Realization of Hermitian Symmetric Spaces as Generalized Half-planes

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1. Introduction

A celebrated result of É. Cartan and Harish-Chandra is that every hermitian symmetric space of non-compact type can be realized as a bounded domain in a complex vector space. These spaces have other realizations which generalize the upper half-plane realization of the unit disc. Some can be realized as tube domains over self-dual cones[1]; M. Koecher [5] has recently developed a theory of these domains. I. I. Pjateckiĭ-Šapiro [7] has given realizations case by case of the irreducible hermitian symmetric spaces not covered by Koecher's theory; he calls them[2] *Siegel domains of type* II, and they can be regarded as further generalizations of the upper half-plane.

In the present paper we shall define a general Cayley transform which carries the bounded domain of the Harish-Chandra realization into a generalized half-plane. Our results will be completely independent of classification theory, and will include all the exceptional and non-irreducible cases. In the respective special cases our Cayley transform yields the domains considered by Koecher and Pjateckiĭ-Šapiro. Our construction is based on the embedding theorems of A. Borel and of Harish-Chandra, and is a direct generalization of the following one-dimensional situation: M^* is the Riemann sphere, its non-compact dual space M is embedded in it as the lower hemisphere. M^* is in correspondence with the complex plane by stereographic projection; the image of M is the unit disc D. The Cayley transform now is given by a rotation of M^* by the angle $\pi/2$ around a horizontal axis. The corresponding holomorphic map in the com-

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[1] A *self-dual cone* is a set $\mathfrak{c}$ in a real vector space V with a positive definite inner product $\langle , \rangle$ such that $x \in \mathfrak{c}$ if and only if $\langle x, y \rangle > 0$ whenever $0 \neq y \in \bar{\mathfrak{c}}$. The *tube* over $\mathfrak{c}$ is the set $\{z \in V^{\mathbf{C}}: \operatorname{Im} z \in \mathfrak{c}\}$ where V and iV are taken to be the real and imaginary parts of the complexification $V^{\mathbf{C}} = V \otimes \mathbf{C}$ of V.

[2] They are defined as follows. Let V_1 and V_2 be real vector spaces, $\mathfrak{c}$ a self-dual cone in V_1, and $\Phi: V_2^{\mathbf{C}} \times V_2^{\mathbf{C}} \to V_1^{\mathbf{C}}$ a hermitian bilinear map which is positive definite in the sense that $\Phi(z_2, z_2) \in \bar{\mathfrak{c}}$ whenever $z_2 \in V_2^{\mathbf{C}}$. Then the *Siegel domain of type II* is given as

$$\{(z_1, z_2) \in V_1^{\mathbf{C}} \oplus V_2^{\mathbf{C}}: \operatorname{Im} z_1 - \Phi(z_2, z_2) \in \mathfrak{c}\} .$$

plex plane carries D into a half-plane, i.e., a tube domain over a one-dimensional self-dual conc.

This study will also clarify the question: which hermitian symmetric spaces can be represented as tube domains over self-dual cones? An answer to this (condition (i) in our Theorem 4.9) has been given by J. L. Koszul (unpublished). We obtain Koszul's result and several other equivalent characterizations of this type of space; the most interesting of these is the condition that the Bergman-Šilov boundary have exactly half the dimension of the whole space. In this case it also turns out that the Bergman-Šilov boundary is a compact symmetric space, and the self-dual cone arising in the realization as a tube is its non-compact dual.

In the general case we also get geometric results about the Bergman-Šilov boundary. We represent it as a fibre space over a hermitian symmetric space with riemannian symmetric fibre; it is also a real projective algebraic variety defined over the rationals. Furthermore, we show that the first Betti number of the Bergman-Šilov boundary is equal to the number of factors of tube type in the decomposition of the domain into irreducible constituents. These results involve the consideration of the isotropy group of the Bergman-Šilov boudary in the full group of holomorphic automorphisms of the domain; in special cases this group has been studied by Koecher and by Pjateckiĭ-Šapiro. Finally, in an appendix we show that in the tube case our Cayley transform agrees up to a constant factor i with a map defined in Jordan algebra terms by Koecher.

We shall need some results about the Bergman-Šilov boundary of bounded symmetric domains that were obtained in unpublished joint work of R. Bott and one of the authors. With the consent of Professor Bott, we give the proofs of these results in § 3.

In a forthcoming paper we define a more general type of Cayley transform, and apply our technique to the realization of hermitian symmetric spaces as *Siegel domains of type* III in the sense of Pjateckiĭ-Šapiro.

2. Notation

In this section we establish our notation and recall the embedding theorems of Harish-Chandra and A. Borel which will be fundamental for the rest of this paper. The results quoted without proof can all be found in [4].

M will be a hermitian symmetric space of non-compact type, G^0 its connected group of isometries, K the isotropy group. So $M = G^0/K$. Then G^0 is a semi-simple Lie group, K its maximal compact subgroup. M is a product of non-compact irreducible hermitian symmetric spaces, G^0 is the product of the corresponding simple groups of isometries.

$\mathfrak{g}^0$, $\mathfrak{k}$ will be the Lie algebras of G^0, K. $\mathfrak{g}^C$ will be the complexification of $\mathfrak{g}^0$; G^C is the adjoint group of $\mathfrak{g}^C$. $\mathfrak{g}$ will be the compact form of $\mathfrak{g}^C$ such that the involution of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ leaves $\mathfrak{g}^0$ invariant. Under this involution $\mathfrak{g}^0$ splits as $\mathfrak{g}^0 = \mathfrak{k} + \mathfrak{p}^0$; with $\mathfrak{p} = i\mathfrak{p}^0$ we also have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We denote by $\mathfrak{k}^C$ the complexification of $\mathfrak{k}$; K^C will be the corresponding analytic subgroup of G^C. G will denote the analytic subgroup of G^C corresponding to $\mathfrak{g}$. G is centerless, and our centerless group G^0 can be regarded as the analytic subgroup of G^C for $\mathfrak{g}^0$. (This follows from the easily checked fact that the analytic subgroup for $\mathfrak{k}$ in the simply connected group with Lie algebra $\mathfrak{g}^C$ contains the center of that simply connected group.)

We choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$. Then $\mathfrak{h}^C$ is a Cartan subalgebra in $\mathfrak{g}^C$. The roots of $\mathfrak{g}^C$ which are also roots of $\mathfrak{k}^C$ are called compact roots. If Φ is a system of simple roots of $\mathfrak{g}^C$, then there is exactly one root $\gamma \in \Phi$ that is non-compact. To each root α we associate in the standard way the elements H_α, E_α of $\mathfrak{g}^C$. For each α we have $iH_\alpha \in \mathfrak{h}$.

We denote by $\mathfrak{z}$ the center of $\mathfrak{k}$. There exists an element $Z \in \mathfrak{z}$ such that, for every non-compact positive root α,

$$[Z, E_\alpha] = -iE_\alpha \ ,$$
$$[Z, E_{-\alpha}] = iE_{-\alpha} \ .$$

For every non-compact positive root α, we define the elements

$$X_\alpha^0 = E_\alpha + E_{-\alpha}$$
$$Y_\alpha^0 = -i(E_\alpha - E_{-\alpha}) \ .$$

These elements span $\mathfrak{p}^0$. The restriction of ad (Z) to $\mathfrak{p}^0$ is a complex structure, which will be denoted by J.

The roots α, β of $\mathfrak{g}^C$ are called strongly orthogonal if $\alpha + \beta$ and $\alpha - \beta$ are not roots. There exists a set Δ of strongly orthogonal non-compact positive roots such that the real subspace $\mathfrak{a}^0$ spanned by the X_α^0 $(\alpha \in \Delta)$ is a maximal abelian subalgebra contained in $\mathfrak{p}^0$. We have

$$JX_\alpha^0 = [Z, X_\alpha^0] = Y_\alpha^0$$
$$JY_\alpha^0 = [Z, Y_\alpha^0] = -X_\alpha^0$$
$$[X_\alpha^0, Y_\alpha^0] = 2iH_\alpha$$

for all $\alpha \in \Delta$.

