GENERALIZED CAYLEY TRANSFORMATIONS OF BOUNDED SYMMETRIC DOMAINS.

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1. Introduction. This paper is a continuation of [7]; its main subject is the study of the realizations of Hermitian symmetric spaces as Siegel domains of type III. The general definition of such a domain was given by Pjateckiĭ-Šapiro [9] as follows.

Let $\mathfrak{v}_1$, $\mathfrak{v}_2$ and $\mathfrak{v}_3$ be complex vector spaces. Let $\mathfrak{u}_1$ be a real form of $\mathfrak{v}_1$, $\mathfrak{c}$ an open cone in $\mathfrak{u}_1$, and D a bounded domain in $\mathfrak{v}_3$. Given any $W \in D$ let $\Lambda_W^{(1)}: \mathfrak{v}_2 \times \mathfrak{v}_2 \rightarrow \mathfrak{v}_1$ be a bilinear form Hermitian with respect to $\mathfrak{u}_1$, let $\Lambda_W^{(2)}: \mathfrak{v}_2 \times \mathfrak{v}_2 \rightarrow \mathfrak{v}_1$ be a complex-symmetric bilinear form, and define $\Lambda_W = \Lambda_W^{(1)} + \Lambda_W^{(2)}$. Then the domain

$$\{E_1 + E_2 + E_3 \in \mathfrak{v}_1 + \mathfrak{v}_2 + \mathfrak{v}_3 : \mathrm{Im.\,E}_1 - \mathrm{Re.}\,\Lambda_{E_3}(E_2, E_2) \in \mathfrak{c}\}$$

is called a Siegel domain of type III.

Pjateckiĭ-Šapiro [9] gave a case by case determination of the realizations of the classical irreducible Hermitian symmetric spaces as Siegel domains of type III. In this paper we will determine those realizations for all Hermitian symmetric spaces by a method which is independent of the classification theory. This is closely related to the study of the boundary structure of bounded symmetric domains. In the classical cases that study is due to Pjateckiĭ-Šapiro [9]; in the general case most of the relevant results have been proved by C. C. Moore [8], who combined our partial Cayley transform with the general theory of boundaries due to Furstenberg [3] and Satake [10]. In this paper (Section 4) we give an explicit direct description of the boundary structure. The greater simplicity of our methods, and the fact that many intermediary results from Section 4 are needed in subsequent discussions, are the reasons why those results are included in this paper.

Our method is an extension of the technique of [7]. Making use of the embedding theorems of Borel and Harish-Chandra, we define partial Cayley

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transformations which carry the bounded domain realization of Harish-Chandra to the various Siegel domains of type III.

In the case of a polycylinder $U^n \subset C^n$ (where U denotes the unit disc in C) a partial Cayley transformation is simply the usual Cayley transformation on some of the factors and the identity transformation on the remaining factors. In the case of a general bounded symmetric domain D in Harish-Chandra realization, it follows from results of Harish-Chandra and is explicitly pointed out by Hermann [5] that D contains a totally geodesic polycylinder U^n with $K \cdot U^n = D$; here n is the rank of D as a symmetric space and K is the isotropy subgroup at the origin of the connected group G^0 of holomorphic automorphisms of D. A partial Cayley transformation of D can be viewed as a natural extension to D of a partial Cayley transformation of U^n.

Sections 2, 3 and 4 contain a considerable amount of expository material, which is included so that the paper can be used by beginners in the subject. In Section 2 we introduce our notation and some definitions. In Section 3 we collect some facts on parabolic subgroups of real Lie groups; these are due to A. Borel and J. Tits [2] and to a conversation between J. Tits and the first-named author. In Section 4 we give an explicit description of the boundary components of D (Theorem 4.8) and compute them in the irreducible cases (Theorem 4.13). We do not reprove [8, Theorems 1 and 2] because Moore's proof is independent of the general theory of boundaries of symmetric spaces.

In Sections 5 and 6 we show that the set of all analytically equivalent ("same type") boundary components is, for each type, both a homogeneous space of K and of G^0; we study the Riemannian geometry and topology of these spaces in some detail. The isotropy subgroup B^Γ of G^0 is transitive on D; this fact is basic in Section 7 where we construct the image of D under the partial Cayley transformation; this Cayley transform is an orbit of a certain conjugate of B^Γ, which we determine explicitly. The resulting domain is a Siegel domain of type III, and in the classical irreducible cases our results specialize to those of Pjateckiĭ-Šapiro.

The results of [7] are degenerate special cases of theorems in the present Sections 5, 6 and 7, but some of our proofs here depend on the results of [7].

2. Notations. As in [7], M will be a Hermitian symmetric space of non-compact type, G^0 its connected group of isometries, and K the isotropy group. G^0 is globally a product of simple Lie groups and M a product of non-compact irreducible Hermitian symmetric spaces. The Lie algebras of G^0 and K are $\mathfrak{g}^0$ and $\mathfrak{k}$, $\mathfrak{g}^C$ is the complexification of $\mathfrak{g}^0$, G^C the adjoint group of $\mathfrak{g}^C$. G^0 is contained in G^C as the analytic group corresponding to $\mathfrak{g}^0$. The symmetry of $\mathfrak{g}^0$ is denoted by σ; under σ we have the splitting $\mathfrak{g}^0 = \mathfrak{k} + \mathfrak{p}^0$. Let $\mathfrak{p} = i\mathfrak{p}^0$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, G the corresponding analytic subgroup of G^C.

$\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{k}$; then $\mathfrak{h}^C$ is a Cartan subalgebra in $\mathfrak{g}^C$. The roots of $\mathfrak{g}^C$ which are also roots of k^C are called compact roots. Given a system of simple roots, if $\mathfrak{g}^C$ is simple, there is a unique non-compact simple root. To each root α we have the standard basis elements H_α, E_α. $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are the abelian subalgebras of $\mathfrak{g}^C$ spanned by the positive (resp. negative) non-compact root vectors E_α; P^+ and P^- are the corresponding analytic groups in G^C.

K^C denoting the analytic subgroup corresponding to $\mathfrak{k}^C$, $K^C \cdot P^+$ is a semidirect product. G/K is identified with $G^C/K^C \cdot P^+$ by the identity map of G into G^C; this space is the compact dual of M, and is denoted by M^*. x denotes the identity coset in $M^* = G^C/K^C \cdot P^+$. The orbit $G^0(x)$ is the image of the holomorphic embedding $gK \to g(x)$ of M into M^*. The map $\xi\colon \mathfrak{p}^- \to M^*$, defined by $\xi(E) = \exp(E) \cdot (x)$ is a holomorphic homeomorphism onto a dense open subset; ξ is $\mathrm{ad}(K^C)$-equivariant. $D = \xi^{-1}(G^0(x))$ is a bounded symmetric domain in $\mathfrak{p}^-$; this is the Harish-Chandra realization of M.

The center $\mathfrak{z}$ of $\mathfrak{k}$ contains an element Z such that $\mathrm{ad}(Z)E = \pm iE$ for $E \in \mathfrak{p}^{\mp}$. $J = \mathrm{ad}(Z)$ is a complex structure on $\mathfrak{p}^0$. A basis of $\mathfrak{p}^0$ is given by the elements

$$X_\alpha{}^0 = E_\alpha + E_{-\alpha}$$
$$Y_\alpha{}^0 = -i(E_\alpha - E_{-\alpha}),$$

where α is non-compact positive. For such α we have the relations

$$JX_\alpha{}^0 = [Z, X_\alpha{}^0] = Y_\alpha{}^0$$
$$JY_\alpha{}^0 = [Z, Y_\alpha{}^0] = -X_\alpha{}^0$$
$$[X_\alpha{}^0, Y_\alpha{}^0] = 2iH_\alpha.$$

We define the elements $X_\alpha = iX_\alpha{}^0$, $Y_\alpha = iY_\alpha{}^0$; these form a basis of $\mathfrak{p}$.

Two roots α and β of $\mathfrak{g}^C$ are called strongly orthogonal if $\alpha \pm \beta$ are not roots. There exists a set Δ of strongly orthogonal positive non-compact roots such that the real span $\mathfrak{a}^0$ of the $X_\alpha{}^0$ $(\alpha \in \Delta)$ is a maximal abelian subalgebra contained in $\mathfrak{p}^0$. $\Delta = \{\delta_1, \cdots, \delta_r\}$ is constructed in [4] as follows: For each j, δ_{j+1} is the lowest positive non-compact root that is strongly orthogonal to $\delta_1, \cdots, \delta_j$. Thus, if $\mathfrak{g}$ is simple, δ_1 is the non-compact simple root. In our proofs we shall calculate with a set Δ constructed in this way. Our results, however, do not depend on the construction of Δ.

For each $\alpha \in \Delta$ we define the 3-dimensional simple subalgebras $\mathfrak{g}_\alpha$, spanned by $\{iH_\alpha, X_\alpha, Y_\alpha\}$, and $\mathfrak{g}_\alpha{}^0$, spanned by $\{iH_\alpha, X_\alpha{}^0, Y_\alpha{}^0\}$. The corresponding analytic subgroups of G^C are G_α and $G_\alpha{}^0$. We define $\mathfrak{h}^- = [\mathfrak{a}^0, J\mathfrak{a}^0]$; then $\mathfrak{h}^- \subset \mathfrak{h}$. $\mathfrak{h}^+$ denotes the orthogonal complement of $\mathfrak{h}^-$ in $\mathfrak{h}$ with respect to the Killing form; $\mathfrak{h}^+$ is the centralizer of $\mathfrak{a}^0$ in $\mathfrak{h}$, and $\mathfrak{t} = \mathfrak{h}^+ + \mathfrak{a}^0$ is a Cartan subalgebra of $\mathfrak{g}^0$, by [7, Proposition 3.8]. We have $Z = Z^0 + Z'$, where $Z^0 = -\frac{1}{2}\sum_{\alpha \in \Delta} H_\alpha \in \mathfrak{h}^-$ and $Z' \in \mathfrak{h}^+$.

For every $\alpha \in \Delta$ we have $c_\alpha = \exp(\pi/4)X_\alpha \in G$; $c = \prod_{\alpha \in \Delta} c_\alpha$ is the *Cayley transform* of M. $\mathrm{ad}(c)$ has order 8 or 4. If it has order 4, M is said to be of *tube type*. This is equivalent to the fact that M can be realized as a tube domain over a self-dual cone (Remark 1 after Theorem 6.8 in [7]). In the general case in [7], Section 4, we described a construction leading to a symmetric subalgebra $\mathfrak{g}_1{}^0 = \mathfrak{k}_1 + \mathfrak{p}_1{}^0$ of $\mathfrak{g}^0$ which is of tube type. In Section 4 of the present paper we shall define certain subalgebras $\mathfrak{g}_\Gamma{}^0$ of $\mathfrak{g}$; the construction leading from $\mathfrak{g}^0$ to $\mathfrak{g}_1{}^0$ can also be performed for $\mathfrak{g}_\Gamma{}^0$, and gives rise to subalgebras $g_{\Gamma,1}{}^0 = k_{\Gamma,1} + p_{\Gamma,1}{}^0$. All these objects will be precisely defined as they occur; here we only wanted to point out the reason for our later notations.

In [7] we determined the Bergman-Šilov boundary $\check{S}$ of the bounded symmetric domain D. In the present paper we make a more detailed study of the boundary of D, and will show that it is a union of boundary components, which we describe explicitly. This notion was introduced Pjateckĭi-Šapiro [9] and is defined as follows: A subset F of the boundary ∂D of D is a *boundary component* if (i) F is locally an analytic set, and (ii) F is minimal with respect to the property that any analytic arc contained in ∂D and having a point in common with F must be entirely contained in F. From our result it follows at once that the Bergman-Šilov boundary of D is exactly the union of all 0-dimensional boundary components.

Index of Notations

Lie algebras and their subsets

$\mathfrak{g}^0$	Lie algebra of G^0, largest connected group of analytic automorphisms of the hermitian symmetric space $M = G^0/K$ of noncompact type
$\mathfrak{k}$	Lie algebra of K
$\mathfrak{p}^0$	(-1)-eigenspace of the symmetry σ on $\mathfrak{g}^0$

$\mathfrak{g}$	$\mathfrak{k}+\mathfrak{p}$, $\mathfrak{p}=i\mathfrak{p}^0$. $\mathfrak{g}$ is the Lie algebra of G where $M^*=G/K$ is the compact dual of M
$\mathfrak{z}$	center of $\mathfrak{k}$
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{k}$, thus also of $\mathfrak{g}^0$ and $\mathfrak{g}$
$\mathfrak{a}^0$	real span of all $X_\alpha{}^0$ $(\alpha\in\Delta)$, Cartan subalgebra of $(\mathfrak{g}^0,\mathfrak{k})$
$\mathfrak{h}^-;\mathfrak{h}^+$	$[\mathfrak{a}^0,J\mathfrak{a}^0]$; orthogonal complement of $\mathfrak{h}^-$ in $\mathfrak{h}$
$\mathfrak{t}$	$\mathfrak{h}^++\mathfrak{a}^0$, maximally split Cartan subalgebra of $\mathfrak{g}^0$
$\mathfrak{g}^C$	complexification of $\mathfrak{g}$, thus also of $\mathfrak{g}^0$

If $\mathfrak{v}$ is a real linear subspace of $\mathfrak{g}^C$:

$\mathfrak{v}^C$	complex span of $\mathfrak{v}$ in $\mathfrak{g}^C$
$\mathfrak{v}^+;\mathfrak{v}^-$	complex span of all positive, or all negative, root vectors in $\mathfrak{v}^C$, except where $\mathfrak{v}=\mathfrak{h},\mathfrak{n}_1{}^\Gamma,\mathfrak{n}_2{}^\Gamma,\mathfrak{n}^\Gamma,\mathfrak{r}_2{}^\Gamma$ or $\mathfrak{r}^\Gamma$.
$\mathfrak{v}^0$	$\mathfrak{v}^C\cap\mathfrak{g}^0$ for $\mathfrak{v}$ such that $\mathfrak{v}=\sigma(\mathfrak{v})$.

If $\mathfrak{v}^0=\sigma(\mathfrak{v}^0)\subset\mathfrak{g}^0$ then $\mathfrak{v}$ denotes $\mathfrak{v}^{0C}\cap\mathfrak{g}$.

If $\alpha\in\Delta$, then $\mathfrak{g}_\alpha$ is the real subalgebra spanned by iH_α, X_α and Y_α.

If Γ is an arbitrary subset of Δ:

$\mathfrak{g}_\Gamma{}^C$	derived algebra of $\mathfrak{h}^C+\sum_{\beta\perp\Delta-\Gamma}E_\beta\cdot C$.
$\mathfrak{k}_\Gamma;\mathfrak{p}_\Gamma;\mathfrak{a}_\Gamma;\mathfrak{h}_\Gamma{}^-$	intersection of $\mathfrak{g}_\Gamma$ with $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{h}^-$
$\mathfrak{p}_{\Gamma,1};\mathfrak{k}_{\Gamma,1};\mathfrak{g}_{\Gamma,1}$	$(+1)$-eigenspace of $\tau_\Gamma{}^2$ on $\mathfrak{p}_\Gamma$; $[\mathfrak{p}_{\Gamma,1},\mathfrak{p}_{\Gamma,1}]$; $\mathfrak{k}_{\Gamma,1}+\mathfrak{p}_{\Gamma,1}$
$\mathfrak{l}_{\Gamma,1};\mathfrak{q}_{\Gamma,1};\mathfrak{k}_{\Gamma,1}{}^*$	(±1)-eigenspaces of τ_Γ on $\mathfrak{k}_{\Gamma,1}$; $\mathfrak{l}_{\Gamma,1}+i\mathfrak{q}_{\Gamma,1}$
$\mathfrak{g}^\Gamma;\mathfrak{k}^\Gamma;\mathfrak{p}^\Gamma$	$(+1)$-eigenspace of $\tau_{\Delta-\Gamma}{}^2$ on $\mathfrak{g}$; $\mathfrak{g}^\Gamma\cap\mathfrak{k}$; $\mathfrak{g}^\Gamma\cap\mathfrak{p}$
$\mathfrak{k}_1{}^\Gamma;\mathfrak{p}_1{}^\Gamma;\mathfrak{g}_1{}^\Gamma$	$[\mathfrak{p}^\Gamma,\mathfrak{p}^\Gamma]$; $\mathfrak{p}^\Gamma$; $\mathfrak{k}_1{}^\Gamma+\mathfrak{p}_1{}^\Gamma$
$\mathfrak{l}_1{}^\Gamma;\mathfrak{q}_1{}^\Gamma;\mathfrak{k}_1{}^{\Gamma*}$	(±1)-eigenspaces of $\tau_{\Delta-\Gamma}$ on $\mathfrak{k}_1{}^\Gamma$; $\mathfrak{l}_1{}^\Gamma+i\mathfrak{q}_1{}^\Gamma$
$\mathfrak{l}_2{}^\Gamma$	centralizer of $\mathfrak{g}_1{}^\Gamma$ in $\mathfrak{g}^\Gamma$
$\mathfrak{q}_2{}^\Gamma;\mathfrak{p}_2{}^\Gamma$	(-1)-eigenspace of $\tau_{\Delta-\Gamma}{}^2$ on $\mathfrak{k}$; on $\mathfrak{p}$
$\mathfrak{l}^\Gamma;\mathfrak{q}^\Gamma;\mathfrak{k}^{\Gamma*}$	$\mathfrak{l}_1{}^\Gamma+\mathfrak{l}_2{}^\Gamma$; $\mathfrak{q}_1{}^\Gamma+\mathfrak{q}_2{}^\Gamma$; $\mathfrak{l}^\Gamma+i\mathfrak{q}_1{}^\Gamma$
$\mathfrak{r}_2{}^{\Gamma\pm};\mathfrak{r}^{\Gamma\pm}$	$\mathfrak{q}_2{}^{\Gamma\mp}+\mathfrak{p}_2{}^{\Gamma\pm}$; $\mathfrak{p}_{\Delta-\Gamma,1}{}^\pm+\mathfrak{r}_2{}^{\Gamma\pm}$
$\mathfrak{n}_2{}^{\Gamma\pm};\mathfrak{n}_1{}^{\Gamma\pm};\mathfrak{n}^{\Gamma\pm}$	$\mathfrak{r}_2{}^{\Gamma\pm}\cap\operatorname{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$; $\mathfrak{p}_{\Delta-\Gamma,1}{}^\pm\cap\operatorname{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$; $\mathfrak{n}_1{}^{\Gamma\pm}+\mathfrak{n}_2{}^{\Gamma\pm}$
$\mathfrak{b}^\Gamma$	Lie algebra of B^Γ
c_Γ	$K_{\Delta-\Gamma,1}{}^*(-io^\Gamma)$

Mappings

σ	symmetry of $\mathfrak{g}$ or of $\mathfrak{g}^0$
$\nu;\nu^0$	complex conjugation of $\mathfrak{g}^C$ over $\mathfrak{g}$; over $\mathfrak{g}^0$
J	$\operatorname{ad}(Z)$
$\tau;\tau_\Gamma$	$\operatorname{ad}(c)^2$; $\operatorname{ad}(c_\Gamma)^2$
ξ	Harish-Chandra's map $\mathfrak{p}^-\to M^*$ given by $E\to\exp(E)(x)$

Subgroups and submanifolds

G^C	adjoint group of $\mathfrak{g}^C$
Capital roman letter corresponding to a small german letter	corresponding analytic subgroup of G^C, with the following exceptions
$L_1{}^\Gamma; L^\Gamma; E^\Gamma$	isotropy subgroup at x^Γ of $K_1{}^\Gamma$; of K; of G^0
B^Γ	subgroup of G^0 preserving the set $c_{\Delta-\Gamma}M_\Gamma$
$M_\Gamma; M_{\Gamma,1}; M^\Gamma$	submanifolds $G_\Gamma{}^0(x)$; $G_{\Gamma,1}{}^0(x)$; $G^{\Gamma 0}(x)$ of M
$M_\Gamma{}^*; M_{\Gamma,1}{}^*; M^{\Gamma *}$	submanifolds $G_\Gamma(x)$; $G_{\Gamma,1}(x)$; $G^\Gamma(x)$ of M^*
$D; D_\Gamma$	$\xi^{-1}(M)$; $D \cap \mathfrak{p}_\Gamma{}^-$
$S^\Gamma; S_D{}^\Gamma$	set of boundary components of type Γ of M; of D
$U^\Gamma; U_D{}^\Gamma$	union of boundary components of type Γ of M; of D
$\check{S}; \check{S}_D$	Bergman-Šilov boundary S^ϕ of M in M^*; $S_D{}^\phi$ of D in $\mathfrak{p}^-$

Group, algebra and manifold elements

$H_\alpha, E_\alpha, \cdots$	standard basis elements of $\mathfrak{g}^C$
Δ	maximal set of strongly orthogonal noncompact positive roots
$X_\alpha{}^0; X_\alpha$	$E_\alpha + E_{-\alpha} \in \mathfrak{p}^0$; $iX_\alpha{}^0 \in \mathfrak{p}$
$Y_\alpha{}^0; Y_\alpha$	$-i(E_\alpha - E_{-\alpha}) \in \mathfrak{p}^0$; $iY_\alpha{}^0 \in \mathfrak{p}$
Z	element of $\mathfrak{z}$ such that $\mathrm{ad}(Z)E = \pm iE$ for $E \in \mathfrak{p}^\mp$
$Z^0; Z'$	$-(i/2)\sum_{\alpha\in\Delta} H_\alpha$; $Z - Z^0$
$X_\Gamma{}^0; Y_\Gamma{}^0; Z_\Gamma{}^0$	$\sum_{\alpha\in\Gamma} X_\alpha{}^0$; $\sum_{\alpha\in\Gamma} Y_\alpha{}^0$; $-i/2 \sum_{\alpha\in\Gamma} H_\alpha$
$X_\Gamma; Y_\Gamma$	$iX_\Gamma{}^0$; $iY_\Gamma{}^0$
$c_\alpha (\alpha \in \Delta)$; c; c_Γ	$\exp((\pi/4)X_\alpha) \in G$; $\prod_{\alpha\in\Delta} c_\alpha$; $\prod_{\alpha\in\Gamma} c_\alpha$
x	identity coset in $M^* = G^C/K^C P^+$
x^Γ	$c_{\Delta-\Gamma}(x)$
$\mathfrak{o}; \mathfrak{o}^\Gamma$	zero elements $\xi^{-1}(x)$ of $\mathfrak{p}^-$; $\xi^{-1}(x^\Gamma)$

3. A theorem on real parabolic groups. We will classify a certain family of real parabolic subgroups of the Lie groups which are the connected groups of analytic automorphisms of the bounded symmetric domains. In Corollary 6.9 it will be seen that those parabolic subgroups are just the stability groups of the various boundary components. We will also need the notion of parabolic group in our proof of Theorem 6.8.

The goal of this section is Theorem 3.4, which resulted from a conversation between J. Tits and one of the authors. All the other results of this section are special cases of theorems of A. Borel and J. Tits [2] on linear algebraic groups.

