  and  $\mathbf{I}(M)$  from  $(G, K)$  is due to Cartan (see [16, § 2]); in the irreducible case  $G_0 = \mathbf{I}_0(M)$ , so  $K_0 = K \cap \mathbf{I}_0(M)$ , and  $\mathbf{I}(M)$  is constructed from the pair  $(G_0, K_0)$  by examining automorphisms of  $K_0$  which extend to  $G_0$ . It turns out that Cartan's idea works for isotropy irreducible spaces:

**17.1 THEOREM.** *Let  $M = G/K$  be a noneuclidean simply connected effective coset space with a  $G$ -invariant riemannian metric, where  $G$  is a connected Lie group and  $K$  is a (necessarily connected) compact subgroup. Suppose that the linear isotropy action  $\chi$  of  $K$  on  $M_x$  is  $\mathbf{R}$ -irreducible. Suppose  $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$ . Then  $G = \mathbf{I}_0(M)$ .*

Let  $\text{Aut}(K)^G$  denote the group of all automorphisms of  $K$  which extend to  $G$ , and let  $\text{Inn}(K)^G$  denote the normal subgroup of finite index consisting of inner automorphisms of  $K$ ; let  $\text{Aut}(K)^G = \bigcup_{i=1}^r k_i \cdot \text{Inn}(K)^G$  be the coset decomposition. Define

$$\begin{aligned} \bar{G} &= G \cup s \cdot G \quad \text{and} \quad \bar{K} = K \cup s \cdot K \quad \text{if } \text{rank } G > \text{rank } K \text{ and } G/K \text{ is symmetric with symmetry } s, \\ \bar{G} &= G \quad \text{and} \quad \bar{K} = K \quad \text{otherwise.} \end{aligned}$$

If  $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$ , then

$$\mathbf{I}(M) = \bigcup_{i=1}^r k_i \cdot \bar{G}, \quad \text{and} \quad \bigcup_{i=1}^r k_i \cdot \bar{K} \quad \text{is the isotropy subgroup at } x.$$

*Remark.* If  $G/K = \mathbf{G}_2/\mathbf{SU}(3)$ , then  $M = \mathbf{S}^6$  must be rewritten as  $\mathbf{SO}(7)/\mathbf{SO}(6)$  to apply the theorem. If  $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$  then  $M = \mathbf{S}^7$  must be re-written as  $\mathbf{SO}(8)/\mathbf{SO}(7)$ .

*Proof.* Let  $A = \mathbf{I}_0(M)$  and let  $B$  be the isotropy subgroup at  $x$ . Then  $M = A/B$ , and  $K \subset B$  shows  $B$  to be  $\mathbf{R}$ -irreducible on  $M_x$ . If  $A \neq G$ , then Theorem 15.2 says that either  $G/K = \mathbf{G}_2/\mathbf{SU}(3)$  with  $A/B = \mathbf{SO}(7)/\mathbf{SO}(6)$ , or  $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$  with  $A/B = \mathbf{SO}(8)/\mathbf{SO}(7)$ . Thus  $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$  implies  $G = \mathbf{I}_0(M)$ .

Now we must prove:

(17.2) *Suppose  $G = \mathbf{I}_0(M)$ . Let  $K'$  be the isotropy subgroup of  $\mathbf{I}(M)$  at  $x$ . Let  $k' \in K'$ . Then  $k' \in \bar{K}$  if and only if  $\text{ad}(k')|_K$  is an inner automorphism of  $K$ .*

(17.2) is meaningful because  $\bar{K} \subset K'$  and  $K$  is the identity component of  $K'$ . If  $k' \in \bar{K}$ , then either  $k' \in K$ , or  $k's \in K$ , where  $s$  is the symmetry. In the latter case  $\text{ad}(k's)|_K = \text{ad}(k')|_K$ . Thus  $\text{ad}(k')|_K$  is inner.

Conversely let  $\text{ad}(k')|_K$  be inner. Then  $K$  has an element  $k$  such that  $\text{ad}(k'k)|_K$  is trivial. Let  $\chi'$  be the isotropy representation of  $K'$  on  $M_x$  and let  $A$  be the commuting algebra of  $\chi'$ . Then  $\chi'(k'k) \in A$ . If  $\chi'(k'k) = I$  then  $k'k = 1$  and  $k' \in \bar{K}$ . If  $\chi'(k'k) = -I$  then  $M$  is symmetric and  $k'k = s \in \bar{K}$ , so  $k' \in \bar{K}$ .