Now $\mathfrak{p}^+$ will be the complex subalgebra of $\mathfrak{g}^C$ spanned by the E_α, α positive non-compact. Similarly, the $E_{-\alpha}$ span the subalgebra $\mathfrak{p}^-$. The analytic subgroups of G^C corresponding to these subalgebras will be P^+ and P^-. The exponential map from $\mathfrak{p}^-$ to P^- is one-to-one.

The group $K^C \cdot P^+$ is a semi-direct product, and is the normalizer of

P^+ in G^C. The compact dual $M^* = G/K$ of the hermitian symmetric space $M = G^0/K$ can be identified with $G^C/K^C \cdot P^+$, through the inclusion of G in G^C. We denote by x the identity coset in $G^C/K^C \cdot P^+$. The orbit $G^0(x)$ is then a holomorphic embedding of M as an open set in M^*.

The mapping $\xi: \mathfrak{p}^- \to M^*$ defined by $\xi(X) = \exp(X)\cdot(x)$ is a one-to-one holomorphic map onto a dense open subset of M^*. $D = \xi^{-1}(G^0(x))$ is a bounded domain in the complex vector space $\mathfrak{p}^-$; this is the Harish-Chandra realization of M as a bounded domain. The ξ-equivariant action of G^0 on D is just the action of the connected group of holomorphic automorphisms of D.

3. Preliminary results on hermitian symmetric spaces

We first discuss the Bergman-Šilov boundary of a bounded domain. We then specialize to the case of a hermitian symmetric space, obtaining a more precise description, and finally establish some results on roots which will be important in the sequel.

3.1. Let D be a starlike bounded homogeneous (under holomorphic automorphisms) domain in a complex vector space. The Bergman-Šilov boundary of D is the minimal closed subset of the topological boundary ∂D, on which the absolute value of any function holomorphic on D and continuous on its closure $\bar{D}$ achieves its maximum. Under our hypotheses, the Bergman-Šilov boundary exists and is unique. For D has a Bergman metric because it is bounded, and the metric is complete because D is homogeneous; thus D is a domain of holomorphy [1, p. 382]. It follows (using [8, Theorem 2.12] for example), that the maximal ideal space of the Banach algebra of functions holomorphic on D and continuous on $\bar{D}$ can be identified with $\bar{D}$. Now the existence and uniqueness of the Bergman-Šilov boundary is a general fact about Banach algebras (see [6, p. 212] for example).

3.2. PROPOSITION (Bott-Korányi). *Let D be a bounded domain in a complex vector space, starlike and circular with respect to the origin. Suppose that D is homogeneous under a group G^0 of holomorphic automorphisms which*

(i) *contains the rotations $z \to e^{i\theta}z$ $(0 \leqq \theta < 2\pi)$, and*

(ii) *extends continuously to ∂D.*

Then the Bergman-Šilov boundary of D is contained in the closure of every G^0-orbit on $\bar{D}$ and is itself a union of G^0-orbits.

PROOF. Let F be a G^0-orbit on ∂D, and choose $\zeta \in F$. Then D contains the set $U = \{z : z = re^{i\theta}\zeta, 0 \leqq r < 1, 0 \leqq \theta < 2\pi\}$, and every function f holomorphic on D and continuous on $\bar{D}$ restricts to a function with the same property on U and $\bar{U}$; also $|f(0)| \leqq \max_{\eta \in \partial U} |f(\eta)|$ by the maximum principle, so that $|f(0)| \leqq \max_{\eta \in \bar{F}} |f(\eta)|$. As G^0 is transitive on D and preserves $\bar{F}$, it follows that $|f(z)| \leqq \max_{\eta \in \bar{F}} |f(\eta)|$ for every $z \in D$. Now $\bar{F}$ is a determining set [6, p. 212], and so contains the Bergman-Šilov boundary.

For the second assertion, we note that G^0 acts as a group of automorphisms of the Banach algebra under consideration of $g: f \to f \cdot g$, so G^0 preserves the Bergman-Šilov boundary, q.e.d.

3.3. COROLLARY. *If, under the hypotheses of Proposition* 3.2, G^0 *has a compact subgroup* K *such that some* K*-orbit on* $\bar{D}$ *is also a* G^0*-orbit, then this orbit is the Bergman-Šilov boundary of* D.

For the orbit is closed because K is compact.

3.4. In the next theorem we apply these results to the bounded symmetric domain $D \subset \mathfrak{p}^-$ described in § 2. Retain the notation of § 2, and let $\check{S}_D$ denote the Bergman-Šilov boundary of D. D is homogeneous, G^0 acting transitively. We will slice D to see that it is starlike.

The map $j: \mathfrak{p}^0 \to \mathfrak{p}^-$ given by $j(U) = (U - iJ(U))/2$ is a vector space isomorphism; it commutes with the adjoint action of K because J does. If α is a positive root, one easily checks that $j(X_\alpha^0) = E_{-\alpha}$ and $j(Y_\alpha^0) = iE_{-\alpha}$. Define $\mathfrak{a}^- = j(\mathfrak{a}^0)$; then $\mathfrak{a}^-$ is the real subspace of $\mathfrak{p}^-$ spanned by $\{E_{-\alpha}: \alpha \in \Delta\}$. Conjugacy of Cartan subalgebras of $(\mathfrak{g}^0, \mathfrak{k})$ now implies that every element of $\mathfrak{p}^-$ is of the form $\mathrm{ad}(k)E$ with $k \in K$ and $E \in \mathfrak{a}^-$. In particular $D = \mathrm{ad}(K)(D \cap \mathfrak{a}^-)$. $D \cap \mathfrak{a}^-$ being the product $\{\sum_{\alpha \in \Delta} b_\alpha E_{-\alpha}: |b_\alpha| < 1\}$ of intervals, (immediate consequence of Lemma 3.5 below), it is starlike with respect to the origin, and this shows that D is starlike from the origin. Now we are assured of the existence and uniqueness of $\check{S}_D$. Observe also, by applying Schur's Lemma on the irreducible factors of M, that the one parameter subgroup tangent to Z in the center of K acts on $\mathfrak{p}^-$ by the rotations $E \to e^{i\theta}E$, $0 \leqq \theta < 2\pi$, so D is circular. Finally notice that the action of G^0 extends continuously to ∂D.

3.5. LEMMA. $\mathfrak{a}^-$ *is invariant under the action of* A^0 $(= \exp(\mathfrak{a}^0) \subset G^0)$. *If* $g = \exp\left(\sum_{\alpha \in \Delta} t_\alpha X_\alpha^0\right)$ *and* $E = \sum_{\alpha \in \Delta} b_\alpha E_{-\alpha}$, *then*

$$g(E) = \sum_{\alpha \in \Delta} \frac{b_\alpha \cosh t_\alpha + \sinh t_\alpha}{b_\alpha \sinh t_\alpha + \cosh t_\alpha} \cdot E_{-\alpha} .$$

PROOF. Let $\mathfrak{g}_\alpha^C$ be the three-dimensional complex subalgebra $\{E_{-\alpha}, H_\alpha, E_\alpha\}$ of $\mathfrak{g}^C$; G_α^C denotes the corresponding analytic subgroup of G^C. Calculating in a given $\mathfrak{g}_\alpha^C$, we may assume $E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $X_\alpha^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This gives

$$\exp(t_\alpha X_\alpha^0)\cdot\exp(b_\alpha E_{-\alpha}) = p_\alpha^- k_\alpha p_\alpha^+ \ ,$$

where

$$p_\alpha^- = \exp\left(\frac{b_\alpha \cosh t_\alpha + \sinh t_\alpha}{b_\alpha \sinh t_\alpha + \cosh t_\alpha}\cdot E_{-\alpha}\right)$$
$$k_\alpha = \exp\{\log(\cosh t_\alpha + b_\alpha \sinh t_\alpha) H_\alpha\}$$
$$p_\alpha^+ = \exp\left(\frac{\sinh t_\alpha}{\cosh t_\alpha + \beta_\alpha \sinh t_\alpha}\cdot E_\alpha\right) .$$

Taking the product over all α in Δ, and using the fact that every element of $G_\alpha^{\mathbf{C}}$ commutes with every element of $G_\beta^{\mathbf{C}}$ $(\alpha \neq \beta;\ \alpha, \beta \in \Delta)$ by strong orthogonality of Δ, it follows that

$$\xi(g(E)) = \exp\left(\sum\nolimits_{\alpha\in\Delta} \frac{b_\alpha \cosh t_\alpha + \sinh t_\alpha}{b_\alpha \sinh t_\alpha + \cosh t_\alpha} E_{-\alpha}\right)\cdot K^{\mathbf{C}}P^+ \ .$$

This proves the lemma, q.e.d.

