SURFACES OF CONSTANT MEAN CURVATURE

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1. Introduction. Let $S$ be a surface immersed in euclidean space $\mathbb{R}^3$ with constant mean curvature $H$. In a recent note [3] we proved that the quadratic differential form $-HI + II$ is a flat Lorentz metric on the complement of the umbilic set of $S$. Here the result is used to set up a certain type of isothermal local coordinate system on $S$. The main consequences are:

(i) an obstruction theory, which tells one when an isometry of connected surfaces of the same constant mean curvature is a congruence;\(^2\)

(ii) Gauss curvature on $S$ is set up as a solution to a nonlinear elliptic boundary value problem; and

(iii) construction of local surfaces of any given constant mean curvature.

2. Notation. $S$ denotes a surface with a fixed immersion $\nu: S \to \mathbb{R}^3$. If $\xi$ is a smooth choice of unit normal defined over an open set $U \subseteq S$, then we recall the fundamental forms of the immersion:

$I = dv \cdot dv$, first fundamental form;

$II = dv \cdot d\xi$, second fundamental form;

$III = d\xi \cdot d\xi$, third fundamental form.

$I = dv^2$ is the riemannian metric induced on $S$ by the immersion. The eigenvalues of $II$ relative to $I$ are the principle curvatures, denoted $k_i$. As usual we have functions $H, K$ on $S$ given by

$$H = \frac{1}{2} \{k_1 + k_2\}, \text{ mean curvature;}$$

$$K = k_1 k_2, \text{ Gauss curvature.}$$

They define the quadratic differential form

$$\Omega = -HI + II, \text{ modified fundamental form.}$$

The eigenvalues of $\Omega$ relative to $I$ are $k_i - \frac{1}{2} (k_1 + k_2) = \pm \frac{1}{2} (k_1 - k_2)$. Thus $\Omega$ is a pseudo-riemannian metric of Lorentz signature (Lorentz metric) on the open subset

$$S_\Omega = \{ x \in S: k_1(x) \neq k_2(x) \}$$

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\(^2\) In other words, when the isometry is the restriction of a rigid motion of the ambient euclidean space $\mathbb{R}^3$. 

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of $S$. We view $S_\Omega$ as a Lorentz surface with metric $\Omega$. Recall that a point $x \in S$ is called umbilic if $k_1(x) = k_2(x)$; thus $S_\Omega$ is the complement of the umbilic set of $S$.

3. Special coordinates on $S_\Omega$. The results of this note are based on the following observation.

3.1. Theorem. Let $S$ be a surface immersed in $\mathbb{R}^3$ with constant mean curvature $H$. Let $K$ denote Gauss curvature and define\(^3\) a function

\[
\lambda = -\frac{1}{2} \log(H^2 - K) \text{ on } S_\Omega.
\]

If $x \in S_\Omega$, then $x$ has a local coordinate neighborhood\(^4\) $(U, u)$ with $U \subset S_\Omega$ and

\[
I = e^\lambda \left( du^1 \otimes du^1 + du^2 \otimes du^2 \right);
\]

\[
II = (He^\lambda + 1) du^1 \otimes du^1 + (He^\lambda - 1) du^2 \otimes du^2;
\]

\[
k_1 = H + e^{-\lambda}, \quad k_2 = H - e^{-\lambda}, \quad K = H^2 - e^{-2\lambda}.
\]

If $(V, v)$ is another local coordinate neighborhood of $x$ with these properties, then $v^i = \pm u^i + c^i$, $c^i$ constant, on each component of $U \cap V$.

Proof. Let the principle curvature be numbered so that $k_1 > k_2$ on $S_\Omega$. Given $x \in S_\Omega$ we choose a neighborhood $W \subset S_\Omega$ of $x$ which carries an $I$-orthonormal moving frame $\{X_1, X_2\}$ such that $X_i$ is a principle vector with principle curvature $k_i$. We have seen [3, Corollary 4.11] that the connection form of the Lorentz surface $S_\Omega$ is identically zero in the $S^2$-orthonormal moving frame $\{F_1, F_2\}$, where $F_i = \left\{ \frac{1}{2}(k_1 - k_2) \right\}^{1/2} X_i$. It follows that $x$ has a local coordinate neighborhood $(U, u)$ such that $U \subset W$ and $\partial/\partial u^i = Y_i$. Now

\[\Omega = du^1 \otimes du^1 - du^2 \otimes du^2 \text{ in } U.\]

