Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces

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1. Introduction. The compact simply connected complex homogeneous contact manifolds M were studied by W. Boothby [1, 2]. He showed that the manifolds M are in one-one correspondence with the compact centerless Lie groups G_u, the correspondence being given by $M = G_u/L$ where L is a certain subgroup unique up to conjugacy. L is a local direct product $L_1 \cdot T$ where T is a circle group which defines the complex structure.

The compact simply connected quaternionic symmetric spaces S can be found in É. Cartan's list [4] of all irreducible Riemannian symmetric spaces by use of our Theorem 3.7. An examination of the list shows that the spaces S are in one-one correspondence with the compact centerless Lie groups G_u, the correspondence being given by $S = G_u/K$ where K is a certain subgroup unique up to conjugacy. K is a local direct product $K_1 \cdot A_1$ where $A_1 \cong Sp(1)$, multiplicative group of unit quaternions.

A further look shows that K_1 and L_1 are locally isomorphic. This suggests the possibility that $K_1 = L_1$, that $T \subset A_1$, and that there is a fibering $M \to S$ given by $G_u/L \to G_u/K$ with fibre $K/L = A_1/T$ isometric to the Riemann sphere. In this paper we will give *a priori* proofs of the suggested relations between manifolds M and spaces S. We also extend these relations to the noncompact case. As a preliminary, we develop the theory of quaternionic structures on Riemannian manifolds.

2. Preliminaries on complex contact structure. Let M be a complex manifold of odd complex dimension, $\dim_C M = 2n + 1$. (All manifolds will be assumed Hausdorff, separable and connected.) A *complex contact structure* on M is a family $\{(U_i, \omega_i)\}$ where

(i) $\{U_i\}$ is an open covering of M,

(ii) ω_i is a holomorphic 1-form on U_i such that $\omega_i \wedge (d\omega_i)^n \neq 0$ at every point of U_i,

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(iii) $\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$ where f_{ij} is a holomorphic function on $U_i \cap U_j$, and

(iv) the family is maximal for these properties.

The ω_i are called the *(local) contact forms* of the structure. A *complex contact manifold* is a complex manifold together with a complex contact structure.

S. Kobayashi [5] has given a line bundle formulation of complex contact structure. Let $\{(U_i, \omega_i)\}$ be a complex contact structure on M, $\omega_i = f_{ij}\omega_j$ on $U_i \cap U_j$. The f_{ji}^{-1} are the transition functions of a complex line bundle $\tilde{\pi} : \tilde{B} \to M$. Let C^* denote the multiplicative group of nonzero complex numbers, so $\pi : B \to M$ is the associated principle C^*-bundle where B is the complement of the zero cross section of $\tilde{B}$ and $\pi = \tilde{\pi}|_B$. Then $\pi^*\omega_i$ is a holomorphic 1-form on $\pi^{-1}(U_i)$ and $\pi^*\omega_i = \pi^*\omega_j$ on $\pi^{-1}(U_i \cap U_j)$; in other words the $\pi^*\omega_i$ are restrictions of a holomorphic 1-form ω on B. Furthermore

(2.1) $(d\omega)^{n+1} \neq 0$ *at every point of* B,

(2.2) *the restriction of* ω *to the fibres is zero,*

and

(2.3) $\tau_z^*\omega = z\omega$ *where* τ_z *is the action of* $z \in C^*$ *on* B.

Conversely, let $\pi : B \to M$ be a holomorphic principle C^*-bundle, $\dim_C M = 2n + 1$, where B has a holomorphic 1-form ω which satisfies (2.1–2.3). If

$$\varphi_i : U_i \times C^* \to \pi^{-1}(U_i)$$

is the maximal family of coordinate functions of the bundle, then we define local sections $s_i : x \to \varphi_i(x, 1)$ over U_i and forms $\omega_i = s_i^*\omega$ on U_i; then $\{(U_i, \omega_i)\}$ is a complex contact structure on M.

Now M has a globally defined contact form if and only if $\pi : B \to M$ has a global section. This is equivalent to triviality of $\pi : B \to M$, so a theorem of Kobayashi [5] gives us:

(2.4) *Let M be a compact complex manifold. Then a complex contact structure on M has a globally defined contact form if and only if the first Chern class $c_1(M) = 0$.*

Let M be a complex contact manifold. Then a complex analytic homeomorphism $g : M \to M$ is a *complex contact automorphism* if it preserves the complex contact structure, *i.e.*, if $g^*\omega_i$ is a local contact form whenever ω_i is a local contact form. The complex contact automorphisms of M are just the transformations induced by bundle automorphisms of B which preserve ω. M is a *homogeneous* complex contact manifold if the group of complex contact automorphisms is transitive on the points of M.

A structure theory for compact simply connected homogeneous complex contact manifolds is given in Boothby's papers [1] and [2]. Combining these results with a theorem on parabolic subgroups, one has the following slightly sharper structure theory.

Let $\mathcal{G}$ be a complex simple Lie algebra, choose a Cartan subalgebra $\mathcal{H}$, and let

$$\mathcal{G} = \mathcal{H} + \sum_{\alpha \varepsilon \psi} \mathcal{G}_\alpha$$

be the root decomposition where Ψ is the set of all roots of $\mathcal{G}$ relative to $\mathcal{H}$. The Killing form is denoted $\langle\ ,\ \rangle$. Given $\alpha \varepsilon \Psi$, $H_\alpha \varepsilon \mathcal{H}$ denotes the element defined by: $\langle H_\alpha , H\rangle = \alpha(H)$ for all $H \varepsilon \mathcal{H}$. Choose a lexicographic ordering on the roots and let ρ be the greatest root. Now define

$$\mathcal{U} = \mathcal{H} + \sum_{\langle \alpha , \rho\rangle \geqq 0} \mathcal{G}_\alpha ,$$

which can be proved to be the sum of the non-negative eigenspaces of $ad(H_\rho)$. $\mathcal{U}$ is a subalgebra of $\mathcal{G}$; let U be the corresponding analytic subgroup of the adjoint group G of $\mathcal{G}$. Define