3.6. THEOREM (Bott-Korányi). $K\left(\sum_{\alpha\in\Delta} E_{-\alpha}\right)$ *is the unique* K*-orbit on* $\bar{D}$ *which is a* G^0*-orbit; thus* $\check{S}_D = K\left(\sum_{\alpha\in\Delta} E_{-\alpha}\right)$ *is the Bergman-Šilov boundary.*

PROOF. The orbit of $0 \in \mathfrak{p}^-$ under A^0 is $\{\sum_{\alpha\in\Delta} b_\alpha E_{-\alpha}\colon |b_\alpha| < 1\}$ by Lemma 3.5. Let $E = \sum_{\alpha\in\Delta} b_\alpha E_{-\alpha} \in \mathfrak{a}^-$. Unless $|b_\alpha| = 1$ for all $\alpha \in \Delta$, Lemma 3.5 shows that $A^0(E)$, and thus $G^0(E)$, contains infinitely many points of $\mathfrak{a}^-$, whence $G^0(E) \supsetneqq K(E)$. But if $|b_\alpha| = 1$ for all $\alpha \in \Delta$, then $K(E) \cap \mathfrak{a}^- = \{\sum_{\alpha\in\Delta} \pm E_{-\alpha}\}$ because K acts unitarily on $\mathfrak{p}^-$, and the Weyl group of $(\mathfrak{g}^0, \mathfrak{k})$ contains all reflections in root planes. As A^0 preserves $\mathfrak{a}^-$ we have $A^0(K(E)\cap\mathfrak{a}^-) = A^0K(E)\cap\mathfrak{a}^-$, so (Lemma 3.5) $A^0(K(E)\cap\mathfrak{a}^-) = K(E)\cap\mathfrak{a}^-$. Take the union over all ad(K)-conjugates of A^0, $\mathfrak{a}_-$; then $A^0K(E) = K(E)$. $G = KA^0K$ now shows $G(E) = KA^0K(E) = KK(E) = K(E)$, and the theorem is proved, q.e.d.

3.7. DEFINITIONS. We define $\mathfrak{h}^- = [\mathfrak{a}^0, J\mathfrak{a}^0]$. By definition of $\mathfrak{a}^0$ and strong orthogonality of Δ, $\mathfrak{h}^-$ is the real span of $\{iH_\alpha\colon \alpha \in \Delta\}$ and $\dim \mathfrak{h}^- = \dim \mathfrak{a}^0$. We define $\mathfrak{h}^+$ to be the orthogonal complement of $\mathfrak{h}^-$ in $\mathfrak{h}$ with respect to the Killing form.

3.8. PROPOSITION. $\mathfrak{h}^+$ *is the centralizer of* $\mathfrak{a}^0$ *in* $\mathfrak{h}$, *and* $\mathfrak{t} = \mathfrak{h}^+ + \mathfrak{a}^0$ *is a Cartan subalgebra of* $\mathfrak{g}^0$.

PROOF. $\mathfrak{h}^+ = \{H \in \mathfrak{h}\colon \langle H, \mathfrak{h}^-\rangle = 0\} = \{H \in \mathfrak{h};\ \alpha(H) = 0 \text{ for every } \alpha \in \Delta\} = \{H \in \mathfrak{h};\ [H, X_\alpha^0] = 0 \text{ for every } \alpha \in \Delta\}$ which is the centralizer of $\mathfrak{a}^0$ in $\mathfrak{h}$. This proves the first statement and shows that $\mathfrak{t}$ is commutative. Now $\dim \mathfrak{a}^0 = \dim \mathfrak{h}^-$ implies $\dim \mathfrak{h} = \dim \mathfrak{t} = \dim(\mathfrak{h}^+ + i\mathfrak{a}^0)$, so $\mathfrak{h}^+ + i\mathfrak{a}^0$ is a Cartan subalgebra of $\mathfrak{g}$; this proves that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^0$, q.e.d.

3.9. DEFINITION. We define elements $X^0 = \sum_{\alpha\in\Delta} X_\alpha^0$, $Y^0 = \sum_{\alpha\in\Delta} Y_\alpha^0$ and

$Z^0 = -(i/2)\sum_{\alpha\in\Delta} H_\alpha$ in $\mathfrak{g}^0$. By strong orthogonality of Δ, they span a three-dimensional simple subalgebra of $\mathfrak{g}^0$.

Now let Z be the central element of $\mathfrak{k}$ defined in § 2.

3.10. PROPOSITION. *If Z is contained in a three-dimensional simple subalgebra of $\mathfrak{g}^0$, then $Z = Z^0$ (so $\{X^0, Y^0, Z\}$ spans such a subalgebra).*

PROOF. $[Z, X_\alpha^0] = Y_\alpha^0 = [Z^0, X_\alpha^0]$ for all $\alpha \in \Delta$ by definition of Z and strong orthogonality of Δ, so $Z - Z^0$ centralizes $\mathfrak{a}^0$. As $Z, Z^0 \in \mathfrak{h}$, this implies $Z - Z^0 \in \mathfrak{h}^+$.

Let $\mathfrak{m}^0$ be a simple three-dimensional subalgebra containing Z. The symmetry σ preserves $\mathfrak{m}^0$ because $\sigma = \operatorname{ad}(\exp \pi Z)$; thus $\mathfrak{m}^0 = (\mathfrak{m}^0 \cap \mathfrak{k}) + (\mathfrak{m}^0 \cap \mathfrak{p}^0)$. Z cannot be central in the simple algebra $\mathfrak{m}^0$; thus $\mathfrak{m}^0 \cap \mathfrak{p}^0 \neq 0$.

Now we replace $\mathfrak{m}^0$ by an $\operatorname{ad}(K)$-conjugate; this does not change Z, but does allow us to assume that we have a non-zero element $X' \in \mathfrak{m}^0 \cap \mathfrak{a}^0$; $X' = \sum_{\alpha\in\Delta} b_\alpha X_\alpha^0$ and $Y' = [Z, X'] = \sum_{\alpha\in\Delta} b_\alpha Y_\alpha^0$. It follows that Z spans $\mathfrak{m}^0 \cap \mathfrak{k}$, $\{X', Y'\}$ spans $\mathfrak{m}^0 \cap \mathfrak{p}^0$, and Z is a non-zero scalar multiple of $[X', Y'] \in [\mathfrak{a}^0, J\mathfrak{a}^0] = \mathfrak{h}^-$, proving $Z \in \mathfrak{p}^-$.

We also have $Z^0 \in \mathfrak{h}^-$, whence $Z - Z^0 \in \mathfrak{h}^-$. As $Z - Z^0 \in \mathfrak{h}^+$, it follows that $Z = Z^0$, q.e.d.

3.11. DEFINITION. The hermitian symmetric space M of non-compact type is *of tube type* if it is holomorphically equivalent to the tube over a self-dual cone in the sense of Koecher [5].

3.12. PROPOSITION. *If M is of tube type, then Z is contained on a three dimensional simple subalgebra of $\mathfrak{g}^0$, thus $Z = Z^0$.*

The proof is immediate from Koecher's results. Any tube domain is the set of all elements z in a complex semi-simple Jordan algebra for which the imaginary part of the left multiplication operator $l(z)$ is positive definite. SL(2, R) acts on the domain by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad z \longrightarrow (az + b)\cdot(cz + d)^{-1}$$

in the Jordan multiplication, and the Lie algebra element

$$\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

is sent to Z by this action, where $z = ie$, and e is the Jordan identity element, q.e.d.