On the other hand, $I$ and $II$ are diagonalized by $\{X_1, X_2\}$, hence also by $\{Y_1, Y_2\} = \{\partial/\partial u^1, \partial/\partial u^2\}$. Thus

\[
I = \sum_{i=1}^{2} g_{i} du^i \otimes du^i \quad \text{and} \quad II = \sum_{i=1}^{2} b_i du^i \otimes du^i
\]

in $U$. This tells us

\[
b_i = k_ig_i, \quad -Hg_1 + b_1 = 1, \quad -Hg_2 + b_2 = -1.
\]

We compute

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\(^3\) Here we must observe that $H^2 - K > 0$ on $S_\Omega$; for $H^2 - K = \frac{1}{2}(k_1 - k_2)^2$.

\(^4\) $U$ is the neighborhood and $u = (u^1, u^2)$ is the local coordinate.
\[ 2H = k_1 + k_2 = \frac{b_1}{g_1} + \frac{b_2}{g_2} = \frac{Hg_1 + 1}{g_1} + \frac{Hg_2 - 1}{g_2} = 2H + \frac{1}{g_1} - \frac{1}{g_2}. \]

Thus \( g_1 = g_2 \), which must be positive because \( I \) is positive definite. Now \( g_1 = g_2 = e^\lambda \) for some function \( \lambda \) defined on \( U \). We compute

\[
\begin{align*}
  b_1 &= Hg_1 + 1 = He^\lambda + 1, \quad k_1 = b_1 / g_1 = H + e^{-\lambda}; \\
  b_2 &= Hg_2 - 1 = He^\lambda - 1, \quad k_2 = b_2 / g_2 = H - e^{-\lambda}; \\
  K &= k_1k_2 = H^2 - e^{-2\lambda}, \quad \text{so} \quad \lambda = -\frac{1}{2} \log(H^2 - K).
\end{align*}
\]

This proves (3.3), (3.4) and (3.5).

For the uniqueness, observe that \( \{ \partial / \partial v^1, \partial / \partial v^2 \} \) diagonalizes \( I \) and \( II \) with first coefficient greater than second in \( II \). Thus \( \partial / \partial v^i \) is a principle vector with principle curvature \( k_i \) on \( S \). As \( \Omega(\partial / \partial v^i, \partial / \partial v^i) = \Omega(\partial / \partial u^i, \partial / \partial u^i) = \pm 1 \neq 0 \), now \( \partial / \partial v^i = \pm \partial / \partial u^i \), so \( dv^i = \pm du^i \). q.e.d.

**4. The Mainardi-Codazzi equations.** Let \((U, u)\) be a connected local coordinate neighborhood on a surface \( S \) immersed in \( \mathbb{R}^3 \). Suppose that the fundamental forms are given by

\[ (4.1) \quad I = \varepsilon^i \{ du^1 \otimes du^1 + du^2 \otimes du^2 \} \quad \text{and} \quad II = \sum_{ij} b_{ij} du^i \otimes du^j. \]

Then the Christoffel symbols are easily computed:

\[ (4.2) \quad \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \frac{1}{2} \frac{\partial \lambda}{\partial u^1}; \quad \Gamma^1_{12} = -\Gamma^2_{11} = \Gamma^2_{22} = \frac{1}{2} \frac{\partial \lambda}{\partial u^2}. \]

Thus the Mainardi-Codazzi equations reduce to

\[ (4.3) \quad \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} = \frac{1}{2} (b_{11} + b_{22}) \frac{\partial \lambda}{\partial u^2} \quad \text{and} \quad \frac{\partial b_{22}}{\partial u^1} - \frac{\partial b_{12}}{\partial u^2} = \frac{1}{2} (b_{11} + b_{12}) \frac{\partial \lambda}{\partial u^1}. \]

Now suppose that our surface \( S \) has constant mean curvature \( H \). Let \( z = u^1 + (-1)^{1/2} u^2 \), complex local coordinate, and define

\[ \phi(z) = (b_{11} - b_{22}) + 2 (-1)^{1/2} b_{12}. \]