$$\mathcal{U}_1 = \{X \varepsilon U : \langle X, H_\rho\rangle = 0\};$$

$\mathcal{U}_1$ is a subalgebra of $\mathcal{G}$, equal to the centralizer in $\mathcal{U}$ of any nonzero element $E_\rho \varepsilon \mathcal{G}_\rho$, and the corresponding subgroup U_1 satisfies $U/U_1 \cong C^*$. Let $B = G/U_1$ and $M = G/U$, and define $\pi : B \to M$ by $\pi(gU_1) = gU$; then $\pi : B \to M$ is a holomorphic principle C^*-bundle. The projection $p : G \to B$ given by $g \to gU_1$ has differential which maps

$$\mathcal{B} = \{H_\rho\} \oplus \sum_{\langle \alpha , \rho\rangle < 0} \mathcal{G}_\alpha$$

isomorphically onto the tangent space to B at $p(1)$. $X \to \langle E_\rho , X\rangle$ is an $ad(U_1)$-invariant linear form on $\mathcal{B}$, so it defines a G-invariant holomorphic 1-form ω on B. Now ω satisfies (2.1–2.3) where n is given by $\dim_C M = 2n + 1$, so ω and $\pi : B \to M$ give M the structure of a homogeneous complex contact manifold. Furthermore M is a compact simply connected Kähler manifold of restricted type (Hodge metric, thus algebraic) because U is a parabolic subgroup of G. Finally, we have

(2.5) *Let $A(M)$ be the group of all complex contact automorphisms of M, endowed with the compact-open topology, and let $A_0(M)$ be the identity component. Then $G = A_0(M)$.*

Proof of (2.5). As observed by Boothby [1, (3.2)], compactness of M implies that $A_0(M)$ is a complex Lie group acting holomorphically on M. An argument of H.-C. Wang [10] implies that a maximal semisimple analytic subgroup G' of $A_0(M)$ is transitive on M, and $M = G'/U'$ for some parabolic subgroup U'. Let $A = A_0(M)$, express M as a coset space A/E, with $U' \subset E$, and let N be the normalizer of E in A. This gives an A-equivariant fibering $M \to A/N$, so $A/N = G'/U''$ where $U'' = N \cap G'$ contains U'. If $g \varepsilon U''$ then g normalizes both E and G', so g normalizes $U' = E \cap G'$; then $g \varepsilon U'$ because a parabolic subgroup is its own normalizer. Now $U'' = U'$, so $M \to A/N$ is one-one and $N = E$. An argument of J. Tits [9, p. 116] shows that the radical R of A has a fixed point on A/N; this proves that R acts trivially on M. It follows that A is semisimple, $A = G'$, and now the proof of [1, (6.3)] proves A simple. As the simple subgroup G is transitive on $M = A/U'$ and as U' is parabolic in A, we must have $G = A$.

q.e.d.

Conversely, if M is a compact simply connected homogeneous complex contact manifold, then M is a manifold G/U described above.

Let M be a compact simply connected homogeneous complex contact manifold. Then we have the description $M = G/U$ in terms of complex Lie groups. Let G_u be a maximal compact subgroup of G; in other words, its Lie algebra $\mathcal{G}_u$ is a compact real form of $\mathcal{G}$. Then G_u is transitive on M. We may assume G_u chosen (just replace by a conjugate if necessary) so that $\mathcal{H}_u = \mathcal{G}_u \cap \mathcal{H}$ is a real form of $\mathcal{H}$; then

$$\mathcal{G}_u = \mathcal{H}_u + \sum_{\alpha>0} \mathcal{G}_u \cap (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha}).$$

We decompose $\mathfrak{U} = \mathfrak{U}_0 + \mathfrak{U}'$ into the sum of the zero and positive eigenspaces; then

$$\mathfrak{U}_0 = \mathcal{H} + \sum_{\langle \alpha, \rho \rangle = 0} \mathcal{G}_\alpha \quad \text{and} \quad \mathfrak{U}' = \sum_{\langle \alpha, \rho \rangle > 0} \mathcal{G}_\alpha .$$

Define

$$\mathcal{L} = \mathcal{H}_u + \sum_{\substack{\alpha>0 \\ \langle \alpha, \rho \rangle = 0}} \mathcal{G}_u \cap (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha}), \tag{2.6}$$

and

$$\mathcal{L}_1 = \{H \in \mathcal{H}_u : \rho(H) = 0\} \oplus \sum_{\substack{\alpha>0 \\ \langle \alpha, \rho \rangle = 0}} \mathcal{G}_u \cap (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha}).$$

Then $\mathcal{L} = \mathfrak{U} \cap \mathcal{G}_u$, $\mathcal{L}$ is a real form of $\mathfrak{U}_0$, $\mathcal{L}$ is the centralizer of H_ρ in $\mathcal{G}_u$, $\mathcal{L}_1 = \mathfrak{U}_1 \cap \mathcal{G}_u$, $\mathcal{L}_1$ is a real form of $\mathfrak{U}_1 \cap \mathfrak{U}_0$, and $\mathcal{L} = \mathcal{L}_1 \oplus \{iH_\rho\}$. If L and L_1 are the analytic subgroups of G_u with Lie algebras $\mathcal{L}$ and $\mathcal{L}_1$, then $L = G_u \cap U$ and $L_1 = G_u \cap U_1$. Now $M = G_u/L$, compact presentation. Furthermore G_u/L_1 is a circle bundle over M, principle bundle corresponding to the reduction of the group of $G/U_1 \to G/U$ from C^* to the unimodular complex numbers.

3. Preliminaries on quaternionic structure. Let V be a real vector space. Then a *quaternion algebra on* V is an algebra of linear transformation of V which is isomorphic to the algebra of real quaternions, and whose unit element is the identity transformation of V. If A is a quaternion algebra on V, then we may view V as a quaternionic vector space by viewing it as a vector space over A. Then the quaternion-linear transformations are just the linear transformations which commute with every element of A.