Now suppose  $\chi'(k'k) \neq \pm I$ . Then  $A \cong \mathbf{R}$ , so  $\chi$  is not absolutely irreducible and  $A \cong \mathbf{C}$ ;  $\chi'(k'k) \in A$  corresponds to a non-real element  $\varepsilon \in \mathbf{C}$  of norm 1. Let  $Z$  be the center of  $K$ . If  $M$  is symmetric, it is hermitian symmetric and  $\chi(Z) \subset A$  corresponds to the set of all elements of norm 1 in  $\mathbf{C}$ ; thus  $k'k \in Z$  and so  $k' \in \bar{K}$ .

Now assume  $M$  nonsymmetric. Let  $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ ,  $\mathfrak{K}^C = \mathfrak{M}_0$  and  $\mathfrak{M}^C = \mathfrak{M}_\varepsilon + \mathfrak{M}_{\bar{\varepsilon}}$ , where  $\text{ad}(k'k)$  is scalar multiplication by  $\alpha$  on  $\mathfrak{M}_\alpha$ . Note that  $\text{ad}(k'k)$  is scalar multiplication by  $\alpha\beta$  on  $[\mathfrak{M}_\alpha, \mathfrak{M}_\beta]$ . Thus  $[\mathfrak{M}_\varepsilon, \mathfrak{M}_{\bar{\varepsilon}}] \subset \mathfrak{K}^C$  and  $[\mathfrak{M}_\varepsilon, \mathfrak{M}_\varepsilon] \subset \mathfrak{M}_{\varepsilon^2}$ . As  $\varepsilon^2 \neq \varepsilon$ , and as  $[\mathfrak{M}^C, \mathfrak{M}^C] \not\subset \mathfrak{K}^C$  by nonsymmetry, now  $\varepsilon^2 = \bar{\varepsilon}$ . Thus  $\varepsilon = e^{\pm 2\pi i/3}$ . If  $\text{rank } G = \text{rank } K$ , then  $Z$  has order 3 and so  $\chi'(k'k) \in \chi(Z)$ ; thus  $k'k \in Z$  and  $k' \in \bar{K}$ .

Finally suppose  $\text{rank } G > \text{rank } K$ .  $G/K$  is  $\mathbf{E}_6/\{\mathbf{SU}(3)/\mathbf{Z}_3\}$  or  $\{\mathbf{Spin}$  or  $\mathbf{SO}\}(n^2 - 1)/\text{ad}\mathbf{SU}(n)$ . If  $\tau$  is an outer automorphism of order 3 on  $G$ , it follows that  $G/K = \mathbf{Spin}(8)/\text{ad}\mathbf{SU}(3)$  and  $\tau$  is triality; then  $\tau$  has fixed point set  $\mathbf{G}_2$  which does not contain the centerless version of  $\mathbf{SU}(3)$ . As  $k'k$  has order 3 it follows that  $\text{ad}(k'k)|_G$  is an inner automorphism. Let  $g \in G$  such that  $\text{ad}(k'k)|_G = \text{ad}(g)$  and let  $L$  be the connected centralizer of  $g$ . Then  $K \subset L$  and  $\text{rank } L = \text{rank } G > \text{rank } K$ ; thus  $L = G$ . Now  $\text{ad}(k'k)|_M = \chi'(k'k) = I$ , so  $k'k = 1$  and  $k' \in \bar{K}$ . This completes the proof of (17.2).

Now  $\bar{K} \subset K' \subset \bigcup_{i=1}^v k_i \cdot \bar{K}$ . Re-ordering the  $k_i$ , it follows that

$$K' = \bigcup_{i=1}^v k_i \cdot \bar{K}, \text{ and thus } \mathbf{I}(M) = \bigcup_{i=1}^v k_i \cdot \bar{G}.$$