As \( 2H = b_{11}e^{-\lambda} + b_{22}e^{-\lambda} = (b_{11} + b_{22})e^{-\lambda} \) is constant, \( (4.3) \) says that \( \partial / \partial z = \frac{1}{2} \{ \partial / \partial u^1 + (-1)^{1/2} \partial / \partial u^2 \} \) annihilates \( \phi \); thus \( \phi \) is a holomorphic function of \( z \). Let \( f \) be the function on \( U \) defined by

\[ b_{11} = He^\lambda + f, \quad b_{22} = He^\lambda - f. \]

Suppose that Gauss curvature satisfies
\[ K = H^2 - e^{-2\lambda}, \text{ i.e., } \lambda = -\frac{1}{2} \log(H^2 - K). \]

Then
\[
H e^{2\lambda} - 1 = K e^{2\lambda} = b_{11}b_{22} - b_{12}^2 = H^2 e^{2\lambda} - (f^2 + b_{12}^2),
\]
so \( f^2 + b_{12}^2 = 1 \). But \( \phi = 2(f + (-1)^{1/2}b_{12}) \) is holomorphic; now the maximum modulus principle says that \( \phi \) is constant; thus \( f \) and \( b_{12} \) are constant.

Notice that \( P \in \Omega \) by the assumption \( H^2 - K = e^{-2\lambda} > 0 \). Cutting \( U \) down if necessary, Theorem 3.1 gives us a local coordinate \( v \) on \( U \) in which \( I = e^\lambda \{ dv^1 \otimes dv^1 + dv^2 \otimes dv^2 \} \) and \( II = (He^\lambda + 1)dv^1 \otimes dv^1 + (He^\lambda - 1)dv^2 \otimes dv^2 \). If \( \alpha \) is the oriented angle from \( \partial/\partial u^1 \) to \( \partial/\partial v^1 \), the two expressions for \( I \) give
\[
dv^1 = \cos \alpha \, du^1 + \sin \alpha \, du^2 \quad \text{and} \quad dv^2 = -\sin \alpha \, du^1 + \cos \alpha \, du^2.
\]
Equating coefficients of \( du^1 \otimes du^1 \) in the two expressions for \( II \),
\[
He^\lambda + f = b_{11} = (He^\lambda + 1) \cos^2 \alpha + (He^\lambda - 1) \sin^2 \alpha = He^\lambda + \{ \cos^2 \alpha - \sin^2 \alpha \}.
\]
Thus \( f = \cos^2 \alpha - \sin^2 \alpha = \cos (2\alpha) \). Similarly \( b_{12} = 2 \cos \alpha \sin \alpha = \sin (2\alpha) \). Now \( \alpha \) is constant, and \( v^1 = \cos \alpha u^1 + \sin \alpha u^2 + c^1 \) and \( v^2 = -\sin \alpha u^1 + \cos \alpha u^2 + c^2 \) for some constants \( c^i \). We summarize as follows.

\textbf{4.4. Theorem.} Let \( S \) be a surface immersed in \( \mathbb{R}^3 \) with constant mean curvature \( H \), and define \( \lambda = -\frac{1}{2} \log(H^2 - K) \) on \( S_\lambda \). Let \( (U, u) \) be a connected local coordinate neighborhood such that \( U \subset S_\lambda \) and
\[
I = e^\lambda \{ du^1 \otimes du^1 + du^2 \otimes du^2 \}.
\]
Then there is a constant \( \alpha \) such that
\[
II = (He^\lambda + \cos 2\alpha)du^1 \otimes du^1 + 2 \sin 2\alpha \, du^1 du^2 + (He^\lambda - \cos 2\alpha)du^2 \otimes du^2.
\]
Let \( c^i \) be constants, \( v^1 = \cos \alpha u^1 + \sin \alpha u^2 + c^1 \) and \( v^2 = -\sin \alpha u^1 + \cos \alpha u^2 + c^2 \). Then \( v = (v^1, v^2) \) is a local coordinate on \( U \), \( \alpha \) is the angle from \( \partial/\partial u^1 \) to \( \partial/\partial v^1 \), and
\[
I = e^\lambda \{ dv^1 \otimes dv^1 + dv^2 \otimes dv^2 \} \quad \text{and} \quad II = (He^\lambda + 1)dv^1 \otimes dv^1 + (He^\lambda - 1)dv^2 \otimes dv^2.
\]

\textbf{5. Obstruction to a congruence.} The following result generalizes the fact that an isometry of small patches of a right circular cylinder is a congruence only when it preserves the direction of the axis of the cylinder.