Let S be a Riemannian manifold. Given $x \in S$, we have a group Ψ_x (the *linear holonomy group*) consisting of all linear transformations of the tangent space S_x

obtained from parallel translation about curves from x to x. A set A_x of linear transformations of S_x is called Ψ_x-invariant if $gA_xg^{-1} = A_x$ for every $g \in \Psi_x$. A set A of fields of linear transformations of all tangent spaces of S is called *parallel* if, given $x, y \in S$ and a curve σ from x to y, the parallel translation τ_σ along σ satisfies $\tau_\sigma A_x \tau_\sigma^{-1} = A_y$. A_x extends to a parallel set of fields of linear transformations of tangent spaces, if and only if A_x is Ψ_x-invariant; in that case the extension is unique, being defined in the notation above by $A_y = \tau_\sigma A_x \tau_\sigma^{-1}$.

A *quaternionic structure* on a Riemannian manifold S is a parallel field A of quaternion algebras A_x on the tangent spaces S_x, such that every unimodular element of A_x is an orthogonal linear transformation S_x. The latter condition says that A is "hermitian" relative to the Riemannian metric. The holonomy reduction theorem [6, p. 37] and the discussion above, show that a quaternionic structure on S is just a reduction of the group of the orthonormal frame bundle from the orthogonal group $O(m)$, $m = \dim. S$, to $Sp(m/4)\cdot Sp(1)$. Here $Sp(m/4)$ is the quaternion unitary group of S_x (over A_x) and $Sp(1)$ is the group of unimodular A_x-scalar transformations.

Let S be a Riemannian manifold with quaternionic structure. Then S has dimension divisible by 4, $\dim. S = 4n$ with $n > 0$. Every orientable Riemannian 4-manifold has quaternionic structure because its holonomy group is contained in the rotation group $SO(4) = Sp(1)\cdot Sp(1)$. Thus the product $S^2 \times S^2$ of 2-spheres has quaternionic structure while the factors do not. On the other hand the product $S^4 \times S^4$ of spheres does not have quaternionic structure (this is proved below) although each factor does. With these curious facts in mind, we will sketch a decomposition theory for quaternionic structure.

Choose $x \in S$ and let A be the quaternionic structure. Then A_x is a Ψ_x-stable quaternion algebra on S_x. In other words, $\Psi_x = \Phi_x \cdot A_x'$ where Φ_x is the centralizer of A_x and A_x' is the intersection with A_x. Φ_x and A_x' are the *A-linear* and *A-scalar parts* of Ψ_x. Notice that $\Phi_x \cap A_x'$ is either $\{1\}$ or $\{\pm 1\}$. We will say that the holonomy groups have *real scalar part* if the elements of A_x' are real scalars (thus 1 or -1), *complex scalar part* if the A_x' lie in complex subfields of the A_x but not in real subfields, *quaternion scalar part* otherwise (this is the case where A_x' spans A_x).

A Riemannian manifold is *locally flat* if it is locally isometric to euclidean space, and is *irreducible* if it is not locally flat and has irreducible holonomy group. According to de Rham [7] any complete simply connected Riemannian manifold is isometric to a product $S_0 \times S_1 \times \cdots \times S_r$ where S_0 is a euclidean space and the S_i are irreducible.

3.1. Lemma. *Let S be a complete simply connected Riemannian manifold with quaternionic structure A and de Rham decomposition $S_0 \times S_1 \times \cdots \times S_r$. If the holonomy of S has real scalar part, then A induces a quaternionic structure A_i on each S_i for which the holonomy of S_i has real scalar part. If the holonomy of S has complex scalar part, then S_0 is a single point; each S_i is a Kähler manifold with nonvanishing Ricci tensor, and the de Rham decomposition is of the form*

$$S = X_1 \times \cdots \times X_l \times (Y_1 \times Y_1') \times \cdots \times (Y_m \times Y_m'),$$

where (i) *A induces a quaternionic structure on each X_i and each $(Y_i \times Y_i')$ for which the holonomy has complex scalar part and* (ii) *each Y_i and each Y_i' has complex dimension* 1. *If the holonomy of S has quaternion scalar part, then S is irreducible ($S = S_i$ for some $i > 0$).*

Proof. Let $x \in S$ and decompose $S_x = V_0 \oplus \cdots \oplus V_r$, where V_i is the tangent space of S_i. The holonomy $\Psi_x = \Psi_0 \times \cdots \times \Psi_r$, where Ψ_i acts on V_i only, $\Psi_0 = \{1\}$, and Ψ_i is irreducible on V_i for $i > 0$. We deompose $\Psi_x = \Phi_x \cdot A_x'$ into A-linear and A-scalar parts. $Sp(1)$ denotes the multiplicative group of unimodular elements of A_x.

If the holonomy of S has real scalar part then A_x centralizes Ψ_x. Thus A_x permutes the V_i, so $A_x V_i = V_i$ by connectedness, and A induces a quaternionic structure A_i on S_i. The scalar part $\Psi_i \cap A_i$ is central in A_i because Ψ centralizes A, so Ψ_i has real A_i-scalar part.

If the holonomy of S has quaternion scalar part then A_x' generates A_x. Thus $A_x V_i \subset \Psi_x V_i = V_i$, so A induces a quaternionic structure A_i on S_i. As A_x' is a simple normal subgroup of Ψ_x it must lie in one of the factors Ψ_i, so A_x is trivial on the other V_j. Thus S is irreducible.

Now let the holonomy of S have complex scalar part. The element $J \in A_x'$ of order 4 is parallel and preserves each V_i because A_x' centralizes Ψ_x. Thus J defines a Kähler structure on each S_i. The Ricci tensor of S_i vanishes if and only if Ψ_i is in the *special* unitary group (relative to J), and that would contradict $J \in \Psi$. Thus S_i has nonzero Ricci tensor, and in particular the euclidean factor S_0 is a single point.