3.1. *Parabolic subgroups of complex Lie groups.* Let E be a complex connected Lie group. Then the maximal solvable subgroups of E are all closed, complex, connected and conjugate; they are called the *Borel subgroups* of E and their Lie algebras are the *Borel subalgebras* of $\mathfrak{e}$. If a complex Lie subgroup of E contains a Borel subgroup, then it and its Lie algebra are called *parabolic.* Every parabolic subgroup $F \subset E$ is connected, for every component of F contains an element which normalizes a Borel subgroup B of F_0 (and thus of E), and it follows that this component must be F_0 because it contains an element which centralizes a Cartan subgroup of E which lies in B. Similarly every parabolic subgroup $F \subset E$ is its own normalizer. As every Borel subgroup of E contains the radical of E, we may pass to a quotient and restrict our study to the case where E is semisimple.

Let E be a connected complex semisimple Lie group. Choose a Cartan subalgebra $\mathfrak{c}$ of the Lie algebra $\mathfrak{e}$, let Λ denote the root system of $\mathfrak{e}$ relative to $\mathfrak{c}$, and choose a simple system Ψ of roots. If $\mathfrak{e}_\lambda$ denotes the root space for $\lambda \in \Lambda$, and if Λ^+ denotes the set of positive roots, then our choices amount to the choice of the Borel subalgebra

$$\mathfrak{b} = \mathfrak{c} + \sum_{\lambda \in \Delta^+} \mathfrak{e}_\lambda$$

of $\mathfrak{e}$. Now let $\Phi \subset \Psi$, and define

$$\Phi^+ = \{\lambda \in \Lambda : \lambda = \sum_{\alpha \in \Psi} a_\alpha \alpha \text{ with } a_\alpha > 0 \text{ for some } \alpha \in \Phi\},$$

$$\Phi^0 = \{\lambda \in \Lambda : \lambda = \sum_{\alpha \in \Psi} a_\alpha \alpha \text{ with } a_\alpha = 0 \text{ for every } \alpha \in \Phi\}, \text{ and}$$

$$\Phi^* = \Phi^0 \cup \Phi^+ = \{\lambda \in \Lambda : \lambda = \sum_{\alpha \in \Psi} a_\alpha \alpha \text{ with } a_\alpha \geqq 0 \text{ for every } \alpha \in \Phi\}.$$

Then

$$\mathfrak{f}_\Phi = \mathfrak{c} + \sum_{\lambda \in \Phi^*} \mathfrak{e}_\lambda$$

is a parabolic subalgebra of $\mathfrak{e}$ which contains $\mathfrak{b}$. $\mathfrak{f}_\Psi = \mathfrak{b}$, $\mathfrak{f}_\phi = \mathfrak{e}$, and $\mathfrak{f}_\Sigma \subset \mathfrak{f}_\Gamma$ for $\Gamma \subset \Sigma \subset \Psi$. Conversely, let $\mathfrak{f}$ be a parabolic subalgebra of $\mathfrak{e}$ which contains $\mathfrak{b}$, and define

$$\Phi = \{\alpha \in \Psi : a_\alpha \geqq 0 \text{ whenever } \mathfrak{e}_\lambda \subset \mathfrak{f} \text{ with } \lambda = \sum_{\beta \in \Psi} a_\beta \beta\}.$$

Then it is routine to check that $\mathfrak{f} = \mathfrak{f}_\Phi$. Now we can specify a conjugacy class of parabolic subgroups of E by marking the elements of $\Psi - \Phi$ on the Dynkin diagram of $\mathfrak{e}$, where $\mathfrak{f}_\Phi$ is the Lie algebra of an element of this class.

Retain the notation just above, and define (for every subset $\Phi \subset \Psi$)

$$\mathfrak{c}_\Phi = \bigcap_{\alpha \in \Psi - \Phi} (\text{kernel of } \alpha),$$

$$\mathfrak{r}_\Phi = \mathfrak{c} + \sum_{\lambda \in \Phi^0} \mathfrak{e}_\lambda, \text{ and}$$

$$\mathfrak{u}_\Phi = \sum_{\lambda \in \Phi^+} \mathfrak{e}_\lambda.$$

Then $\mathfrak{f}_\Phi = \mathfrak{r}_\Phi + \mathfrak{u}_\Phi$ (semidirect sum) and is the normalizer of $\mathfrak{u}_\Phi$ in $\mathfrak{e}$. $\mathfrak{u}_\Phi$ is nilpotent, $\mathfrak{r}_\Phi$ is reductive in $\mathfrak{e}$ because it is the centralizer of $\mathfrak{c}_\Phi$ in $\mathfrak{e}$, and $\mathfrak{c}_\Phi$ is the center of $\mathfrak{r}_\Phi$. Let R_Φ, U_Φ and F_Φ denote the analytic subgroups of E for the subalgebras $\mathfrak{r}_\Phi$, $\mathfrak{u}_\Phi$ and $\mathfrak{f}_\Phi$ of $\mathfrak{e}$. Then $F_\Phi = R_\Phi \cdot U_\Phi$ semidirect product. We may view E as a linear algebraic group because it is complex, connected and semisimple, and then this semidirect product decomposition of P_Φ is the Chevalley decomposition into reductive and unipotent parts.

3.2. *Parabolic subgroups of real Lie groups.* Let E' be a connected semisimple real Lie group embedded in its complexification. In other words there is a complex connected semisimple Lie group E and a real form $\mathfrak{e}'$ of the complex Lie algebra $\mathfrak{e}$ such that E' is the real analytic subgroup of E with Lie algebra $\mathfrak{e}'$. We will say that a subgroup $F' \subset E'$ is a *parabolic subgroup* of E' if there exists a parabolic subgroup $F \subset E$ such that (i) $F' = E' \cap F$ and (ii) $\mathfrak{f}$ is the complexification of $\mathfrak{f}'$. If F' is a parabolic subgroup of E' and F_0' denotes the identity component, then any element $f \in F'$ can be altered by an element of F_0' to centralize a Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{e}'$ contained in $\mathfrak{f}'$, and it follows that $F' = (C \cap E') \cdot F_0'$ where $C = \exp(\mathfrak{c})$ and $\mathfrak{c}$ is the Cartan subalgebra of $\mathfrak{e}$ which is the complexification of $\mathfrak{c}'$.

Let $\mathfrak{c}'$ be a Cartan subalgebra of $\mathfrak{e}'$. Then there is a canonical decomposition $\mathfrak{c}' = \mathfrak{c}_t + \mathfrak{c}_v$ where the roots are real valued on $\mathfrak{c}_v$ and take pure imaginary values on $\mathfrak{c}_t$. To obtain this decomposition, consider the Cartan subgroup $C' = \exp(\mathfrak{c}') \subset E'$. C' has a unique maximal compact subgroup C_t, and $\mathfrak{c}_t$ is the corresponding subalgebra; it is clear that the roots take pure imaginary values on $\mathfrak{c}_t$. Let $\mathfrak{c}_v$ be the orthogonal complement of $\mathfrak{c}_t$ in $\mathfrak{c}'$ under the Killing form. Then $C_v = \exp(\mathfrak{c}_v)$ is a vector subgroup of C'. If $\mathfrak{c}_v$ and and C_v were not diagonable in $\mathrm{ad}(\mathfrak{e}')$ on $\mathfrak{e}'$, C_t would not be maximal. Thus the roots are real valued on $\mathfrak{c}_v$. C_t (resp. $\mathfrak{c}_t$) and C_v (resp. $\mathfrak{c}_v$) are the *totally non-split* and the *split* parts of C' (resp. $\mathfrak{c}'$). C' and $\mathfrak{c}'$ are *maximally* *split* if $\dim . \mathfrak{c}_v$ is maximal among the dimensions of the split parts of the Cartan subalgebra of $\mathfrak{e}'$.

LEMMA. *The parabolic subgroups of E' are just the subgroups $F' = F \cap E'$ for which there exist* (a) *a maximally split Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{e}'$,* (b) *a system Ψ of simple roots of $\mathfrak{e}$ for $\mathfrak{c} = \mathfrak{c}'^{C}$ and* (c) *a subset $\Phi \subset \Psi$, such that* (i) *$\mathfrak{c}'_\Phi$ is a real form of $\mathfrak{c}_\Phi$,* (ii) *$\mathfrak{c}'_\Phi$ and its split part $\mathfrak{c}'_\Phi \cap \mathfrak{c}_v$ have the same centralizer in $\mathfrak{e}$,* (iii) *$\mathfrak{f}_\Phi$ is the sum of the non negative weight spaces of $\mathrm{ad}(\mathfrak{c}'_\Phi \cap \mathfrak{c}_v)$ on $\mathfrak{e}$,* and (iv) *$F = F_\Phi$.*

Proof. Let ν be conjugation of $\mathfrak{e}$ over $\mathfrak{e}'$. Given $\mathfrak{c}'$, Ψ and Φ satisfying (i)-(iv), we recall that the roots of $\mathfrak{e}$ are real-valued on $\mathfrak{c}_v$. Thus every weight space of $\mathrm{ad}(\mathfrak{c}'_\Phi \cap \mathfrak{c}_v)$ on $\mathfrak{e}$ is stable under ν. Now (iii) says that $\mathfrak{f}_\Phi = (\mathfrak{f}_\Phi \cap \mathfrak{e}')^{C}$, so (iv) tells us that $F' = F \cap E'$ is parabolic in E'.

Let F' be a parabolic subgroup of E'. Then $F' = F \cap E'$ for some parabolic subgroup F of E, and $\nu(\mathfrak{f}) = \mathfrak{f}$. ν preserves the maximal nilpotent normal subalgebra $\mathfrak{u}$ of $\mathfrak{f}$ and we choose a ν-invariant reductive complement $\mathfrak{r}$. If $\mathfrak{c}_*$ denotes the center of $\mathfrak{r}$, then $\nu(\mathfrak{c}_*) = \mathfrak{c}_*$ so $\mathfrak{c}'_* = \mathfrak{c}_* \cap \mathfrak{e}'$ is a real form of $\mathfrak{c}_*$. There is a lexicographic ordering on the dual space of the real form $\mathfrak{c}_{*v} + i\mathfrak{c}_{*t}$ of $\mathfrak{c}_*$, such that $\mathfrak{u}$ is the sum of the positive weight spaces. We extend $\mathfrak{c}_*$ to a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{e}$ for which $\mathfrak{c}' = \mathfrak{c} \cap \mathfrak{e}'$ is a real form, in such a manner as to maximize the dimension of the split part of $\mathfrak{c}'$; we then extend the ordering of weights on $\mathfrak{c}_*$ to an ordering of the $\mathfrak{c}$-roots of $\mathfrak{e}$. Let Ψ be the corresponding system of simple roots. Now $F = F_\Phi$ for some subset $\Phi \subset \Psi$.

We must check that the $\mathfrak{c}'$, Ψ and Φ just constructed satisfy the conditions (i), (ii) and (iii) and that $\mathfrak{c}'$ is maximally split. Condition (i) is immediate because $\mathfrak{c}_* = \mathfrak{c}_\Phi$ from the construction of complex parabolic algebras. The split part $\mathfrak{c}_{*v} = \mathfrak{c}'_\Phi \cap \mathfrak{c}_v$ of $\mathfrak{c}'_\Phi$ is nonzero because the sum of the elements of Φ induces a positive linear functional on it. Now $\mathfrak{e} \supsetneqq \mathfrak{r}_* \supset \mathfrak{r}$ where $\mathfrak{r}_*$ is the centralizer of $\mathfrak{c}_{*v}$ and $\mathfrak{r}$, the reductive part of $\mathfrak{f}$, is the centralizer of $\mathfrak{c}_\Phi$. If $\mathfrak{r}_* \neq \mathfrak{r}$ then $\mathfrak{u} \cap \mathfrak{r}_* \neq 0$, so $\mathfrak{u} \cap \mathfrak{r}_*$ is a nontrivial sum of root spaces. The roots which enter into this sum belong to Φ^+, because $\mathfrak{u} \cap \mathfrak{r}_* \subset \mathfrak{u}$, so their negatives do not appear. Thus the roots which enter the sum must vanish on $\mathfrak{c}_{*t}$, and thus on $\mathfrak{c}'_* = \mathfrak{c}_{*t} + \mathfrak{c}_{*v}$. That is impossible. Now $\mathfrak{r}_* = \mathfrak{r}$ and (ii) is proved. (iii) follows by our ordering of roots so that $\mathfrak{c}_{*v}$ precedes $i\mathfrak{c}_{*t}$. From (ii) we also see that $\mathfrak{r}'$ contains a maximally split Cartan subalgebra of $\mathfrak{e}'$, so the maximality condition in our choice of $\mathfrak{c}$ implies that $\mathfrak{c}'$ is maximally split in e'. *Q.E.D.*

Lemma 3.2 shows that real parabolic groups are determined by the split parts of the centers of their reductive parts. Let F' be any parabolic subgroup of E'. In the notation of Lemma 3.2, let B' be the group $B \cap E'$ where $\mathfrak{b}$ is the sum of the non negative weight spaces of $\mathrm{ad}(\mathfrak{c}_v)$ on $\mathfrak{e}$. Then B' is a parabolic subgroup of E', $B' \subset F'$, and every parabolic subgroup of E' contains a conjugate of B'. B' is a *minimal parabolic subgroup* of E'. There is an Iwasawa decomposition $E' = K \cdot A \cdot N$ such that $B' = L \cdot A \cdot N$ where $L \subset K$ is the centralizer of A. Furthermore there is a subset $\Sigma \subset \Psi$ defined by $B = F_\Sigma$, such that $\Phi \subset \Sigma$ whenever $F_\Phi \cap E'$ is parabolic in E'.

3.3. *The action of the Galois group.* The Galois group of C over R acts on $\mathfrak{e}$ as the conjugation ν of $\mathfrak{e}$ over $\mathfrak{e}'$. This is a real automorphism of $\mathfrak{e}$ and induces a real automorphism of E. The fixed point set of ν on E is E'. Let F be a parabolic subgroup of E. Now $F \cap E'$ is parabolic in E' if and only if $\nu(F) = F$, and this is equivalent to $\nu(\mathfrak{f}) = \mathfrak{f}$.

Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{e}$ which is the complexification of a maximality split Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{e}'$, and let Ψ be a system of simple roots. Then the Galois group $\{1, \nu\}$ acts on Ψ as follows. The subsets of Ψ are in one-one correspondence with the conjugacy classes of parabolic subgroups of E, a subset Φ corresponding to the class of F_Φ. Given Φ, $\nu(F_\Phi)$ is conjugate to some F_Σ and we define $\Sigma = \Phi^\nu$. This transformation on the subsets of Ψ is induced by its restriction to the one-point subsets, so ν acts on Ψ. If $E' \cap F_\Phi$ is parabolic in E', then $\nu(F_\Phi) = F_\Phi$, and so $\Phi^\nu = \Phi$. The converse is:

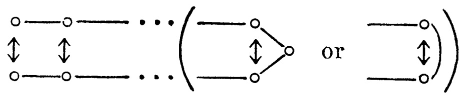
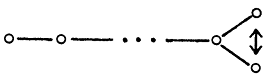
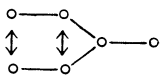
LEMMA. *Let Ψ be a system of simple roots of $\mathfrak{e}$ for the complexification of a maximally split Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{e}'$, and let Σ be the subset of Ψ such that $E' \cap F_\Sigma$ is a minimal parabolic subgroup of E'. Then the parabolic subgroups of E' are just the conjugates of the groups $E' \cap F_\Phi$ for which $\Phi \subset \Sigma$ and $\Phi^\nu = \Phi$.*

Proof. The remark above and the results of § 3.2 show that $\Phi \subset \Sigma$ and $\Phi^\nu = \Phi$ in case $E' \cap F_\Phi$ is parabolic in E'.

Let $\Phi \subset \Sigma$ and $\Phi^\nu = \Phi$; we will check that $E' \cap F_\Phi$ is parabolic in E'. If two parabolic subgroups of E are conjugate and contain the same Borel subgroup, then they must be the same. Now if two parabolic subgroups of E are conjugate and contain F_Σ they must be the same, for F_Σ contains a Borel subgroup. $\nu(F_\Sigma) = F_\Sigma$, and $F_\Sigma \subset F_\Phi$ because $\Phi \subset \Sigma$; thus F_Φ and $\nu(F_\Phi)$ contain F_Σ. $\Phi = \Phi^\nu$ says that F_Φ is conjugate to $\nu(F_\Phi)$. Thus $F_\Phi = \nu(F_\Phi)$. This proves that $E' \cap F_\Phi$ is parabolic in E'. *Q.E.D.*

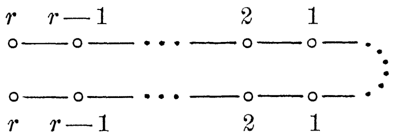
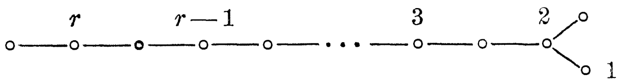
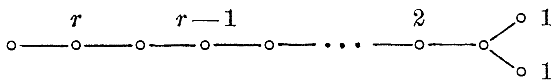
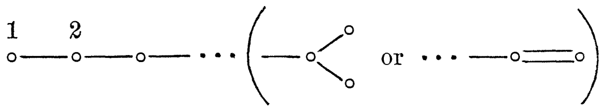
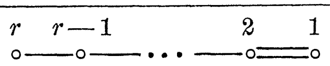
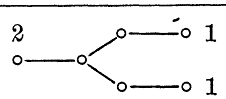
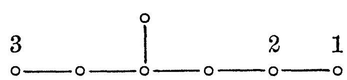
In order to apply the Lemma, one must know the action of the Galois group on Ψ.

COMPLEMENT TO LEMMA. *Let $\mathfrak{e}'$ be a real simple Lie algebra with simple complexification $\mathfrak{e}$, let ν be the nontrivial element of the Galois group of C over R, and let Ψ be a system of simple roots of $\mathfrak{e}$ for the complexification of a maximally split Cartan subalgebra of $\mathfrak{e}'$. Then the action of ν on Ψ is trivial except in the following cases.*

$\mathfrak{e}'$	action of ν on Dynkin diagram
$\mathfrak{su}^k(n)$	
$\mathfrak{so}^*(4n+2)$ *or* $\mathfrak{so}^{2k}(4n+2)$ *or* $\mathfrak{so}^{2k+1}(4n)$	
$\mathfrak{e}_{6(-14)}$ *or* $\mathfrak{e}_{6(+2)}$	

Proof. $\mathfrak{e}'$ has a maximal compactly embedded subalgebra $\mathfrak{k}$ such that, if σ denotes the symmetry of the symmetric pair $(\mathfrak{e}', \mathfrak{k})$ and $\mathfrak{e}' = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition, $\mathfrak{k} \cap \mathfrak{c}' = \mathfrak{c}_t$ and $\mathfrak{p} \cap \mathfrak{c}' = \mathfrak{c}_v$. The root vectors are in $i\mathfrak{c}_t + \mathfrak{c}_v$, so ν is -1 on their $\mathfrak{c}_t$ projections and is $+1$ on their $\mathfrak{c}_v$ projections. σ is complex linear on $\mathfrak{e}$, $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$; thus $\sigma\nu$ sends each root vector to its negative. Thus (i) if σ is an inner automorphism, then ν is trivial on Ψ precisely in case $-I$ is in the Weyl group of $\mathfrak{e}$, and (ii) if σ is an outer automorphism, then ν is trivial on Ψ precisely in case $-I$ is not in the Weyl group. As $-I$ is in the Weyl group in all cases except $\mathfrak{e} = A_n$ $(n > 1)$, D_{2n+1} $(n > 1)$, or E_6 [12, Theorem 4.1], the result follows from the classification of the real simple Lie algebras. *Q.E.D.*

3.4. THEOREM. *Let M be an irreducible hermitian symmetric space of noncompact type; let G^0 be the connected group of analytic automorphisms of M, embedded in its complexification G^C; let $\mathfrak{u}$ be a maximally split Cartan subalgebra of $\mathfrak{g}^0$ which is preserved by the symmetry at a point $x \in M$, and let Ψ be a system of simple roots of g^C for $\mathfrak{u}^C$. If r is the rank of M, then there is a unique sequence $\{\Phi_0, \Phi_1, \cdots, \Phi_r\}$ of subsets of Ψ such that* (i) *$F_{\Phi_i}{}^0 = G^0 \cap F_{\Phi_i}$ is a parabolic subgroup of G^0,* (ii) *the reductive part of $F_{\Phi_i}{}^0$ has a simple normal subgroup $G_i{}^0$ such that $G_i{}^0(x)$ is a hermitian symmetric subspace of rank i in M, and* (iii) *the $G_i{}^0$ can be chosen so that $G_0{}^0 \subset G_1{}^0 \subset \cdots \subset G_r{}^0$. Furthermore, $G_r{}^0 = F_{\Phi_r}{}^0 = G^0$ and Φ_i consists of the elements of Ψ numbered $\{i+1, \cdots, r-1, r\}$ in the chart below.*

$\mathfrak{g}^0$	*Dynkin diagram*
$\mathfrak{su}^r(n)$ $(2r \leqq n)$	
$\mathfrak{so}^*(4r)$	
$\mathfrak{so}^*(4r+2)$	
$\mathfrak{so}^2(n+2)$ $(r=2)$	
$\mathfrak{sp}(r, R)$	
$\mathfrak{e}_{6(-14)}$ $(r=2)$	
$\mathfrak{e}_{7(-25)}$ $(r=3)$	

Proof. To satisfy (ii) and (iii) we must have $\Phi_r \subsetneq \Phi_{r-1} \subsetneq \cdots \subsetneq \Phi_0$. Let $\Sigma \subset \Psi$ so that $G^0 \cap F_\Sigma$ is a minimal parabolic subgroup of G^0. Then $\Phi_0 \subset \Sigma$ by (i). The Galois group has exactly r orbits on Σ, and each Φ_i is a union of orbits by (iii). It follows that we can number the orbits as $\Sigma_1, \cdots, \Sigma_r$ so that $\Phi_i = \Sigma_{i+1} \cup \cdots \cup \Sigma_r$; $\Phi_r \neq \phi$ and $\Phi_0 = \Sigma$. Now a case by case check, using the fact that $\mathfrak{g}_i{}^0$ must be one of the algebras listed for the $\mathfrak{g}^0$ with $i = r$, and using the fact that the Dynkin diagram of $\mathfrak{g}_i{}^0$ must be a connected component of the complement of $\Sigma_{i+1} \cup \cdots \cup \Sigma_r$ in the diagram of $\mathfrak{h}^0$, shows that (ii) and (iii) imply that Σ_i must consist of the points numbered i in the Dynkin diagram, and that then (i), (ii) and (iii) are satisfied. The Theorem follows. *Q.E.D.*

4. The boundary components of a bounded symmetric domain. Let D be a bounded symmetric domain embedded in $\mathfrak{p}^-$ as described in § 2. Retain the notation of § 2, and let Γ be an arbitrary subset of the maximal set Δ of strongly orthogonal noncompact roots. We will see that every $\Gamma \subsetneq \Delta$ corresponds to a certain boundary component of D, the empty set ϕ corresponding to a point of the Bergman-Šilov boundary, and Δ corresponding to D. It turns out that two subsets of Δ give analytically equivalent boundary components if and only if they contain the same number of roots from each irreducible factor of M.