4. Dimension of the Bergman-Šilov boundary and its representation by symmetric spaces

We will define a generalized Cayley transform $c \in G$, and check that $c(x)$ is in the Bergman-Šilov boundary $\check{S}$ of $M = G^0(x)$ in M^*. Then $\check{S} = K(c(x)) \cong K/L$ where L is the isotropy subgroup at $c(x)$. Next we decompose the Lie algebras $\mathfrak{g}$, $\mathfrak{g}^C$, $\mathfrak{k}$ and $\mathfrak{k}^C$ under conjugation by c or its powers. We are able to compare the various spaces involved in these decompositions. This yields a formula for the dimension of $\check{S}$ and a detailed description of the structure of $\check{S}$ as a submanifold of M^*.

4.1. Let $\mathfrak{g}_\alpha^0$ denote the subalgebra of $\mathfrak{g}^0$ spanned by $\{iH_\alpha, X_\alpha^0, Y_\alpha^0\}$ for $\alpha \in \Delta$; $\mathfrak{g}_\alpha^C$ will denote its complexification. Strong orthogonality of Δ implies $[\mathfrak{g}_\alpha^C, \mathfrak{g}_\beta^C] = 0$ for $\alpha \neq \beta$.

Define $X_\alpha = iX_\alpha^0 \in \mathfrak{g}$ and $c_\alpha = \exp(\pi/4)X_\alpha \in G$. Let $X = iX^0 = \sum_{\alpha\in\Delta} X_\alpha \in \mathfrak{g}$. Define the *Cayley transform* $c \in G$ by

$$c = \exp\frac{\pi}{4}X .$$

Observe that $c = \prod_{\alpha\in\Delta} c_\alpha$.

4.2. LEMMA. $c(x) = \xi\left(i\sum_{\alpha\in\Delta} E_{-\alpha}\right) \in \check{S}$.

PROOF. By Theorem 3.6 and the fact that K contains the transformations $E \to e^{i\theta}E$ of $\mathfrak{p}^-$, it suffices to prove that the map $\xi\colon \mathfrak{p}^- \to M^*$ carries $i\sum_{\alpha\in\Delta} E_{-\alpha}$ to $c(x)$.

By definition $c = \exp(\pi/4)\sum_{\alpha\in\Delta} X_\alpha$ where $-iX_\alpha = X_\alpha^0 = E_\alpha + E_{-\alpha}$, and we have $c_\alpha = \exp(\pi/4)X_\alpha$ for $\alpha \in \Delta$. We wish to prove

$$c_\alpha = \exp(iE_{-\alpha})\cdot k_\alpha \cdot \exp(iE_\alpha)$$

with k_α in the complex analytic group determined by $[X_\alpha, Y_\alpha]$. Then strong orthogonality of Δ implies that each k_α and each $\exp(iE_{\pm\alpha})$ commutes with each k_β and each $\exp(iE_{\pm\beta})$ for $\alpha \neq \beta$, implying $c \in \left(\prod_{\alpha\in\Delta}\exp(iE_{-\alpha})\right)\cdot K^C P^+ = \left(\exp i\sum_{\alpha\in\Delta} E_{-\alpha}\right)\cdot K^C P^+$, whence $c(x) = \xi\left(i\sum_{\alpha\in\Delta} E_{-\alpha}\right)$. We have reduced the problem to the case of one complex dimension.

Let $\dim_C M = 1$. Then we have

$$c_\alpha = \exp\frac{\pi}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \cosh\frac{\pi i}{4} & \sinh\frac{\pi i}{4} \\ \sinh\frac{\pi i}{4} & \cosh\frac{\pi i}{4} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = akb ;$$

here

$$a = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \exp i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \exp iE_{-\alpha}$$

and similarly $b = \exp iE_\alpha$. Thus c_α has the required form, q.e.d.

4.3. LEMMA. *Let $\alpha \in \Delta$. Then*

$$\mathrm{ad}\,(c)\colon X_\alpha^0 \longrightarrow X_\alpha^0, \quad Y_\alpha^0 \longrightarrow -H_\alpha, \quad H_\alpha \longrightarrow Y_\alpha^0 \,.$$

PROOF. $[\mathfrak{g}_\alpha^C, \mathfrak{g}_\beta^C] = 0$, for $\alpha \neq \beta$ shows that $\mathrm{ad}\,(c)\,|\mathfrak{g}_\alpha^C = \mathrm{ad}\,(c_\alpha)\,|\mathfrak{g}_\alpha^C$, so we may calculate in $\mathfrak{g}_\alpha^C$. There

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$c_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The result follows, q.e.d.

4.4. PROPOSITION. *Let τ be the automorphism* $\mathrm{ad}\,(c)^2$ *of* $\mathfrak{g}^C$. *Then* $\tau^2(\mathfrak{k}) = \mathfrak{k}$, $\tau^4 = 1$, *and the following statements are equivalent:*

(i) *Z is in a three-dimensional simple subalgebra,*

(ii) $\tau(Z) = -Z$,

(iii) $\tau(\mathfrak{k}) = \mathfrak{k}$,

(iv) $\tau^2 = 1$,

(v) *csc^{-1} normalizes K where s is the symmetry at x.*

PROOF. Let $\mathfrak{m}$ be the three-dimensional simple subalgebra of $\mathfrak{g}$ with basis $\{Z^0, X, Y\}$, $X = iX^0$, and $Y = iY^0$. Under the adjoint action of $\mathfrak{m}$, $\mathfrak{g} = \mathfrak{f}_0 + \sum_{j=1}^r \mathfrak{f}_j$ where $\mathrm{ad}\,(\mathfrak{m})$ is trivial on $\mathfrak{f}_0$, non-trivial and irreducible on the other $\mathfrak{f}_j$. Lemma 4.3 implies

$$\mathrm{ad}\,(c)\colon\ X \longrightarrow X, \quad Y \longrightarrow 2Z^0, \quad 2Z^0 \longrightarrow -Y\,.$$

This shows that $\mathrm{ad}\,(c)$ has order 4 on $\mathfrak{f}_j$ if $\dim \mathfrak{f}_j$ is odd; order 8 if $\dim \mathfrak{f}_j$ is even ($j \geqq 1$). Thus $c^8 = 1$, so $\tau^4 = 1$. If Z is in a three-dimensional simple subalgebra, so $Z^0 = Z$ by Proposition 3.10, then each $\mathfrak{f}_j$ has dimension 3 because 0, i and $-i$ are the only eigenvalues of $\mathrm{ad}\,(Z)$; in that case $c^4 = 1$, so $\tau^2 = 1$; thus (i) implies (iv).

K is the identity component of the fixed point set in G of $\sigma = \mathrm{ad}\,(s)$; thus $\mathrm{ad}\,(g)\mathfrak{k} = \mathfrak{k}$ whenever $g \in G$ with $\sigma(g) = g$. Now recall $\sigma(c) = c^{-1}$ because c is a transvection of M^* from x. If $\tau^{2a} = 1$, then $\sigma(c^{2a}) = c^{-2a} = c^{2a}$, so $\tau^a(\mathfrak{k}) = \mathfrak{k}$; if $\tau^a(\mathfrak{k}) = \mathfrak{k}$, then $\tau^a(Z) = \pm Z$, so $c^{2a}sc^{-2a} = s^{\pm 1} = s$, whence $c^{2a} = \sigma(c^{2a}) = c^{-2a}$, implying $c^{4a} = 1$ and $\tau^{2a} = 1$. As $\tau^4 = 1$, let $a = 2$ to see $\tau^2(\mathfrak{k}) = \mathfrak{k}$. Let $a = 1$ to see that (iii) is equivalent to (iv). For equivalence of (iii) and (v), we observe that $K = \exp(\mathfrak{k})$, and that $\mathrm{ad}\,(csc^{-1})\mathfrak{k} = \mathrm{ad}\,(csc^{-1})\,\mathrm{ad}\,(s^{-1})\mathfrak{k} = \mathrm{ad}\,(c\cdot sc^{-1}s^{-1})\mathfrak{k} = \mathrm{ad}\,(c^2)\mathfrak{k} = \tau(\mathfrak{k})$.

If $\tau(Z) = Z$, then $[X, Z] = 0$ by Lemma 4.3; thus $\tau(Z) = \pm Z$ implies $\tau(Z) = -Z$, proving that (iii) implies (ii). Now recall the proof of Proposition 3.10; we saw that $Z = Z^0 + Z'$ with $Z^0 \in \mathfrak{h}^-$ and $Z' \in \mathfrak{h}^+$. τ is the identity on $\mathfrak{h}^+$ by Proposition 3.8, and τ is $-I$ on $\mathfrak{h}^-$ by Lemma 4.3; thus (ii) implies (i).