Define groups  $K'' = \bigcup_{i=1}^v k_i \cdot \bar{K}$  and  $G'' = \bigcup_{i=1}^v k_i \cdot \bar{G}$ . Then  $G$  is transitive on  $G''/K''$  and  $K = G \cap K''$ . Thus we identify  $M$  with  $G''/K''$ . Let  $V$  be the kernel of the action of  $G''$  on  $M$ . Then  $V \subset K''$  is normalized by  $K$ , and  $V$  is finite because  $V \cap K = V \cap K_0'' = \{1\}$ . As  $K$  is connected it follows that  $V$  centralizes  $K$ . Now  $V \subset \bar{K} \subset K'$ , so  $V = \{1\}$ . Thus  $G''$  is effective on  $M$ . As  $K''$  is compact,  $M$  has a  $G''$ -invariant riemannian metric. That metric is  $G$ -invariant, hence proportional to the original one. Thus  $G'' \subset \mathbf{I}(M)$ . But we just saw  $\mathbf{I}(M) \subset G''$ , so now  $\mathbf{I}(M) = G''$  and  $K' = K''$ , q.e.d.

A similar result holds for isometries which preserve an almost-complex structure. The symmetric case is due to Cartan.

**17.3 THEOREM.** *Let  $M = G/K$  be a noneuclidean simply connected effective coset space, with a  $G$ -invariant riemannian metric  $ds^2$  and a  $G$ -invariant almost complex structure  $J$ . Suppose that  $G$  is a connected Lie group and  $K$  is a compact subgroup whose linear isotropy action is  $\mathbf{R}$ -irreducible. Given  $z \in M$  and tangent vectors  $X, Y \in M_z$ , define  $\omega_z(X, Y) = ds_z^2(X, J_z Y)$  and define  $h_z = ds_z^2 + i\omega_z$ . Then  $h$  is an almost-hermitian metric on  $M$ .*

*Let  $\mathbf{H}(M)$  be the group of all almost-hermitian isometries of  $M$  and let  $\mathbf{H}_0(M)$  be the identity component. Then  $G = \mathbf{H}_0(M)$ . If  $G/K \cong \text{ad}(\mathbf{E}_6)/\mathbf{A}_2 \cdot \mathbf{A}_2 \cdot \mathbf{A}_2$ , then  $G = \mathbf{H}(M)$ . If  $G/K = \text{ad}(\mathbf{E}_6)/\mathbf{A}_2 \cdot \mathbf{A}_2 \cdot \mathbf{A}_2$ , then  $\mathbf{H}(M) = G \cup \varphi \cdot G$ , where  $\text{ad}(\varphi)|_G$  is an involutive outer automorphism with fixed point set  $\mathbf{F}_4$ .  $M$  has an isometry  $\lambda$  which sends  $J$  to  $-J$ . If  $G/K \cong \mathbf{G}_2/\text{SU}(3)$ , then  $\mathbf{I}(M) = \mathbf{H}(M) \cup \lambda \cdot \mathbf{H}(M)$ .*

*Proof.* Let  $\chi$  be the representation of  $K$  on  $M_x$ . Then the commuting algebra of  $\chi$  is  $\mathbf{C}$  and  $J_x$  is one of its two elements of square  $-I$ . The unimodular elements of  $\mathbf{C}$  are in  $\text{SO}(M_x)$ . Thus  $h_x$  is a hermitian inner product on  $M_x$ . Now  $h$  is an almost-hermitian metric on  $M$ .

Let  $\mathbf{H}(M)$  be the group of all almost-hermitian isometries of  $M$ . Then  $G \subset \mathbf{H}(M) \subset \mathbf{I}(M)$ . If  $G/K \cong \mathbf{G}_2/\text{SU}(3)$ , then  $G = \mathbf{I}_0(M)$  and so  $G = \mathbf{H}_0(M)$ . If  $G/K = \mathbf{G}_2/\text{SU}(3)$ , then  $\mathbf{I}_0(M) = \text{SO}(7)$  has  $G = \mathbf{G}_2$  as a maximal connected subgroup, and  $\mathbf{I}_0(M) \not\subset \mathbf{H}(M)$ ; it follows that  $G = \mathbf{H}_0(M)$ . Now  $G = \mathbf{H}_0(M)$  in general.