\textbf{5.1. Theorem.} Let \( S, S' \) and \( S'' \) be connected surfaces embedded in
$R^3$ with the same constant mean curvature $H$, which are not open subsets of a plane or a sphere. Then to every isometry \( f: S \to S' \) we have a real number $\alpha(f)$, defined up to addition of an integral multiple of $\pi$, specified by the property: if $x \in S$ and $(U, u)$ is a local coordinate neighborhood of $x$ given by Theorem 3.1 then $f^* II' = \{ He^\lambda + \cos 2\alpha(f) \} du^1 \otimes du^1 + 2 \sin 2\alpha(f) du^1 du^2 + \{ He^\lambda - \cos 2\alpha(f) \} du^2 \otimes du^2$ in $U$.

$\alpha$ has the properties:

(i) $f$ extends to a rigid motion of $R^3$, if and only if $\alpha(f) \equiv 0 \mod \pi$;
(ii) $\alpha(f^{-1}) = -\alpha(f)$;
(iii) if $g: S' \to S''$ is an isometry, then $\alpha(g \cdot f) = \alpha(f) + \alpha(g)$;
(iv) two isometries $f, g: S \to S'$ differ by a rigid motion of $R^3$, if and only if $\alpha(f) \equiv \alpha(g) \mod \pi$.

Given $x \in S$ and a real number $b$, there is a neighborhood $V$ of $x$, a surface $W$ of constant mean curvature $H$, and an isometry $h: V \to W$, such that $\alpha(h) = b$.

The proof is based on the standard fact [2]:

5.2. Lemma. Let $S$ be a connected surface immersed in $R^3$ with constant mean curvature. If some point of $S$ is not umbilic, then the umbilics of $S$ are isolated.

The function $\phi$ of §4 is holomorphic, and the umbilics of $S$ in the domain of $\phi$ are just the zeroes of $\phi$.

Now the surfaces $S$, $S'$ and $S''$ have all umbilics isolated; for an all-umbilic surface is an open subset of a plane or a sphere. Thus $S$ (resp. $S'$, resp. $S''$) is arcwise connected and dense in $S$ (resp. $S'$, resp. $S''$). Let $K$, $K'$ and $K''$ denote their Gauss curvature functions and let $f: S \to S'$ be an isometry. Then $f(S) = S'$ because $K'(f(x)) = K(x)$, $S = \{ x \in S: H^2 - K(x) > 0 \}$ and $S' = \{ f(x) \in S': H^2 - K'(f(x)) > 0 \}$. Similarly, the functions $\lambda = -\frac{1}{2} \log (H^2 - K)$ on $S$ and $\lambda' = -\frac{1}{2} \log (H^2 - K')$ on $S'$ are related by $\lambda = \lambda' \cdot f$.

Let $x \in S$ and choose a connected local coordinate neighborhood $(U, u)$ according to Theorem 3.1. Then $I = e^u \{ du^1 \otimes du^1 + du^2 \otimes du^2 \}$ and $II = (He^\lambda + 1) du^1 \otimes du^1 + (He^\lambda - 1) du^2 \otimes du^2$. Let $W = f(U)$, $w^i(f(z)) = u^i(z)$ for $z \in U$; then $(W, w)$ is a connected local coordinate neighborhood of $f(x)$. $U \subset S$ implies $W \subset S'_{f*} \cap S''_{f*} \equiv I$ because $f$ is an isometry, so $f^* I' = e^w \{ dw^1 \otimes dw^1 + dw^2 \otimes dw^2 \} = e^{\lambda'} \{ d(w^1 \cdot f) \otimes d(w^1 \cdot f) + d(w^2 \cdot f) \otimes d(w^2 \cdot f) \}$; thus $I' = e^{\lambda'} \{ dw^1 \otimes dw^1 + dw^2 \otimes dw^2 \}$. Applying Theorem 4.4 to $S'$, we have a number $\alpha(f)(x)$ such that $II' = He^\lambda + \cos 2\alpha(f)(x) dw^1 \otimes dw^1 + 2 \sin 2\alpha(f)(x) dw^1 dw^2 + (He^\lambda - \cos 2\alpha(f)(x)) dw^2 \otimes dw^2$. Thus $f^* II' = (He^\lambda + \cos 2\alpha(f)(x)) du^1 \otimes du^1 + 2 \sin 2\alpha(f)(x) du^1 du^2 + (He^\lambda - \cos 2\alpha(f)(x)) du^2 \otimes du^2$. This specifies $\alpha_f(x)$ up to an integral.
multiple of $\pi$. The uniqueness part of Theorem 3.1 says that $\alpha_f(x)$ is well defined up to an integral multiple of $\pi$.