View Ψ_i as a group of complex-linear (relative to J) transformations of V_i and choose $K \in Sp(1)$ of square -1 which anticommutes with J. K centralizes Φ_x and sends complex subspaces of V to complex subspaces; thus K permutes the V_i. As $K^2 = -1$ the decomposition

$$S = X_1 \times \cdots \times X_l \times (Y_1 \times Y_1') \times \cdots \times (Y_m \times Y_m')$$

results; there the X_i are S_a for which $KV_a = V_a$, and the $\{Y_j, Y_j'\}$ are pairs $\{S_a, S_b\}$ for which K interchanges V_a and V_b. Assertion (i) is immediate. Now let $\{S_a, S_b\}$ be a pair $\{Y_j, Y_j'\}$. Then $\Psi_{ax} = \Phi_{ax} \cdot E_{ax}'$, where Φ_{ax} is the centralizer of A_x in Ψ_{ax}. It follows that E_{ax}' is the circle group of unimodular complex scalar transforms of V_a. Thus the action of Φ_{ax} on V_a is complex-irreducible where the complex structure comes from J, for Ψ_{ax} is real-irreducible on V_a. As A centralizes Φ_{ax} on V, it preserves the trivial representation space. The latter is $\sum_{i \neq a} V_i$ if $\Phi_{ax} \neq \{1\}$, so A_x preserves V_a if $\Phi_{ax} \neq \{1\}$. Thus $\Phi_{ax} = \{1\}$ and Ψ_{ax} is E_{ax}'. Now irreducibility implies $\dim_C V_a = 1$. Thus $\dim_C Y_j = \dim_C Y_j^i = 1$. *q.e.d.*

3.2. Remark. Let S_1 and S_2 be Kähler manifolds of complex dimension 1 which are not flat. Then the S_i are irreducible and have holonomy $\Psi_i = U(1)$, unitary group. $S_1 \times S_2$ has two quaternionic structures because it has real dimension 4, and Lemma 3.1 shows that the holonomy has complex scalar part in each of these structures.

3.3. Lemma. *Let S be a complete simply connected irreducible Kähler manifold. Assume that S has nontrivial Ricci tensor, so the holonomy $\Psi = \Phi \cdot E'$ where E' is the group of unimodular complex scalar transformations of the tangent space and Φ is in the special unitary group. Then S has a quaternionic structure A for which E' is the (necessarily complex) scalar part of Ψ, if and only if S has complex dimension* $\dim_C S = 2$.

Proof. If $\dim_C S = 2$, then $\Phi_x = SU(2) = Sp(1)$ by irreducibility of S, and the centralizer of Φ_x (in the algebra of all real-linear endomorphisms of S_x) is a quaternion algebra A_x on S_x; then $A = \{A_x : x \in S\}$ is a quaternion structure on S for which Ψ has complex scalar part E'.

Conversely, let A be a quaternion structure on S for which Ψ has complex scalar part E'. If S is not symmetric, then [0, Théorème 3 of Ch. III] Berger's classifications shows first that $\Phi_x = Sp(n)$ where $\dim_C S = 2n$, and then that $n = 1$. Now we may assume S symmetric. Given $x \in S$ we have $S = G/K$ where G is the largest connected group of isometries and $K = \{g \in G : g(x) = x\}$, and Ψ_x is just the linear isotropy action of K on S_x. Furthermore $K = K_1 \cdot T$ where T is a circle group, center of K, and K_1 is semisimple; thus $\Phi_x = K_1$ and $E' = T$. K_1 is complex linear and complex irreducible on S_x, and has an invariant antisymmetric complex-bilinear invariant form on S_x because it centralizes A_x. In particular K_1 has real character on S_x and has center of order 1 or 2. According to É. Cartan, $S = G/K$ is one of the spaces

(α) $ad(E_7)/E_6 \cdot T$

(β) $ad(E_6)/\text{Spin}\ (10) \cdot T$,

(γ) $SO(2n)/U(n), \quad n > 2$,

(δ) $Sp(n)/U(n)$,

(ϵ) $SO(n + 2)/SO(n) \times SO(2)$,

(φ) $SU(m + n)/\{SU(m + n) \cap [U(m) \times U(n)]\}$,

or a noncompact "dual" which has the same holonomy. We eliminate (α) because there $K_1 = E_6$ has center of order 3, (β) because there the representation of $K_1 = \text{Spin}\ (10)$ is a half-spin representation which has no bilinear invariant, (γ) because there $K_1 = SU(n)$ acts by antisymmetrisation (which has no bilinear invariant) of the usual representation of degree n, (δ) for $n > 2$ as for (γ), (δ) for $n < 2$ by dimension, (ε) because there $K_1 = SO(n)$ has a symmetric bilinear invariant (the representation is by real matrices) and thus no antisymmetric one, and (φ) unless $m, n \leqq 2$ because the representation of K_1 has no bilinear invariant, (φ) for $m = n = 2$ because there the bilinear invariant is symmetric, and (φ) for $m = n = 1$ by dimension. Thus S has complex dimension 2, being given by (δ) for $n = 2$, (φ) for $m = 1$, $n = 2$, or the dual to one of these. *q.e.d.*

Our results so far, allow analysis of quaternionic structure. The following lemma allows the synthesis.

3.4. Lemma. *Let $\{S_\alpha\}$ be a finite collection of complete simply connected Riemannian manifolds where S_α has quaternionic structure A_α and holonomy Ψ_α, and let S be the product of the S_α. If each Ψ_α has real* (*resp. complex*) *A_α-scalar part, then S has a quaternionic structure A which induces A_α on S_α and for which the holonomy has real* (*resp. complex*) *A-scalar part. If there is more than one S_α, and if some Ψ_β has quaternionic A_β-scalar part, then S does not have a quaternionic structure which induces A_α on S_α for each α.*

Proof. The first statement is proved by embedding a quaternion algebra A^* in $(\prod_\alpha A_\alpha)_x$ by means of an isomorphism on each factor. For the second statement, observe that no element of a quaternionic structure on S inducing A_β could centralize Ψ_β; thus the existence of the structure would imply irreducibility of S by Lemma 3.1. *q.e.d.*

We can now prove our decomposition theorem for quaternionic structure.