Let  $L$  be the isotropy subgroup of  $\mathbf{H}(M)$  at  $x$ . Then  $K = L_0$ . If  $k' \in L$ , then  $k'_*$  commutes with  $J_x$ , so  $\text{ad}(k')|_K$  does not interchange the two irreducible summands of  $\chi$ . If  $G/K = \text{ad}(\mathbf{E}_6)/\mathbf{A}_2 \cdot \mathbf{A}_2 \cdot \mathbf{A}_2$ , assume further that  $\text{ad}(k')|_G$  is inner. Then a case by case check shows that  $\text{ad}(k')|_K$  is inner. Let  $k \in K$  so that  $\text{ad}(k'k)|_K$  is trivial. As in the proof of Theorem 17.1, it follows that  $k'k$  is central in  $K$ , so  $k' \in K$ . On the other hand, as noted in the proof of Theorem 13.6,  $\text{ad}(\mathbf{E}_6)/\mathbf{A}_2 \cdot \mathbf{A}_2 \cdot \mathbf{A}_2$  has an almost complex involutive automorphism  $\varphi$  such that  $\text{ad}(\varphi)$  is outer on  $\mathbf{E}_6$ . Thus  $G = \mathbf{H}(M)$  for  $G/K \cong \text{ad}(\mathbf{E}_6)/\mathbf{A}_2 \cdot \mathbf{A}_2 \cdot \mathbf{A}_2$  and  $\mathbf{H}(M) = G \cup \varphi \cdot G$  in the exceptional case.

We find  $\lambda$ . First suppose  $G$  compact with  $\text{rank } G > \text{rank } K$ . If  $G/K = \mathbf{E}_6/\text{ad } \text{SU}(3)$  then  $\lambda$  is the outer automorphism of  $G$  which preserves  $K$ . If  $G/K = \text{SO}(n^2 - 1)/\text{ad } \text{SU}(n)$ , then  $\lambda$  is the outer automorphism of  $\mathfrak{SU}(n)$  viewed as an element of  $\text{GL}(n^2 - 1, \mathbf{R})$  which normalizes  $\text{SO}(n^2 - 1)$ . Now suppose  $G$  compact with  $\text{rank } G = \text{rank } K$ . Embed the center  $Z$  of  $K$  in a maximal torus  $T \subset K$ , and let  $\lambda$  be the automorphism of order 2 on  $G$  which is  $-I$  on  $\mathfrak{L}$ .  $\lambda$  preserves  $K$  because  $K$  is the connected centralizer of  $Z$ . Finally suppose  $G$  noncompact. Then  $G/K$  is hermitian symmetric of noncompact type, and we have an automorphism  $\lambda_u$  of the compact form  $G_u$  which preserves  $K$  and is inversion on  $Z$ . Extend  $\lambda_u$  from  $\mathfrak{G}_u$  to  $\mathfrak{G}^{\mathbf{C}}$  by linearity and let  $\lambda$  be its restriction to  $\mathfrak{G}$ .

Finally suppose  $G/K \cong \mathbf{G}_2/\mathbf{SU}(3)$ . If  $k$  is in the isotropy subgroup of  $\mathbf{I}(M)$  at  $x$ , then  $k_*$  either commutes or anticommutes with  $J_x$ , for  $k_*Jk_*^{-1}$  is another almost complex structure on  $M$ . In the commuting case,  $k \in \mathbf{H}(M)$ . In the anticommuting case,  $k\lambda \in \mathbf{H}(M)$ . Thus  $\mathbf{I}(M) = \mathbf{H}(M) \cup \lambda \cdot \mathbf{H}(M)$ , q.e.d.

The analysis of  $\mathbf{H}(M)$  allows us to study the group  $\mathbf{A}(M)$  of all almost-complex diffeomorphisms of  $M$ :

**17.4 THEOREM.** *Let  $M = G/K$  be a noneuclidean effective coset space with a  $G$ -invariant almost-complex structure, where  $G$  is a connected Lie group and  $K$  is a compact subgroup whose identity component is  $\mathbf{R}$ -irreducible on the tangent space. Choose a  $G$ -invariant riemannian metric on  $M$ . Then:*

1. *If  $G/K$  is noncompact then  $\mathbf{A}(M) = \mathbf{H}(M) = \mathbf{I}_0(M) = G$ .*
2. *If  $G/K$  is compact, then  $\mathbf{A}(M)$  is a simple Lie group with maximal compact subgroup  $\mathbf{H}(M)$ , and  $\mathbf{A}_0(M)$  is a simple Lie group with finite center and maximal compact subgroup  $G = \mathbf{H}_0(M)$ .*
3. *If  $G/K$  is compact with  $\text{rank } G = \text{rank } K$ , then either  $G/K$  is hermitian symmetric with  $\mathbf{A}(M) = \mathbf{H}(M)^c = G^c$ , or  $G/K$  is nonsymmetric with  $\mathbf{A}(M) = \mathbf{H}(M)$  and  $\mathbf{A}_0(M) = G$ .*