4.1. We have $c_\alpha = \exp(\pi/4) X_\alpha \in G$ for every $\alpha \in \Delta$, and the Cayley transform on M was defined to be $c = \prod_{\alpha \in \Delta} c_\alpha$. We now define *partial Cayley transforms* by

$$c_\Gamma = \prod_{\alpha \in \Gamma} c_\alpha, \quad \Gamma \subset \Delta,$$

so $c_\Delta = c$ and $c_\phi = 1$. Similarly we define

$$X_\Gamma{}^0 = \sum_{\alpha \in \Gamma} X_\alpha \text{ and } X_\Gamma = iX_\Gamma{}^0,$$

$$Y_\Gamma{}^0 = \sum_{\alpha \in \Gamma} Y_\alpha{}^0 \text{ and } Y_\Gamma = iY_\Gamma{}^0, \text{ and}$$

$$Z_\Gamma{}^0 = -\tfrac{1}{2} \sum_{\alpha \in \Gamma} H_\alpha \text{ and } \mathfrak{h}_\Gamma{}^- = \sum_{\alpha \in \Gamma} iH_\alpha \cdot R.$$

By definition of H_β the centralizer of $\mathfrak{h}_{\Delta-\Gamma}{}^-$ in $\mathfrak{g}^C$ is

$$\mathfrak{h}^C + \sum_{\beta \perp \Delta - \Gamma} E_\beta \cdot C. \tag{4.1.1}$$

The centralizer of $\sum_{\alpha \in \Delta - \Gamma} \mathfrak{g}_\alpha$ in $\mathfrak{g}^C$ is

$$\mathfrak{h}^{+C} + \mathfrak{h}_\Gamma{}^{-C} + \sum_{\beta \perp \Delta - \Gamma} E_\beta \cdot C. \tag{4.1.2}$$

For this, it suffices to show that $\beta \perp \Delta - \Gamma$ implies $[E_\beta, E_{\pm\alpha}] = 0$ whenever $\alpha \in \Delta - \Gamma$. Here we may assume $\beta > 0$; then the assertion is trivial for β noncompact and known [4, Lemma 13] for β compact.

The algebras (4.1.1) and (4.1.2) are reductive and have the same derived algebra. We denote this derived algebra by $\mathfrak{g}_\Gamma{}^C$, and $\mathfrak{g}_\Gamma = \mathfrak{g} \cap \mathfrak{g}_\Gamma{}^C$ and $\mathfrak{g}_\Gamma{}^0 = \mathfrak{g}_\Gamma{}^0 \cap \mathfrak{g}_\Gamma{}^C$ are real forms of $\mathfrak{g}_\Gamma{}^C$; they are semisimple. We have $\mathfrak{g}_\Gamma = \mathfrak{k}_\Gamma + \mathfrak{p}_\Gamma$ and $\mathfrak{g}_\Gamma{}^0 = \mathfrak{k}_\Gamma + \mathfrak{p}_\Gamma{}^0$ where $\mathfrak{k}_\Gamma = \mathfrak{g}_\Gamma \cap \mathfrak{k} = \mathfrak{g}_\Gamma{}^0 \cap \mathfrak{k}$, $\mathfrak{p}_\Gamma = \mathfrak{g}_\Gamma \cap \mathfrak{p}$ $\mathfrak{p}_\Gamma{}^0 = \mathfrak{g}_\Gamma{}^0 \cap \mathfrak{p}^0$, and $\mathfrak{p}_\Gamma{}^0 = i\mathfrak{p}_\Gamma$. $\mathfrak{p}_\Gamma{}^\pm$ denotes $\mathfrak{p}_\Gamma{}^C \cap \mathfrak{p}^\pm$. G_Γ and $G_\Gamma{}^0$ denote the respective analytic subgroups of G^C with Lie algebras $\mathfrak{g}_\Gamma$ and $\mathfrak{g}_\Gamma{}^0$, and K_Γ denotes their common intersection with K.

Further, we define $M_\Gamma = G_\Gamma{}^0(x)$, $M_\Gamma{}^* = G_\Gamma(x)$, and $x^\Gamma = c_{\Delta-\Gamma}(x)$. In the special case where Γ is empty, M_Γ and $M_\Gamma{}^*$ are just $\{x\}$ and x^Γ is the point $c(x)$ on the Bergman-Šilov boundary. More generally, we will eventually see that $c_{\Delta-\Gamma}(M_\Gamma) = G_\Gamma{}^0(x^\Gamma)$ and is a typical boundary component of M in M^*.

Finally define $D_\Gamma = D \cap \mathfrak{p}_\Gamma{}^-$ and $\mathfrak{o}^\Gamma = \xi^{-1}(x^\Gamma)$; $\mathfrak{o}$ will denote the origin, $\mathfrak{o} = \mathfrak{o}^\Delta = 0$, of $\mathfrak{p}^-$.

4.2. LEMMA. *M_Γ is a complex totally geodesic submanifold of M, thus being a sub hermitian symmetric space of M; the same is true for $M_\Gamma{}^*$ in M^*, and $M_\Gamma \subset M_\Gamma{}^*$ is the Borel embedding. Γ is a maximal set of strongly orthogonal noncompact roots of $\mathfrak{g}_\Gamma{}^0$, $\{X_\alpha{}^0\}_{\alpha\in\Gamma}$ spans a Cartan subalgebra $\mathfrak{a}_\Gamma{}^0 = \mathfrak{a}^0 \cap \mathfrak{g}_\Gamma{}^0$ of $(\mathfrak{g}_\Gamma{}^0, \mathfrak{k}_\Gamma)$, and c_Γ is the Cayley transform of M_Γ. Let $\xi^{-1}: M \to \mathfrak{p}^-$ be the Harish-Chandra embedding as a bounded domain $D = \xi^{-1}(M)$; then $D_\Gamma = \xi^{-1}(M_\Gamma)$ and $\xi^{-1}: M_\Gamma \to \mathfrak{p}_\Gamma{}^-$ is the Harish-Chandra embedding. $c_{\Delta-\Gamma}(M_\Gamma) \subset \xi(\mathfrak{p}^-)$, $\xi^{-1}c_{\Delta-\Gamma}\xi$ acts on D_Γ by $E \to E + i \sum_{\beta\perp\Delta-\Gamma} E_{-\alpha}$.*

Proof. The algebras 4.1.1 and 4.1.2 are preserved by $\operatorname{ad}(\mathfrak{h})$, and thus by $\operatorname{ad}(Z)$, so $\operatorname{ad}(Z)$ preserves $\mathfrak{g}_\Gamma{}^C$; as $\operatorname{ad}(Z)$ preserves $\mathfrak{g}$ and $\mathfrak{g}^0$, it must preserve $\mathfrak{g}_\Gamma$ and $\mathfrak{g}_\Gamma{}^0$; now $\operatorname{ad}(\exp(tZ))$ preserves G_Γ and $G_\Gamma{}^0$, so $M_\Gamma{}^* \subset M^*$ and $M_\Gamma \subset M$ are sub hermitian symmetric spaces. $M_\Gamma \subset M_\Gamma{}^*$ is the Borel embedding by construction.

The definitions of $\mathfrak{g}_\Gamma{}^0$ and $\mathfrak{g}_{\Delta-\Gamma}{}^0$ give us

$$\sum_{\alpha\in\Gamma} X_\alpha{}^0 \cdot R \subset \mathfrak{a}_\Gamma{}^0, \quad \Big(\sum_{\alpha\in\Gamma} X_\alpha{}^0 \cdot R\Big) \cap \mathfrak{a}_{\Delta-\Gamma}{}^0 = 0, \text{ and } \sum_{\alpha\in\Delta-\Gamma} X_\alpha{}^0 \cdot R \subset \mathfrak{a}_{\Delta-\Gamma}{}^0;$$

linear independence of the $X_\alpha{}^0$ now shows that $\mathfrak{a}_\Gamma{}^0$ has $\{X_\alpha{}^0\}_{\alpha\in\Gamma}$ for a basis. If $\mathfrak{a}_\Gamma{}^0$ is not a Cartan subalgebra of $(\mathfrak{g}_\Gamma{}^0, \mathfrak{k}_\Gamma)$, then it is properly contained in one, say in $\mathfrak{e}$. $[\mathfrak{e}, \mathfrak{a}_{\Delta-\Gamma}{}^0] = 0$ by definition of $\mathfrak{g}_\Gamma{}^0$, so we have

$$\mathfrak{a}^0 = \mathfrak{a}_\Gamma{}^0 + \mathfrak{a}_{\Delta-\Gamma}{}^0 \subsetneqq \mathfrak{e} + \mathfrak{a}_{\Delta-\Gamma}{}^0 \subset \mathfrak{f}$$

where $\mathfrak{f}$ is a Cartan subalgebra of $\mathfrak{g}_\Gamma{}^0$. Then $\dim.\mathfrak{f} = \dim.\mathfrak{a}^0$, which is a contradiction. Now $\mathfrak{a}_\Gamma{}^0$ is a Cartan subalgebra of $(\mathfrak{g}_\Gamma{}^0, \mathfrak{k}_\Gamma)$, and the assertions on Γ and c_Γ follow. It also follows that the Harish-Chandra embedding of M as $D \subset \mathfrak{p}^-$ induces that of M_Γ as $D_\Gamma \subset \mathfrak{p}_\Gamma{}^-$.

As $c_{\Delta-\Gamma}$ commutes with every element of $G_\Gamma{}^0$, the proof of the last statement reduces to proving that $\xi^{-1}c_{\Delta-\Gamma}\xi\colon \mathfrak{o} \to \sum_{\alpha\in\Delta-\Gamma} iE_{-\alpha}$, i.e., that $\xi(i\sum_{\alpha\in\Delta-\Gamma} E_{-\alpha}) = x^\Gamma$; this is a calculation contained in the proof of [7, Lemma 4.2]. *Q.E.D.*

The following is the first step toward relating the M_Γ to the boundary components of D.

4.3. LEMMA. *∂D is the union of all sets of the form*

4.3.1. $$\mathrm{ad}(k)[\xi^{-1}c_{\Delta-\Sigma}M_\Sigma],\ k\in K,\ \Sigma \subsetneqq \Delta.$$

and every boundary component of D is a union of sets of that form.

Proof. $A^0 = \exp(\mathfrak{a}^0)$ consists of transvections of M, so 1 is the only element of $\xi^{-1}A^0\xi$ with a fixed point on D. From the action of the latter on $\mathfrak{a}^-$ [7, Lemma 3.5] it follows that $\partial D \cap \mathfrak{a}^-$ consists of all $\sum_{\alpha\in\Delta} b_\alpha E_{-\alpha}$ with $-1 \leqq b_\alpha \leqq 1$ where at least one $|b_\alpha| = 1$. In particular, every $\sum_{\alpha\in\Sigma} E_{-\alpha} \in \partial D$; as $\xi^{-1}\exp(tZ)\xi$ acts on D and ∂D by unimodular complex scalars, we have $\mathfrak{o}^\Sigma = i\sum_{\alpha\in\Sigma} E_{-\alpha} \in \partial D$. Applying $G_\Sigma{}^0$, $\xi^{-1}c_{\Delta-\Sigma}M_\Sigma = (\xi^{-1}G_\Sigma{}^0\xi)(\mathfrak{o}^\Sigma) \subset \partial D$. Thus ∂D contains every set of the form (4.3.1).

We wish to show that ∂D is the union of the sets (4.3.1). As $\partial D = \mathrm{ad}(K)[\partial D \cap i\mathfrak{a}^-]$, it suffices to show that every point of $\partial D \cap i\mathfrak{a}^-$ lies in a set of that form. Every such element has expression $E' = i\sum_{\alpha\in\Delta-\Sigma} \pm E_{-\alpha} + i\sum_{\alpha\in\Sigma} b_\alpha E_{-\alpha}$ where $-1 < b_\alpha < 1$ and $\Sigma \subsetneqq \Delta$; applying an element of $\mathrm{ad}(K \cap \exp \sum_{\alpha\in\Delta-\Sigma} \mathfrak{g}_\alpha)$ we bring it to $E = i\sum_{\alpha\in\Delta-\Sigma} E_{-\alpha} + i\sum_{\alpha\in\Sigma} b_\alpha E_{-\alpha}$, which is in $\xi^{-1}c_{\Delta-\Sigma}M_\Sigma$ by Lemma 4.2. This completes the proof that ∂D is the union of all the sets (4.3.1).

As M_Σ is a hermitian symmetric space of noncompact type by Lemma 4.2, any two of its points can be joined by an analytic arc. It follows that any two points of $\mathrm{ad}(k)[\xi^{-1}c_{\Delta-\Sigma}M_\Sigma]$ can be joined by an analytic arc in ∂D. This completes the proof. *Q.E.D.*

4.4. LEMMA.[3] *The restriction of* $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ *to* $\mathfrak{p}^C$ *has only the eigen-*

[3] This lemma and a sharpened form (Lemma 6.3) will be used repeatedly. The apparently elaborate notation will be re-introduced and motivated in § 5.4.

values 0, $\pm i$ *and* $\pm i/2$; *the respective eigenspaces are* $\mathfrak{p}_\Gamma{}^C$, *the centralizer* $\mathfrak{p}_{\Delta-\Gamma,1}{}^C$ *of* $c_{\Delta-\Gamma}{}^4$ *in* $\mathfrak{p}_{\Delta-\Gamma}{}^C$, *and the* (-1)*-eigenspace of* $\mathrm{ad}(c_{\Delta-\Gamma})^4$ *in* $\mathfrak{p}^C$.

Proof. Let $\mathfrak{e}$ and $\mathfrak{f}$ be the $+1$ and -1 eigenspaces of $\mathrm{ad}(c_{\Delta-\Gamma})^4$ on $\mathfrak{p}^C$. As $c_{\Delta-\Gamma}$ is a transvection of order 4 or 8 in M^*, so $(c_{\Delta-\Gamma}{}^4)^2 = 1$ and $\sigma(c_{\Delta-\Gamma}{}^4) = (c_{\Delta-\Gamma}{}^4)^{-1}$, it follows that $\mathrm{ad}(c_{\Delta-\Gamma})^4$ preserves $\mathfrak{p}^C$ and that $\mathfrak{p}^C = \mathfrak{e} \oplus \mathfrak{f}$.

We have $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0) \cdot \mathfrak{p}_\Gamma{}^C = 0$ and $\mathfrak{p}_\Gamma{}^C \subset \mathfrak{e}$, by definition of $\mathfrak{g}_\Gamma{}^C$. An application of [7, Lemma 5.3] to $(\mathfrak{g}_{\Delta-\Gamma}{}^0, \mathfrak{k}_{\Delta-\Gamma})$ shows that $\mathfrak{p}_{\Delta-\Gamma,1}{}^C$ is spanned by $(\pm i)$-eigenvectors of $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$, that $\mathfrak{p}_{\Delta-\Gamma}{}^C \cap \mathfrak{f}$ is spanned by $(\pm i/2)$-eigenvectors, and that $\mathfrak{p}_{\Delta-\Gamma}{}^C = \mathfrak{p}_{\Delta-\Gamma,1}{}^C + (\mathfrak{f} \cap \mathfrak{p}_{\Delta-\Gamma}{}^C)$. Let $\mathfrak{d}$ be the complexification of the orthogonal complement of $\mathfrak{p}_\Gamma + \mathfrak{p}_{\Delta-\Gamma}$ in $\mathfrak{p}$; it remains only to show that $\mathfrak{d} \subset \mathfrak{f}$ and that $\mathfrak{d}$ is spanned by $(\pm i/2)$-eigenspaces of $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$. As $\mathfrak{e}$ (resp. $\mathfrak{f}$) is the intersection with $\mathfrak{p}^C$ of the sum of the odd (resp. even) dimensional irreducible representation spaces of

$$\mathrm{ad}(\{X_{\Delta-\Gamma}, Y_{\Delta-\Gamma}, Z_{\Delta-\Gamma}{}^0\}),$$

we need only prove $\mathfrak{d}$ to be spanned by $(\pm i/2)$-eigenspaces, and then $\mathfrak{d} \subset \mathfrak{f}$ will follow.

$\mathfrak{d}$ is spanned by root vectors $E_{\pm\beta}$, β positive noncompact; thus we need only prove that $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0) \cdot E_\beta = \mp (i/2) E_\beta$ for every noncompact positive root β with $E_\beta \in \mathfrak{d}$. By definition of $Z_{\Delta-\Gamma}{}^0$ and $\mathfrak{d}$, this is equivalent to the proof that, for every noncompact positive root β which is orthogonal neither to Γ nor to $\Delta - \Gamma$, we have $\sum_{\alpha \in \Delta-\Gamma} \langle \alpha, \beta \rangle = \pm 1$. This has been proved by Harish-Chandra [4, Lemmas 13-16]. *Q.E.D.*

4. 5. Let ν^0 be conjugation of $\mathfrak{g}^C$ over $\mathfrak{g}^0$. As $\mathfrak{p}^0$ is spanned by the $X_\beta{}^0 = E_\beta + E_{-\beta}$ and the $Y_\beta{}^0 = -i(E_\beta - E_{-\beta})$ for the noncompact roots β, ν^0 exchanges E_β and $E_{-\beta}$, so $(I + \nu^0) i E_{-\beta} = -Y_\beta{}^0$ and $(I + \nu^0) E_{-\beta} = X_\beta{}^0$.

Let ν be the conjugation of $\mathfrak{g}^C$ over $\mathfrak{g}$ and observe that $\langle U, V \rangle_\nu = -\langle U, \nu V \rangle$ is a positive definite hermitian form on $\mathfrak{g}^C$ where $\langle\ ,\ \rangle$ denotes the Killing form. Let $\|\ \ \|$ denote operator norm relative to $\langle\ ,\ \rangle_\nu$ for linear transformations of $\mathfrak{g}^C$.

The following result is included for completeness. It was proved by C. C. Moore [8, Lemma 4.5] in a somewhat different manner. The idea of using operator norms is due to R. Hermann.

4. 6. LEMMA. *Let* $\mathfrak{p}^0$ *be given the complex structure defined by* $\mathrm{ad}\,(Z)$

and define $\psi\colon \mathfrak{p}^- \to \mathfrak{p}^0$ *by* $\psi(E) = \frac{1}{2}(E + \nu^0 E)$. *Then* ψ *is an isomorphism of complex vector spaces, and*

$$\psi(D) = \{U \in \mathfrak{p}^0 \colon \| \operatorname{ad}(U) \| < 1\}.$$

Proof. The first statement is clear. ψ is $\operatorname{ad}(K)$-equivariant because $K \subset \exp(\mathfrak{g})$, and $\psi(\mathfrak{a}^-) = \mathfrak{a}^0$. Thus we need only prove $E \in D$ if and only if $\| \operatorname{ad} \psi(E) \| < 1$ for every $E \in \mathfrak{a}^-$. As $\mathfrak{a}^-$ consists of all $E = \sum_{\alpha \in \Delta} b_\alpha E_{-\alpha}$ with b_α real, and $E \in D$ if and only if each $| b_\alpha | < 1$, we need only prove that $\| \operatorname{ad}(\frac{1}{2} \sum b_\alpha X_\alpha{}^0) \| < 1$ is equivalent to the condition that each $| b_\alpha | < 1$.

Each $\mathfrak{g}_\beta{}^0$, $\beta \in \Delta$, has a nonzero element W_β with $[X_\beta{}^0, W_\beta] = 2W_\beta$; now $\operatorname{ad}(\frac{1}{2} \sum b_\alpha X_\alpha{}^0) \cdot W_\beta = b_\beta W_\beta$. Thus $\| \operatorname{ad}(\frac{1}{2} \sum b_\alpha X_\alpha{}^0) \| < 1$ implies that each $| b_\alpha | < 1$.

Suppose that each $| b_\alpha | < 1$; we will see that $\| \operatorname{ad}(\frac{1}{2} \sum b_\alpha X_\alpha{}^0) \| < 1$. As $Y \in \mathfrak{g}$, $\operatorname{ad}(\exp(\pi/4) Y)$ preserves operator norm; that element sends each $X_\alpha{}^0$ to H_α as seen by calculating in $\mathfrak{g}_\alpha{}^C$, so we need only prove $\| \operatorname{ad}(\frac{1}{2} \sum b_\alpha H_\alpha) \| < 1$. In other words, we need $| \sum_{\alpha \in \Delta} b_\alpha \langle \alpha, \beta \rangle | < 2$ for every root β. This now follows from [4, Lemmas 13-16] which say that, if $\langle \alpha, \beta \rangle \neq 0$ for some $\alpha \in \Delta$, then either $\langle \alpha, \beta \rangle = \pm 1$ and $\langle \alpha', \beta \rangle \neq 0$ for at most one other $\alpha' \in \Delta$, or $\langle \alpha, b \rangle = \pm 2$ and $\langle \alpha', \beta \rangle = 0$ for $\alpha \neq \alpha' \in \Delta$. *Q.E.D.*

We can now take the main step toward relating M_Γ to the boundary components of D. Here $\mathfrak{p}^-$ is endowed with the positive definite hermitian form $\langle\ ,\ \rangle_\nu$.

4.7. LEMMA. *Let* $\Gamma \subsetneq \Delta$, *and define* $\mathfrak{e}_\Gamma{}^R$ *and* $\mathfrak{e}_\Gamma{}^C$ *to be the respective real and complex hyperplanes*[4] *in* $\mathfrak{p}^-$ *in which* $\mathfrak{o}^\Gamma$ *is the point nearest to the origin. Then*

4.7.1. $$\bar{D} \cap [\mathfrak{o}^\Gamma + \mathfrak{p}_\Gamma{}^-] = \bar{D} \cap \mathfrak{e}_\Gamma{}^C = \bar{D} \cap \mathfrak{e}_\Gamma{}^R;$$

this set is the closure of $\xi^{-1} c_{\Delta - \Gamma} M_\Gamma$ *in* ∂D *and is a union of boundary components of* D; *it is the union of all sets of the form*

4.7.2. $$\operatorname{ad}(k) [\xi^{-1} c_{\Delta - \Sigma} M_\Sigma], \ k \in K_\Gamma, \Sigma \subset \Gamma.$$

Proof. This proof is close to an argument of Moore. $\psi = \frac{1}{2}(I + \nu^0)$ is a unitary transformation of $\mathfrak{p}^-$ onto $\mathfrak{p}^0$, so $\psi(\mathfrak{e}_\Gamma{}^R)$ consists of all

$$V = -\tfrac{1}{2} \sum_{\alpha \in \Delta - \Gamma} Y_\alpha{}^0 + U$$

where U is real-orthogonal to the first summand. Now decompose U

[4] real (resp. complex) affine subspaces of real (resp. complex) codimension 1.

