EXOTIC METRICS ON IMMERSED SURFACES

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To A. A. Albert on his sixtieth birthday

1. **Introduction.** In a series of articles ([1], [2], [3], [4]), Tilla Klotz studied immersed surfaces by examining riemannian metrics constructed as linear combinations $s_1\boldsymbol{I}+s_2\boldsymbol{II}+s_3\boldsymbol{III}$ of the fundamental forms of the immersion. Here we study the Gauss curvature of pseudo-riemannian metrics $s_1\boldsymbol{I}+s_2\boldsymbol{II}+s_3\boldsymbol{III}$. If s_i are constants, and if mean and Gauss curvature satisfy $s_1+s_2H+s_3K=0$, then we show that the metric $s_1\boldsymbol{I}+s_2\boldsymbol{II}+s_3\boldsymbol{III}$ is flat where it is nondegenerate. In particular we prove that $\boldsymbol{II}$ is a flat Lorentz metric on the complement of the umbilic set of a minimal surface.

2. **The structure equations.** Let S be a pseudo-riemannian 2-manifold with metric $d\nu^2$. This means that S is a 2-dimensional differentiable manifold and $d\nu^2$ is a smooth[2] family of nondegenerate inner products on the tangent planes of S. If the inner products are all positive definite, then S is a *riemannian* 2-manifold. Given $x\in S$ we write S_x for the tangent plane at x. If $X\in S_x$, then $d\nu_x^2$ denotes the inner product on S_x, and $\|X\|^2$ denotes $d\nu_x^2(X, X)$. Let $\{X_1, X_2\}$ be a moving frame on an open set $U\subset S$. This means that the X_i are smooth tangent vector fields on U which are linearly independent at every point. Then the "dual co-frame" is the pair $\{\theta^1, \theta^2\}$ of linear differential forms on U defined by $\theta^i(a^1X_1+a^2X_2)=a^i$; the metric has local expression $d\nu^2=\sum_{i,j} g_{ij}\theta^i\otimes\theta^j$ where the "coefficients" are the functions $g_{ij}(x)=d\nu_x^2(X_{ix}, X_{jx})$.

The moving frame $\{X_1, X_2\}$ is called *orthonormal* if $g_{ij}=\pm\delta_{ij}$. This means that $\|X_i\|^2=e_i=\pm 1$ and $d\nu^2(X_1, X_2)\equiv 0$, and it says that $d\nu^2=\sum_i e_i\theta^i\otimes\theta^i$. An obvious modification of the Gram-Schmidt process constructs an orthonormal moving frame from an arbitrary moving frame.

Let $\{X_1, X_2\}$ be an orthonormal moving frame on an open set $U\subset S$. Then the dual coframe $\{\theta^1, \theta^2\}$ is also called "orthonormal," and we have the signs $e_i=\|X_i\|^2=\pm 1$. New forms are defined on U by

$$\theta_i = e_i\theta^i, \tag{2.1}$$

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[2] "Smooth" means "sufficiently differentiable." The reader can easily keep count.

and we define functions $a_i(x)$ on U by

$$d\theta_i = a_i\theta_1 \wedge \theta_2. \tag{2.2}$$

Now the *connection forms* are the linear differential forms ω_{ij} defined on U by

$$-\omega_{12} = +\omega_{21} = e_2a_1\theta_1 + e_1a_2\theta_2, \qquad \omega_{11} = 0 = \omega_{22}. \tag{2.3}$$

They are characterized by the *structure equations*

$$d\theta^i = \sum_j \theta^j \wedge \omega^i_j,\ \omega^i_j = e_i\omega_{ij}, \qquad \omega_{ij} + \omega_{ji} = 0. \tag{2.4}$$

The connection forms ω_{ij} are specified by ω_{12}, and the structure equations can be written

$$d\theta_1 = e_2\theta_2 \wedge \omega_{12} \quad \text{and} \quad d\theta_2 = \omega_{12} \wedge e_1\theta_1. \tag{2.5}$$

Gauss curvature is a function $K(x)$ on S. In the notation above, it is defined on the open set $U \subset S$ by the equation

$$d\omega_{12} = K\theta_1 \wedge \theta_2. \tag{2.6}$$

One can check [**5**, Theorem 2.2.1] that this defines K independently of the choice of orthonormal moving frame, and we will note in Lemma 4.5 that it is equivalent to the classical definition for surfaces immersed in $\mathbf{R}^3$.