We have proved $\tau^4 = 1$, $\tau^2(\mathfrak{k}) = \mathfrak{k}$, (i) $\to$ (iv) $\leftrightarrow$ (iii) $\leftrightarrow$ (v) and (iii) $\to$ (ii) $\to$ (i), q.e.d.

REMARK. Conditions (i) through (v) of Proposition 4.4 are equivalent to the absence of an $\mathfrak{h}$-root β of $\mathfrak{g}^0$ whose restriction to $\mathfrak{h}^-$ is $\alpha/2$, $\alpha \in \Delta$. For if $\beta\,|\,\mathfrak{h}^- = \alpha/2$, then $\mathrm{ad}\,(Z^0)E_\beta = \pm(i/2)E_\beta$ and so $Z \neq Z^0$. If $Z \neq Z^0$, then $0 \neq \mathrm{ad}\,(Z^0)E_\beta = \pm iE_\beta$ for some root β. Therefore $\beta\,|\,\mathfrak{h}^- \neq \pm\,(\alpha_1 \pm \alpha_2)/2$ with $\alpha_i \in \Delta$; then $\beta\,|\,\mathfrak{h}^- = \pm\alpha/2$ for some $\alpha \in \Delta$ by a result of Harish-Chandra [3].

4.5. Definitions. Let $\tau = \mathrm{ad}\,(c)^2$, viewed as an automorphism both of $\mathfrak{g}$ and of $\mathfrak{g}^C$. We define: $\mathfrak{g}'$ is the set of elements of $\mathfrak{g}$ fixed under τ^2; $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}'$, $\mathfrak{k}_1 = [\mathfrak{p}_1, \mathfrak{p}_1]$, and $\mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1$; (We write $\mathfrak{p}_1$ rather than $\mathfrak{p}'$ because we use $\mathfrak{g}_1$ more often than $\mathfrak{g}'$.) $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$, and $\mathfrak{l}_2$ is the centralizer of $\mathfrak{g}_1$ in $\mathfrak{g}'$. Now $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}_1$ because τ^2 preserves $\mathfrak{k}$ and $\mathfrak{p}$, and $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}'$. We have $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{l}_2$ and $\mathfrak{k}' = \mathfrak{k}_1 \oplus \mathfrak{l}_2$, direct sums of ideals, from the theory of orthogonal involutive Lie algebras. Everything is preserved by τ.

τ preserves and has square I on $\mathfrak{k}_1$, and τ^2 preserves and has square I on $\mathfrak{k}$ and on $\mathfrak{p}$. This leads us to define:

$\mathfrak{l}_1$ is the $(+1)$-eigenspace of τ on $\mathfrak{k}_1$,

$\mathfrak{q}_1$ is the (-1)-eigenspace of τ on $\mathfrak{k}_1$,

$\mathfrak{q}_2$ is the (-1)-eigenspace of τ^2 on $\mathfrak{k}$,

$\mathfrak{p}_2$ is the (-1)-eigenspace of τ^2 on $\mathfrak{p}$.

Now we have $\mathfrak{k}_1 = \mathfrak{l}_1 + \mathfrak{q}_1$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$. It will be convenient to define $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ and $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$ so $\mathfrak{k} = \mathfrak{l} + \mathfrak{q}$.

The subalgebras $\mathfrak{g}_1$, $\mathfrak{k}_1$ and $\mathfrak{g}'$ of $\mathfrak{g}$ yield subalgebras $\mathfrak{g}_1^0 = \mathfrak{k}_1 + i\mathfrak{p}_1$, $\mathfrak{k}_1^* = \mathfrak{l}_1 + i\mathfrak{q}_1$ and $\mathfrak{g}'^0 = \mathfrak{k}' + i\mathfrak{p}_1$ of $\mathfrak{g}^C$. The corresponding analytic subgroups of G^C will be denoted G_1, G_1^0, K_1, K_1^*, G' and G'^0. Observe that $c \in G_1$, so $\mathrm{ad}(c)$ and its square τ are trivial on $\mathfrak{l}_2$; it is immediate that $\mathfrak{l}$ is the fixed point set of τ on $\mathfrak{k}$.

4.6. LEMMA. *$\mathfrak{g}_1$ is semi-simple, $(\mathfrak{g}_1^0, \mathfrak{k}_1)$ is a hermitian symmetric pair, $Z^0 \in \mathfrak{k}_1$, Z^0 gives the complex structure to G_1^0/K_1, and $Z' \in \mathfrak{l}_2$.*

PROOF. We have $Z \in \mathfrak{k}'$; $\mathrm{ad}\,(Z)$ has no eigenvalue $+1$ on $\mathfrak{p}_1$, hence none on $\mathfrak{p}_1$. Thus $\mathfrak{p}_1$ has no non-zero element central in $\mathfrak{g}'$, hence none central in $\mathfrak{g}_1$. As

G_1 is compact, this shows that G_1/K_1 has no locally euclidean factor. The decomposition theory of orthogonal involutive Lie algebras shows that $\mathfrak{k}_1$ is effective on $\mathfrak{p}_1$. It follows that $\mathfrak{g}_1$ is semi-simple. The restriction of $\mathrm{ad}(Z)^2$ to $\mathfrak{p}_1$ is $-I$ there, so G_1/K_1 is hermitian symmetric.

We still have $c \in G_1$ as the Cayley transform on G_1/K_1, and $\tau^2 = 1$ on $\mathfrak{g}_1$. $Z \in \mathfrak{k}'$, so $Z = Z_1 + Z_2$ with $Z_1 \in \mathfrak{k}_1$ and $Z_2 \in \mathfrak{l}_2$; Z and Z_1 both give the complex structure of $G'/K' = G_1/K_1$. As $\tau^2 = 1$ on $\mathfrak{g}_1$, Proposition 4.4 says $\tau(Z_1) = -Z_1$; on the other hand, $\tau(Z_2) = Z_2$ because τ is trivial on $\mathfrak{l}_2$. The proof of Proposition 3.10 yields the decomposition $Z = Z^0 + Z'$, $\tau(Z') = Z'$ by Proposition 3.8, and $\tau(Z^0) = -Z^0$ by Lemma 4.3. It follows that $Z_1 = Z^0$ and $Z_2 = Z'$. Thus $Z^0 \in \mathfrak{k}_1$, Z^0 gives the complex structure of G_1/K_1, and $Z' \in \mathfrak{l}_2$, q.e.d.

4.7. LEMMA. *τ interchanges $\mathfrak{p}_2$ and $\mathfrak{q}_2$,* $\mathrm{ad}(c)$ *interchanges $\mathfrak{q}_1$ with the (-1)-eigenspace of τ on $\mathfrak{p}_1$, and J interchanges the (± 1)-eigenspaces of τ on $\mathfrak{p}_1$.*

PROOF. $V \in \mathfrak{p}_2$ and $R \in \mathfrak{q}_2$ gives $\sigma\tau V = \tau^{-1}\sigma V = -\tau^{-1} V = -\tau\tau^2 V = \tau V$ and $\sigma\tau R = \tau^{-1}\sigma R = \tau^{-1} R = \tau\tau^2 R = -\tau R$; thus $\tau(\mathfrak{p}_2) \subset \mathfrak{k}$ and $\tau(\mathfrak{q}_2) \subset \mathfrak{p}$. Now $\tau\tau^2 = \tau^2\tau$ gives $\tau(\mathfrak{p}_2) \subset \mathfrak{q}_2$ and $\tau(\mathfrak{q}_2) \subset \mathfrak{p}_2$. Equality follows from dimension, proving the first statement. The proof of the second statement is similar.

Lemma 4.6 gives $\mathrm{ad}(Z) = \mathrm{ad}(Z^0)$ on $\mathfrak{p}_1$, so J anti-commutes with τ on $\mathfrak{p}_1$ because $J = \mathrm{ad}(Z)|\mathfrak{p}$ and $\tau(Z^0) = -Z^0$. The last assertion follows, q.e.d.