$= U_1 + U_2$ where U_1 is real-orthogonal to $Y_\alpha{}^0$ for every $\alpha \in \Delta - \Gamma$, and where $U_2 = \sum_{\alpha \in \Delta - \Gamma} u_\alpha Y_\alpha{}^0$. The condition on U_1 gives

$$U_1 = \sum a_\beta X_\beta{}^0 + b_\beta Y_\beta{}^0$$

where the sum runs over all positive noncompact roots β, and where $b_\beta = 0$ in case $\beta \in \Delta - \Gamma$. As each $Y_\alpha{}^0$ has the same length, the condition on U implies $\sum_{\alpha \in \Delta - \Gamma} u_\alpha = 0$. Finally, by Lemma 4.6, $V \in \psi(\bar{D})$ if and only if $\| \mathrm{ad}(V) \| \leqq 1$.

Let $V \in \psi(\bar{D})$. We define $W_\alpha = X_\alpha{}^0 - 2Z_\alpha{}^0$ and $W = \sum_{\alpha \in \Delta - \Gamma} W_\alpha$. Now $[-\frac{1}{2} Y_\alpha{}^0, W] = W_\alpha$. This gives

$$\mathrm{ad}(V) \cdot W = W + [U_1, W],$$

and $[U_1, W]$ is real-orthogonal to W by definition of W and by $b_\beta = 0$ for $\beta \in \Delta - \Gamma$. As $\| \mathrm{ad}(V) \| \leqq 1$, we must have $\| \mathrm{ad}(V) \| = 1$ and

$$0 = [U_1, W] = 2\, \mathrm{ad}(Z_{\Delta - \Gamma}{}^0) \cdot U_1 + F$$

where $F \in \mathfrak{k}$. This yields $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0) \cdot U_1 = 0$; now $U_1 \in \mathfrak{p}_\Gamma{}^0$ by Lemma 4.4. As $U_2 \in \mathfrak{p}_\Gamma{}^0$ by construction, this proves $U \in \mathfrak{p}_\Gamma{}^0$. Thus $V \in \psi(\mathfrak{o}^\Gamma + \mathfrak{p}_\Gamma{}^-)$. We have just proved $\bar{D} \cap \mathfrak{e}_\Gamma{}^R \subset \bar{D} \cap [\mathfrak{o}^\Gamma + \mathfrak{p}_\Gamma{}^-]$; therefore (4.7.1) follows immediately. Lemma 4.2 shows that this set in the image by $\xi^{-1} c_{\Delta-\Gamma} \xi$ of the closure of $\xi^{-1} M_\Gamma$ in $\mathfrak{p}_\Gamma{}^-$, and the set lies in ∂D by the observation $\| \mathrm{ad}(V) \| = 1$ above; it follows that the set is the closure of $\xi^{-1} c_{\Delta-\Gamma} M_\Gamma$ in ∂D.

Let b be the complex linear functional on $\mathfrak{p}^-$ such that $b(E) = 1$ is the equation of $\mathfrak{e}_\Gamma{}^C$, and notice from the above paragraph that $\mathfrak{e}_\Gamma{}^C$ does not meet D. As D is preserved by the rotations $e^{i\theta}$, we then have $| b(E) | \leqq 1$ for every $E \in \bar{D}$. Now let $\mu \colon U \to \mathfrak{p}^-$ be an analytic arc in ∂D such that $\mu(U)$ meets $\mathfrak{e}_\Gamma{}^C$. Then the holomorphic function $b \circ \mu$ on U is bounded by 1 and this bound is achieved; thus $b \circ \mu$ is constant by the maximum modulus principle; in other words, $\mu(U) \subset \mathfrak{e}_\Gamma{}^C$. This proves that the set (4.7.1) is a union of boundary components of D.

The last statement follows by application of Lemma 4.3 to ∂D_Γ, $D_\Gamma = \xi^{-1}(M_\Gamma)$, and by the observation that $c_{\Delta - \Sigma} = c_{\Delta - \Gamma} \cdot c_{\Delta - \Sigma}$ for every $\Sigma \subset \Gamma$.

Q.E.D.

4.8. THEOREM. *The boundary components of D in $\mathfrak{p}^-$ are just the sets*

$$\mathrm{ad}(k)[\xi^{-1} c_{\Delta-\Gamma} M_\Gamma], \; k \in K, \; \Gamma \subsetneq \Delta.$$

The boundary components of M in M^* are the sets

$$k(c_{\Delta-\Gamma}(M_\Gamma)), \; k \in K, \; \Gamma \subsetneq \Delta.$$

Proof. The two statements are equivalent because ξ is an $\mathrm{ad}(K)$-equivariant complex analytic homeomorphism of a neighborhood of $\bar{D}$ in $\mathfrak{p}^-$ onto a neighborhood of $\bar{M}$ in M^*, carrying D onto M.

Every boundary component of D is a union of sets $\mathrm{ad}(k)[\xi^{-1}c_{\Delta-\Sigma}M_\Sigma]$, $k \in K$, $\Sigma \subsetneqq \Delta$, and every such set lies in a boundary component of D, by Lemma 4.3. Thus it suffices to prove, given an analytic arc $\mu\colon U \to \mathfrak{p}^-$ in ∂D such that $\mu(U)$ meets $\xi^{-1}c_{\Delta-\Gamma}M_\Gamma$, that $\mu(U) \subset \xi^{-1}c_{\Delta-\Gamma}M_\Gamma$. Lemma 4.7 says that the closure of $\xi^{-1}c_{\Delta-\Gamma}M_\Gamma$ is a union of boundary components of D, so $\mu(U)$ is contained in that closure. Now define $\beta = \xi^{-1} \cdot c_{\Delta-\Gamma}{}^{-1} \cdot \xi \cdot \mu$; then $\beta\colon U \to \mathfrak{p}_\Gamma{}^-$ is an analytic arc in $\bar{D}_\Gamma$ which meets $D_\Gamma = \xi^{-1}(M_\Gamma)$, and we wish to prove that $\beta(U) \subset D_\Gamma$.

Suppose that $\beta(U)$ contains a point E of ∂D_Γ. Applying Lemma 4.3 to D_Γ we see that E is contained in a set $\mathrm{ad}(k)[\xi^{-1}c_{\Gamma-\Sigma}M_\Sigma]$, $k \in K_\Gamma$, $\Sigma \subsetneqq \Gamma$. Applying Lemma 4.7 to D_Γ, we obtain a complex linear functional b on $\mathfrak{p}_\Gamma{}^-$ whose restriction to $\bar{D}_\Gamma$ attains its maximum at E; b is the linear functional specifying $\mathrm{ad}(k)\mathfrak{e}_\Sigma{}^C$. Now $b \circ \beta$ is a holomorphic function on U which attains its maximum, so $b \circ \beta$ is constant by the maximum modulus principle, whence

$$\beta(U) \subset \{F \in \bar{D}_\Gamma\colon b(F) = b(E)\} \subset \partial D_\Gamma.$$

This contradicts the fact that $\beta(U)$ meets D_Γ. This shows $\beta(U) \subset D_\Gamma$, and the Theorem is proved. *Q.E.D.*

4.9. COROLLARY. *The boundary components of D in $\mathfrak{p}^-$ are bounded symmetric domains in Harish-Chandra embedding, where the ambient space is a complex affine subspace of $\mathfrak{p}^-$ and the domain is the interior of the intersection of the ambient space with $\bar{D}$. The boundary components of M in M^* are hermitian symmetric spaces of noncompact type in Borel embedding, where the ambient space is a complex totally geodesic submanifold of M^* and the noncompact space is the interior of the intersection of the ambient space with $\bar{M}$.*

The first statement is immediate from Theorem 4.8 and Lemmas 4.2 and 4.7; the second statement follows upon mapping by ξ and applying Lemma 4.2.

4.10. COROLLARY. *Let $D = D_1 \times \cdots \times D_r$ be the decomposition of D as a product of irreducible domains. Then the boundary components of D are just the sets $F = F_1 \times \cdots \times F_r \neq D$ with F_j either equal to D_j or a boundary component of D_j; each of the bounded symmetric domains F_j is irreducible. The analogous result holds for the boundary components of M.*

Proof. Let F be a boundary component of D; then without loss of generality we may assume $F=\xi^{-1}c_{\Delta-\Gamma}M_\Gamma$ with $\Gamma \subsetneqq \Delta$. Let $\mathfrak{g}^0=\mathfrak{g}_1{}^0\oplus\cdots\oplus\mathfrak{g}_r{}^0$ be the decomposition as a sum of simple ideals, ordered so that the analytic subgroup of G^0 for $\mathfrak{g}_j{}^0$ is the connected group of analytic automorphisms of D_j. Then $\Delta=\Delta_1\cup\cdots\cup\Delta_r$ (disjoint) where Δ_j is a maximal set of strongly orthogonal noncompact roots of $\mathfrak{g}_j{}^0$. Define

$$\Gamma_j=\Gamma\cap\Delta_j \text{ and } \mathrm{F}_j=\xi^{-1}c_{\Delta_j-\Gamma_j}(M_j\cap M_\Gamma).$$

Then $F=F_1\times\cdots\times F_r$, $F_j=D_j$ if $\Gamma_j=\Delta_j$, and F_j is a boundary component of D_j if $\Gamma_j\neq\Delta_j$.

Let $F=F_1\times\cdots\times F_r\neq D$ where F_j is D_j or a boundary component of D_j. The last part of the proof of Theorem 4.8 consisted of showing that D_Γ is an analytic arc component of $\bar{D}_\Gamma$; thus F_j is an analytic arc component of $\bar{D}_j$, so F is an analytic arc component of $\bar{D}$. Now $F\subset\partial D$ by construction, so F is an analytic arc component of ∂D, i.e., a boundary component of D.

To prove F_j irreducible we may assume D irreducible and $F_j=\xi^{-1}c_{\Delta-\Gamma}M_\Gamma$ with $\Gamma\subset\Delta$, and we need only prove that the effective part of $\mathfrak{g}_\Gamma{}^0$ is simple. It suffices to prove that the effective part of $\mathfrak{g}_\Gamma$ is simple. For this, we define W_Δ to be the subgroup of the Weyl group of G relative to $\mathfrak{h}$ consisting of the elements which preserve Δ as a set, and we define W_Γ to be the subgroup of W_Δ consisting of the elements which fix every element of $\Delta-\Gamma$. A result of C. C. Moore [8, Theorem 2] says that W_Δ induces the full group of permutations of Δ; thus W_Γ is transitive on Γ. Let U be the centralizer of $\mathfrak{h}_{\Delta-\Gamma}{}^-$ in G. $\exp(\mathfrak{h}_{\Delta-\Gamma}{}^-)$ is a torus because it is closed in $\exp(\mathfrak{h})$, so U is the centralizer of a torus. Now U is connected, the Weyl group of U relative to $\mathfrak{h}$ contains W_Γ (by definition of U), and G_Γ is the semisimple part of U (by definition of $\mathfrak{g}_\Gamma$); it follows that the Weyl group of G_Γ relative to $\mathfrak{h}\cap\mathfrak{g}_\Gamma$ is transitive on Γ. This proves that $\mathfrak{g}_\Gamma$ is simple. *Q.E.D.*

4.11. COROLLARY. *If M is of tube type, then each of its boundary components is of tube type. If M is irreducible and has a positive-dimensional boundary component of tube type, then M is of tube type.*

Proof. If M is of tube type, then the Cayley transform $c=c_\Delta$ has order 4. As $c=c_\Gamma\cdot c_{\Delta-\Gamma}$ and $\operatorname{ad}(c_{\Delta-\Gamma})|_{\mathfrak{g}_\Gamma}=1$, this implies $\operatorname{ad}(c_\Gamma)^4|_{\mathfrak{g}_\Gamma}=1$, and it follows that M_Γ is of tube type. Thus each boundary component is of tube type.

Before proving the second statement, we must check that M is of tube type whenever, for some $\Gamma\subset\Delta$, both M_Γ and $M_{\Delta-\Gamma}$ are of tube type. To see this, we write $\mathfrak{p}=\mathfrak{p}_\Gamma+\mathfrak{p}_{\Delta-\Gamma}+\mathfrak{d}$ where $\mathfrak{d}$ is the orthogonal complement of $\mathfrak{p}_\Gamma+\mathfrak{p}_{\Delta-\Gamma}$. By Lemma 4.4, both $\mathrm{ad}(c_{\Delta-\Gamma})^4$ and $\mathrm{ad}(c_\Gamma)^4$ are -1 on $\mathfrak{d}$, so $\mathrm{ad}(c)^4|_{\mathfrak{d}}=1$. By hypothesis and the argument of the preceding paragraph, $\mathrm{ad}(c)$ is 1 on $\mathfrak{p}_\Gamma$ and on $\mathfrak{p}_{\Delta-\Gamma}$. Now $\mathrm{ad}(c)^4$ is 1 on $\mathfrak{p}$, and thus also on $\mathfrak{k}=[\mathfrak{p},\mathfrak{p}]$, so $c^4=1$ and M is of tube type.

Let M be irreducible with a positive-dimensional boundary component of tube type. Then some M_Γ, $\phi\neq\Gamma\subsetneqq\Delta$, is of tube type. Let $\alpha\in\Delta-\Gamma$ and $\beta\in\Gamma$, and define $\Phi=\Gamma\cup\{\alpha\}$ and $\Psi=\Phi-\{\beta\}$. A result of C. C. Moore [8, Theorem 2] shows that an element of the subgroup preserving $\mathfrak{h}^-$ in the Weyl group of $\mathfrak{g}^0$ send Γ to Ψ; thus M_Ψ is of tube type. Applying the first part of this Lemma to M_Ψ we see that $M_{\{\alpha\}}$ is of tube type. Applying the above paragraph to M_Φ with the decomposition $\Phi=\Gamma\cup\{\alpha\}$, now M_Φ is of tube type. Iterating the argument, $M_\Delta=M$ is seen to be of tube type.

Q.E.D.

4.12. COROLLARY. *For a bounded symmetric domain in Harish-Chandra embedding, a boundary component of a boundary component is a boundary component.*

This is immediate from Theorem 4.8 and from (4.7.2) in Lemma 4.7.

Lemma 4.4, Theorem 4.8 and Corollaries 4.11 and 4.12 allow us to list the boundary components. Here we say that two boundary components are of the *same type* if an element of G^0 sends one to the other.

4.13. THEOREM. *Let D be an irreducible bounded symmetric domain of rank m in Harish-Chandra embedding. For each integer r, $0\leqq r<m$, there is just one type D_r of boundary component of D which has rank r as a symmetric space. D_0 is a single point and the other D_r are given as follows.*

4.13.1. $D=SU^m(2m+k)/S(U(m)\times U(m+k))$, $k\geqq 0$. *Then* $D_r=SU^r(2r+k)/S(U(r)\times U(r+k))$.

4.13.2. $D=SO^*(4m)/U(2m)$. *Then* $D_r=SO^*(4r)/U(2r)$.

4.13.3. $D=SO^*(4m+2)/U(2m+1)$.
Then $D_r=SO^*(4r+2)/U(2r+1)$.

4.13.4. $D=Sp(m,R)/U(m)$. *Then* $D_r=Sp(r,R)/U(r)$.

4.13.5. $D=SO^2(n+2)/SO(2)\times SO(n)$, $n>2$; *here* $m=2$. D_1 *is the unit disc in* C^1.

4.13.6. $D=E_6/SO(10)\cdot SO(2)$; *here* $m=2$. M_1 *is the open unit ball in* C^5.

4.13.7. $D = E_7/E_6 \cdot SO(2)$; *here* $m = 3$. D_1 *is the unit disc in* C^1, *and* $D_2 = SO^2(12)/SO(2) \times SO(10)$.

Remark. For the classical domains the statement is due to L. K. Hua and K. H. Look [6], and the proof is due to Satake [10]. The result is new for the exceptional domains.

Remark. There are some duplications. For example $SO^*(8)/U(4) = SO^2(8)/SO(2) \times SO(6)$ and $SO^*(4)/U(2)$ is the unit disc in C^1.

Remark. The classification is not necessary for the first assertion. That assertion follows from Theorem 4.8 and transitivity of the small Weyl group on the collection of all subsets of r elements in $\mathrm{ad}(c)^2\Delta$.

Proof. The domains D listed exhaust the class of irreducible noncompact non-Euclidean hermitian symmetric spaces, according to É. Cartan. Here the domains of tube type are (4.13.1) for $k = 0$, (4.13.2), (4.13.4), (4.13.5) and (4.13.7). Now assertion (4.13.5) is immediate from Corollary 4.11 because the unit disc is the only tube-type domain of rank 1.

Let $\alpha \in \Delta$ and define $\Gamma = \Delta - \{\alpha\}$. The fixed point set of $\mathrm{ad}(c_\alpha)^4$ on $\mathfrak{g}_{\{\alpha\}}$ is of the form $\mathfrak{g}_{\{\alpha\},1} \oplus \mathfrak{l}_{\{\alpha\},2}$ where the second summand is in $\mathfrak{k}$ and the first is equal to $[\mathfrak{p}_{\{\alpha\},1}, \mathfrak{p}_{\{\alpha\},1}] + \mathfrak{p}_{\{\alpha\},1}$. Part 4 of [7, Theorem 4.9] shows that $\mathfrak{g}_{\{\alpha\},1}$ is of Cartan classification type $\mathfrak{a}_1$. Now Lemma 4.4 gives a direct sum decomposition $\mathfrak{p} = \mathfrak{p}_{\{\alpha\},1} + \mathfrak{p}_1 + \mathfrak{f}$ where $\mathrm{ad}(c_\alpha)^4$ is $+1$ on the first two summands and -1 on the third. Consider the decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{v}$ into $+1$ and -1 eigenspaces of $\mathrm{ad}(c_\alpha)^4$; it follows that $\mathfrak{u} = \mathfrak{a}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{w}$ (direct sum of ideals) with $\mathfrak{w} \subset \mathfrak{k}$. As $\mathrm{ad}(c_\alpha)^4$ is an inner automorphism of $\mathfrak{g}$, we have proved: $\mathfrak{g}$ *has a symmetric subalgebra* $\mathfrak{u}$ *of maximal rank which has* $\mathfrak{a}_1$ *and* $\mathfrak{g}_\Gamma$ *as distinct simple ideals.*

Let $D = SU^m(2m+k)/S(U(m) \times U(m+k))$. Then $\mathfrak{g} = \mathfrak{a}_{2m+k-1}$, so the only possibility is $\mathfrak{u} = \mathfrak{a}_1 \oplus \mathfrak{a}_{2m+k-3} \oplus$ (1-dimensional abelian). Thus $\mathfrak{g}_\Gamma = \mathfrak{a}_{2(m-1)+k-1}$. As D_Γ has rank $m-1$, (4.13.1) follows.

Let $D = SO^*(2n)/U(n)$ where $n = 2m$ or $2m+1$. Then $\mathfrak{g} = \mathfrak{d}_n$ and $\mathfrak{u} = \mathfrak{d}_2 \oplus \mathfrak{d}_{n-2}$ is the only possibility; here observe that $\mathfrak{d}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_1$. If $\mathfrak{g}_\Gamma = \mathfrak{a}_1$, then $m - 1 = 1$, and $n = 2m$ by Corollary 4.11, so $n = 4$ and $D_1 =$ (unit disc) $= SO^*(4)/U(2)$; conversely, if $n = 4$, then $\mathfrak{d}_{n-2} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$ so $\mathfrak{g}_\Gamma = \mathfrak{d}_{n-2}$. We must check that

$$D_\Gamma = SO^*(2[n-2])/U(n-2).$$

If $m - 1 > 2$ then this is true because the rank of D_Γ is too large to allow D_Γ to be of type (4.13.5); it is true if $m-1 = 2$ and $n = 2m+1$, for then

D_Γ cannot be of type (4.13.5) by Corollary 4.11; it is true for $m-1=2$ and $n=2m$ because $SO^*(8)/U(4)=SO^2(8)/SO(2)\times SO(6)$. Now (4.13.2) and (4.13.3) are proved.

Let $D=Sp(m,R)/U(m)$. Then $\mathfrak{g}=\mathfrak{c}_m$ so $\mathfrak{u}=\mathfrak{c}_1\oplus\mathfrak{c}_{m-1}$ and $\mathfrak{g}_\Gamma=\mathfrak{c}_{m-1}$. Thus $D_\Gamma=Sp(m-1,R)/U(m-1)$. This proves (4.13.4).

Let $D=E_6/SO(10)\cdot SO(2)$. Then $\mathfrak{g}=\mathfrak{e}_6$ so $\mathfrak{u}=\mathfrak{a}_1\oplus\mathfrak{a}_5$; thus $\mathfrak{g}_\Gamma=\mathfrak{a}_5$. As D_Γ is of rank 1, (4.13.6) is proved.

Let $D=E_7/E_6\cdot SO(2)$. Then $\mathfrak{g}=\mathfrak{e}_7$ so $\mathfrak{u}=\mathfrak{a}_1\oplus\mathfrak{d}_6$; thus $\mathfrak{g}_\Gamma=\mathfrak{d}_6$. As D_Γ is of tube type, or because $SO^*(12)/U(6)$ has rank 3, now D_Γ must be $SO^2(12)/SO(2)\times SO(10)$. This proves the statement on D_2; the statement on D_1 follows either from Corollary 4.11 or from Corollary 4.12.

Q.E.D.

The following result shows how the boundary components are related to the limit points of geodesic rays. We work in M and M^* for convenience, but the result translates immediately to D and $\mathfrak{p}^-$.

4.14. THEOREM. *Given $y\in M$ and a boundary component F of M in M^*, there is a unique point $f\in F$ such that some geodesic ray of M from y tends to f.*

Proof. Let $U=\{k(x^\Gamma): k\in K,\ \Gamma\subsetneqq\Delta\}$, and define V to be the set of all limit points in M^* of geodesic rays of M with initial point x. If $X'=\sum_{\alpha\in\Delta}t_\alpha X_\alpha\in\mathfrak{a}^0$, then the geodesic ray $\{\exp(sX')\cdot x\}_{s\geqq 0}$ is given by $\exp(sX')\cdot x=\xi(\sum_{\alpha\in\Delta}\tanh(t_\alpha s)\cdot E_{-\alpha})$. Thus the limit point of the geodesic ray is $\xi(\sum_{\alpha\in\Delta}\epsilon_\alpha E_{-\alpha})$ where ϵ_α is 0, 1 or -1 as t_α is 0, positive or negative. We can find $k\in K$ such that $\mathrm{ad}(k)X'=\sum|t_\alpha|\cdot X_\alpha$; now the limit point is $\mathrm{ad}(k)^{-1}\cdot x^\Gamma$ where $\Gamma=\{\alpha\in\Delta: t_\alpha=0\}$. This proves $U=V$.