3. **Metrics associated to quadratic differential forms.** S denotes a fixed riemannian 2-manifold with (positive definite) metric dv^2, and we study the geometry of a smooth family $\Phi = \{\Phi_x\}_{x\in S}$ of inner products on the tangent planes of S. Eventually S will be an immersed surface and Φ will be a linear combination of its fundamental forms.

If $x \in S$, then we diagonalize Φ_x relative to dv_x^2; so Φ_x has matrix

$$\begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix}$$

in some orthonormal basis of S_x. Generalizing the case where Φ is the second fundamental form of an immersion, we define

$f_i(x)$: the *principle* Φ*-curvatures* at x,
$H_\Phi(x)$: the *mean* Φ*-curvature* $\frac{1}{2}\{f_1(x)+f_2(x)\}$,
$K_\Phi(x)$: the *Gauss* Φ*-curvature* $f_1(x)\cdot f_2(x)$,
Φ*-elliptic point*: point x with $K_\Phi(x) > 0$,
Φ*-parabolic point*: point x with $K_\Phi(x) = 0$,
Φ*-hyperbolic point*: point x with $K_\Phi(x) < 0$,
Φ*-umbilic*: point x with $f_1(x) = f_2(x)$.

The set of all Φ-umbilics, and the set of all Φ-parabolic points, are closed in S. Thus the set

$$(3.1) \qquad S_\Phi = \{x \in S: x \text{ is neither } \Phi\text{-umbilic nor } \Phi\text{-parabolic}\}$$

is open in S, and Φ restricts to a pseudo-riemannian metric ds^2_Φ on S_Φ. We will study the pseudo-riemannian manifold S_Φ with metric ds^2_Φ.

3.2. THEOREM. *Let $x \in S_\Phi$. Then x has an open neighborhood $U \subset S_\Phi$ which carries linear differential forms θ^i such that*

$$(3.3) \quad dv^2 = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 \quad \textit{and} \quad ds_\Phi{}^2 = f_1\theta^1 \otimes \theta^1 + f_2\theta^2 \otimes \theta^2 \textit{ in } U.$$

Define functions $r_i > 0$ in U by $r_i^2 = \epsilon_i f_i$, $\epsilon_i = \pm 1$, so the $\phi^i = r_i\theta^i$ are ds^2_Φ-orthonormal. Let ω_{12} and β_{12} be the respective connection forms for dv^2 relative to $\{\theta^1, \theta^2\}$ and ds^2_Φ relative to $\{\phi^1, \phi^2\}$. Then

$$(3.4) \qquad \beta_{12} = -\ |f_1f_2|^{-1/2}\{(f_1a_1 - \tfrac{1}{2}f_{1;2})\theta^1 + (f_2a_2 + \tfrac{1}{2}f_{2;1})\theta^2\}$$

where $df_i = \sum_j f_{i;j}\theta^j$ and $\omega_{12} = -(a_1\theta^1 + a_2\theta^2)$.

PROOF. For the first assertion we choose an orthonormal frame $\{X_1, X_2\}$ on a neighborhood U of x such that Φ_z is diagonal relative to $\{X_{1z}, X_{2z}\}$ for every $z \in U$. Then (3.3) follows with $\{\theta^1, \theta^2\}$ dual to $\{X_1, X_2\}$ because $f_1 \neq f_2$ throughout S_Φ.