3.5. Theorem. *The building blocks for complete simply connected Riemannian manifolds with quaternionic structure are the following classes of spaces:*

1. *Euclidean spaces of dimension $4n$, $n > 0$.*
2. *Complete simply connected Riemannian manifolds of dimension $4n$, $n > 0$, with holonomy*[2] *$Sp(n)$.*
3. *Products $S_1 \times S_2$ of complete simply connected Kähler manifolds of complex dimension* 1 *which are not locally flat.*
4. *Complete simply connected irreducible Kähler manifolds of complex dimension* 2 *with Ricci tensor not identically zero.*
5. *Complete simply connected irreducible Riemannian manifolds with holonomy $\Psi = \Phi \cdot E'$ where $E' = Sp(1)$ generates a quaternion algebra on each tangent space.*

A complete simply connected Riemannian manifold has a quaternionic structure if and only if (a) *it is a product of spaces from classes* 1 *and* 2, *or* (b) *it is a product of spaces from classes* 3 *and* 4, *or* (c) *it is a space from class* 5. *The scalar part of the holonomy is real, complex, or quaternion, respectively, in cases* (a), (b) *and* (c).

Proof. Let S be irreducible with a quaternionic structure A for which the holonomy Ψ has real scalar part. Then S is not Riemannian symmetric [11, Lemma 2.4.3]. Thus $x \in S$ implies transitivity of Ψ_x on the unit sphere in S_x [8, Theorem 4]. Now $\Psi_x \subset Sp(n)$, dim. $S = 4n$, implies $\Psi_x = Sp(n)$. With this in mind, the theorem is immediate from the lemmas and remarks above. *q.e.d.*

[2] No examples are known, nor are examples known for $n > 1$, even if the condition of completeness is dropped. But some 4–manifolds with holonomy $Sp(1) = SU(2)$ are known.

3.6. Remark. Let S be a Riemannian 4-manifold with holonomy $\Psi = U(2)$, unitary group acting in the usual way. Then S plays a dual role, being in classes 4 and 5 above. More precisely, S has distinct quaternionic structures A_1 and A_2 where A_{1x} is spanned by the derived group $[\Psi_x, \Psi_x] = SU(2)$ and A_{2x} is the centralizer of $SU(2)$ in the algebra of real-linear endomorphisms of S_x. The A_1-scalar part of Ψ is complex while the A_2-scalar part of Ψ is quaternion.

This also explains the dual role of the complex projective and hyperbolic planes in the following corollary to Theorem 3.5.

3.7. Theorem. *The complete simply connected Riemannian symmetric spaces with quaternionic structure are the following:*

(a) *Euclidean spaces of dimension* $4n$, $n > 0$. *Here the holonomy has real scalar part.*

(b) *Products* $S_1 \times \cdots \times S_r$ *where each* S_i *is either* (b1) *the complex projective or hyperbolic plane with the quaternionic structure for which the holonomy has complex scalar or* (b2) *a product* $S_i' \times S_i''$ *where each factor is either a complex projective or hyperbolic line. Here the holonomy has complex scalar part.*

(c) *Complete simply connected irreducible Riemannian symmetric spaces with holonomy* $\Psi = \Phi \cdot E'$ *where* $E' = Sp(1)$ *and* E' *generates a quaternion algebra on each tangent space. Here the holonomy has quaternion scalar part.* {*These spaces are completely described in Theorem* 5.4 *below.*}

Proof. Theorem 3.5 and Remark 3.6 show that the spaces listed have the requisite quaternionic structures. Referring to the five classes of Theorem 3.5, now, we need only check (i) no space of class 2 is symmetric, (ii) the complex projective and hyperbolic lines are the only hermitian symmetric spaces of complex dimension 1, beside the Gaussian plane and (iii) the complex projective and hyperbolic planes are the only irreducible hermitian symmetric spaces of complex dimension 2. Assertion (i) follows from [11, Lemma 2.4.3], (ii) is a triviality, and (iii) is both classical (via constant holomorphic sectional curvature) and proved in the proof of Theorem 3.5. *q.e.d.*

Although we have used classification in the proof of Theorem 3.7, our proofs of our main results will remain independent of classifications, for they will not depend on Corollary 3.7. More precisely, we will work directly with symmetric spaces with quaternionic structure for which the holonomy has quaternion scalar part. The next step is to describe those spaces in terms of maximal roots, which were the key to the description of homogeneous complex contact manifolds.

4. Characterization of the maximal root.

We will need to be able to recognize a maximal root in a simple Lie algebra.

Let $\mathfrak{G}$ be a complex simple Lie algebra. We choose a Cartan subalgebra $\mathfrak{H}$ and have the decomposition

$$\mathfrak{G} = \mathfrak{H} + \sum_{\alpha \in \Psi} \mathfrak{G}_\alpha ,$$

where Ψ is the set of roots of $\mathcal{G}$ for $\mathcal{H}$ and $\mathcal{G}_\alpha$ is the one-dimensional space characterized by

$$[H, E] = \alpha(H)E \quad \text{for all} \quad H \in \mathcal{H}, \qquad E \in \mathcal{G}_\alpha .$$

The Killing form on $\mathcal{G}$ is denoted $\langle\ ,\ \rangle$, and $H_\alpha (\alpha \in \Psi)$ denotes the element of $\mathcal{H}$ characterized by: $\langle H_\alpha , H\rangle = \alpha(H)$ for all $H \in \mathcal{H}$. Finally, we will say "ordering of the roots" for an ordering defined by a lexicographic ordering on the real span of Ψ in the dual space of $\mathcal{H}$.

We will abbreviate $\langle H_\alpha , H_\beta\rangle$ by $\langle \alpha, \beta\rangle$ for $\alpha, \beta \in \Psi$. An *α-chain from β* is a series of *roots* $\beta + t\alpha$ where t ranges over a set of consecutive integers, say $-r \leqq t \leqq s$ where r and s are positive integers. The chain is *maximal* if it cannot be enlarged, *i.e.*, if neither $\beta - (r + 1)\alpha$ nor $\beta + (s + 1)\alpha$ is a root. It is standard that:

$$\frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} = r - s, \tag{4.1}$$

where the maximal α-chain from β runs from $\beta - r\alpha$ to $\beta + s\alpha$.