Let $C$ be the circle which is the real numbers modulo $\pi$. We have a map $\alpha_f: S_\varnothing \to C$. If $x \in S_\varnothing$ then $\alpha_f$ is constant on a neighborhood of $x$. As $S_\varnothing$ is connected, now $\alpha_f$ is constant. Let $\alpha(f)$ denote its value. We have proved the existence of a number $\alpha(f)$ defined modulo $\pi$ and specified by $f^*I' = II$ as required.

If $\alpha(f) \equiv 0 \mod \pi$ if and only if $\cos 2\alpha(f) = 1$ and $\sin 2\alpha(f) = 0$, which is equivalent to $f^*I' = I$. In that case $f: S \to S'$ is a diffeomorphism of connected surfaces in $R^3$ such that $f^*I' = I$ and $f^*II' = II$, and a classical theorem says that $f$ extends to a rigid motion of $R^3$. This proves (i).

For (iii) we have a local coordinate $v$ on $W \subset S'$ given by $(\alpha_f = \alpha(f))$

$v^1 = \cos(\alpha_f)w^1 + \sin(\alpha_f)w^2$ and $v^2 = -\sin(\alpha_f)w^1 + \cos(\alpha_f)w^2$, and $II' = (He^{\lambda'} + 1)dv^1 \otimes dv^1 + (He^{\lambda'} - 1)dv^2 \otimes dv^2$ on $W$. Let $X = g(W) \subset S''_\varnothing$ and let $x$ be the local coordinate on $X$ with $x \cdot g = v$. Define $y$ on $X$ by

$y^1 = \cos(\alpha_g)x^1 + \sin(\alpha_g)x^2$ and $y^2 = -\sin(\alpha_g)x^1 + \cos(\alpha_g)x^2$; then $II'' = (He^{\lambda''} + 1)dy^1 \otimes dy^1 + (He^{\lambda''} - 1)dy^2 \otimes dy^2$ on $X$. We compute

$u^1 = \cos(\alpha_f + \alpha_g)(y^1 \cdot g \cdot f) + \sin(\alpha_f + \alpha_g)(y^2 \cdot g \cdot f)$.

$u^2 = -\sin(\alpha_f + \alpha_g)(y^1 \cdot g \cdot f) + \cos(\alpha_f + \alpha_g)(y^2 \cdot g \cdot f)$.

Thus $\alpha_{g \cdot f} = \alpha_g + \alpha_f$.

Now (ii) and (iv) are immediate.

Let $x \in S_\varnothing$ and $b \in R$. Choose a local coordinate neighborhood $(U, u)$ of $x$ as in Theorem 3.1. $D = u(U) \subset R^2$ is the parameter domain. We define functions $g_{11}(u(z)) = g_{22}(u(z)) = e^\lambda(z)$, $g_{12} = g_{21} = 0$, $b_{11}(u(z)) = He^{\lambda(z)} + \cos(2b)$, $b_{22}(u(z)) = He^{\lambda(z)} - \cos(2b)$ and $b_{12}(u(z)) = b_{21}(u(z)) = \sin(2b)$. Then the forms

$I_0 = \sum b_{ij}(u)du^i \otimes du^j$ and $II_0 = \sum b_{ij}(u)du^i \otimes du^j$

satisfy the Mainardi-Codazzi equations (4.3). As $S$ satisfies the Gauss equation a priori, and as $I_0 = I$ and $\det (b_{ij}) = H^2e^{2\lambda} - 1$ as for $S$, now $I_0$ and $II_0$ satisfy the Gauss equation. Thus Bonnet's existence theorem says that every $m \in D$ has a neighborhood $V(m) \subset D$ which is parameter domain for a local surface $W \subset R^3$ with first and second fundamental forms $I_0$ and $II_0$. Let $w$ be the coordinate on $W$, $m = u(x)$, $V = u^{-1}(V(m))$. Then $h = u^{-1} \cdot u|_V$ is a diffeomorphism of $V$ onto $W$ such that $h^*I_0 = I_0 = I$ (so $h$ is an isometry) and $h^*II_0 = II_0$ (so $\alpha(h) = b$). q.e.d.