*Proof.* If  $G/K$  is noncompact it is a hermitian symmetric space of noncompact type. Then, in the Harish-Chandra realization as a bounded domain with Bergman metric, every analytic automorphism is an isometry; so  $\mathbf{A}(M) = \mathbf{H}(M)$  and our assertions follow from Theorem 17.3

Now assume  $G/K$  compact. Let  $A$  denote  $\mathbf{A}_0(M)$  and let  $B$  denote the isotropy subgroup; so  $G \subset A$  and  $K = G \cap B$ .  $\mathbf{A}(M)$  is a Lie group [2]; now  $\mathbf{H}(M)$  must be a maximal compact subgroup. In particular  $G$  is a maximal compact subgroup of  $A$ . Whenever  $S$  is a closed connected subgroup of  $A$  normalized by  $G$ , we have  $G/K = (G \cdot S)/(K \cdot S)$ ; if  $G \not\subset S$  then simplicity of  $G$  and effectiveness of  $A$  show that  $S = \{1\}$ . Take  $S$  to be the connected radical of  $A$ ; now  $A$  is semisimple. If  $A$  has two simple factors take  $S$  to be one which does not contain  $G$ ; now  $A$  is simple. If  $A$  has infinite center take  $\mathfrak{S}$  to be the one dimensional vector group orthogonal to  $\mathfrak{G}$  in a maximal compactly embedded subalgebra of  $\mathfrak{A}$ ; now  $A$  has finite center. Thus  $A$  is a simple linear group with  $G$  as maximal compact subgroup.

Suppose  $\text{rank } K = \text{rank } G$ . As  $\mathfrak{K}$  is its own normalizer in  $\mathfrak{G}$ , it is an algebraic subalgebra of  $\mathfrak{G}$ ; thus  $A$  has an Iwasawa decomposition  $GSN$  for some maximal  $\mathbf{R}$ -split algebraic torus  $S$ , such that  $B = KSN$ . If  $A = G^c$ , then  $A$  has a complex Cartan subgroup  $H$  such that  $H \cap K$  is a maximal torus  $T \subset K$  and  $H = T \cdot S$ . Now  $B$  contains the Borel subgroup  $T'SN$

of  $A$ , so  $B$  is a parabolic subgroup of  $A$  and  $M = A/B$  has a natural  $A$ -invariant complex structure. Of course this structure is  $G$ -invariant. Now Theorem 13.1 says that  $G/K$  is hermitian symmetric. Conversely  $G^C \subset \mathbf{A}(M)$  if  $M = G/K$  is hermitian symmetric. Thus  $M$  is (hermitian) symmetric if and only if  $A = G^C$

Let  $G/K$  be  $G_2/A_2$ ,  $E_6/A_2A_2A_2$ ,  $E_7/A_2A_5$ ,  $E_8/A_8$  or  $E_8/A_2E_6$ . Then  $A \neq G^C$  and there is no noncompact absolutely simple group with  $G$  as maximal compact subgroup. Thus  $A$  is compact. Now  $A = G$ .

Let  $G/K = F_4/A_2A_2$ . Then  $E_{6,F_4}$  is the only noncompact absolutely simple group with  $G$  as maximal compact subgroup. Suppose  $A = E_{6,F_4}$  and let  $\sigma$  be the involutive automorphism with fixed point set  $G$ . Then  $L = (B \cap \sigma B)_0$  is reductive in  $A$  with maximal compact subgroup  $K$ . If  $L$  is almost effective on  $L/K$ , it follows that the semisimple part  $L' = K^C$ , so  $\text{rank } L \geq 8$  in contradiction to  $L \subset A$ . Now  $L = L_1 \cdot A_2$ , where  $A_2$  is the second factor in  $K = A_2 \cdot A_2$ , and  $L/K = L_1/A_2$ , where the  $A_2$  is the first factor. If  $L' = K$  then  $L = K \cdot S = K \times S$  in the notation of the Iwasawa decomposition  $A = GSN$ , and  $K$  is the semisimple part of the centralizer of  $S$ . That is false.<sup>(1)</sup> Thus  $L = A^C \cdot A_2$ . Now  $A/L$  is the noncompact almost complex isotropy irreducible space derived from  $E_6/A_2A_2A_2$  by the involution  $\sigma$ . Thus  $L$  is irreducible on the orthocomplement of  $\mathfrak{Q}$  in  $\mathfrak{A}$ . As  $L \subset B$  normalizes  $B$ , and as  $B \neq A$ , that says  $L = B$ ; but then  $\dim A/B = 54 > 36 = \dim G/K$  contradicts  $A/B \cong M \cong G/K$ . This proves  $A \neq E_{6,F_4}$ . As  $A \neq G^C$  we conclude  $A = G$ , q.e.d.