Define a_i and b_i by $d\phi_i = b_i\phi_1 \wedge \phi_2$ and $d\theta_i = a_i\theta_1 \wedge \theta_2$. Then (2.3) says

$$\omega_{12} = -\ (a_1\theta_1 + a_2\theta_2) \quad \text{and} \quad \beta_{12} = -\ (\epsilon_2 b_1\phi_1 + \epsilon_1 b_2\phi_2).$$

Now compute

$$\begin{aligned} d\phi_i &= d(\epsilon_i\phi^i) = \epsilon_i d\phi^i = \epsilon_i(dr_i \wedge \theta^i + r_i d\theta^i), \\ \epsilon_i dr_i \wedge \theta^i &= \epsilon_i r_i d(\log r_i) \wedge \theta^i = d(\log r_i) \wedge \phi_i \\ &= \tfrac{1}{2}d(\log r_i^2) \wedge \phi_i = \tfrac{1}{2}r_i^{-2}d(\epsilon_i f_i) \wedge \phi_i \\ &= \tfrac{1}{2}f_i^{-1}(f_{i;1}\theta^1 + f_{i;2}\theta^2) \wedge \phi_i = \frac{1}{2f_i}\left\{\frac{\epsilon_1}{r_1}f_{i;1}\phi_1 + \frac{\epsilon_2}{r_2}f_{i;2}\phi_2\right\} \wedge \phi_i, \end{aligned}$$

$$\epsilon_i r_i d\theta^i = \epsilon_i r_i d\theta_i = \epsilon_i r_i a_i \theta_1 \wedge \theta_2 = \epsilon_i\epsilon_1\epsilon_2 \frac{r_i a_i}{r_1 r_2}\phi_1 \wedge \phi_2.$$

Thus

$$b_1 = \frac{\epsilon_2}{r_2}\left\{a_1 - \frac{f_{1;2}}{2f_1}\right\} \quad \text{and} \quad b_2 = \frac{\epsilon_1}{r_1}\left\{a_2 + \frac{f_{2;1}}{2f_2}\right\},$$

so

$$\beta_{12} = -\left\{\frac{1}{r_2}\left(a_1 - \frac{f_{1;2}}{2f_1}\right)\phi_1 + \frac{1}{r_1}\left(a_2 + \frac{f_{2;1}}{2f_2}\right)\phi_2\right\}$$
$$= -\left\{\frac{\epsilon_1 r_1}{r_2}\left(\frac{2f_1a_1 - f_{1;2}}{2f_1}\right)\theta^1 + \frac{\epsilon_2 r_2}{r_1}\left(\frac{2f_2a_2 + f_{2;1}}{2f_2}\right)\theta^2\right\}$$
$$= -\frac{1}{r_1r_2}\left\{\left(f_1a_1 - \frac{1}{2}f_{1;2}\right)\theta^1 + \left(f_2a_2 + \frac{1}{2}f_{2;1}\right)\theta^2\right\}.$$

The assertion follows from $f_i = \epsilon_i r_i^2$. q.e.d.

A pseudo-riemannian 2-manifold is called *flat* if its Gauss curvature is identically zero.

3.5. Corollary. *If* $2a_1f_1 = f_{1;2}$ *and* $2a_2f_2 + f_{2;1} = 0$, *then* S_Φ *is flat.*

4. **Metrics defined by immersions.** An *immersed surface* is a pair (S, ν) where S is a two dimensional differentiable manifold and $\nu: S \to \boldsymbol{R}^3$ is a differentiable map with nowhere vanishing Jacobian determinant. Thus $\nu(S)$ is a smooth surface in $\boldsymbol{R}^3$ which has no singularities but may have self intersections. The inner products on the tangent planes of $\nu(S)$ define a riemannian metric $d\nu^2 \equiv d\nu \cdot d\nu$ on S, and we view S as a riemannian 2-manifold with that metric.

Let ξ be a smooth choice of unit normal to $\nu(S)$, defined over an open set $U \subset S$. Then we recall the classical quadratic differential forms

$$\begin{aligned} I &= d\nu \cdot d\nu, \textit{ first fundamental form};\\ II &= d\nu \cdot d\xi, \textit{ second fundamental form};\\ III &= d\xi \cdot d\xi, \textit{ third fundamental form}. \end{aligned}$$

Of course II is only defined up to sign unless we have an orientation on S. Principle, mean and Gauss curvature of (S, ν), and elliptic, parabolic, hyperbolic and umbilic points, are classically defined as in §3 for the case $\Phi = II$.