We have $g\in G^0$ with $g(y)=x$, and $k\in K$ with $k(gF)=c_{\Delta-\Gamma}M_\Gamma$ for some $\Gamma\subsetneqq\Delta$, so we may assume that $y=x$ and $F=c_{\Delta-\Gamma}M_\Gamma$. Now we need only prove that $c_{\Delta-\Gamma}M_\Gamma\cap\{k(x^\Sigma\}: x^\Gamma$ is the only element of $k\in K,\Sigma\subsetneqq\Delta\}$. If $k(x^\Sigma)\in c_{\Delta-\Gamma}M_\Gamma$, then $c_{\Delta-\Gamma}M_\Gamma$ must coincide with $kc_{\Delta-\Sigma}M_\Sigma$, for both are boundary components containing $k(x^\Sigma)$. Lemma 4.7 shows that $\mathfrak{o}^\Gamma=\xi^{-1}(x^\Gamma)$ is closer to the origin of $\mathfrak{p}^-$ than any other point of $\xi^{-1}c_{\Delta-\Gamma}M_\Gamma$, and $\xi^{-1}(kx^\Sigma)$ is closer to the origin of $\mathfrak{p}^-$ than any other point of $\xi^{-1}kc_{\Delta-\Sigma}M_\Sigma$. Thus $k(x^\Sigma)=x^\Gamma$. *Q.E.D.*

5. The space of boundary components of a given type.

5.1. We say that two boundary components of D (or M) are of the *same type* if an element of G^0 carries one to the other, and we say that a boundary component is of *type* Γ $(\Gamma \subsetneqq \Delta)$ if it is of the same type as $(\xi^{-1}c_{\Delta-\Gamma}\xi)D_\Gamma$ (or $c_{\Delta-\Gamma}M_\Gamma$). Here we remark

LEMMA. *Let* $\Gamma \subsetneqq \Delta$ *and* $\Sigma \subsetneqq \Delta$, *and suppose that* F *is a boundary component of type* Γ. *Then the following statements are equivalent.*

(i) *F is of type* Σ

(ii) $\mathrm{ad}(c)^2\Sigma$ *is equivalent to* $\mathrm{ad}(c)^2\Gamma$ *under the small Weyl group of* $(\mathfrak{g}^0, \mathfrak{k})$ *rel.* $\mathfrak{a}^0$

(iii) *For every simple ideal of* $\mathfrak{g}^0$, *both* Σ *and* Γ *contain the same number of roots of that ideal.*

Proof. Corollary 4.10 reduces the proof to the case where D is irreducible. Then (i) implies (iii) because Lemma 4.2 shows that the symmetric space rank of a component of type Γ is the number of elements of Γ. (iii) implies (ii) by [8, Theorem 2], and it is obvious that (ii) implies (i). *Q.E.D.*

Let $\Gamma \subsetneqq \Delta$. We define S^Γ to be the set of all boundary components of M of type Γ, and we define $U^\Gamma \subset \partial M$ to be the union of all boundary components of type Γ. Similarly, $S_D{}^\Gamma$ denotes the set of boundary components of D of type Γ, and $U_D{}^\Gamma \subset \partial D$ is the union. These two notions coincide in the case where Γ is the empty set ϕ; there we have

$S^\emptyset = U^\emptyset = \check{S}$, Bergman-Šilov boundary of M in M^*

$S_D{}^\emptyset = U_D{}^\emptyset = \check{S}_D$, Bergman-Šilov boundary of D in $\mathfrak{p}^-$.

Theorem 4.8 and the Lemma above show that K acts transitively on S^Γ (resp. $S_D{}^\Gamma$). Let L^Γ denote the isotropy subgroup of K on $c_{\Delta-\Gamma}M_\Gamma \in S^\Gamma$ (resp. on $\xi^{-1}c_{\Delta-\Gamma}M_\Gamma \in S_D{}^\Gamma$; it is the same subgroup). Then L^Γ is the set of all elements of K which preserve the closure of $c_{\Delta-\Gamma}M_\Gamma$ in the compact set ∂M, and it follows that L^Γ is closed in K. Now we have identifications $S^\Gamma \cong K/L^\Gamma \cong S_D{}^\Gamma$, so S^Γ and $S_D{}^\Gamma$ are real analytic manifolds, homogeneous spaces of K.

As a final preliminary remark we observe that K cannot be transitive on U^Γ or $U_D{}^\Gamma$ for $\Gamma \neq \phi$, because any orbit of K is compact and Lemma 4.7 gives us the closures

$$\overline{U^\Gamma} = \bigcup_{\Sigma \subset \Gamma} U^\Sigma \text{ and } \overline{U_D{}^\Gamma} = \bigcup_{\Sigma \subset \Gamma} U_D{}^\Sigma.$$

5.2. LEMMA. *Let* $k \in K$. *If* k *preserves* $c_{\Delta-\Gamma}M_\Gamma$, *then* $k(x^\Gamma) = x^\Gamma$.

Proof. L^Γ is a compact group of isometries of $c_{\Delta-\Gamma}M_\Gamma$, so it has a stationary point. But $K_\Gamma \subset L^\Gamma$, and x^Γ is the unique stationary point of K_Γ on $c_{\Delta-\Gamma}M_\Gamma$. This proves that x^Γ is stationary under L^Γ. *Q.E.D.*

5.3. Let $\pi\colon U^\Gamma \to S^\Gamma$ be the natural projection. This is a differentiable bundle with fibre M_Γ and group $G_\Gamma{}^0$, and K is transitive on the base. Lemma 5.2 may be paraphrased as: $kc_{\Delta-\Gamma}M_\Gamma \to k(x^\Gamma)$ *is a K-equivariant global section of the bundle* $U^\Gamma \to S^\Gamma$. Lemma 5.2 also allows us to identify S^Γ with $K(x^\Gamma)$.

5.4. *Definitions.* We will decompose $\mathfrak{g}^0$ under $c_{\Delta-\Gamma}$ in order to study S^Γ. Let $\tau_{\Delta-\Gamma} = \operatorname{ad}(c_{\Delta-\Gamma})^2$. We define:

$\mathfrak{g}^\Gamma$: the set of all elements of $\mathfrak{g}$ fixed under $\tau_{\Delta-\Gamma}{}^2$;
$\mathfrak{k}^\Gamma = \mathfrak{g}^\Gamma \cap \mathfrak{k}$;
$\mathfrak{p}_1{}^\Gamma = \mathfrak{g}^\Gamma \cap \mathfrak{p}$;
$\mathfrak{k}_1{}^\Gamma = [\mathfrak{p}_1{}^\Gamma, \mathfrak{p}_1{}^\Gamma]$;
$\mathfrak{g}_1{}^\Gamma = \mathfrak{k}_1{}^\Gamma + \mathfrak{p}_1{}^\Gamma$;
$\mathfrak{l}_2{}^\Gamma$: the centralizer of $\mathfrak{g}_1{}^\Gamma$ in $\mathfrak{g}^\Gamma$.

Here $\mathfrak{g}^\Gamma$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{g}^\Gamma = \mathfrak{k}^\Gamma + \mathfrak{p}_1{}^\Gamma$ because $\tau_{\Delta-\Gamma}{}^2$ preserves both $\mathfrak{k}$ and $\mathfrak{p}$. $\mathfrak{l}_2{}^\Gamma$ is the centralizer of $\mathfrak{p}_1{}^\Gamma$ in $\mathfrak{g}^\Gamma$, by the Jacobi identity; the decomposition theory of orthogonal involutive Lie algebras now implies that $\mathfrak{k}^\Gamma = \mathfrak{k}_1{}^\Gamma + \mathfrak{l}_2{}^\Gamma$ and $\mathfrak{g}^\Gamma = \mathfrak{g}_1{}^\Gamma \oplus \mathfrak{l}_2{}^\Gamma$, direct sums of ideals.

$\tau_{\Delta-\Gamma}$ preserves and has square I on $\mathfrak{k}_1{}^\Gamma$; its square preserves and has square I on $\mathfrak{k}$ and $\mathfrak{p}$. Thus we define:

$\mathfrak{l}_1{}^\Gamma$: the $(+1)$-eigenspace of $\tau_{\Delta-\Gamma}$ on $\mathfrak{k}_1{}^\Gamma$;
$\mathfrak{q}_1{}^\Gamma$: the (-1)-eigenspace of $\tau_{\Delta-\Gamma}$ on $\mathfrak{k}_1{}^\Gamma$;
$\mathfrak{q}_2{}^\Gamma$: the (-1)-eigenspace of $\tau_{\Delta-\Gamma}{}^2$ on $\mathfrak{k}$;
$\mathfrak{p}_2{}^\Gamma$: the (-1)-eigenspace of $\tau_{\Delta-\Gamma}{}^2$ on $\mathfrak{p}$;

$\mathfrak{l}^\Gamma = \mathfrak{l}_1{}^\Gamma + \mathfrak{l}_2{}^\Gamma$ and $\mathfrak{q}^\Gamma = \mathfrak{q}_1{}^\Gamma + \mathfrak{q}_2{}^\Gamma$.

Now we have $\mathfrak{k}_1{}^\Gamma = \mathfrak{l}_1{}^\Gamma + \mathfrak{q}_1{}^\Gamma$, $\mathfrak{k}^\Gamma = \mathfrak{l}^\Gamma + \mathfrak{q}_1{}^\Gamma$, $\mathfrak{k} = \mathfrak{l}^\Gamma + \mathfrak{q}^\Gamma$, and $\mathfrak{p} = \mathfrak{p}_1{}^\Gamma + \mathfrak{p}_2{}^\Gamma$.

We finally define some related subalgebras

$$\begin{aligned}
\mathfrak{g}_1{}^{\Gamma,0} &= \mathfrak{k}_1{}^\Gamma + i\mathfrak{p}_1{}^\Gamma \\
\mathfrak{g}^{\Gamma,0} &= \mathfrak{k}^\Gamma + i\mathfrak{p}_1{}^\Gamma \\
\mathfrak{k}_1{}^{\Gamma*} &= \mathfrak{l}_1{}^\Gamma + i\mathfrak{q}_1{}^\Gamma \\
\mathfrak{k}^{\Gamma*} &= \mathfrak{l}^\Gamma + i\mathfrak{q}_1{}^\Gamma = \mathfrak{l}_2{}^\Gamma \oplus \mathfrak{k}_1{}^{\Gamma*}
\end{aligned}$$

of $\mathfrak{g}^C$.

Latin letters denote the corresponding analytic subgroups of G^C, except that L^Γ was already defined to be the isotropy subgroup of K at x^Γ and

$L_1{}^\Gamma$ will be the isotropy subgroup of $K_1{}^\Gamma$ at x^Γ. We will justify this exception by checking that $\mathfrak{l}^\Gamma$ is the Lie algebra of L^Γ. As $L^\Gamma \subset G$ and $c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1}$ is the symmetry of M^* at x^Γ, this check is reduced to seeing that $\mathfrak{l}^\Gamma$ is the fixed point set of $\mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1})$ on $\mathfrak{k}$. To prove the latter, we first observe that $\mathfrak{l}^\Gamma$ is the fixed point set of $\tau_{\Delta-\Gamma}$ on $\mathfrak{k}$, for the fixed point set is in $\mathfrak{l}^\Gamma$ by definition of $\mathfrak{q}^\Gamma$, the fixed point set contains $\mathfrak{l}_1{}^\Gamma$ by definition and the fixed point set contains $\mathfrak{l}_2{}^\Gamma$ as a consequence of $c_{\Delta-\Gamma} \in G_1{}^\Gamma$. Now let $V \in \mathfrak{k}$ and observe that

$$\begin{aligned}\mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1}) \cdot V &= \mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1}) \cdot \mathrm{ad}(s^{-1}) \cdot V \\ &= \mathrm{ad}(c_{\Delta-\Gamma}) \cdot \mathrm{ad}(\mathrm{ad}(s) c_{\Delta-\Gamma}) \cdot V = \tau_{\Delta-\Gamma}(V).\end{aligned}$$

Our assertion follows.

5.5. Lemma. *$M^\Gamma = G^{\Gamma,0}(x)$ is a hermitian symmetric subspace of M. $L_2{}^\Gamma$ is the identity component of the kernel of the action of $G^{\Gamma,0}$ on M^Γ, $G_1{}^{\Gamma,0}$ is (locally) the connected group of analytic automorphisms of M^Γ, and $\mathfrak{g}_1{}^\Gamma$ and $\mathfrak{g}_1{}^{\Gamma,0}$ are semisimple. The Cayley transform on M^Γ is $c = c_\Delta \in G_1{}^\Gamma$, M^Γ is of tube type if $\Gamma = \phi$, and M^Γ is of tube type if and only if M_Γ is of tube type when $\Gamma \neq \phi$.*

Proof. $\sum_{\alpha\in\Delta} \mathfrak{g}_\alpha \subset \mathfrak{g}^\Gamma$ by construction. Z is the sum of its projections Z' and Z^0 on $\mathfrak{h}^+$ and $\mathfrak{h}^-$, $\mathfrak{h}^- \subset \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, and $\mathfrak{h}^+$ is centralized by each c_α. This proves $Z \in \mathfrak{g}^\Gamma$, so $Z \in \mathfrak{g}^{\Gamma,0}$ and it follows that M^Γ is a sub hermitian symmetric space of M. The statement on $L_2{}^\Gamma$ is immediate from the definition of $\mathfrak{l}_2{}^\Gamma$, and the assertions on $G_1{}^{\Gamma,0}$, $\mathfrak{g}_1{}^{\Gamma,0}$ and $\mathfrak{g}_1{}^\Gamma$ follow. The next statement follows from $\mathfrak{a}^0 \subset \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha{}^0 \subset \mathfrak{g}_1{}^{\Gamma,0}$, which is a consequence of $\sum_{\alpha\in\Delta} \mathfrak{g}_\alpha \subset \mathfrak{g}^\Gamma$ and the fact that $\mathfrak{g}_\alpha \cap \mathfrak{p}$ generates $\mathfrak{g}_\alpha$. The remaining assertions are immediate from Lemma 4.4. *Q.E.D.*

5.6. Lemma. *$\tau_{\Delta-\Gamma}$ interchanges $\mathfrak{p}_2{}^\Gamma$ and $\mathfrak{q}_2{}^\Gamma$; $\mathrm{ad}(c_{\Delta-\Gamma})$ interchanges $\mathfrak{q}_1{}^\Gamma$ with the (-1)-eigenspace of $\tau_{\Delta-\Gamma}$ on $\mathfrak{p}_1{}^\Gamma$; $\mathfrak{p}_1{}^\Gamma = \mathfrak{p}_\Gamma + \mathfrak{p}_{\Delta-\Gamma,1}$ where $\tau_{\Delta-\Gamma}$ is $+1$ on $\mathfrak{p}_\Gamma$ and has square 1 on $\mathfrak{p}_{\Delta-\Gamma,1}$, and where $J = \mathrm{ad}\, Z$ interchanges the (± 1)-eigenspaces of $\tau_{\Delta-\Gamma}$ on $\mathfrak{p}_{\Delta-\Gamma,1}$.*

Proof. Let $V \in \mathfrak{p}_2{}^\Gamma$ and $R \in \mathfrak{q}_2{}^\Gamma$. Then

$$\sigma\tau_{\Delta-\Gamma} V = \tau_{\Delta-\Gamma}{}^{-1}\sigma V = -\tau_{\Delta-\Gamma}{}^{-1} V = -\tau_{\Delta-\Gamma}\,\tau_{\Delta-\Gamma}{}^{2} V = \tau_{\Delta-\Gamma} V$$

and

$$\sigma\tau_{\Delta-\Gamma} R = \tau_{\Delta-\Gamma}{}^{-1}\sigma R = \tau_{\Delta-\Gamma}{}^{-1} R = \tau_{\Delta-\Gamma}\,\tau_{\Delta-\Gamma}{}^{2} R = -\tau_{\Delta-\Gamma} R.$$

Thus $\tau_{\Delta-\Gamma}(\mathfrak{p}_2{}^\Gamma) \subset \mathfrak{k}$ and $\tau_{\Delta-\Gamma}(\mathfrak{q}_2{}^\Gamma) \subset \mathfrak{p}$. Now $\tau_{\Delta-\Gamma}$ comutes with its own square, and this implies $\tau_{\Delta-\Gamma}(\mathfrak{p}_2{}^\Gamma) \subset \mathfrak{q}_2{}^\Gamma$ and $\tau_{\Delta-\Gamma}(\mathfrak{q}_2{}^\Gamma) \subset \mathfrak{p}_2{}^\Gamma$. Equality follows from dimension considerations. This proves the interchange statement for $\tau_{\Delta-\Gamma}$. The proof of the interchange statement for $\mathrm{ad}(c_{\Delta-\Gamma})$ is similar.

Lemma 4.4 shows $\mathfrak{p} = \mathfrak{p}_\Gamma + \mathfrak{p}_{\Delta-\Gamma,1} + \mathfrak{p}_2{}^\Gamma$, direct sum; thus we need only check that $J = \mathrm{ad}(Z)$ interchanges the (± 1)-eigenspaces of $\mathfrak{p}_{\Delta-\Gamma,1}$. This follows from the fact that Lemma 4.2 allows us to apply [7, Lemma 4.7] to $M_{\Delta-\Gamma}$. *Q.E.D.*

5.7. THEOREM. *Let L^Γ and $L_1{}^\Gamma$ be the isotropy subgroups of K and $K_1{}^\Gamma$ at $x^\Gamma = c_{\Delta-\Gamma}(x)$. Then:*

5.7.1. $S^\Gamma = K(x^\Gamma) \cong K/L^\Gamma$ *and* $\dim. S^\Gamma = \dim. \mathfrak{p}_2{}^\Gamma + \frac{1}{2}\dim. \mathfrak{p}_{\Delta-\Gamma,1}$.

5.7.2. $U^\Gamma = G^0(x^\Gamma)$ *and* $\dim. U^\Gamma = \dim. \mathfrak{p}_2{}^\Gamma + \frac{1}{2}\dim. \mathfrak{p}_{\Delta-\Gamma,1} + \dim. \mathfrak{p}_\Gamma$.

5.7.3. *$K(c_{\Delta-\Gamma}{}^2(x))$ is a complex totally geodesic submanifold of M^*, and is thus a compact herimitian symmetric space; K^Γ is the isotropy subgroup of K at $c_{\Delta-\Gamma}{}^2(x)$, so $K(c_{\Delta-\Gamma}{}^2(x)) \cong K/K^\Gamma$.*

5.7.4. *The map $k(x^\Gamma) \to k(c_{\Delta-\Gamma}{}^2(x))$ is a fibering of S^Γ over $K(c_{\Delta-\Gamma}{}^2(x))$; the fibre over $k(c_{\Delta-\Gamma}{}^2(x))$ is $kK_1{}^\Gamma(x^\Gamma)$, which is totally geodesic in M^*, Riemannian symmetric and isometric to $K_1{}^\Gamma/L_1{}^\Gamma$.*

5.7.5. *The following statements are equivalent:*

(i) *The partial Cayley transform $c_{\Delta-\Gamma}$ has order* 4, i.e., $\mathfrak{g} = \mathfrak{g}^\Gamma$, *i.e., $K(c_{\Delta-\Gamma}{}^2(x))$ is a single point.*

(ii) *S^Γ is a totally geodesic submanifold of M^* (in which case it is Riemannian symmetric and K induces the largest connected group of isometries).*

(iii) *Let $M = M_1 \times \cdots \times M_r$ be the decomposition into irreducible factors, and let $c_{\Delta-\Gamma}M_\Gamma = F_1 \times \cdots \times F_r$ be the corresponding decomposition of the boundary component $c_{\Delta-\Gamma}M_\Gamma$. Then for each j, either $F_j = M_j$, or M_j is of tube type and F_j is a point on its Bergman-Šilov boundary.*

Proof. $S^\Gamma = K(x^\Gamma)$ was observed in § 5.3, and $K(x^\Gamma) \cong K/L^\Gamma$ by definition of L^Γ. Now $\dim. S^\Gamma = \dim. K - \dim. L^\Gamma = \dim. \mathfrak{k} - \dim. \mathfrak{l}^\Gamma = \dim. \mathfrak{q}^\Gamma = \dim. \mathfrak{q}_1{}^\Gamma + \dim. \mathfrak{q}_2{}^\Gamma$. Lemma 5.6 shows that $\dim. \mathfrak{q}_2{}^\Gamma = \dim. \mathfrak{p}_2{}^\Gamma$ and $\dim. \mathfrak{q}_1{}^\Gamma = \frac{1}{2}\dim. \mathfrak{p}_{\Delta-\Gamma,1}$. This proves (5.7.1).

$U^\Gamma = G^0(M_\Gamma) = G^0(G_\Gamma(x^\Gamma)) = G^0(x^\Gamma)$, and Lemma 5.2 shows that $\dim. U^\Gamma - \dim. S^\Gamma = \dim. M_\Gamma = \dim. \mathfrak{p}_\Gamma$. Now (5.7.2) follows from (5.7.1).

The isotropy subalgebra of $\mathfrak{k}$ at $c_{\Delta-\Gamma}{}^2(x)$ is the fixed piont set in $\mathfrak{k}$ of conjugation by the symmetry $c_{\Delta-\Gamma}{}^2 s c_{\Delta-\Gamma}{}^{-2} = c_{\Delta-\Gamma}{}^4 s$ there; thus $\mathfrak{k}^\Gamma$ is the isotropy subalgebra of $\mathfrak{k}$ at $c_{\Delta-\Gamma}{}^2(x)$. On the other hand, $\mathfrak{k} = \mathfrak{k}^\Gamma + \mathfrak{q}_2{}^\Gamma$, and conjugation by the symmetry is -1 on $\mathfrak{q}_2{}^\Gamma$; thus the orbit $K(c_{\Delta-\Gamma}{}^2(x))$ is totally geodesic in M^*. The complex structure operator at $c_{\Delta-\Gamma}{}^2(x)$ is $\tau_{\Delta-\Gamma}(Z)$. $[\tau_{\Delta-\Gamma}Z, \mathfrak{q}_2{}^\Gamma] = \tau_{\Delta-\Gamma}{}^{-1}[Z, \tau_{\Delta-\Gamma}\mathfrak{q}_2{}^\Gamma] = \tau_{\Delta-\Gamma}{}^{-1}[Z, \mathfrak{p}_2{}^\Gamma] = \tau_{\Delta-\Gamma}{}^{-1}(\mathfrak{p}_2{}^\Gamma) = \mathfrak{q}_2{}^\Gamma$. Thus $K(c_{\Delta-\Gamma}{}^2(x))$ is a complex submanifold of M^*. $K(c_{\Delta-\Gamma}{}^2(x))$ is simply connected; this is seen in the irreducible case because a local toral factor would be a coset space of the (one real dimensional) center of K, and the assertion follows in general. Now the isotropy subgroup of K at $c_{\Delta-\Gamma}{}^2(x)$ is connected; as its Lie algebra is $\mathfrak{k}^\Gamma$, it must be the analytic group K^Γ. This proves (5.7.3).