*Remark.* The method shows that  $\mathbf{A}_0(E_6/\text{ad } \text{SU}(3))$  is  $E_6$  or  $E_6^C$ , and that  $\mathbf{A}_0(\text{SO}(n^2 - 1)/\text{ad } \text{SU}(n))$  is  $\text{SO}(n^2 - 1)$ ,  $\text{SO}(n^2 - 1, \mathbf{C})$ ,  $\text{SL}(n^2 - 1, \mathbf{R})$  or  $\text{SO}^1(n^2)$ . It seems probable that  $\mathbf{A}_0(G/K) = G$  in both cases, just as for the non-integrable almost complex spaces of equal rank.

### 13. Local structure

Let  $M$  be a riemannian manifold. If  $x \in M$ , then  $K^{(x)}$  denotes the group of all isometries of neighborhoods of  $x$  which fix  $x$ , where we identify two isometries if they coincide on a neighborhood of  $x$ .  $K^{(x)}$  is called the *group of local isometries at  $x$* .  $K^{(x)}$  is a compact Lie

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<sup>(1)</sup>  $E_6$  has simple roots  $\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \circ - \circ - \circ - \circ - \circ \\ | \\ \circ \\ \alpha_6 \end{matrix}$ ,  $F_4$  has simple roots  $\{\frac{1}{2}(\alpha_1 + \alpha_5), \frac{1}{2}(\alpha_2 + \alpha_4), \alpha_3, \alpha_6\}$ ,

and  $S$  is spanned by  $\{\alpha_1 - \alpha_5, \alpha_2 - \alpha_4\}$ . Thus the centralizer of  $S$  in  $E_6$  has positive roots  $\{\alpha_3, \alpha_6, \alpha_3 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\}$ ,

so it has semisimple part of type  $D_4$  with simple roots  $\begin{matrix} & & \circ & & \\ & & / & & \\ \circ - \circ & & \circ & & \circ \\ & & \backslash & & \\ & & \circ & & \circ \\ \alpha_3 & \alpha_6 & & & \alpha_2 + \alpha_3 + \alpha_4 \end{matrix}$

group, for its linear isotropy representation  $\chi$  on  $M_x$  is faithful, and  $\chi(K^{(x)})$  consists of all the orthogonal linear transformations of  $M_x$  which preserve all covariant differentials  $(\nabla^m R)_x$ ,  $m \geq 0$ , of the curvature tensor at  $x$ . We say that  $M$  is *locally isotropy irreducible* at  $x$  if  $\chi(K_0^{(x)})$  is  $\mathbf{R}$ -irreducible on  $M_x$ .  $M$  is said to be *locally isotropy irreducible* if it is locally isotropy irreducible at each of its points.

Let  $\mathfrak{G}^{(x)}$  denote the Lie algebra of germs of Killing vector fields at  $x$ . Then  $\mathfrak{K}^{(x)}$  is naturally identified with the subalgebra consisting of all elements of  $\mathfrak{G}^{(x)}$  which vanish at  $x$ . If  $\varphi$  is a local isometry carrying  $x$  to  $z$ , then  $\varphi$  sends  $\mathfrak{G}^{(x)}$  isomorphically onto  $\mathfrak{G}^{(z)}$  and carries  $K^{(x)}$  to  $K^{(z)}$ . If  $M$  is locally homogeneous, it follows that the isomorphism classes of the pair  $(\mathfrak{G}^{(x)}, \mathfrak{K}^{(x)})$  and the group  $K^{(x)}$  do not depend on the choice of  $x$ .