Let $\{v_1, v_2, v_3\}$ be a Darboux frame on an open set $U \subset S$. This means that $\{v_1, v_2\}$ is a moving orthonormal frame and v_3 is a smooth unit normal. Viewing ν as position vector, now

(4.1) $\quad d\nu = \theta^1 v_1 + \theta^2 v_2$ where $\{\theta^1, \theta^2\}$ is dual to $\{v_1, v_2\}$.

We define forms ω_j^i on S by

$$dv_j = \sum_{i=1}^{3} \omega_j^i v_i. \tag{4.2}$$

Writing out $0 = d(d\nu)$ and $0 = d(dv_j)$, one has

$$(4.3) \quad d\overset{i}{\theta} = \sum_{j=1}^{2} \overset{j}{\theta} \wedge \overset{i}{\omega_j}, \quad 0 = \sum_{j=1}^{2} \overset{j}{\theta} \wedge \overset{3}{\omega_j}, \quad d\overset{i}{\omega_j} = \sum_{k=1}^{3} \overset{k}{\omega_j} \wedge \overset{i}{\omega_k}.$$

As $\|v_i\|^2 = 1$ now $\omega_{ij} = \omega_j^i$, and differentiation of $v_i \cdot v_j = \delta_{ij}$ gives $\omega_{ij} + \omega_{ji} = 0$. Now (4.3) yields

$$(4.4) \qquad d\theta_1 = \theta_2 \wedge \omega_{12}, \quad d\theta_2 = \omega_{12} \wedge \theta_1, \quad d\omega_{12} = \omega_{32} \wedge \omega_{13};$$

and (2.4) shows that ω_{12} is the connection form.

4.5. LEMMA. *Let k_i be the principle curvatures on (S, ν) and suppose that $x \in S$ is not an umbilic. Then x has an open neighborhood $U \subset S$ which carries a Darboux frame $\{v_1, v_2, v_3\}$ in which*

$$I = \sum_{i=1}^{2} \overset{i}{\theta} \otimes \overset{i}{\theta}, \quad II = \sum_{i=1}^{2} k_i \overset{i}{\theta} \otimes \overset{i}{\theta} \quad \text{and} \quad III = \sum_{i=1}^{2} k_i^2 \overset{i}{\theta} \otimes \overset{i}{\theta}.$$

In this frame $\omega_{3i} = k_i\theta_i$, so Gauss curvature $K = k_1k_2$.

The result is standard. x has a neighborhood U_1 of nonumbilics, which contains a smaller neighborhood U carrying a smooth unit normal v_3. Order the k_i on U with $k_1 > k_2$, and let $\{v_1, v_2\}$ give the corresponding principle directions. That constructs the Darboux frame, and I and II have the required form. It follows that $dv_3 = \sum_{i=1}^2 k_i\theta^i v_i$, so $\omega_{3i} = k_i\theta_i$ and III has the required form. Now the structure equations give

$$K\theta_1 \wedge \theta_2 = d\omega_{12} = \omega_{32} \wedge \omega_{13} = \omega_{31} \wedge \omega_{32} = k_1k_2\theta_1 \wedge \theta_2$$

so $K = k_1k_2$. q.e.d.

4.6. THEOREM. *Let (S, ν) be an immersed surface with principle curvatures k_i, mean curvature $H = \frac{1}{2}(k_1 + k_2)$ and Gauss curvature $K = k_1k_2$. Let s_i be differentiable functions on S and define Φ to be the quadratic differential form $s_1I + s_2II + s_3III$. Choose a Darboux frame $\{v_1, v_2, v_3\}$ satisfying Lemma 4.5 and define functions a_i, $k_{i;j}$ and $s_{i;j}$ by $d\theta^i = a_i\theta^1 \wedge \theta^2$, $dk_i = k_{i;1}\theta^1 + k_{i;2}\theta^2$ and $ds_i = s_{i;1}\theta^1 + s_{i;2}\theta^2$. If*

$$2a_1(s_1 + s_2H + s_3K) - (s_{1;2} + s_{2;2}k_1 + s_{3;2}k_1^2) = 0$$

and

$$2a_2(s_1 + s_2H + s_3K) + (s_{1;1} + s_{2;1}k_2 + s_{3;1}k_2^2) = 0,$$

then S_Φ is flat.