4.8. LEMMA. *Let L be the isotropy subgroup of K at $c(x)$. Then the set $\mathfrak{l}$ of all elements of $\mathfrak{k}$ fixed under τ is the Lie algebra of L, and*

$$\dim \mathfrak{k} - \dim \mathfrak{l} = \dim \mathfrak{p}_2 + \frac{1}{2}\dim \mathfrak{p}_1 \geqq \frac{1}{2}\dim M$$

with equality if and only if $\tau^2 = 1$.

PROOF. The isotropy algebra of $\mathfrak{k}$ at $c(x)$ is the fixed point set in $\mathfrak{k}$ for conjugation by the symmetry csc^{-1} at $c(x)$. If $U \in \mathfrak{k}$, then $\mathrm{ad}(csc^{-1})U = \mathrm{ad}(csc^{-1})\cdot\mathrm{ad}(s^{-1})U = \mathrm{ad}(c)\cdot\mathrm{ad}(sc^{-1}s^{-1})U = \mathrm{ad}(c)^2 U = \tau(U)$.

We have $\mathfrak{k} = \mathfrak{l} + \mathfrak{q}_1 + \mathfrak{q}_2$, and Lemma 4.7 gives $\dim \mathfrak{q}_2 = \dim \mathfrak{p}_2$ and $\dim \mathfrak{q}_1 = (1/2)\dim \mathfrak{p}_1$. As $\dim M = \dim \mathfrak{p} = \dim \mathfrak{p}_1 + \dim \mathfrak{p}_2$, this gives $\dim \mathfrak{k} - \dim \mathfrak{l} = \dim \mathfrak{p}_2 + (1/2)\dim \mathfrak{p}_1 \geqq (1/2)\dim M$ with equality if and only if $\mathfrak{p}_2 = 0$. $\mathfrak{p}_2 = 0$ if and only if $\tau(\mathfrak{p}) = \mathfrak{p}$, as seen from Lemma 4.7; $\tau(\mathfrak{p}) = \mathfrak{p}$ if and only if $\tau(\mathfrak{k}) = \mathfrak{k}$; $\tau(\mathfrak{k}) = \mathfrak{k}$ if and only if $\tau^2 = 1$, by Proposition 4.4, q.e.d.

4.9. THEOREM. *Let $\check{S}$ be the Bergman-Šilov boundary of $M = G^0(x)$ in M^*; let L and L_1 be the isotropy subgroups of K and K_1 at $c(x)$. Then:*

1. *$\check{S} = K(c(x)) \cong K/L$ and* $\dim \check{S} = \dim \mathfrak{p}_2 + (1/2)\dim \mathfrak{p}_1 \geqq (1/2)\dim M$.
2. *$K(c^2(x))$ is a totally geodesic complex submanifold of M^*, and is thus a compact hermitian symmetric space; the isotropy subgroup of K at $c^2(x)$ is the analytic subgroup K' with Lie algebra $\mathfrak{k}'$, so $K(c^2(x)) \cong K/K'$.*

3. *The map* $k(c(x)) \to k(c^2(x))$ *is a fibering of* $\check{S}$ *over* $K(c^2(x))$; *the fibre over* $k(c^2(x))$ *is* $k(K_1(c(x)))$, *which is totally geodesic in* M^*, *riemannian symmetric and isometric to* K_1/L_1.

4. *The following statements are equivalent*:

(i) Z *is in a three-dimensional simple subalgebra.*

(ii) *The Cayley transform* c *has order* 4, i.e., $\mathfrak{g} = \mathfrak{g}_1$.

(iii) $\check{S}$ *is a totally geodesic submanifold of* M^*, *so* $\check{S}$ *is riemannian symmetric and the largest connected group of isometries consists of the motions induced by* K.

(iv) *The dimension of* $\check{S}$ *is half the real dimension of* M.

(v) $K(c^2(x))$ *is a single point.*

PROOF. We have $c(x) \in \check{S}$ by Lemma 4.2 and K transitive on $\check{S}$ by Theorem 3.6; thus $\check{S} = K(c(x)) \cong K/L$. Lemma 4.8 now yields statement (1) and proves the equivalence of (ii) and (iv) in (4).

The symmetry at $c^2(x)$ is $c^2sc^{-2} = c^2sc^{-2}s^{-1}s = c^4s$, which, normalizes K because $\tau^2(\mathfrak{k}) = \mathfrak{k}$; this proves $K(c^2(x))$ totally geodesic in M^*. The almost-complex structure of M^* at $c^2(x)$ acts on the space of transvections there by $\mathrm{ad}(c^2)\cdot\mathrm{ad}(Z)\cdot\mathrm{ad}(c^{-2}) = \mathrm{ad}(\mathrm{ad}(c^2)Z) = \mathrm{ad}(\tau Z) = \mathrm{ad}(\tau Z^0 + \tau Z') = \mathrm{ad}(-Z^0 + Z') = \mathrm{ad}(-Z + 2Z')$. The space of transvections of $K(c^2(x))$ at $c^2(x)$ is $\mathfrak{q}_2$, $\mathrm{ad}(Z)$ acts trivially on $\mathfrak{q}_2$, and $\mathrm{ad}(Z')$ preserves $\mathfrak{q}_2$; this proves that $K(c^2(x))$ is a complex submanifold of M^*. The isotropy subalgebra of $\mathfrak{k}$ at $c^2(x)$ is the fixed point set $\mathfrak{k}'$ of τ^2 on $\mathfrak{k}$, by construction; thus K' is the identity component of the isotropy subgroup of K at $c^2(x)$. On the other hand that isotropy group is connected because $K(c^2(x))$ is simply connected, being a compact hermitian symmetric space without euclidean factor. This proves (2).

The map $\check{S} \to K(c^2(x))$ given by $k(c(x)) \to k(c^2(x))$ is just the map $K/L \to K/K'$ given by $kL \to kK'$; thus it is a well defined fibering. The fibre over $k(c^2(x))$ is $(kK'c)(x)$; the normal subgroup of K' with Lie algebra $\mathfrak{l}_2$ leaves $c(x)$ fixed, so the fibre is $(kK_1c)(x)$. $\mathrm{ad}(csc^{-1})K_1 = \tau(K_1) = K_1$ so $K_1(c(x))$ is totally geodesic in M^*; thus $k(K_1(c(x)))$ is totally geodesic. This proves (3).

Proposition 4.4 shows that (i), (ii) and (v) are equivalent in (4), and that they are equivalent to K being normalized by the symmetry $s' = csc^{-1}$ at $c(x)$. If $\mathrm{ad}(s')K = K$, then $\check{S} = K(c(x))$ is totally geodesic in M^*. On each irreducible factor of M, the corresponding ideal of $\mathfrak{k}$ is a maximal subalgebra of the corresponding ideal of $\mathfrak{g}$; it follows that K is the identity component of

$$E = \{g \in G\colon g(\check{S}) = \check{S}\} .$$

Now if $\check{S}$ is totally geodesic, then $s'(\check{S}) = \check{S}$, so $\mathrm{ad}(s')E = E$, and it follows that $\mathrm{ad}(s')K = K$. We have proved (4).

This completes the proof of Theorem 4.9, q.e.d.

4.10. EXAMPLE. Let M be the open unit ball in complex euclidean space $\mathbf{C}^n$. Then M^* is the complex projective space $P^n(\mathrm{C})$ with the Fubini-Study metric, $\check{S}$ is the full topological boundary S^{2n-1} of M in M^*, $K(c^2(x))$ is the polar hyperplane $P^{n-1}(\mathrm{C})$ to x in M^*, and the fibering $\check{S} \to K(c^2(x))$ is the usual circle bundle $S^{2n-1} \to P^{n-1}(\mathrm{C})$.