The map $S^\Gamma \to K(c_{\Delta-\Gamma}{}^2(x))$ is given by the map $kL^\Gamma \to kK^\Gamma$ of K/L^Γ onto K/K^Γ; to prove it to be well defined, we must check that $L^\Gamma \subset K^\Gamma$ (although we do not yet know that L^Γ is connected). K^Γ is the identity component of V, where V is the full centralizer of $\tau_{\Delta-\Gamma}{}^2$ in K. As K/V is hermitian symmetric without locally euclidean factor, as checked in the paragraph above, it is simply connected. Thus V is connected, and now $K^\Gamma = V$. On the other hand, $L^\Gamma = K \cap \mathrm{ad}(c_{\Delta-\Gamma})K$ is contained in the centralizer of $\tau_{\Delta-\Gamma}$ in K. Thus $L^\Gamma \subset K^\Gamma$. Now $S^\Gamma \to K(c_{\Delta-\Gamma}{}^2(x))$ is a well-defined fibering. The fibre over $k(c_{\Delta-\Gamma}{}^2(x))$ is $kK^\Gamma(x^\Gamma) = k\cdot K_1{}^\Gamma \cdot L_2{}^\Gamma(x^\Gamma) = kK_1{}^\Gamma(x^\Gamma)$. $K_1{}^\Gamma(x^\Gamma)$ is totally geodesic in M^*, because $c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1}$ is the symmetry at x^Γ, and because $\mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1})K_1{}^\Gamma = \tau_{\Delta-\Gamma}K_1{}^\Gamma = K_1{}^\Gamma$. Now $kK_1{}^\Gamma$ is totally geodesic in M^*. We have proved (5.7.4).

Let $c_{\Delta-\Gamma}{}^4 = 1$. Then $\mathfrak{g} = \mathfrak{g}^\Gamma$, so in particular $\mathfrak{k} = \mathfrak{k}^\Gamma$ and $K(c_{\Delta-\Gamma}{}^2(x)) \cong K/K^\Gamma$ is a single point. If $\mathfrak{k} = \mathfrak{k}^\Gamma$, then $\mathfrak{q}_2{}^\Gamma = 0$, so $\mathfrak{p}_2{}^\Gamma = 0$ by Lemma 5.6, whence $\mathfrak{g} = \mathfrak{g}^\Gamma$ and $c_{\Delta-\Gamma}{}^4 = 1$. Now the conditions of (i) of (5.7.5) are equivalent.

Assume (i). Then s commutes with $c_{\Delta-\Gamma}{}^2$ because $c_{\Delta-\Gamma}{}^4 = 1$, so $\tau_{\Delta-\Gamma}(\mathfrak{k}) = \mathfrak{k}$. As $\tau_{\Delta-\Gamma}$ coincides with $\mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1})$ on $\mathfrak{k}$, $S^\Gamma = K(x^\Gamma)$ is totally geodesic in M^*, which is (ii). Assume (ii). If M is irreducible then K is the largest connected subgroup of G which preserves $K(x^\Gamma)$, by maximality of $\mathfrak{k}$ in $\mathfrak{g}$; now $\mathfrak{k} = \mathrm{ad}(c_{\Delta-\Gamma} s c_{\Delta-\Gamma}{}^{-1})\mathfrak{k} = \tau_{\Delta-\Gamma}(\mathfrak{k})$ by (ii), and (i) follows via Lemma 5.2 from $\mathfrak{q}_2{}^\Gamma = 0 = \mathfrak{p}_2{}^\Gamma$. Now (i) is equivalent to (ii) in (5.7.5).

For the equivalence of (i) and (iii) we may assume M irreducible. Assume (iii), then M_Γ is a point and M is of tube type, so $c_{\Delta-\Gamma} = c$ and [7, Theorem 4.9] $c^4 = 1$, proving (i). Assume (i). Then $M = M^\Gamma$. As $M^\Gamma = M_\Gamma \times M_{\Delta-\Gamma,1}$ by $\mathfrak{p}_1{}^\Gamma = \mathfrak{p}_\Gamma + \mathfrak{p}_{\Delta-\Gamma,1}$, and as M is irreducible, we must have $\Gamma = \phi$ or $\Gamma = \Delta$; (iii) follows. Now (i) and (iii) are equivalent in (5.7.5). *Q.E.D.*

5.8. COROLLARY. *The fundamental group $\pi_1(S^\Gamma)$ is the direct product of a finite abelian group and a group which is free abelian with one generator for each tube type irreducible factor of M whose Bergman-Šilov boundary is a direct factor of $c_{\Delta-\Gamma}M_\Gamma$. In particular the first Betti number of S^Γ is the number of irreducible tube type factors of M whose Bergman-Šilov boundary is a factor of $c_{\Delta-\Gamma}M_\Gamma$.*

Proof. We may assume M irreducible. Now $Z \in \mathfrak{q}_1{}^\Gamma$ if and only if M is of tube type and $\Gamma = \phi$, for $Z = Z' + Z_\Gamma{}^0 + Z_{\Delta-\Gamma}{}^0$ where $Z' + Z_\Gamma{}^0 \in \mathfrak{l}^\Gamma$, $Z_{\Delta-\Gamma}{}^0 \in \mathfrak{q}_1{}^\Gamma$, and $Z' = 0$ if and only if M is of tube type. Let $\mathfrak{k}_{ss}$ be the derived algebra of $\mathfrak{k}$. As $\dim.\mathfrak{k} - \dim.\mathfrak{k}_{ss} = 1$, $\mathfrak{k}_{ss} \perp Z$ under the Killing form of $\mathfrak{g}$, and $\mathfrak{l}^\Gamma \perp \mathfrak{q}_1{}^\Gamma$, it follows that

(i) if M is of tube type and $\Gamma = \phi$ then $\mathfrak{l}^\Gamma \subset \mathfrak{k}_{ss}$, and

(ii) otherwise $\mathfrak{k}_{ss} + \mathfrak{l}^\Gamma = \mathfrak{k}$.

We also have

(iii) $\pi_1(S^\Gamma)$ is abelian

as in [7, Theorem 4.11] because S^Γ is fibered over a hermitian symmetric space of a semisimple group with symmetric fibre. Now our assertion follows from some homotopy sequences as in [7, Theorem 4.11]. *Q.E.D.*

6. The stability group of a boundary component. G^0 is transitive both on the set S^Γ of boundary components of type Γ and on the union U^Γ of these boundary components. Let B^Γ be the set of all elements of G^0 which preserve $c_{\Delta-\Gamma}M_\Gamma \in S^\Gamma$ and define E^Γ to be the isotropy subgroup of G^0 at $x^\Gamma = c_{\Delta-\Gamma}(x)$. Now

$$S^\Gamma \cong G^0/B^\Gamma \text{ and } U^\Gamma \cong G^0/E^\Gamma.$$

We will study S^Γ and U^Γ by examining B^Γ and E^Γ.

6.1. We have $L^\Gamma \subset E^\Gamma \subset B^\Gamma$ because L^Γ is the isotropy subgroup of K at x^Γ and $x^\Gamma \in c_{\Delta-\Gamma}M_\Gamma$, and $L^\Gamma = K \cap B^\Gamma$ by Lemma 5.2. K is transitive on S^Γ so $G^0 = KB^\Gamma$; now $G^0 = B^\Gamma \cdot K$ and B^Γ is transitive on M, $B^\Gamma/L^\Gamma = M$. M being connected and acyclic, it follows that L^Γ is a maximal compact subgroup of B^Γ.

E^Γ is in general not transitive on M. For if $E^\Gamma(x) = M$, then $\dim.E^\Gamma = \dim.B^\Gamma$ because $K \cap B^\Gamma = L^\Gamma = K \cap E^\Gamma$, whence $E^\Gamma = B^\Gamma$ because it meets every component. Then $c_{\Delta-\Gamma}M_\Gamma = x^\Gamma$ because $G_\Gamma{}^0 \subset B^\Gamma$ and so $\Gamma = \phi$.

Now L^Γ is maximal compact both in B^Γ and E^Γ, and these groups are generated by L^Γ and their respective identity components $B_0{}^\Gamma$ and $E_0{}^\Gamma$. This brings the study of B^Γ and E^Γ down to the study of their Lie algebras $\mathfrak{b}^\Gamma$ and $\mathfrak{e}^\Gamma$. We will need some definitions in order to calculate these Lie algebras.

(6.1.1) If $\mathfrak{u}$ is a subspace of $\mathfrak{g}$ or $\mathfrak{g}^0$ and $\mathfrak{u}^C$ is the sum of $\mathfrak{h}^C$-root spaces, then $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) denotes the sum of the positive (resp. negative) root spaces in $\mathfrak{u}^C$. This defines $\mathfrak{p}_i{}^{\Gamma\pm}$, $\mathfrak{q}_2{}^{\Gamma\pm}$, $\mathfrak{p}_\Sigma{}^\pm$ and $\mathfrak{p}_{\Sigma,i}{}^\pm$ $(i=1,2;\Sigma=\Gamma,\Delta-\Gamma)$.
(6.1.2) $\mathfrak{r}_2{}^{\Gamma\pm}=\mathfrak{q}_2{}^{\Gamma\mp}+\mathfrak{p}_2{}^{\Gamma\pm}$ and $\mathfrak{r}^{\Gamma\pm}=\mathfrak{p}_{\Delta-\Gamma,1}{}^\pm+\mathfrak{r}_2{}^{\Gamma\pm}$, complex subspaces of $\mathfrak{g}^C$.
(6.1.3) $\mathfrak{n}_2{}^{\Gamma\pm}=\mathfrak{r}_2{}^{\Gamma\pm}\cap\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$, $\mathfrak{n}_1{}^{\Gamma\pm}=\mathfrak{p}_{\Delta-\Gamma,1}{}^\pm\cap\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ and

$$\mathfrak{n}^{\Gamma\pm}=\mathfrak{n}_1{}^{\Gamma\pm}+\mathfrak{n}_2{}^{\Gamma\pm}, \text{ real subspaces of } \mathfrak{g}^C.$$

(6.1.4) Recall $\mathfrak{k}_1{}^{\Gamma*}=\mathfrak{l}_1{}^\Gamma+i\mathfrak{q}_1{}^\Gamma$ and $\mathfrak{k}^{\Gamma*}=\mathfrak{l}^\Gamma+i\mathfrak{q}_1{}^\Gamma$.

Convention. From now on we assume that $\alpha>\beta$ for $\alpha\in\Gamma$ and $\beta\in\Delta-\Gamma$. This causes no loss of generality because [8, Theorem 2] on each irreducible factor of M the small Weyl group induces all permutations on the strongly orthogonal roots.

6.2. LEMMA. *$\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ coincides with $\frac{1}{2}\mathrm{ad}(Z)$ on $\mathfrak{p}_2{}^{\Gamma C}$ and $\tau_{\Delta-\Gamma}$ interchanges $\mathfrak{p}_2{}^{\Gamma\pm}$ with $\mathfrak{q}_2{}^{\Gamma\pm}$.*

Proof. $Z=(Z'+Z_\Gamma{}^0)+Z_{\Delta-\Gamma}{}^0$ where $\tau_{\Delta-\Gamma}$ is $+1$ on the first summand and -1 on the second. Now $\tau_{\Delta-\Gamma}Z=Z-2Z_{\Delta-\Gamma}{}^0$. Let $E\in\mathfrak{p}_2{}^{\Gamma C}$; $E=\tau_{\Delta-\Gamma}Q$ with $Q\in\mathfrak{q}_2{}^{\Gamma C}$ by Lemma 5.6, and

$$(\mathrm{ad}(Z)-2\mathrm{ad}(Z_{\Delta-\Gamma}{}^0))E=\mathrm{ad}(\tau_{\Delta-\Gamma}Z)(\tau_{\Delta-\Gamma}Q)=\tau_{\Delta-\Gamma}\,\mathrm{ad}(Z)Q=0.$$

This proves the first statement.

We may now assume $\mathfrak{g}^0$ simple. Let $\Delta=\{\delta_1,\cdots,\delta_r\}$ with $\delta_1<\delta_2<\cdots<\delta_r$, so $\Gamma=\{\delta_{i+1},\cdots,\delta_r\}$ by hypothesis on the ordering of roots. It is known [8, Theorem 1] that the compact simple roots have restrictions 0, $\frac{1}{2}(\delta_2-\delta_1),\cdots,\frac{1}{2}(\delta_r-\delta_{r-1})$, and perhaps also $-\frac{1}{2}\delta_r$, to $\mathfrak{h}^-$. Thus $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)\cdot E_\beta$ is 0 or $(i/2)E_\beta$. It follows that $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)\cdot E_\gamma=ia_\gamma E_\gamma$ with $a_\gamma\geqq 0$ for every compact positive root γ. The first statement says that $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ is $\mp i/2$ on $\mathfrak{p}_2{}^{\Gamma\pm}$. As $\tau_{\Delta-\Gamma}{}^0(Z_{\Delta-\Gamma}{}^0)=-Z_{\Delta-\Gamma}{}^0$, it follows that $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ is $\pm i/2$ on $\tau_{\Delta-\Gamma}(\mathfrak{p}_2{}^{\Gamma\pm})$. Thus $\tau_{\Delta-\Gamma}(\mathfrak{p}_2{}^{\Gamma\pm})\subset\mathfrak{q}_2{}^{\Gamma\pm}$ and the interchange statement follows. *Q.E.D.*

6.3. LEMMA. *The eigenvalues and eigenspaces of $\mathrm{ad}(-Y_{\Delta-\Gamma}{}^0)$ are:*

eigenvalue	*eigenspace on* $\mathfrak{g}^C$	*eigenspace on* $\mathfrak{g}^0$
0	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma C}+\mathfrak{p}_\Gamma{}^C$	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma*}+\mathfrak{p}_\Gamma{}^0$
± 1	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{r}_2{}^{\Gamma\pm}$	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{n}_2{}^{\Gamma\pm}$
± 2	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{p}_{\Delta-\Gamma,1}{}^\pm$	$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{n}_1{}^{\Gamma\pm}$

In particular, $\mathfrak{k}^{\Gamma *}=\mathfrak{l}_2^{\Gamma}\oplus\mathfrak{k}_1^{\Gamma *}=\mathfrak{k}^{\Gamma C}\cap \operatorname{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ *and is a real form of* $\mathfrak{k}^{\Gamma C}$, $\mathfrak{n}_2^{\Gamma\pm}$ *is a real form of* $\mathfrak{r}_2^{\Gamma\pm}$, *and* $\mathfrak{n}_1^{\Gamma\pm}$ *is a real form of* $\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}$.

Proof. $\mathfrak{g}^C=\mathfrak{k}^{\Gamma C}+\mathfrak{q}_2^{\Gamma C}+\mathfrak{p}_2^{\Gamma C}+\mathfrak{p}_\Gamma{}^C+\mathfrak{p}_{\Delta-\Gamma,1}{}^C$. $\operatorname{ad}(Z_{\Delta-\Gamma}{}^0)$ is $\mp i$ on $\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}$ by Lemma 4.4, $\pm i/2$ on $\mathfrak{q}_2^{\Gamma\pm}+\mathfrak{p}_2^{\Gamma\mp}$ by Lemma 6.2, and 0 on $\mathfrak{p}_\Gamma{}^C$ by definition of $\mathfrak{g}_\Gamma{}^C$. $\mathfrak{k}^{\Gamma C}=\mathfrak{l}_2^{\Gamma C}+(\mathfrak{k}_\Gamma{}^C+\mathfrak{k}_{\Delta-\Gamma,1}{}^C)$ for $\mathfrak{p}^{\Gamma C}=\mathfrak{p}_\Gamma{}^C+\mathfrak{p}_{\Delta-\Gamma,1}{}^C$ by Lemma 4.4. Now $\operatorname{ad}(Z_{\Delta-\Gamma}{}^0)\mathfrak{k}^{\Gamma C}=0$ because $Z_{\Delta-\Gamma}{}^0$ is central in $\mathfrak{k}_{\Delta-\Gamma,1}{}^{0C}$, centralizes $\mathfrak{k}_\Gamma{}^C$ by construction, and centralizes $\mathfrak{l}_2^{\Gamma C}$ by $[\mathfrak{l}_2^\Gamma,\mathfrak{k}_1^\Gamma]=0$ and $\mathfrak{k}_{\Delta-\Gamma,1}\subset\mathfrak{k}_1^\Gamma$. Now we know the eigenvalues and eigenspaces of $\operatorname{ad}(Z_{\Delta-\Gamma}{}^0)$ on $\mathfrak{g}^C$, and the assertions for $\operatorname{ad}(-Y_{\Delta-\Gamma}{}^0)$ follow from

$$\operatorname{ad}(c_{\Delta-\Gamma})^{-1}(2iZ_{\Delta-\Gamma}{}^0)=-Y_{\Delta-\Gamma}{}^0.$$

As $-Y_{\Delta-\Gamma}{}^0\in\mathfrak{g}^0$ and $\operatorname{ad}(-Y_{\Delta-\Gamma}{}^0)$ is a semisimple linear transformation with all eigenvalues real, every eigenspace of $\operatorname{ad}(-Y_{\Delta-\Gamma}{}^0)$ on $\mathfrak{g}^C$ is the complexification of its intersection with $\mathfrak{g}^0$. Thus we need only prove that

$$\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}\cap\mathfrak{g}^0=\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{n}_1^{\Gamma\pm},$$

that

$$\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{r}_2^{\Gamma\pm}\cap\mathfrak{g}^0=\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{n}_2^{\Gamma\pm},$$

and that

$$(\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma C}+\mathfrak{p}_\Gamma{}^C)\cap\mathfrak{g}^0=\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma *}+\mathfrak{p}_\Gamma{}^0.$$

The first two equalities are immediate from the definitions of the $\mathfrak{n}_i^{\Gamma\pm}$, and $\mathfrak{p}_\Gamma{}^C\cap\mathfrak{g}^0=\mathfrak{p}_\Gamma{}^0$ by construction. Thus we need only prove that

$$\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma C}\cap\mathfrak{g}^0=\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma *}.$$

As $\mathfrak{k}^{\Gamma *}$ is a real form of $\mathfrak{k}^{\Gamma C}$, it suffices to check that $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma *}\subset\mathfrak{g}^0$.

$\mathfrak{k}^{\Gamma *}=\mathfrak{l}^\Gamma+i\mathfrak{q}_1^\Gamma$ and $\operatorname{ad}(c_{\Delta-\Gamma})$ is trivial on $\mathfrak{l}^\Gamma$. Thus $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{l}^\Gamma=\mathfrak{l}^\Gamma\subset\mathfrak{k}\subset\mathfrak{g}^0$. Lemma 5.6 says that $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}(i\mathfrak{q}_1^\Gamma)\subset i\mathfrak{p}_1^\Gamma\subset\mathfrak{p}^0\subset\mathfrak{g}^0$. Now $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma *}\subset\mathfrak{g}^0$ and the Lemma is proved. *Q.E.D.*

6.4. LEMMA. $[\mathfrak{q}_2^{\Gamma\mp},\mathfrak{p}_2^{\Gamma\pm}]\subset\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}$, $\mathfrak{r}^{\Gamma\pm}$ *is a complex nilpotent subalgebra of degree 2 which is unipotent in the adjoint representation of* $\mathfrak{g}^C$, *and* $\mathfrak{n}^{\Gamma\pm}$ *is a real form of* $\mathfrak{r}^{\Gamma\pm}$.

Proof. $[\mathfrak{q}_2^{\Gamma\mp},\mathfrak{p}_2^{\Gamma\pm}]\subset\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}$ by addition of eigenvalues of $\operatorname{ad}(Z_{\Delta-\Gamma}{}^0)$ and because $[\mathfrak{k}^C,\mathfrak{p}^{\pm}]\subset\mathfrak{p}^{\pm}$. $[\mathfrak{p}_2^{\Gamma\pm},\mathfrak{p}_2^{\Gamma\pm}]=0$ and $[\mathfrak{q}_2^{\Gamma\mp},\mathfrak{q}_2^{\Gamma\mp}]=0$ now by Lemma 6.2. Finally $[\mathfrak{r}^{\Gamma\pm},\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}]=0$ by addition of eigenvalues of $\operatorname{ad}(Z_{\Delta-\Gamma}{}^0)$. Thus $\mathfrak{r}^{\Gamma\pm}$ is nilpotent of degree 2.

$\mathfrak{n}^{\Gamma\pm}$ is a real form of $\mathfrak{r}^{\Gamma\pm}$, and $\operatorname{ad}(\mathfrak{r}^{\Gamma\pm})$ is unipotent on $\mathfrak{g}^C$ by addition of eigenvalues, by Lemma 6.3. *Q.E.D.*

6.5. THEOREM. $\mathfrak{b}^\Gamma=\mathfrak{p}_\Gamma{}^0+\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^{\Gamma *}+\mathfrak{n}^{\Gamma -})$; $\mathfrak{b}^\Gamma$ *is the sum of the nonpositive eigenspaces of* $\operatorname{ad}(-Y_{\Delta-\Gamma}{}^0)$ *on* $\mathfrak{g}^0$ *and is the normalizer of* $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{n}^{\Gamma -}$ *in* $\mathfrak{g}^0$. $\mathfrak{e}^\Gamma$ *is the subalgebra* $\operatorname{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^\Gamma+\mathfrak{n}^{\Gamma -})$ *of* $\mathfrak{b}^\Gamma$.

Proof. The isotropy subalgebra of $\mathfrak{g}^C$ at x^Γ is $\mathrm{ad}(c_{\Delta-\Gamma})\cdot(\mathfrak{k}^C+\mathfrak{p}^+)$, which we decompose as

$$\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{k}^C+\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{q}_2{}^{\Gamma C} + \mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{p}_2{}^{\Gamma+}+\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{p}_{\Delta-\Gamma,1}{}^{+}+\mathfrak{p}_\Gamma{}^{+}.$$

As $\tau_{\Delta-\Gamma}\mathfrak{k}^\Gamma=\mathfrak{k}^\Gamma$ we have $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{k}^{\Gamma C}=\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{k}^{\Gamma C}$. Lemma 6.2 gives us

$$\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{r}_2{}^{\Gamma-}=\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{r}_2{}^{\Gamma+}\subset\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{q}_2{}^{\Gamma C}+\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{p}_2{}^{\Gamma+}.$$

Finally $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\mathfrak{p}_{\Delta-\Gamma,1}{}^{-}=\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{p}_{\Delta-\Gamma,1}{}^{+}$. Thus the isotropy subalgebra of $\mathfrak{g}^C$ at x^Γ contains $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^{\Gamma C}+\mathfrak{r}^{\Gamma-})$. Now Lemma 6.3 say that $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^{\Gamma *}+\mathfrak{n}^{\Gamma-})$ lies in the isotropy subalgebra of $\mathfrak{g}^0$ at x^Γ.