We can now characterize the maximal root.

4.2. Theorem. *Let β be a root of a complex simple Lie algebra $\mathcal{G}$ relative to a Cartan subalgebra $\mathcal{H}$. Then β is the maximal root for some ordering, if and only if the eigenvalues of $ad(H_\beta)$ are $-\frac{1}{2}\,|\beta|^2$, 0 and $\frac{1}{2}\,|\beta|^2$ off of $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$. In that case the centralizer of H_β in $\mathcal{G}$ is a direct sum $\mathfrak{z}_1 \oplus \{H_\beta\}$ of ideals where $\mathfrak{z}_1$ centralizes $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$.*

Proof. Let β be maximal for some ordering of the roots. If α is a positive root, $\alpha \neq \beta$, then: $\alpha + \beta$ is not a root, α is a root, $\alpha - \beta$ might be a root and $\alpha - 2\beta$ is not a root. Now (4.1) says $2\langle \alpha, \beta\rangle/\langle \beta, \beta\rangle = 0$ or 1, so $\langle \alpha, \beta\rangle$ is 0 or c where $c = \frac{1}{2}|\beta|^2$. Thus $ad(H_\beta)E_\alpha$ is 0 or $\frac{1}{2}|\beta|^2E_\alpha$ and $ad(H_\beta)E_{-\alpha}$ is 0 or $-\frac{1}{2}\,|\beta|^2E_{-\alpha}$. As $ad(H_\beta)\mathcal{H} = 0$, our eigenvalue assertion is proved.

Let $-\frac{1}{2}\,|\beta|^2$, 0 and $\frac{1}{2}\,|\beta|^2$ be the only eigenvalues of $ad(H_\beta)$ off of $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$. We order the roots so that H_β is in the closure of the positive Weyl chambre; in other words, $\alpha(H_\beta) = \langle \alpha, \beta\rangle \geqq 0$ for all roots $\alpha > 0$. If β is not maximal then we have a root $\alpha > 0$ such that $\beta + \alpha$ is a root, and $ad(H_\beta)$ is multiplication by 0 or $\frac{1}{2}\,|\beta|^2$ on $\mathcal{G}_{\alpha+\beta}$. If $\langle \alpha, \beta\rangle \neq 0$ then $ad(H_\beta)E_{\alpha+\beta} = \frac{3}{2}\,|\beta|^2E_{\alpha+\beta}$, which is impossible. If $\langle \alpha, \beta\rangle = 0$, then $ad(H_\beta)E_{\alpha+\beta} = |\beta|^2E_{\alpha+\beta}$, which is also impossible. Thus β is maximal.

Let $\mathfrak{z}$ be the centralizer of H_β in $\mathcal{G}$. Then

$$\mathfrak{z} = \mathcal{H} + \sum_{\alpha \in \Phi} \mathcal{G}_\alpha$$

where Φ consists of all roots α with $\langle \alpha, \beta\rangle = 0$. Now $\mathfrak{z} = \mathfrak{z}_1 \oplus \{H_\beta\}$ direct sum of ideals where

$$\mathfrak{z}_1 = \mathcal{H}_1 + \sum_{\alpha \in \Phi} \mathcal{G}_\alpha ,$$

and $\mathfrak{H}_1$ is the orthocomplement of H_β in $\mathfrak{H}$. $\mathfrak{H}_1$ centralizes $\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$ by definition. Now suppose that β is maximal and let $\alpha \in \Phi$. We may assume $\alpha > 0$, so $\beta + \alpha$ and $-\beta - \alpha$ are not roots. Now (4.1) says that $\beta - \alpha$ and $\alpha - \beta$ are not roots. Thus $[\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}] = 0$, proving that $\mathfrak{z}_1$ centralizes $\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$.

q.e.d.

5. Characterization of quaternionic symmetric spaces. Theorem 4.2 allows us to prove the following Theorem 5.4, which completes our decomposition of quaternionic symmetric spaces (Theorem 3.7) to a structure theorem.

Let G_u be a compact centerless simple lie group. Choose a maximal torus T and let $\mathfrak{g}_u$ and $\mathfrak{T}$ be the respective Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{H}$ be the respective complexifications of $\mathfrak{g}_u$ and $\mathfrak{T}$; $\mathfrak{H}$ is a Cartan subalgebra of the complex simple Lie algebra $\mathfrak{g}$. Order the roots and let β be the maximal root. We now define

$$(5.1) \qquad \mathfrak{L}_1 = \{H \in \mathfrak{T} : \beta(H) = 0\} + \sum_{\substack{\alpha > 0 \\ \langle \alpha, \beta \rangle = 0}} \mathfrak{g}_u \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}),$$

$$(5.2) \qquad \mathfrak{A}_1 = \mathfrak{g}_u \cap (\{H_\beta\} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}),$$

and

$$(5.3) \qquad \mathfrak{K} = \mathfrak{L}_1 + \mathfrak{A}_1 .$$

Theorem 4.2 says that $\mathfrak{L}_1 \oplus \mathfrak{g}_u \cap \{H_\beta\}$ is the centralizer of H_β in $\mathfrak{g}_u$, and that $\mathfrak{K} = \mathfrak{L}_1 \oplus \mathfrak{A}_1$ direct sum of ideals. Let L_1, A_1 and $K = L_1 \cdot A_1$ be the corresponding analytic subgroups of G_u.

5.4. Theorem. *G_u/K is a compact simply connected irreducible Riemannian symmetric space. The holonomy $K = L_1 \cdot A_1$ where $A_1 \cong Sp(1)$ and A_1 generates a quaternion algebra on the tangent space. This quaternion algebra parallel translates over G_u/K to give a quaternionic structure A in which the holonomy has quaternion scalar part.*

Conversely if S is a compact simply connected Riemannian symmetric space with a quaternionic structure A_S in which the holonomy has quaternion scalar part, then there is an isometry of S onto a manifold G_u/K described above, which carries A_S to A. Furthermore, the noncompact Riemannian symmetric spaces with quaternionic structure in which the holonomy has quaternion scalar part, are just the noncompact duals of the spaces G_u/K described above.