PROOF. Following Corollary 3.5, we find the condition for $2a_1f_1$

$-f_{1;2}=0=2a_2f_2+f_{2;1}$. Here Lemma 4.5 shows that $f_i=s_1+s_2k_i+s_3k_i^2$; thus

$$f_{i;j} = s_{1;j} + s_{2;j}k_i + s_{3;j}k_i^2 + s_2k_{i;j} + 2s_3k_ik_{i;j}. \tag{4.7}$$

To evaluate this we compute

$$\begin{aligned}(a_1k_1 - k_{1;2})\theta_1 \wedge \theta_2 = dk_1 \wedge \theta_1 + k_1d\theta_1 = d(k_1\theta_1) = d\omega_{13} = \omega_{23} \wedge \omega_{12}\\ = k_2\theta_2 \wedge \omega_{12} = a_1k_2\theta_1 \wedge \theta_2\end{aligned}$$

and similarly

$$(a_2k_2 + k_{2;1})\theta_1 \wedge \theta_2 = d(k_2\theta_2) = a_2k_1\theta_1 \wedge \theta_2,$$

so

$$k_{1;2} = a_1(k_1 - k_2) \quad \text{and} \quad k_{2;1} = a_2(k_1 - k_2). \tag{4.8}$$

Combining (4.7) and (4.8) we have

$$f_{1;2} = s_{1;2} + s_{2;2}k_1 + s_{3;2}k_1^2 + (s_2 + 2s_3k_1)a_1(k_1 - k_2),$$

$$f_{2;1} = s_{1;1} + s_{2;1}k_2 + s_{3;1}k_2^2 + (s_2 + 2s_3k_2)a_2(k_1 - k_2).$$

It follows that

$$2a_1f_1 - f_{1;2} = 2a_1(s_1 + s_2H + s_3K) - (s_{1;2} + s_{2;2}k_1 + s_{3;2}k_1^2),$$

$$2a_2f_2 + f_{2;1} = 2a_2(s_1 + s_2H + s_3K) + (s_{1;1} + s_{2;1}k_2 + s_{3;1}k_2^2).$$

Now our assertion follows from Corollary 3.5. q.e.d.

The most tractable special case is when the s_i are constants. Then $s_{i;j}=0$ and Theorem 4.6 simplifies to:

4.9. Theorem. *Let (S, ν) be an immersed surface with mean curvature H and Gauss curvature K. Let s_i be constants and suppose*

$$s_1 + s_2H + s_3K \equiv 0 \text{ on } S. \tag{4.10}$$

Then $S_{s_1I+s_2II+s_3III}$ is flat.

Condition (4.10) specifies an interesting class of Weingarten surfaces, including the surfaces of constant mean curvature and the surfaces of constant Gauss curvature. For those surfaces we have:

4.11. Corollary. *Let (S, ν) be an immersed surface of mean curvature H and Gauss curvature K, and let b be a nonzero real number.*

1. *$S_{-bHI+bII}$ is the set of nonumbilic points of S. If H is constant then $S_{-bHI+bII}$ is a flat Lorentz*[3] *2-manifold.*

2. *$S_{-bKI+bIII}$ is the set of nonumbilic nonparabolic points of S where $H \neq 0$. If K is constant then $S_{-bI+bIII}$ is flat.*

Recall that *minimal surface* means an immersed surface with mean curvature $H \equiv 0$.

4.12. Corollary. *If (S, ν) is an immersed minimal surface, then S_{II} is a flat Lorentz 2-manifold.*

In the context of Corollaries 4.11 and 4.12, we note that one combines (3.4) and (4.8) to see that S_{II} has connection form

$$\beta_{12} = \frac{H}{|K|^{1/2}}\omega_{12}. \tag{4.13}$$

[3] Here, Lorentz signature means that the pseudo-reimannian metric is neither positive definite nor negative definite.

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