The isotropy subalgebra of $\mathfrak{g}^0$ at x^Γ has dimension $\dim. G^0-\dim. U^\Gamma$, and this is equal to $\dim.\mathfrak{k}+\frac{1}{2}\dim.\mathfrak{p}_{\Delta-\Gamma,1}$ by Theorem 5.7. Now

$$\begin{aligned}\dim.\mathfrak{k}+\tfrac{1}{2}\dim.\mathfrak{p}_{\Delta-\Gamma,1}&=\dim.\mathfrak{k}^\Gamma+\dim.\mathfrak{q}_2{}^\Gamma+\dim.\mathfrak{n}_1{}^{\Gamma-}\\&=\dim.\mathfrak{k}^{\Gamma *}+\dim.\mathfrak{n}^{\Gamma-}=\dim.\mathrm{ad}(c_{\Delta-\Gamma})^{-1}(\mathfrak{k}^{\Gamma *}+\mathfrak{n}^{\Gamma-}).\end{aligned}$$

The final assertion of the Theorem is proved. As $\mathfrak{p}_\Gamma{}^0\subset\mathfrak{b}^\Gamma$, as

$$\mathfrak{p}_\Gamma{}^0\cap\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^{\Gamma *}+\mathfrak{n}^{\Gamma-})=0,$$

and as $\dim.\mathfrak{p}_\Gamma{}^0=\dim. M_\Gamma=\dim. U^\Gamma-\dim. S^\Gamma$, it follows that

$$\mathfrak{b}^\Gamma=\mathfrak{p}_\Gamma{}^0+\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\cdot(\mathfrak{k}^{\Gamma *}+\mathfrak{n}^{\Gamma-}),$$

which is our main assertion.

The eigenspace assertion now follows from Lemma 6.3, and the normalizer assertion is immediate. *Q.E.D.*

6.6. *Remarks on* $\mathfrak{b}^\Gamma$. The linear transformation $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ is 0 on $\mathfrak{p}_\Gamma$ and $\pm i$ on $\mathfrak{p}_{\Delta-\Gamma,1}$; thus $[\mathfrak{p}_\Gamma,\mathfrak{p}_{\Delta-\Gamma,1}]$ is in the $(\pm i)$-eigenspace of $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$ on $\mathfrak{k}$, which is zero. We conclude that

$$[\mathfrak{g}_\Gamma{}^C,\mathfrak{g}_{\Delta-\Gamma,1}{}^C]=0. \tag{6.6.1}$$

Recall that

$$\mathfrak{p}_1{}^\Gamma=\mathfrak{p}_\Gamma+\mathfrak{p}_{\Delta-\Gamma,1},\ \mathfrak{k}_1{}^\Gamma=[\mathfrak{p}_1{}^\Gamma,\mathfrak{p}_1{}^\Gamma],\ \mathfrak{k}_\Gamma=[\mathfrak{p}_\Gamma,\mathfrak{p}_\Gamma]$$

and $\mathfrak{k}_{\Delta-\Gamma,1}=[\mathfrak{p}_{\Delta-\Gamma,1}\mathfrak{p}_{\Delta-\Gamma,1}]$. With (6.6.1) this gives $\mathfrak{k}_1{}^\Gamma=\mathfrak{k}_\Gamma\oplus\mathfrak{k}_{\Delta-\Gamma,1}$ (direct sum of ideals); now it follows that

$$\mathfrak{k}_1{}^{\Gamma *}=\mathfrak{k}_\Gamma\oplus\mathfrak{k}_{\Delta-\Gamma,1}{}^{*}. \tag{6.6.2}$$

Now (6.6.2) and $\mathfrak{g}^\Gamma=\mathfrak{k}^\Gamma+\mathfrak{p}^\Gamma=\mathfrak{l}_2{}^\Gamma\oplus\mathfrak{g}_1{}^\Gamma$ yield

$$\mathfrak{k}^{\Gamma *}+\mathfrak{p}_\Gamma{}^0=\mathfrak{g}_\Gamma{}^0\oplus\mathfrak{l}_2{}^\Gamma\oplus\mathfrak{k}_{\Delta-\Gamma,1}{}^{*}. \tag{6.6.3}$$

The algebra (6.6.3) is a reductive subalgebra of $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ which is a complement to $\mathfrak{n}^{\Gamma-}$ in $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{b}^\Gamma$. It follows that

(6.6.4) $\mathfrak{n}^{\Gamma-}$ is the nilradical of $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{b}^{\Gamma}$,

(6.6.5) $\mathfrak{g}_{\Gamma}{}^{0} \oplus \mathfrak{l}_2{}^{\Gamma} \oplus \mathfrak{k}_{\Delta-\Gamma,1}{}^{*}$ is a reductive complement to $\mathfrak{n}^{\Gamma-}$ in $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{b}^{\Gamma}$, and

(6.6.6) $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{b}^{\Gamma} = (\mathfrak{g}_{\Gamma}{}^{0} \oplus \mathfrak{l}_2{}^{\Gamma} \oplus \mathfrak{k}_{\Delta-\Gamma,1}{}^{*}) + \mathfrak{n}^{\Gamma-}$, semidirect sum.

$\mathfrak{e}^{\Gamma}$ denotes the isotropy subalgebra of $\mathfrak{g}^0$ at x^{Γ}. As above we see that

(6.6.7) $\mathfrak{n}^{\Gamma-}$ is the nilradical of $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{e}^{\Gamma}$,

(6.6.8) $\mathfrak{k}_{\Gamma} \oplus \mathfrak{l}_2{}^{\Gamma} \oplus \mathfrak{k}_{\Delta-\Gamma,1}{}^{*}$ is a reductive complement to $\mathfrak{n}^{\Gamma-}$ in $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{e}^{\Gamma}$, and

(6.6.9) $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{e}^{\Gamma} = (\mathfrak{k}_{\Gamma} \oplus \mathfrak{l}_2{}^{\Gamma} \oplus \mathfrak{k}_{\Delta-\Gamma,1}{}^{*}) + \mathfrak{n}^{\Gamma-}$, semidirect sum.

6.7. In order to describe B^{Γ} we define

(6.7.1) $P_{\Delta-\Gamma,1}{}^{\pm} = \exp(\mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm}) \subset G^C$ and $N_1{}^{\Gamma\pm} = \mathrm{ad}(c_{\Delta-\Gamma})G^0 \cap P_{\Delta-\Gamma,1}{}^{\pm}$,

(6.7.2) $R^{\Gamma\pm} = \exp(\mathfrak{r}^{\Gamma\pm}) \subset G^C$ and $N^{\Gamma\pm} = \mathrm{ad}(c_{\Delta-\Gamma})G^0 \cap R^{\Gamma\pm}$.

Lemma 6.4 says that every element of $\mathrm{ad}(\mathfrak{r}^{\pm})$ is a nilpotent linear transformation of $\mathfrak{g}^C$. Thus $P_{\Delta-\Gamma,1}{}^{\pm}$ and $R^{\Gamma\pm}$ are unipotent subgroups of G^C, and

$$\exp: \mathfrak{p}_{\Delta-\Gamma,1}{}^{\pm} \to P_{\Delta-\Gamma,1}{}^{\pm} \text{ and } \exp: \mathfrak{r}^{\Gamma\pm} \to R^{\Gamma\pm}$$

are one-one onto. In particular, the groups $P_{\Delta-\Gamma,1}{}^{\pm}$ and $R^{\Gamma\pm}$ are connected simply connected nilpotent Lie groups. Now let η be conjugation of G^C over $\mathrm{ad}(c_{\Delta-\Gamma})G^0$; η induces involutive automorphisms of the real groups $P_{\Delta-\Gamma,1}{}^{\pm}$ and $R^{\Gamma\pm}$ with respective fixed point sets $N_1{}^{\Gamma\pm}$ and $N^{\Gamma\pm}$. It follows that

(6.7.3) $N_1{}^{\Gamma\pm}$ and $N^{\Gamma\pm}$ are the analytic subgroups of $\mathrm{ad}(c_{\Delta-\Gamma})G^0$ with Lie algebras $\mathfrak{n}_1{}^{\Gamma\pm}$ and $\mathfrak{n}^{\Gamma\pm}$.

In particlar, $N_1{}^{\Gamma\pm}$ and $N^{\Gamma\pm}$ are connected simply connected nilpotent Lie groups.

6.8. THEOREM. *B^{Γ} is a parabolic subgroup of G^0 and is the normalizer of* $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}N^{\Gamma-}$ *in G^0. The identity component of B^{Γ} is given by*

$$B_0{}^{\Gamma} = \{G_{\Gamma}{}^{0} \cdot L_2{}^{\Gamma} \cdot \mathrm{ad}(c_{\Delta-\Gamma})^{-1}K_{\Delta-\Gamma,1}{}^{*}\} \cdot \mathrm{ad}(c_{\Delta-\Gamma})^{-1}N^{\Gamma-}$$

semidirect product; this is the Chevalley decomposition into reductive and unipotent parts.

Remark 1. *$G^0/B^{\Gamma} = S^{\Gamma}$ is a real projective variety defined over the rational number field.* For $B^{\Gamma} = B^{\Gamma C} \cap G^0$ for a parabolic subgroup $B^{\Gamma C}$ of G^C, $G^C/B^{\Gamma C}$ is a complex projective variety defined over the rationals, and a result of Borel [1, Proposition 3.7] gives the conjugation of G^C over G^0 defined over the rationals.

Remark 2. The reductive part of $B_0{}^\Gamma$ admits

$$G_\Gamma{}^0 \times L_2{}^\Gamma \times \mathrm{ad}(c_{\Delta-\Gamma})^{-1}K_{\Delta-\Gamma,1}{}^*$$

as a covering group.

Remark 3. B^Γ is the subgroup of G^0 which preserves the boundary component $\xi^{-1}c_{\Delta-\Gamma}M_\Gamma = (\xi^{-1}c_{\Delta-\Gamma}\xi)D_\Gamma$ of D in $\mathfrak{p}^-$.

Proof of Theorem. Let $B^{\Gamma C}$ denote the analytic subgroup of G^C whose Lie algebra is the complexification $\mathfrak{b}^{\Gamma C}$ of $\mathfrak{b}^\Gamma$. As $-Y_{\Delta-\Gamma}{}^0$ is a basis of the Lie algebra of a split algebraic torus of G^0, and as $\mathfrak{b}^{\Gamma C}$ (resp $\mathfrak{b}^\Gamma$) is the sum of the nonpositive weight spaces of $\mathrm{ad}(-Y_{\Delta-\Gamma}{}^0)$ on $\mathfrak{g}^C$ (resp. on $\mathfrak{g}^0$), $B^{\Gamma C}$ is a parabolic subgroup of G^C and $B^{\Gamma C} \cap G^0$ is parabolic in G^0.

Let $B_1{}^\Gamma$ denote $G^0 \cap B^{\Gamma C}$. Now $B_0{}^\Gamma$ is the identity component both of B^Γ and $B_1{}^\Gamma$, and $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}N^{\Gamma-}$ is the unipotent radical of all three by (6.6.4) and (6.7.3). $B_1{}^\Gamma$ is the full normaliser of $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}N^{\Gamma-}$ in G^0 because $B^{\Gamma C}$ is the full normaliser of $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}R^{\Gamma-}$ in G^C; now $B^\Gamma \subset B_1{}^\Gamma$. On the other hand $\mathfrak{b}^{\Gamma C} \subset \mathrm{ad}(c_{\Delta-\Gamma})(\mathfrak{k}^C + \mathfrak{p}^+)$ as in Theorem 6.5 so $B^{\Gamma C} \subset \mathrm{ad}(c_{\Delta-\Gamma})(K^C \cdot P^+)$; thus $B_1{}^\Gamma = B^{\Gamma C} \cap G^0 \subset \mathrm{ad}(c_{\Delta-\Gamma})((K^C \cdot P^+) \cap G^0) = B^\Gamma$. Now $B^\Gamma = B_1{}^\Gamma$, parabolic subgroup of G^0 which is the normalizer of $\mathrm{ad}(c_{\Delta-\Gamma})^{-1}N^{\Gamma-}$ in G^0.

The assertions on $B_0{}^\Gamma$ now follow from Theorem 6.5, (6.6.5) and (6.6.6). *Q.E.D.*

6.9. COROLLARY. *If M is irreducible and of rank r, and if Γ has precisely t elements, then B^Γ is conjugate in G^0 to the group $F_{\Phi_t}{}^0$ of Theorem* 3.4.

Remark 1. This corollary identifies B^Γ for reducible M by means of Corollary 4.10.

Remark 2. It is instructive to compare Corollary 6.9 with Theorem 4.13.

Proof of Corollary. Recall the maximally split Cartan subalgebra $\mathfrak{t} = \mathfrak{h}^- + \mathfrak{a}^0$ of $\mathfrak{g}^0$. Now $\mathfrak{u} = \mathfrak{h}^- + J\mathfrak{a}^0 = \mathrm{ad}(\exp(\pi/4)Z)\mathfrak{t}$ is a maximally split Cartan subalgebra of $\mathfrak{g}^0$; $J\mathfrak{a}^0$, the span of $\{Y_\delta{}^0\}_{\delta\in\Delta}$, is the split part of $\mathfrak{u}$. $\Delta = \{\delta_1, \cdots, \delta_r\}$ with $\delta_1 < \cdots < \delta_r$. Define $\Delta(a)$ to be the last a elements of Δ so $\Gamma = \Delta(t)$. Let $Y(a) = -Y_{\Delta-\Delta(a)}{}^0$; $\{Y(1), \cdots, Y(r)\}$ is a basis of the split part $J\mathfrak{a}^0$ of $\mathfrak{u}$. We order the dual space of $J\mathfrak{a}^0$ lexicographically by values on this basis. Let β be a $\mathfrak{t}^C$-root of $\mathfrak{g}^C$. Then $\mathrm{ad}(c_\Delta)^*\beta = \beta^*$ is an $\mathfrak{h}^C$-root. If β^* is noncompact positive, then $\beta^*(Z_{\Delta-\Delta(a)}{}^0)$ is 0, $-i/2$ or $-i$; as

$$\mathrm{ad}(c_\Delta{}^{-1})(2iZ_{\Delta-\Delta(a)}{}^0) = \mathrm{ad}(c_{\Delta-\Delta(a)})^{-1}(2iZ_{\Delta-\Delta(a)}{}^0) = Y(a).$$

we then have $\beta(Y(a))$ equal to 0, 1 or 2. If β^* is a compact simple root we similarly have $\beta(Y(a))$ equal to 0 or -1, so $\beta(Y(a)) \geqq 0$ if β^* is compact negative. Now $\beta\,|_{J\mathfrak{a}^0} > 0$ implies either β^* is noncompact positive or β^* is compact negative. $\mathrm{ad}(c_{\Delta-\Delta(a)})\mathfrak{b}^{\Delta(a)C}$ contains every noncompact negative and every compact positive $\mathfrak{h}^C$-root space; thus the $\mathfrak{b}^{\Delta(a)C}$ are parabolic for the split torus $J\mathfrak{a}^0$. Now our assertion is the content of Theorem 3.4. *Q.E.D.*

7. The partial Cayley transforms of *D*. In this section we shall apply the partial Cayley transformation $c_{\Delta-\Gamma}$, where Γ is a subset of Δ of the type considered in Section 6, to the domain D embedded in $\mathfrak{p}^-$. It will turn out that the result of this transformation is a Siegel domain of type III, which we shall describe explicitly by determining the action of $\mathrm{ad}(c_{\Delta-\Gamma})B^\Gamma$ on $\mathfrak{p}^-$.

7.1. ν and ν^0 denote the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ and $\mathfrak{g}^0$, respectively; $\langle\ ,\ \rangle$ denoting the Killing form, we define a positive definite Hermitian form by $\langle U, V\rangle_\nu = -\langle U, \nu V\rangle$ on $\mathfrak{g}^C$. The adjoint of a linear transformation $\mathrm{ad}(V)$ $(V \in \mathfrak{g}^C)$ with respect to this form is given by $\mathrm{ad}(V)^* = -\mathrm{ad}(\nu V)$ (cf. [7], § 6.1). We have $\mathfrak{p}^\pm = \mathfrak{p}_{\Delta-\Gamma,1}{}^\pm + \mathfrak{p}_2{}^{\Gamma\pm} + \mathfrak{p}_\Gamma{}^\pm$. ν is a complex antilinear map of $\mathfrak{p}^\pm$ onto $\mathfrak{p}^\mp$ preserving this direct decomposition.

For any $E \in \mathfrak{p}^-$ we denote by E_1, E_2 and E_3 the projections of E onto $\mathfrak{p}_{\Delta-\Gamma,1}{}^-$, $\mathfrak{p}_2{}^{\Gamma-}$ and $\mathfrak{p}_\Gamma{}^-$, respectively. So $E = E_1 + E_2 + E_3$.

By Lemma 6.3, $\mathfrak{n}_1{}^{\Gamma-}$ is a real form of $\mathfrak{p}_{\Delta-\Gamma,1}{}^-$. The terms "real," "imaginary," "Hermitian" will always refer to this real form. As in Section 4, we have $\mathfrak{o}^\Gamma = \xi^{-1}(c_{\Delta-\Gamma}(x)) = i \sum_{\alpha\in\Delta-\Gamma} E_{-\alpha} \in i\mathfrak{n}_1{}^{\Gamma-}$. By [7, Proposition 6.2] applied to the pair $(\mathfrak{g}_{\Delta-\Gamma}{}^0, \mathfrak{k}_{\Delta-\Gamma})$, the orbit $K_{\Delta-\Gamma,1}{}^*(-i\mathfrak{o}^\Gamma)$ is a self-dual cone in $\mathfrak{n}_1{}^{\Gamma-}$; we shall denote it by $\mathfrak{c}^\Gamma$.

7.2. Lemma. *For all $U \in \mathfrak{p}_2{}^{\Gamma+}$, we have $\tau_{\Delta-\Gamma}(U) = -[U, \mathfrak{o}^\Gamma]$.*

Proof. First we show that, restricted to $\mathfrak{p}_2{}^{\Gamma C} + \mathfrak{q}_2{}^{\Gamma C}$, $\tau_{\Delta-\Gamma}$ and $\mathrm{ad}(X_{\Delta-\Gamma})$ coincide. By Lemma 6.3, $\mathfrak{p}_2{}^{\Gamma C} + \mathfrak{q}_2{}^{\Gamma C}$ is the sum of the (± 1)-eigenspaces of $\mathrm{ad}(Y_{\Delta-\Gamma}{}^0)$ on $\mathfrak{g}^C$. We have $X_{\Delta-\Gamma} = iX_{\Delta-\Gamma}{}^0 = -i\,\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)(Y_{\Delta-\Gamma}{}^0)$, and $\mathfrak{p}_2{}^{\Gamma C} + \mathfrak{q}_2{}^{\Gamma C}$ is invariant under $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$. It follows that $\mathfrak{p}_2{}^{\Gamma C} + \mathfrak{q}_2{}^{\Gamma C}$ is the sum of the $(\pm i)$-eigenspaces of $\mathrm{ad}(X_{\Delta-\Gamma})$. Now, if $\mathrm{ad}(X_{\Delta-\Gamma})U = \pm iU$, then $\tau_{\Delta-\Gamma}(U) = (\exp(\pi/2)\mathrm{ad}(X_{\Delta-\Gamma}))(U) = e^{\pm i\pi/2}U = \pm iU$, proving the assertion.

To prove the Lemma, let $U \in \mathfrak{p}_2{}^{\Gamma+}$. Then

$$\tau_{\Delta-\Gamma}(U) = -[U, X_{\Delta-\Gamma}] = -[U, i\sum_{\alpha\in\Delta-\Gamma} E_\alpha] - [U, i\sum_{\alpha\in\Delta-\Gamma} E_{-\alpha}] = -[U, \mathfrak{o}^\Gamma],$$

since $[U, E_\alpha] = 0$ for all $\alpha \in \Delta - \Gamma$, $\mathfrak{p}^+$ being abelian. *Q.E.D.*

Definitions. For all $W \in D_\Gamma$ we define the linear transformation $\mu(W) : \mathfrak{p}_2{}^{\Gamma -} \to \mathfrak{p}_2{}^{\Gamma -}$ by

$$\mu(W)U = \operatorname{ad}(W)\tau_{\Delta-\Gamma}\,\nu(U),$$

For all $V \in \mathfrak{p}_2{}^{\Gamma -}$ we define the linear function $f_V : \mathfrak{p}_\Gamma{}^- \to \mathfrak{p}_2{}^{\Gamma -}$ by

$$f_V(W) = (I + \mu(W))V.$$

Finally, for all $W \in D_\Gamma$ we define the vector-valued bilinear form $\Lambda_W : \mathfrak{p}_2{}^{\Gamma -} \times \mathfrak{p}_{2\Gamma}{}^- \to \mathfrak{p}_{\Delta-\Gamma,1}{}^-$ by

$$\Lambda_W(U,V) = -(i/2)[U, \tau_{\Delta-\Gamma}(\nu(I+\mu(W)))^{-1}V].$$

It is easy to see that these definitions are meaningful; for the definition of Λ_W we only have to note that $\|\mu(W)\| < 1$ in the operator norm with respect to the real part of $\langle\ ,\ \rangle_\nu$ restricted to $\mathfrak{p}_2{}^{\Gamma -}$. In fact, $\tau_{\Delta-\Gamma}$ and ν are isometric transformations on $\mathfrak{g}^C$ in this norm; $\tau_{\Delta-\Gamma}\,\nu$ maps $\mathfrak{p}_2{}^{\Gamma -}$ onto $\mathfrak{q}_2{}^{\Gamma +}$, and on $\mathfrak{q}_2{}^{\Gamma +}$ we have $\|\operatorname{ad}(W)\| < 1$ for all $W \in D_\Gamma$ by Lemma 4.6.

7.3. LEMMA.

(i) *For all* $k \in K^{\Gamma *}$, $W \in D_\Gamma$ *and* $U, V \in \mathfrak{p}_2{}^{\Gamma -}$,
$$\operatorname{ad}(k)\Lambda_W(U,V) = \Lambda_{\operatorname{ad}(k)W}(\operatorname{ad}(k)U, \operatorname{ad}(k)V).$$

(ii) *For all* $W \in D_\Gamma$ *and* $U, V \in \mathfrak{p}_2{}^{\Gamma -}$,
$$\Lambda_0(U, \mu(W)V)) = \Lambda_0(V, \mu(W)U).$$

(iii) *For all* $W \in D_\Gamma$ *we have* $\Lambda_W = \Lambda_W{}^{(1)} + \Lambda_W{}^{(2)}$ *where*

$$\Lambda_W{}^{(1)}, \Lambda_W{}^{(2)} : \mathfrak{p}_2{}^{\Gamma -} \times \mathfrak{p}_2{}^{\Gamma -} \to \mathfrak{p}_{\Delta-\Gamma,1}{}^-$$

are defined by

$$\Lambda_W{}^{(1)}(U,V) = -(i/2)[U, \tau_{\Delta-\Gamma}\,\nu\,(1-\mu(W)^2)^{-1}V];$$
$$\Lambda_W{}^{(2)}(U,V) = (i/2)[U, \tau_{\Delta-\Gamma}\,\nu(1-\mu(W)^2)^{-1}\mu(W)V].$$

$\Lambda_W{}^{(1)}$ *is Hermitian bilinear and such that* $\Lambda_W{}^{(1)}(U,U) \in \overline{\mathfrak{c}^\Gamma}$ *for all* $U \in \mathfrak{p}_2{}^{\Gamma -}$; $\Lambda_W{}^{(2)}$ *is complex bilinear symmetric.*

(iv) *For any* $W \in D_\Gamma$, Λ_W *is nondegenerate in the sense that if* $\Lambda_W(U, V_0) = 0$ *for all* $U \in \mathfrak{p}_2{}^{\Gamma -}$, *then* $V_0 = 0$.