Proof. Let $S = G_u/K$. Then S is compact because G_u is compact, and S is simply connected because K is connected and of maximal rank. $ad(H_\beta)$ has only $-c$, 0 and c for eigenvalues off of $\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$, $c > 0$, by Theorem 4.2. Define $X = (i/c)H_\beta \in \mathfrak{T}$, $i^2 = -1$. Now $s = \exp \pi X \in A_1$ has order 2 and our constructions imply $\mathfrak{g}_u = \mathfrak{K} + \mathfrak{P}$, where

$$\mathfrak{P} = \sum_{\substack{\beta \neq \alpha > 0 \\ \langle \alpha, \beta \rangle \neq 0}} \mathfrak{g}_u \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}),$$

and $ad(s) = -1$ on $\mathcal{P}$. $ad(s) = 1$ on $\mathcal{K}_1$ by construction, and $ad(s) = 1$ on $\mathcal{A}_1$ because $c = \frac{1}{2}\langle\beta, \beta\rangle$ and $ad(H_\beta)$ has eigenvalues $\pm\langle\beta, \beta\rangle$ on $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$; thus $ad(s) = 1$ on $\mathcal{K}$. Now S is Riemannian symmetric with symmetry s. Irreducibility of S follows from simplicity of G_u .

$A_1 \cong Sp(1)$ by a glance at the eigenvalues of $ad(X)$. Now all elements of order 4 in A_1 have square s and are conjugate to $j = \exp \frac{1}{2}\pi X$. This gives $j, k \in A_1$ such that $ad(j)$ and $ad(k)$ restrict to an anticommuting pair of transformations of square -1 on $\mathcal{P}$. In other words, $ad(A_1)|_{\mathcal{P}}$ generates a quaternion algebra on $\mathcal{P}$. As $ad(K)|_{\mathcal{P}}$ is the holonomy as a linear group, our assertions on S are immediate.

Let S be a compact simply connected Riemannian symmetric space with a quaternionic structure A_S in which the holonomy has quaternion scalar part. Then $S = G_u/K$, G_u compact semisimple and K connected, where $K = L_1 \cdot A_1$ and A_1 generates a quaternion algebra on the tangent space. S is irreducible by Lemma 3.1. If G_u were not simple, S would be a simple group manifold of the group K. Simplicity of K would give $L_1 = \{1\}$ and $\dim S = 3$; thus G_u is simple. Conjugation by the element of order 2 in A_1 is the symmetry because it is central and of order 2 in K. Now K contains the symmetry, so rank $G_u =$ rank K; in other words, K contains a maximal torus T of G_u . In particular G_u is centerless.

Script denotes Lie algebras. Let $\mathcal{G}$ and $\mathcal{K}$ be the complexifications of $\mathcal{G}_u$ and $\mathfrak{I}$. $\mathfrak{I}$ normalizes $\mathcal{A}_1$, so the complexification

$$\mathcal{A}_1^C = \{H_\beta\} + \mathcal{G}_\beta + \mathcal{G}_{-\beta} ,$$

for some root β. Let $c = \frac{1}{2}\langle\beta, \beta\rangle$ and $X = (i/c)H_\beta \in \mathfrak{I} \cap \mathcal{A}_1$. Then $\exp tX$ is a rotation through an angle of t on every plane $\mathcal{P}_\alpha$; here $\mathcal{G}_u = \mathcal{K} + \mathcal{P}$ with $ad(s) = 1$ on $\mathcal{P}$, and $\mathcal{P}$ is a sum of planes $\mathcal{P}_\alpha = \mathcal{G}_u \cap (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha})$. $\exp tX$ is rotation by $2t$ on $\mathcal{G}_u \cap (\mathcal{G}_\beta + \mathcal{G}_{-\beta})$ and by 0 on $\mathcal{L}_1$. Thus $ad(H_\beta)$ has only $-c$, 0 and c for eigenvalues off of $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$, and $\mathcal{K}_1^C + \{H_\beta\}$ is the centralizer of H_β in $\mathcal{G}$. As $\mathcal{K}_1$ centralizes $\mathcal{G}_\beta + \mathcal{G}_{-\beta}$, we may apply Theorem 4.2 and assume the roots ordered so that β is the maximal root. Now $\mathcal{L}_1$, $\mathcal{A}_1$ and $\mathcal{K}$ are given by (5.1), (5.2) and (5.3), so S has the form asserted. A_S goes to A under this realization because we chose A_1 to generate the image of A_S .

For the last remark we observe that a pair of dual symmetric spaces have equivalent linear holonomy groups. *q.e.d.*

We now have precisely one compact space and one noncompact space for each compact simple Lie algebra, in class (c) of Theorem 3.7.

6. The correspondence between contact and quaternionic structures. Comparison of Boothby's structure theorem for homogeneous complex contact manifolds with Theorem 5.4 yields our main result.

6.1. Theorem. *There is a one to one correspondence $M \leftrightarrow S$ between the compact simply connected homogeneous complex contact manifolds M and the* *compact simply connected Riemannian symmetric spaces S with quaternionic structure in which the holonomy has quaternion scalar part. The correspondence is given by a bundle $\Psi : M \to S$ where the fibres are 2-spheres. More precisely,*

$$\Psi : G_u/L_1 \cdot T^1 \to G_u/L_1 \cdot A_1 ,$$

where G_u is a centerless simple compact Lie group, L_1 and A_1 are the analytic subgroups with Lie algebras given by (5.1) *and* (5.2), *and T^1 is a circle subgroup of A_1 ; T^1 defines the complex contact structure on $M = G_u/L_1 \cdot T^1$ and A_1 defines the quaternionic structure on $S = G_u/L_1 \cdot A_1$.*

Proof. Given M, let G be the identity component of the group of all complex contact automorphisms of M. Then G is a centerless simple complex Lie group by (2.5), and $M = G_u/L$ as described in §2. Now $S = G_u/K$ is determined by Theorem 5.4. Given S, let G_u be the largest connected group of isometries. Then $S = G_u/K$ as described in Theorem 5.4. Now $M = G_u/L$ is determined as described in §2. *q.e.d.*

We will now extend this theorem to the noncompact case.