Theorem 5.1 requires $S_\varnothing$ to be nonempty. Thus we remark:

5.3. Complement to Theorem 5.1. Let $S$ and $S'$ be connected all-
umbilic surfaces in \( R^3 \). Then any isometry \( f: S \to S' \) extends to a rigid motion of \( R^3 \).

For \( f \) is a diffeomorphism with \( f^* I' = I \), and we need only check that \( f^* I'' = I'' \). An all-umbilic surface is an open subset of a plane or a sphere. In the first case \( II = 0 \) and \( K = 0 \). In the second case \( II = (1/r)I \) and \( K = 1/r^2 \) where \( r \) is the radius of the sphere. As \( f \) preserves Gauss curvature, now \( f^* I'' = I'' \). q.e.d.

6. The Gauss Equation. The Gauss equation for a surface \( S \) with first fundamental form \( e^\lambda \left\{ du^1 \otimes du^1 + du^2 \otimes du^2 \right\} \) says that Gauss curvature is given by

\[
(6.1) \quad K = \frac{1}{2} e^{-\lambda} \Delta \lambda, \quad \Delta = \frac{\partial^2}{\partial u^1 \partial u^1} + \frac{\partial^2}{\partial u^2 \partial u^2}.
\]

Now suppose that \( S \) has constant mean curvature \( H \) and that our local coordinate neighborhood \( (U, u) \) is given by Theorem 3.1. Then \( K = H^2 - e^{-2\lambda} \), so (6.1) becomes a nonlinear elliptic equation

\[
(6.2) \quad \Delta \lambda = 2(e^{-\lambda} - H^2 e^{\lambda}).
\]

We view this as an equation\(^6\) for \( K = H^2 - e^{-2\lambda} \).

We regard (6.2) as a boundary value problem on a disc \( D \) of radius \( r > 0 \) in \( R^2 \). Let \( b \) be a continuous function on the boundary \( \partial D \) of the disc; we look for a solution \( \lambda(u^1, u^2) \) to (6.2) on \( D \), continuous on the closure and with values \( b \) on \( \partial D \). Let \( h(u^1, u^2) \) be the harmonic function on \( D \) with boundary values \( b \). Then we write (6.2) in the form (this defines \( F \))

\[
(6.3) \quad \Delta \eta = 2(e^{-(\pi \lambda + \pi)} - H^2 e^{\pi + \lambda}) \equiv F(\eta), \quad \eta = \lambda - h,
\]

and we want a solution \( \eta \) on \( D \) vanishing on \( \partial D \).

Following Courant-Hilbert ([1, Appendix to Chapter 4]), such solutions \( \eta \) exist provided that certain bounds \( c, m \) satisfy \( (r + r^2)cm \leq 1/4 \). Here \( c \) and \( m \) are defined as follows. Let \( C^2(D) \) denote the set of all continuous functions on the closure of \( D \) which are twice con-\(^5\)

\(^5\) We remark that this has an interesting expression in complex notation. There \( z = u^1 + (-1)^{1/2} u^2 \) is the variable, so \( ds = du^1 + (-1)^{1/2} du^2 \) and \( \partial \bar{z} = du^1 - (-1)^{1/2} du^2 \), and the vector fields dual to these forms are \( \partial / \partial z = (1/2) \{ \partial / \partial u^1 - (-1)^{1/2} \partial / \partial u^2 \} \) and \( \partial / \partial \bar{z} = (1/2) \{ \partial / \partial u^1 + (-1)^{1/2} \partial / \partial u^2 \} \). The exterior derivative \( d = d' + d'' \) where by definition \( d'(f) = (\partial f / \partial z) dz \) and \( d''(f) = (\partial f / \partial \bar{z}) d\bar{z} \) on functions. In particular \( d'd''f = \frac{1}{2} \Delta f dz \wedge d\bar{z} \). The element of area on \( S \) is given by \( dA = e^\lambda du^1 \wedge du^2 = (1/2)(-1)^{1/2} e^\lambda \cdot ds \wedge d\bar{z} \). Thus (6.1) can be written in coordinate free form \( K dA = -(1)^{1/2} d'd''\lambda \).