4.11. THEOREM. *Let $\check{S}$ be the Bergman-Šilov boundary of M in M^*. Then the fundamental group $\pi_1(\check{S})$ is the product of a finite abelian group and a group which is free abelian with one generator for each irreducible factor of tube type in M. In particular the first Betti number of $\check{S}$ is the number of irreducible factors of tube type in M.*

PROOF. We may assume M irreducible. Then Z spans the center $\mathfrak{z}$ of $\mathfrak{k}$ and the circle group $U = \exp(\mathfrak{z})$ is the center of K. Let $\perp$ refer to the restriction of the Killing form from $\mathfrak{g}$ to $\mathfrak{k}$, so the derived algebra $\mathfrak{k}_{ss}$ of $\mathfrak{k}$ is $Z^\perp$. $Z' \in \mathfrak{l}_2$ by Lemma 4.6. Also $\mathfrak{l}_2 \subset \mathfrak{l} = \mathfrak{q}^\perp$ and $Z^0 \in \mathfrak{q}_1 \subset \mathfrak{q}$. Thus $\mathfrak{l} \perp Z$ if and only if $Z' = 0$, i.e., if and only if M is of tube type. We conclude from $\mathfrak{k}_{ss} = Z^\perp$ and $\dim \mathfrak{k} - \dim \mathfrak{k}_{ss} = 1$ that

(i) if M is of tube type then $\mathfrak{l} \subset \mathfrak{k}_{ss}$, and

(ii) if M is not of tube type then $\mathfrak{k}_{ss} + \mathfrak{l} = \mathfrak{k}$.

We also have

(iii) $\pi_1(\check{S})$ is abelian.

For $\pi_1(K'/L)$ is abelian because [11, Theorem 6.4] $K'/L = K_1/L_1$ is symmetric by Theorem 4.9 (3), K/K' is simply connected by Theorem 4.9 (2), and $\pi_1(K'/L) \to \pi_1(\check{S}) \to \pi_1(K/K')$ is exact.

Let K_{ss} be the derived group of K; $K = K_{ss} \cdot U$. If M is not of tube type, then K_{ss} is transitive on $\check{S}$ by (ii), so $\pi_1(\check{S})$ is finite because $\pi_1(K_{ss}) \to \pi_1(\check{S}) \to \pi_0(L \cap K_{ss})$ is exact. Then $\pi_1(\check{S})$ is finite abelian by (iii). If M is of tube type, then the inclusion $L_0 \subset K$ factors through K_{ss} by (i), so $\pi_1(L)$ has finite image in $\pi_1(K)$. Then exactness of $\pi_1(L) \to \pi_1(K) \to \pi_1(\check{S}) \to \pi_0(L)$ and (iii) show that $\pi_1(\check{S})$ is the product of a finite abelian group and an infinite cyclic group, q.e.d.

5. The Bergman-Šilov boundary as a coset space of the connected group of analytic automorphisms

5.1. Let B denote the isotropy subgroup of G^0 at $c(x)$, so the Bergman-Šilov boundary $\check{S} = G^0(c(x)) \cong G^0/B$. We will examine B by decomposing its Lie algebra $\mathfrak{b}$. For this we need the definitions:

$\mathfrak{p}_2^+$ (resp. $\mathfrak{p}_2^-$) is the sum of all positive (resp. negative) $\mathfrak{h}^C$-root spaces in $\mathfrak{p}_2^C$.

$\mathfrak{q}_2^+$ (resp. $\mathfrak{q}_2^-$) is the sum of all positive (resp. negative) $\mathfrak{h}^C$-root spaces in $\mathfrak{q}_2^C$.

$\mathfrak{n}_2^\pm = (\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm) \cap \mathrm{ad}\,(c)\mathfrak{g}^0$, $\mathfrak{n}_1^\pm = \mathfrak{p}_1^\pm \cap \mathrm{ad}\,(c)\mathfrak{g}^0$, and $\mathfrak{n}^\pm = \mathfrak{n}_1^\pm + \mathfrak{n}_2^\pm$:

$$\mathfrak{r}^\pm = \mathfrak{q}_2^\pm + \mathfrak{p}^\pm$$

$$\mathfrak{k}_1^* = \mathfrak{l}_1 + i\mathfrak{q}_1 \quad \text{and} \quad \mathfrak{k}'^* = \mathfrak{l} + i\mathfrak{q}_1 .$$

We observe that $\mathfrak{p}_2^C = \mathfrak{p}_2^+ + \mathfrak{p}_2^-$ and $\mathfrak{q}_2^C = \mathfrak{q}_2^+ + \mathfrak{q}_2^-$ because τ^2 is $-I$ on them while it is $+I$ on $\mathfrak{h}^C$.

5.2. LEMMA. $\mathrm{ad}\,(Z^0) = (1/2)\,\mathrm{ad}\,(Z)$ *on* $\mathfrak{p}_2^C$, *and* τ *interchanges* $\mathfrak{p}_2^\pm$ *with* $\mathfrak{q}_2^\mp$.

PROOF. $\tau Z = \tau Z' + \tau Z^0 = Z' - Z^0 = Z - 2Z^0$, so $Q \in \mathfrak{q}_2^C$ gives

$$\big(\mathrm{ad}\,(Z) - 2\,\mathrm{ad}\,(Z^0)\big)(\tau Q) = \mathrm{ad}\,(\tau Z)(\tau Q) = \tau\,\mathrm{ad}\,(Z)Q = 0\,;$$

thus $\mathrm{ad}\,(Z) = 2\,\mathrm{ad}\,(Z^0)$ on $\tau(\mathfrak{q}_2^C)$, which is $\mathfrak{p}_2^C$ by Lemma 4.7.

Let $Q \in \mathfrak{q}_2^C$ be in the root space for a root α; then $\tau Q \in \mathfrak{p}_2^C$ is in the root space for a root β where $\alpha = \tau^*\beta$, because $\tau(\mathfrak{h}^C) = \mathfrak{h}^C$ and $\tau(\mathfrak{q}_2^C) = \mathfrak{p}_2^C$. β is positive or negative depending on whether the real number $i\beta(Z)$ is positive or negative. We may assume the real span $i\mathfrak{h}$ of the roots to be ordered with $i\mathfrak{h}^-$ first; now $i\alpha(Z^0) > 0$ implies $\alpha > 0$, and $i\alpha(Z^0) < 0$ implies $\alpha < 0$. As $\alpha(Z^0)\cdot\tau Q = \tau[Z^0, Q] = [\tau Z^0, \tau Q] = -[Z^0, \tau Q] = (-(1/2)\beta(Z))\cdot\tau Q$, it follows that $\tau(\mathfrak{q}_2^+) \subset \mathfrak{p}_2^-$ and $\tau(\mathfrak{q}_2^-) \subset \mathfrak{p}_2^+$. These inclusions are equalities because $\tau(\mathfrak{q}_2^C) = \mathfrak{p}_2^C$, $\mathfrak{q}_2^C = \mathfrak{q}_2^+ + \mathfrak{q}_2^-$ (direct), and $\mathfrak{p}_2^C = \mathfrak{p}_2^+ + \mathfrak{p}_2^-$ (direct). Now from the fact that τ^2 is scalar on $\mathfrak{q}_2^C$, we have $\tau(\mathfrak{p}_2^\mp) = \mathfrak{q}_2^\pm$, q.e.d.

5.3. LEMMA. *The eigenvalues of* $\mathrm{ad}\,(-Y^0)$ *on* $\mathfrak{g}^C$ (*resp.* $\mathfrak{g}^0$) *are* 0, ± 1 *and* ± 2; *the corresponding eigenspaces are* $\mathrm{ad}\,(c)^{-1}\mathfrak{k}'^C$ (*resp.* $\mathrm{ad}\,(c)^{-1}\mathfrak{k}'^*$), $\mathrm{ad}\,(c)^{-1}(\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm)$ (*resp.* $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_2^\pm$), *and* $\mathrm{ad}\,(c)^{-1}\mathfrak{p}_1^\pm$ (*resp.* $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_1^\pm$). *Furthermore,* $\mathfrak{k}'^* = \mathfrak{k}_1^* + \mathfrak{l}_2 = \mathfrak{k}'^C \cap \mathrm{ad}\,(c)^{-1}\mathfrak{g}^0$ *and is a real form of* $\mathfrak{k}'^C$, $\mathfrak{n}_2^\pm$ *is a real form of* $\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm$, *and* $\mathfrak{n}_1^\pm$ *is a real form of* $\mathfrak{p}_1^\pm$.