Let $S = G_u/K$ be a compact simply connected Riemannian symmetric space with G_u simple. Then the Lie algebra decomposes as $\mathfrak{G}_u = \mathcal{K} + \mathcal{P}$ where $\mathcal{K}$ is the $(+1)$-eigenspace of the symmetry and is the Lie algebra of K, and where $\mathcal{P}$ is the (-1)-eigenspace of the symmetry. Then $\mathfrak{G}_* = \mathcal{K} + i\mathcal{P}$, $i^2 = -1$, is a Lie algebra whose complexification is the complexification $\mathfrak{G}$ of $\mathfrak{G}_u$. Let G_* be the centerless group with Lie algebra $\mathfrak{G}_*$ and let K be the analytic subgroup for $\mathcal{K}$. Then $S_* = G_*/K$ is a noncompact Riemannian symmetric space. S and S_* are *dual*. As remarked in Theorem 5.4, S and S_* have the same (if any) quaternionic structures. For their holonomy groups are equivalent, being $ad(K)$ on $\mathcal{P}$ and $i\mathcal{P}$ respectively.

We must now find a suitable notion of duality for complex contact manifolds.

Retain the notation above and let V be a toral subgroup of G_u with centralizer L contained in K. Then $V \subset T \subset L$ for some maximal torus T of G_u . The complexification $\mathcal{H}$ of $\mathfrak{J}$ is a Cartan subalgebra of the complex simple Lie algebra $\mathfrak{G}$. Ordering the roots of $\mathfrak{G}$ relative to $\mathcal{H}$, we order the real linear forms on $i\mathcal{V}$. Let $\mathcal{U}$ be the sum of the non-negative weight spaces of $\mathcal{V}$, in this ordering, for the adjoint representation of $\mathcal{V}^C$ on $\mathfrak{G}$, and let U be the analytic subgroup of G with Lie algebra $\mathcal{U}$. Then U is a parabolic subgroup of G. We view G_u and G_* as real analytic subgroups of G. Then

$$G_u \cap U = L = G_* \cap U. \tag{6.2}$$

Now $\pi_u : G_u/L \to G/U$ and $\pi_* : G_*/L \to G/U$ are defined by

$$\pi_u(gL) = gU \quad \text{and} \quad \pi_*(gL) = gU. \tag{6.3}$$

π_u and π_* both are local homeomorphisms by (6.2). π_u is surjective because G_u/L is compact; it follows that π_u is a covering. As G/U is simply connected, we have

$$\pi_u : G_u/L \to G/U \text{ is a diffeomorphism.} \tag{6.4}$$

Now G/U gives complex structures to G_u/L and G_*/L by the embeddings π_u and π_*, and

(6.5) $\pi_u^{-1} \circ \pi_* : G_*/L \to G_u/L$ *embeds* G_*/L *as an open set in* G_u/L.

If we repeat the construction of G_*/L from G_u/L, starting with G_*/L, then we come back to G_u/L. Thus we will say that G_u/L and G_*/L are *dual* (relative to K). This duality is due to A. Borel [3]; it generalizes duality of hermitian symmetric spaces. In the hermitian symmetric case, (6.5) is the celebrated "Borel embedding", generalizing the embedding of the disc as a cap on the Riemann sphere.

Now let $M = G_u/L$ be a compact simply connected homogeneous complex contact manifold. The associated quaternionic symmetric space $S = G_u/K$ satisfies $L \subset K$. As L is the centralizer of a toral subgroup $V \subset K$ (here $L = L_1 \cdot V$ and $K = L_1 \cdot A_1$ where V is a circle subgroup of A_1), we have the dual $M_* = G_*/L$ of M relative to K as constructed above. The Borel embedding $\xi : M_* \to M$ given by (6.5) embeds M_* as an open submanifold of M and sends G_* to a group of complex contact automorphisms of M. Thus M_* inherits a complex contact structure from M which is homogeneous under G_*. We will say that M_* and M are *dual complex contact manifolds.*

Let $M_* = G_*/L$ be the dual complex contact manifold to a compact simply connected homogeneous complex contact manifold $M = G_u/L$. We have the 2-sphere bundle $\Psi : M \to S$ of M over the associated quaternionic symmetric space. Here $S = G_u/K$ and $\Psi(gL) = gK$. We now have a bundle

(6.6) $$\Psi_* : M_* \to S_*, \quad \text{by} \quad gL \to gK,$$

where $S_* = G_*/K$ is the noncompact dual of S. Ψ_* has fibre L/K, which is a two-dimensional sphere.

Nevertheless in the diagram

$$\begin{tikzcd} M_* \arrow[r, "\xi"] \arrow[d, "\Psi_*"] & M \arrow[d, "\Psi"] \\ S_* \arrow[r, dashed, "\eta"] & S \end{tikzcd}$$

where ξ is the inclusion given by Borel embedding of M_*, we can never define η so that the diagram is commutative. For the condition for existence of η is that $\Psi_* x = \Psi_* y$ imply $\Psi\xi x = \Psi\xi y$; the latter is equivalent to the requirement that K normalize U, while $K \not\subset U$ and U is its own normalizer in G. In particular, in the cases where S is hermitian symmetric, ξ does not cover the Borel embedding $S_* \subset S$.

6.7. Theorem. *There is a one to one correspondence $M_* \leftrightarrow S_*$ between the dual complex contact manifolds M_* to the compact simply connected homogeneous complex contact manifolds, and the noncompact Riemannian symmetric spaces* S_*

with quaternionic structure in which the holonomy has quaternion scalar part. The correspondence is given by a bundle $\Psi_* : M_* \rightarrow S_*$ *whose fibre is a 2-sphere.*

This is immediate from the preceding discussion, Theorem 6.1, and the fact that S_* is automatically complete, simply connected and irreducible.

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