\(^6\) Writing it out in terms of \( K \), one obtains \( \Delta K = \left\{ (\partial K / \partial u^1)^2 + (\partial K / \partial u^2)^2 \right\} (H^2 - K) + 4(H^2 - K)^{3/2} - 4H^2 (H^2 - K)^{1/2}, K < H^2 \), which is more difficult to study than is (6.2).
tinuously differentiable in \( D \). Then \( c \) is specified by
\[
\max \left| \frac{\partial f}{\partial u^i} \right| \leq c r \text{ l.u.b.} \quad |\Delta f| \quad \text{for} \quad f \in C^2(D),
\]
and \( c \) is independent of choice of \( r \) or \( D \). Define the norm
\[
\|f\| = \max |f| + \sum \max \left| \frac{\partial f}{\partial u^i} \right|
\]
on \( C^2(D) \). Then \( m \) is any common bound for all \( |F(f)| \) and all \( |dF/df| \) with \( \|f\| \leq 1 \). A glance at the form of \( F \) in (6.3) shows now that \( m \) is any common bound for \( 2(e^{-1-h} + H^2 e^{+h}) \) and \( 2(e^{1-h} + H^2 e^{h-1}) \).

As \( h \) achieves its maximum on \( \partial D \), now we may take
\[
(6.4) \quad m = \max \{2(e^{-1-\beta} + H^2 e^{1+\beta}), \quad 2(e^{1-\beta} + H^2 e^{\beta-1})\}, \quad \beta = \max |b|.
\]
In summary, and using the fact that solutions to elliptic equations are analytic,

6.5. Theorem. Let \( D \) be a disc of radius \( r > 0 \) in \( \mathbb{R}^2 \), let \( b \) be a continuous function on \( \partial D \), and let \( B = \max_{\partial D} |b| \). If
\[
(6.6) \quad (r + r^2)^{-1} \geq 8c \cdot \max \{e^{-1-\beta} + H^2 e^{1+\beta}, \quad e^{1-\beta} + H^2 e^{\beta-1}\},
\]
then there exists a continuous function \( \lambda \) on the closure of \( D \), real analytic on \( D \), which satisfies (6.2) and has values \( b \) on \( \partial D \).

The usual uniqueness condition for an equation \( \Delta f = A(f, u) \) for given boundary values is \( \partial A/\partial f \geq 0 \). But in our case (6.2) we always have \( \partial A/\partial f < 0 \).

We can now describe the construction of umbilic-free local surfaces of constant mean curvature. Such a surface is specified up to congruence by its first and second fundamental forms, and the condition for two candidates
\[
I = \sum g_{ij} du^i \otimes du^j \quad \text{and} \quad II = \sum b_{ij} du^i \otimes du^j
\]
to give a surface, is that the \( g_{ij} \) and \( b_{ij} \) satisfy the Mainardi-Codazzi equations and the Gauss equation.

If \( H \) is the mean curvature of the desired surfaces, then Theorem 3.1 allows us to take \( I \) and \( II \) in the forms
\[
I = e^\lambda \{du^1 \otimes du^1 + du^2 \otimes du^2\} \quad \text{and} \quad II = \left\{ He^\lambda + \cos(2t) \right\} du^1 \otimes du^1 + 2 \sin(2t) du^1 du^2 \]
\[
+ \left\{ He^\lambda - \cos(2t) \right\} du^2 \otimes du^2
\]
for \( t=0 \). For any constant \( t \), the Mainardi-Codazzi equations (4.3) are satisfied for \( I \) and \( II_t \), and the Gauss equation (6.2) is

\[
\Delta \lambda = 2(e^{-\lambda} - H^2 e^{\lambda}), \quad K = H^2 - e^{-2\lambda}.
\]

Theorem 6.5 gives the existence of many solutions. Given a local solution \( \lambda \), there corresponds a well defined congruence class of local surfaces \( S_{\lambda,t} \) with \( I \) as specified and \( II = II_t \); Theorem 5.1 shows that the natural map\(^7\) \( f_{*,t} : S_{\lambda,s} \rightarrow S_{\lambda,t} \) is an isometry and \( \alpha(f_{*,t}) = t - s \).

**References**


\(^7\) \( f_{*,t} \) is given by \( u^t(f_{*,t}(x)) = u^t(x) \).