**18.1 THEOREM.** *Let  $M$  be a connected locally isotropy irreducible riemannian manifold. Then  $M$  is locally homogeneous.*

Choose  $x \in M$ . Let  $G/K$  be the simply connected effective coset space with  $\mathfrak{G} = \mathfrak{G}^{(x)}$ ,  $\mathfrak{K} = \mathfrak{K}^{(x)}$  and  $G$  connected. Then  $K$  is  $\mathbf{R}$ -irreducible on the tangent space of  $G/K$ . For  $X \in \mathfrak{G}^{(x)}$  near 0, define  $f(\exp_G(X)K) = \exp_M(X) \cdot x$ . Then there is a  $G$ -invariant riemannian metric on  $G/K$  such that  $f$  is an isometry of a neighborhood of  $K \in G/K$  onto a neighborhood of  $x \in M$ . If  $M$  is complete, then  $f$  extends to a riemannian covering.

Before proving this theorem we note some consequences.

**18.2 COROLLARY.** *Let  $M$  be a complete connected simply connected locally isotropy irreducible riemannian manifold. Then  $M$  is homogeneous, so  $M = G/K$  with  $G = \mathbf{I}_0(M)$ , and*

- (i)  $M$  is a euclidean space; or
- (ii)  $M$  is an irreducible riemannian symmetric space; or
- (iii)  $G/K$  is listed in Theorem 11.1.

For  $G/K$  coincides with the coset space of Theorem 18.1.

**18.3 COROLLARY.** *Let  $M_1$  and  $M_2$  be complete connected locally isotropy irreducible riemannian manifolds, with  $M_1$  simply connected. Let  $f: U_1 \rightarrow U_2$  be an isometry, where  $U_1$  is a connected simply connected open submanifold of  $M_1$ . Then  $f$  extends to a riemannian covering  $\tilde{f}: M_1 \rightarrow M_2$ . If  $M_2$  is simply connected then  $\tilde{f}$  is an isometry.*

For let  $\pi: M'_2 \rightarrow M_2$  be the universal riemannian covering and let  $f': U_1 \rightarrow U'_2 = f'(U_1) \subset M'_2$  be a lift of  $f$ . Then  $f'$  is an isometry. Let  $M_1 = G_1/K_1$  and  $M'_2 = G_2/K_2$  as in Corollary 18.2; now  $f'$  induces an isomorphism of  $G_1$  onto  $G_2$  which carries  $K_1$  to  $K_2$ , and thus  $f'$  induces an isometry  $\tilde{f}': M_1 \rightarrow M'_2$ . Define  $\tilde{f} = \pi \cdot \tilde{f}'$  and the assertions follow.



*Proof of theorem.* If  $z \in M$ , then  $V_z = \{X_z: X \in \mathfrak{G}^{(z)}\}$  is a  $K^{(z)}$ -stable subspace of  $M_z$ . By local isotropy irreducibility, either  $V_z = 0$  or  $V_z = M_z$ . If  $V_z = 0$  we choose  $w \in M$  such that  $z \neq w$  but  $z$  is in a normal neighborhood of  $w$ . Let  $\{w_t\}_{0 \leq t \leq 1}$  be the unique minimizing geodesic from  $w$  to  $z$ , and let  $W \in M_w$  be its tangent vector at  $w$ . If  $K_0^{(w)}(z) = z$  then  $K_0^{(w)}$  fixes  $W$ , contradicting isotropy irreducibility at  $w$ . Now  $K_0^{(w)}$  has a one parameter subgroup  $\exp(tX)$ ,  $X \in \mathfrak{K}^{(w)}$ , which moves  $z$ , and so  $0 \neq X_z \in V_z$ . This proves  $V_z = M_z$  for every  $z \in M$ .

Let  $x_0, x_1 \in M$ . Let  $\{x_t\}$  be a smooth curve in  $M$  from  $x_0$  to  $x_1$ . Given  $0 \leq t \leq 1$ , every element of  $M_{x_t}$  is the value of some Killing vector field on a neighborhood of  $x_t$ , so there is an open set  $U_t \ni x_t$  consisting of images of  $x_t$  under local isometries. The Heine–Borel Theorem gives  $0 = t_0 < \dots < t_k = 1$  such that  $\bigcup_{i=0}^k U_{t_i}$  contains the curve  $\{x_t\}$ . Now a composition of  $k$  local isometries carries  $x_0$  to  $x_1$ . This proves that  $M$  is locally homogeneous.