(v) *For any fixed* $U, V \in \mathfrak{p}_2{}^{\Gamma -}$; $\Lambda_W(U, f_V(W))$ *is a constant vector, independent of* W.

Proof. Since $\|\mu(W)\| < 1$ for all $W \in D_\Gamma$, we have the convergent series expansions $(I + \mu(W))^{-1} = \sum_{n=0}^{\infty} (-\mu(W))^n$ and $(I - \mu(W)^2)^{-1} = \sum_{n=0}^{\infty} \mu(W)^{2n}$. These will be used several times in the proof.

To prove (i) we note that $\mathfrak{k}^{\Gamma *} = \mathfrak{l}^1 + i\mathfrak{q}_1{}^\Gamma$. $\tau_{\Delta-\Gamma}$ and ν are both trivial on $\mathfrak{l}^{\Gamma}$ and equal to $-I$ on $i\mathfrak{q}_1{}^{\Gamma}$. Hence $\tau_{\Delta-\Gamma}\,\nu$ is trivial on $\mathfrak{k}^{\Gamma*}$, and thus $\mathrm{ad}(k)$ commutes with $\tau_{\Delta-\Gamma}\,\nu$ for all $k\in K^{\Gamma*}$. Also, $\mathrm{ad}(k)$ preserves $\mathfrak{p}_{\Delta-\Gamma,1}{}^{-}$, $\mathfrak{p}_2{}^{\Gamma-}$ and $\mathfrak{p}_{\Gamma}{}^{-}$ by Lemma 6.3.

It follows that

$$\begin{aligned}\mathrm{ad}(k)\Lambda_W(U,V) &= -(i/2)\,ad(k)\,[U,\tau_{\Delta-\Gamma}\,\nu\,(I+\mu(W))^{-1}V]\\ &= -(i/2)\,[\mathrm{ad}(k)U,\tau_{\Delta-\Gamma}\,\nu\,(I+\mu(\mathrm{ad}(k)W))^{-1}\,\mathrm{ad}(k)V]\\ &= \Lambda_{\mathrm{ad}(k)W}(\mathrm{ad}(k)U,\mathrm{ad}(k)V).\end{aligned}$$

To prove (ii) we use the definition of Λ_0, the fact that $\tau_{\Delta-\Gamma}$ commutes with ν (by definition of $\tau_{\Delta-\Gamma}$), then the Jacobi identity and the fact that $[U,V]=0$:

$$\begin{aligned}\Lambda_0(U,\mu(W)V) &= -(i/2)\,[U,\tau_{\Delta-\Gamma}\,\nu(V)]]\\ &= -(i/2)\,[U,[\tau_{\Delta-\Gamma}\,\nu(W),\,-V]]\\ &= -(i/2)\,[V,[\tau_{\Delta-\Gamma}\,\nu(W),\,-U]]\\ &= \Lambda_0(V,\mu(W)U).\end{aligned}$$

To prove (iii) we note that

$$\begin{aligned}\Lambda_{W^{(1)}}(U,V) &= \Lambda_0(U,\,(I-\mu(W)^2)^{-1}V),\\ \Lambda_{W^{(2)}}(U,V) &= \Lambda_0(U,\,(I-\mu(W)^2)^{-1}\mu(W)V).\end{aligned}$$

Hence $\Lambda_W=\Lambda_{W^{(1)}}+\Lambda_{W^{(2)}}$ is immediate.

Now we prove that $\Lambda_0=\Lambda_0{}^{(1)}$ is Hermitian bilinear and $\Lambda_0(U,U)\in\overline{\mathfrak{c}^{\Gamma}}$ for all $U\in\mathfrak{p}_2{}^{\Gamma-}$. Since Λ_0 is linear in the first and antilinear in the second argument, it suffices to show that $\Lambda_0(U,U)\in\overline{\mathfrak{c}^{\Gamma}}$ for all U. Since $\overline{\mathfrak{c}^{\Gamma}}$ is a self-dual cone, for this we only have to show that $\langle\Lambda_0(U,U),V\rangle_\nu\geqq 0$ for all $U\in\mathfrak{p}_2{}^{\Gamma-}$, $V\in\mathfrak{c}^{\Gamma}$.

Given any such U and V, there exists an element $k\in K_{\Delta-\Gamma,1}{}^{*}$ such that $\mathrm{ad}(k)V=-i\mathfrak{o}^{\Gamma}$. Denoting $U'=\mathrm{ad}(k)U$ and using (i) we have

$$\langle\Lambda_0(U,U),V\rangle_\nu=\langle\Lambda_0(U',U'),-i\mathfrak{o}^{\Gamma}\rangle_\nu.$$

Now note that by Lemma 7.2 we have

$$\tau_{\Delta-\Gamma}\,\nu(U')=-[\nu(U'),\mathfrak{o}^{\Gamma}]=-\mathrm{ad}(\nu(U'))\mathfrak{o}^{\Gamma}=\mathrm{ad}(U')^{*}\mathfrak{o}^{\Gamma}.$$

Hence

$$\Lambda_0(U',U')=-(i/2)\,[U',\tau_{\Delta-\Gamma}\,\nu(U')]=-(i/2)\cdot\mathrm{ad}(U')\mathrm{ad}(U')^{*}\mathfrak{o}^{\Gamma}.$$

Therefore,

$$\begin{aligned}\langle\Lambda_0(U,V),V\rangle_\nu &= \langle-(i/2)\ \mathrm{ad}(U')\mathrm{ad}(U')^{*}\mathfrak{o}^{\Gamma},-i\mathfrak{o}^{\Gamma}\rangle_\nu\\ &= \tfrac{1}{2}\langle\mathrm{ad}(U')^{*}\mathfrak{o}^{\Gamma},\mathrm{ad}(U')^{*}\mathfrak{o}^{\Gamma}\rangle_\nu\geqq 0,\end{aligned}$$

proving the assertion.

To prove the desired properties of $\Lambda_W^{(1)}$ for arbitrary $W \in D_\Gamma$, we first note that $\Lambda_0(U, \mu(W)^2 V) = \Lambda_0(\mu(W)V, \mu(W)U) = \Lambda_0(\mu(W)^2 U, V)$ for all U, V by (ii) and by hermiticity of Λ_0. Repeated application of these identities gives $\Lambda_0(U, \mu(W)^{2n}U) = \Lambda_0(\mu(W)^n U, \mu(W)^n U)$ for all $n \geqq 0$. Now we have

$$\Lambda_W^{(1)}(U, U) = \sum_{n=0}^{\infty} \Lambda_0(U, \mu(W)^{2n}U) = \sum_{n=0}^{\infty} \Lambda_0(\mu(W)^n U, \mu(W)^n U).$$

By what we just proved, each term of the last sum is in $\overline{\mathfrak{c}^\Gamma}$; hence $\Lambda_W(U, U) \in \overline{\mathfrak{c}^\Gamma}$ for all $U \in \mathfrak{p}_2^{\Gamma-}$. Since $\Lambda_W^{(1)}$ is linear in the first, antilinear in the second argument (by complex linearity of $\mu(W)^2$), this also shows that $\Lambda_W^{(1)}$ is Hermitian, as we had to prove.

$\Lambda_W^{(2)}$ is clearly complex bilinear for any $W \in D_\Gamma$. To prove that it is symmetric, we use the definition of $\Lambda_W^{(2)}$, hermiticity of $\Lambda_0(U, (I - \mu(W)^2)^{-1}V) = \Lambda_W^{(1)}(U, V)$ in U and V, hermiticity of Λ_0, then (ii) and again the definition of $\Lambda_W^{(2)}$; denoting the conjugation of $\mathfrak{p}_{\Delta-\Gamma,1}^{-}$ with respect to $\mathfrak{n}_1^{\Gamma-}$ by ρ, we have

$$\begin{aligned}
\Lambda_W^{(2)}(U, V) &= \Lambda_0(U, (I - \mu(W)^2)^{-1}\mu(W)V) \\
&= \rho\Lambda_0(\mu(W)V, (I - \mu(W)^2)^{-1}U) \\
&= \Lambda_0((I - \mu(W)^2)^{-1}U, \mu(W)V) \\
&= \Lambda_0(V, \mu(W)(I - \mu(W)^2)^{-1}U) \\
&= \Lambda_W^{(2)}(V, U).
\end{aligned}$$

This finishes the proof of (iii).

In order to prove (iv) it is enough to show that Λ_0 is non-degenerate. The relation $\Lambda_W(U, V) = \Lambda_0(U, (I + \mu(W))^{-1}V)$ will then imply that Λ_W is non-degenerate. We show that $\Lambda_0(U, U) = 0$ implies $U = 0$.

Suppose $\Lambda_0(U, U) = 0$ for some $U \in \mathfrak{p}_2^{\Gamma-}$. As in the proof of (iii), we have by Lemma 7.2,

$$0 = \langle \Lambda_0(U, U), -i\mathfrak{o}^\Gamma \rangle_\nu = \tfrac{1}{2}\langle \operatorname{ad}(U)^*\mathfrak{o}^\Gamma, \operatorname{ad}(U)^*\mathfrak{o}^\Gamma \rangle_\nu.$$

Since $\langle\ ,\ \rangle_\nu$ is positive definite, this implies $\operatorname{ad}(U)^*\mathfrak{o}^\Gamma = 0$. This means $[-\nu(U), \mathfrak{o}^\Gamma] = 0$. Now, since $\mathfrak{o}^\Gamma = \sum_{\alpha \in \Delta-\Gamma} E_{-\alpha}$, it follows that $[\nu(U), Y_{\Delta-\Gamma}{}^0] = 0$, i.e. $\operatorname{ad}(Y_{\Delta-\Gamma}{}^0(\nu(U))) = 0$. Since $\nu(U) \in \mathfrak{p}_2^{\Gamma+}$, by Lemma 6.3 it follows that $\nu(U) = 0$. Hence $U = 0$, as we had to show.

The proof of (v) is trivial from the definitions; we have

$$\begin{aligned}
\Lambda_W(U, f_V(W)) &= -(i/2)[U, \tau_{\Delta-\Gamma}\,\nu(I + \mu(W))^{-1}(I + \mu(W))V] \\
&= -(i/2)[U, \tau_{\Delta-\Gamma}\,\nu(V)],
\end{aligned}$$

which is independent of W. *Q.E.D.*

7.4. LEMMA. *The map $I - \tau_{\Delta-\Gamma}\,\nu$ is a real linear isomorphism of $\mathfrak{p}_2^{\Gamma-}$*

onto $\mathfrak{n}_2^{\Gamma-}$; *so every element of* $\mathfrak{n}_2^{\Gamma-}$ *can uniquely be written as* $V - \tau_{\Delta-\Gamma}\,\nu(V)$, *with* $V \in \mathfrak{p}_2^{\Gamma-}$.

Proof. If $V \in \mathfrak{p}_2^{\Gamma-}$, then $\tau_{\Delta-\Gamma}\,\nu(V) \in \mathfrak{q}_2^{\Gamma+}$. $\mathfrak{n}_2^{\Gamma-} = (\mathfrak{p}_2^{\Gamma-} + \mathfrak{q}_2^{\Gamma+}) \cap \mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ is a real form of $\mathfrak{p}_2^{\Gamma-} + \mathfrak{q}_2^{\Gamma+}$, hence $\dim.\mathfrak{n}_2^{\Gamma-} = \dim.\mathfrak{p}_2^{\Gamma-} = \dim.\mathfrak{q}_2^{\Gamma+}$. Now it suffices to prove that $V - \tau_{\Delta-\Gamma}\,\nu(V) \in \mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ for all $V \in \mathfrak{p}_2^{\Gamma-}$.

The involution of $\mathfrak{g}^C$ with respect to $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$ is

$$\begin{aligned}\mathrm{ad}(c_{\Delta-\Gamma})\nu^0\,\mathrm{ad}(c_{\Delta-\Gamma})^{-1} &= \mathrm{ad}(c_{\Delta-\Gamma})\sigma\nu\,\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\\ &= \sigma\,\mathrm{ad}(c_{\Delta-\Gamma})^{-1}\nu\,\mathrm{ad}(c_{\Delta-\Gamma})^{-1} = \sigma\nu\tau_{\Delta-\Gamma}^{-1}.\end{aligned}$$

For $V \in \mathfrak{p}_2^{\Gamma-}$ we have

$$\begin{aligned}\sigma\nu\tau_{\Delta-\Gamma}^{-1}(V) &= -\sigma\nu\tau_{\Delta-\Gamma}(V) = -\tau_{\Delta-\Gamma}\,\nu(V),\\ \sigma\nu\tau_{\Delta-\Gamma}^{-1}(\tau_{\Delta-\Gamma}\,\nu(V)) &= \sigma(V) = -V.\end{aligned}$$

Hence $V - \tau_{\Delta-\Gamma}\,\nu(V)$ is invariant under $\sigma\nu\tau_{\Delta-\Gamma}^{-1}$, and so is contained in $\mathrm{ad}(c_{\Delta-\Gamma})\mathfrak{g}^0$. *Q.E.D.*

7. 5. PROPOSITION. *$N^{\Gamma-}$ acts on $\mathfrak{p}^-$ by*

$$g(E) = E + U + f_V(E_3) + 2i\Lambda_{E_3}(E_2, f_V(E_3)) + i\Lambda_{E_3}(f_V(E_3), f_V(E_3))$$

where $g = \exp(U + (I - \tau_{\Delta-\Gamma}\,\nu)(V))$, $U \in \mathfrak{n}_1^{\Gamma-}$, $V \in \mathfrak{p}_2^{\Gamma-}$. *$K^{\Gamma*}$ acts on $\mathfrak{p}^-$ by the adjoint representation; it preserves $\mathfrak{p}_{\Delta-\Gamma,1}^-$, $\mathfrak{p}_2^{\Gamma-}$ and $\mathfrak{p}_\Gamma^-$. On $\mathfrak{p}_{\Delta-\Gamma,1}^-$, $K_{\Delta-\Gamma,1}{}^*$ is real, K_Γ and L_2^Γ are trivial. These actions are ξ-equivariant; in particular $K^{\Gamma*}\cdot N^-$ preserves $\xi(\mathfrak{p}^-)$.*

Proof. It is easy to see that K^C and P^- act on $\mathfrak{p}^-$ in a ξ-equivariant way by the adjoint representation and by translations, respectively. Now let g be any element of $N^{\Gamma-}$; it can be written in the given form by Lemma 7.4. By the Campbell-Hausdorff formula we have

$$\begin{aligned}g &= \exp(U + V - \tau_{\Delta-\Gamma}\,\nu(V))\\ &= \exp(U)\cdot\exp(\tfrac{1}{2}[V, \tau_{\Delta-\Gamma}\,\nu(V)])\cdot\exp(V)\cdot\exp(-\tau_{\Delta-\Gamma}\,\nu(V))\end{aligned}$$

since, by Lemmas 6.3 and 6.4, all other brackets vanish. Now $-\tau_{\Delta-\Gamma}\,\nu(V) \in \mathfrak{q}_2^{\Gamma+} \subset \mathfrak{k}^C$, so $\exp(-\tau_{\Delta-\Gamma}\,\nu(V))$ acts on $\mathfrak{p}^-$ by the adjoint action;

$$\begin{aligned}&\exp(-\tau_{\Delta-\Gamma}\,\nu(V))(E)\\ &\quad= E - [\tau_{\Delta-\Gamma}\,\nu(V), E] + \tfrac{1}{2}[\tau_{\Delta-\Gamma}\,\nu(V), [\tau_{\Delta-\Gamma}\,\nu(V), E]]\\ &\quad= E + [E_2, \tau_{\Delta-\Gamma}\,\nu(V)] + [E_3, \tau_{\Delta-\Gamma}\,\nu(V)]\\ &\qquad\quad + \tfrac{1}{2}[[E_3, \tau_{\Delta-\Gamma}\,\nu(V)], \tau_{\Delta-\Gamma}\,\nu(V)],\end{aligned}$$

since all other brackets vanish, again by Lemmas 6.3 and 6.4. The other factors in the expression of g are in P^-, so they act on $\mathfrak{p}^-$ by translations. Using the definition of Λ_W the assertion about the action of g follows.

$K^{\Gamma *}$ commutes with $\mathrm{ad}(Z_{\Delta-\Gamma}{}^0)$, therefore preserves its eigenspaces $\mathfrak{p}_{\Delta-\Gamma,1}{}^-$, $\mathfrak{p}_2{}^{\Gamma-}$ and $\mathfrak{p}_\Gamma{}^-$ in $\mathfrak{p}^-$. $K_{\Delta-\Gamma,1}{}^*$ is real on $\mathfrak{p}_{\Delta-\Gamma,1}{}^-$ by [7, Proposition 6.6] applied to the pair $(\mathfrak{g}_{\Delta-\Gamma}{}^0, \mathfrak{k}_{\Delta-\Gamma})$. $\mathfrak{l}_2{}^\Gamma$ and $\mathfrak{k}_\Gamma$ centralize $\mathfrak{p}_{\Delta-\Gamma,1}{}^-$, hence $L_2{}^\Gamma$ and K_Γ act trivially on it. *Q.E.D.*

7.6. We define the partial Cayley transform of D by

$$c_{\Delta-\Gamma}D = \xi^{-1}(c_{\Delta-\Gamma}G^0(x)).$$

This is the image of D under $\xi^{-1}c_{\Delta-\Gamma}\xi$ in $\mathfrak{p}^-$. To see that this definition is meaningful, we note that by (6.6.6) we have a local semidirect product $\mathrm{ad}(c_{\Delta-\Gamma})B_0{}^\Gamma = G_\Gamma{}^0 \cdot (K_{\Delta-\Gamma,1}{}^* \cdot L_2{}^\Gamma N^{\Gamma-})$. Using this, we have

$$\begin{aligned} c_{\Delta-\Gamma}(G^0(x)) &= c_{\Delta-\Gamma}(B_0{}^\Gamma(x)) = (\mathrm{ad}(c_{\Delta-\Gamma})B_0{}^\Gamma)(c_{\Delta-\Gamma}x) \\ &= (N^{\Gamma-}\cdot L_2{}^\Gamma \cdot K_{\Delta-\Gamma,1}{}^*)(G_\Gamma{}^0(c_{\Delta-\Gamma}x)) \\ &= (N^{\Gamma-}\cdot L_2{}^\Gamma \cdot K_{\Delta-\Gamma,1}{}^*)(c_{\Delta-\Gamma}M_\Gamma) \subset \xi(\mathfrak{p}^-), \end{aligned}$$

by Lemma 4.2 and Proposition 7.5.

7.7. THEOREM. *The partial Cayley transform* $c_{\Delta-\Gamma}D$ *of* D *is the domain* $\{E : \mathrm{Im}\, E_1 - \mathrm{Re}\, \Lambda_{E_3}(E_2, E_2) \in \mathfrak{c}^\Gamma, E_3 \in D_\Gamma\}$.

Proof. Let us denote by S the domain defined in the text of the Theorem. First we show that $c_{\Delta-\Gamma}D \subset S$. By 7.6 we have

$$c_{\Delta-\Gamma}D = (N^{\Gamma-}\cdot L_2{}^\Gamma \cdot K_{\Delta-\Gamma,1}{}^*)(G_\Gamma{}^0(\mathfrak{o}^\Gamma)).$$

Now $G_\Gamma{}^0(\mathfrak{o}^\Gamma) = \{\mathfrak{o}^\Gamma + E_3 : E_3 \in D_\Gamma\} \subset S$, and in order to see that $c_{\Delta-\Gamma}D \subset S$, it suffices to show that $L_2{}^\Gamma \cdot K_{\Delta-\Gamma,1}{}^*$ and $N^{\Gamma-}$ map S into itself. To show this, let $E \in S$ and let $k \in L_2{}^\Gamma \cdot K_{\Delta-\Gamma,1}{}^*$. We denote $E' = \mathrm{ad}(k)E$. By Lemma 7.3(i) and Proposition 7.5 we have

$$\mathrm{Im}\, E'_1 - \mathrm{Re}\, \Lambda_{E_3}(E'_2, E'_2) = \mathrm{ad}(k)(\mathrm{Im}\, E_1 - \mathrm{Re}\, \Lambda_{E_3}(E_2, E_2)) \in \mathrm{ad}(k)\mathfrak{c}^\Gamma = \mathfrak{c}^\Gamma.$$

Now let $g = \exp(U + (I - \tau_{\Delta-\Gamma}\,\nu)(V)) \in N^{\Gamma-}$ with $U \in \mathfrak{n}_1{}^{\Gamma-}$, $V \in \mathfrak{p}_2{}^{\Gamma-}$, and let $E' = g(E)$. By Proposition 7.5 we have

$$\begin{aligned} &\mathrm{Im}\, E'_1 - \mathrm{Re}\, \Lambda_{E'_3}(E'_2, E'_2) \\ &\quad = \mathrm{Im}(E_1 + U + 2i\Lambda_{E_3}(E_2, f_V(E_3)) + i\Lambda_{E_3}(f_V(E_3), f_V(E_3))) \\ &\qquad\quad - \mathrm{Re}\, \Lambda_{E_3}(E_2 + f_V(E_3), E_2 + f_V(E_3))) \\ &\quad = \mathrm{Im}\, E_1 - \mathrm{Re}\, \Lambda_{E_3}(E_2, E_2), \end{aligned}$$

proving the assertion.

Next we prove that $S \subset c_{\Delta-\Gamma}D$. Let $E \in S$, it is sufficient to show that E can be transformed into the element $\mathfrak{o}^\Gamma \in c_{\Delta-\Gamma}D$ by an element of $\mathrm{ad}(c_{\Delta-\Gamma})B_0{}^\Gamma$. Let $V = -(I + \mu(E_3))^{-1}E_2$; then

$$n_2 = \exp((I - \tau_{\Delta-\Gamma}\,\nu)(V)) \in N^{\Gamma-}.$$

carries E into an element $E' = E'_1 + 0 + E_3$. Now let $U = -\operatorname{Re} E'_1$; then $n_1 = \exp(U) \in N^{\Gamma-}$ carries E' into $E'' = iF + 0 + E_3$, with F real. As we showed above, $N^{\Gamma-}$ preserves S, so we have

$$\operatorname{Im} E''_1 - \operatorname{Re} \Lambda_{E''_3}(E''_2, E''_2) = F \in \mathfrak{c}^{\Gamma}.$$

Now there exists an element $k \in K_{\Delta-\Gamma,1}{}^*$ such that $k \cdot F = -i\mathfrak{o}^{\Gamma}$; k carries E'' into $E'' = \mathfrak{o}^{\Gamma} + 0 + E_3$. Finally, since $E_3 \in D_{\Gamma}$, there exists $g \in G_{\Gamma}{}^0$ such that $g \cdot E_3 = 0$. It follows that $gkn_1n_2 \cdot E = \mathfrak{o}^{\Gamma}$, and $gkn_1n_2 \in \operatorname{ad}(c_{\Delta-\Gamma})B^{\Gamma}$.

Q.E.D.

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