$\mathfrak{G}^{(z)} = \mathfrak{K}^{(z)} \oplus \mathfrak{M}$  where  $\mathfrak{M}$  is an  $\text{ad}(\mathfrak{K}^{(z)})$ -stable subspace identified with  $M_x$  under  $X \rightarrow X_x$ . As  $K_0^{(z)}$  is  $\mathbf{R}$ -irreducible on  $M_x$ , now  $\mathfrak{K}^{(z)}$  is  $\mathbf{R}$ -irreducible on  $\mathfrak{M}$ , and thus  $K$  is  $\mathbf{R}$ -irreducible on the tangent space of  $G/K$ . Lift the metric from  $M_x$  to  $\mathfrak{M}$ ; then it defines a  $G$ -invariant riemannian metric on  $G/K$ . Choose a convex open neighborhood  $\mathfrak{S}$  of 0 in  $\mathfrak{G}^{(z)}$  such that  $\exp_M(-Y_2)$  is defined at  $\exp_M(Y_1) \cdot x$  whenever  $Y_1, Y_2 \in \mathfrak{S}$ . Then we have linear isometries

$$(G/K)_{\exp_G(Y) \cdot K} \xrightarrow{\exp_G(-Y)} (G/K)_K \xrightarrow{f_*} M_x \xrightarrow{\exp_M(Y)} M_{\exp_M(Y) \cdot x}$$

and  $f_*: (G/K)_{\exp_G(Y) \cdot K} \rightarrow M_{\exp_M(Y) \cdot x}$  is their composition. Thus  $f$  is an isometry on neighborhoods.

Let  $M$  be complete and let  $\pi: M' \rightarrow M$  denote the universal riemannian covering. We cut  $f$  down to an isometry  $g: U \rightarrow V$  of simply connected neighborhoods and then lift it to an isometry  $g': U \rightarrow g(U) = V' \subset M'$ . As  $G = \mathbf{I}_0(G/K)$  we can develop  $g'$  along smooth curves. As  $G/K$  is real analytic it follows that  $M$  is real analytic. Now ([10], p. 256)  $g'$  extends to an isometry  $\bar{g}$ , and  $\pi \cdot \bar{g}$  is a riemannian covering.  $\pi \cdot \bar{g}$  agrees with  $f$  on the domain of  $g$ , so they agree on the domain of  $f$  by analyticity. Thus  $\pi \cdot \bar{g}$  extends  $f$ , q.e.d.

#### Added in Proof

On 8 August 1967, Professor C. T. C. Wall informed me of the following generalization of Corollary 10.2. Let  $S = A/B$  be a compact simply connected irreducible symmetric space,  $n = \dim S$  and  $A = \mathbf{I}_0(S)$ ,  $S$  not a real or quaternionic Grassmann manifold.  $\beta, B \rightarrow \mathbf{SO}(n)$  is the linear isotropy representation. Decompose  $B = K \cdot L$ ,  $\beta = \pi \otimes \omega$ ,  $\pi: K \rightarrow G$  where (i)  $S$  is neither hermitian nor quaternionic [18],  $B = K$  and  $G = \mathbf{SO}(n)$ ; or (ii)  $S$  is hermitian,  $L$  is a circle,  $K = [B, B]$  and  $G = \mathbf{SU}(n/2)$ ; or (iii)  $S$  is quaternionic,  $L = \mathbf{SU}(2)$  with  $\omega: \circ$  and  $G = \mathbf{Sp}(n/4)$ . Then  $G/\pi(K)$  is a nonsymmetric isotropy irreducible coset space,  $G$  classical,  $\mathbf{SO}(7)/\mathbf{G}_2 \neq G/\pi(K) \neq \mathbf{SO}(20)/[\mathbf{SU}(4)/\mathbf{Z}_4]$ ; and conversely every nonsymmetric isotropy irreducible coset space  $G/\pi(K)$ ,  $G$  classical,  $\mathbf{SO}(7)/\mathbf{G}_2 \neq G/\pi(K) \neq \mathbf{SO}(20)/[\mathbf{SU}(4)/\mathbf{Z}_4]$ , is constructed as above

from a compact irreducible symmetric space  $S$  which is not a real or quaternionic grassmannian. This observation is checked by classification. An *a priori* proof will be valuable, but it will also be difficult because of the exceptions.

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