PROOF. $\mathrm{ad}\,(Z^0)$ coincides with $\mathrm{ad}\,(Z)$ on $\mathfrak{p}_1^C$ and is thus $\mp i$ on $\mathfrak{p}_1^\pm$. $\mathrm{ad}\,(Z^0)$ is $(1/2)\mathrm{ad}\,(Z)$ on $\mathfrak{p}_2^C$ and thus is $\mp i/2$ on $\mathfrak{p}_2^\pm$. Applying τ and using Lemma 5.2, $\mathrm{ad}\,(Z^0)$ is $\pm i/2$ on $\mathfrak{q}_2^\mp$. Z^0 is central in $\mathfrak{k}'$, so $\mathrm{ad}\,(Z^0)$ is 0 on $\mathfrak{k}'^C$. Thus the eigenvalues for $\mathrm{ad}\,(Z^0)$ on $\mathfrak{g}^C$ are 0, $\mp i/2$ and $\mp i$ with eigenspaces $\mathfrak{k}'^C$, $\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm$ and $\mathfrak{p}_1^\pm$. As $2i\,\mathrm{ad}\,(c)^{-1}Z^0 = iY = -Y^0$, we apply $\mathrm{ad}\,(c)$ and obtain the assertions on the eigenspaces of $\mathrm{ad}\,(-Y^0)$ on $\mathfrak{g}^C$.

$-Y^0 \in \mathfrak{g}^0$ implies $\mathrm{ad}\,(-Y^0)\mathfrak{g}^0 \subset \mathfrak{g}^0$, so the eigenspaces of $\mathrm{ad}\,(-Y^0)$ on $\mathfrak{g}^C$ are the complexifications of their intersections with $\mathfrak{g}^0$. As $\mathfrak{g}^0$ contains no complex subspace of $\mathfrak{g}^C$, the intersection with $\mathfrak{g}^0$ is a real form for any eigenspace of $\mathrm{ad}\,(-Y^0)$. Therefore $\mathrm{ad}\,(c)^{-1}\mathfrak{k}'^C \cap \mathfrak{g}^0$ is a real form of $\mathrm{ad}\,(c)^{-1}\mathfrak{k}'^C$, $\mathrm{ad}\,(c)^{-1}(\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm) \cap \mathfrak{g}^0$ is a real form of $\mathrm{ad}\,(c)^{-1}(\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm)$, and $\mathrm{ad}\,(c)^{-1}\mathfrak{p}_1^\pm \cap \mathfrak{g}^0$ is a real form of $\mathrm{ad}\,(c)^{-1}\mathfrak{p}_1^\pm$. Applying $\mathrm{ad}\,(c)$, Lemma 5.2, and the fact that $\tau(\mathfrak{k}') = \mathfrak{k}'$, $\mathfrak{k}'^C \cap \mathrm{ad}\,(c)^{-1}\mathfrak{g}^0$ is a real form of $\mathfrak{k}'^C$, $\mathfrak{n}_2^\mp$ is a real form of $\mathfrak{q}_2^\mp + \mathfrak{p}_2^\mp$, and $\mathfrak{n}_1^\mp$ is a real from of $\mathfrak{p}_1^\mp$.

By construction $\mathfrak{k}'^*$ is a real form of $\mathfrak{k}'^C$, so we need only check $\mathfrak{k}'^* \subset \operatorname{ad}(c)^{-1}\mathfrak{g}^0$. We have $\mathfrak{k}'^* = \mathfrak{l} + i\mathfrak{q}_1$. Lemma 4.7 gives

$$i\mathfrak{q}_1 \subset i \operatorname{ad}(c)^{-1}\mathfrak{p}_1 \subset \operatorname{ad}(c)^{-1}(i\mathfrak{p}) \subset \operatorname{ad}(c)^{-1}\mathfrak{g}^0 .$$

The analysis of the three-dimensional algebra $\mathfrak{m}$ in the proof of Proposition 4.4 shows that c and c^2 have the same centralizer in $\mathfrak{g}^C$; thus triviality of τ on $\mathfrak{l}$ implies triviality of $\operatorname{ad}(c)^{-1}$ on $\mathfrak{l}$; now $\mathfrak{l} = \operatorname{ad}(c)^{-1}\mathfrak{l} \subset \operatorname{ad}(c)^{-1}\mathfrak{k} \subset \operatorname{ad}(c)^{-1}\mathfrak{g}^0$, q.e.d.

5.4. Lemma. $[\mathfrak{q}_2^\pm, \mathfrak{p}_2^\mp] = 0$, $[\mathfrak{q}_2^\pm, \mathfrak{p}_2^\pm] \subset \mathfrak{p}_1^\pm$, $\mathfrak{r}^\pm$ *is a complex subalgebra of* $\mathfrak{g}^C$ *which is nilpotent of degree* 2 *and spanned by positive* (*or negative*) *root spaces*, $\mathfrak{n}^\pm = \mathfrak{r}^\pm \cap \operatorname{ad}(c)^{-1}\mathfrak{g}^0$, *and* $\mathfrak{n}^\pm$ *is a real form of* $\mathfrak{r}^\pm$.

Proof. $[\mathfrak{q}_2^C, \mathfrak{p}_2^\pm] \subset \mathfrak{p}_1^\pm$ by evaluation of roots on Z^0, proving the second statement and $[\mathfrak{q}_2^\pm, \mathfrak{p}_2^\mp] \subset \mathfrak{p}_1^\mp$. Applying τ to the latter, $[\mathfrak{p}_2^\mp, \mathfrak{q}_2^\pm] \subset \mathfrak{p}_1^\pm$; the first statement follows.

$\mathfrak{r}^\pm = \mathfrak{q}_2^\pm + \mathfrak{p}^\pm$, $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0$ and $\tau\mathfrak{q}_2^\pm \subset \mathfrak{p}^\mp$; also, $[\mathfrak{q}_2^\pm, \mathfrak{p}_1^\pm] = 0$ by applying $\operatorname{ad}(c)$ to Lemma 5.3; this gives $[\mathfrak{r}^\pm, \mathfrak{r}^\pm] \subset \mathfrak{p}_1^\pm \subset \mathfrak{r}^\pm$ and $[\mathfrak{r}^\pm, \mathfrak{p}_1^\pm] = 0$, proving $\mathfrak{r}^\pm$ to be a subalgebra which is nilpotent of degree 2. $\mathfrak{r}^\pm$ is complex and spanned by positive (or negative) root spaces, by construction.

For the last statement we use that $\mathfrak{n}^\pm = \mathfrak{n}_1^\pm + \mathfrak{n}_2^\pm$, that $\mathfrak{n}_1^\pm$ is a real form of $\mathfrak{p}_1^\pm$, and that $\mathfrak{n}_2^\pm$ is a real form of $\mathfrak{q}_2^\pm + \mathfrak{p}_2^\pm$, q.e.d.

5.5. Proposition. $\mathfrak{b} = \operatorname{ad}(c)^{-1}\mathfrak{k}'^* + \operatorname{ad}(c)^{-1}\mathfrak{n}^-$, *is the sum of the non-positive eigenspaces of* $\operatorname{ad}(-Y^0)$ *on* $\mathfrak{g}^0$, *and is the normalizer of* $\operatorname{ad}(c)^{-1}\mathfrak{n}^-$ *in* $\mathfrak{g}^0$; $\operatorname{ad}(c)\mathfrak{b} = \mathfrak{k}'^* + \mathfrak{n}^-$, *and is the normalizer of* $\mathfrak{n}^-$ *in* $\operatorname{ad}(c)\mathfrak{g}^0$.

Proof. The isotropy subgroup of G^C at $c(x)$ is $\operatorname{ad}(c)(K^C P^+)$; thus $\mathfrak{b} = (\operatorname{ad}(c)\mathfrak{k}^C + \operatorname{ad}(c)\mathfrak{p}^+) \cap \mathfrak{g}^0$. According to Lemma 5.3, using $\tau(\mathfrak{k}'^*) = \mathfrak{k}'^*$,

$$\operatorname{ad}(c)\mathfrak{k}'^* = \operatorname{ad}(c)^{-1}\mathfrak{k}'^* \subset \mathfrak{g}^0 .$$

Also,

$$\begin{aligned}\operatorname{ad}(c)^{-1}\mathfrak{n}^- &= \operatorname{ad}(c)^{-1}\mathfrak{n}_2^- + \operatorname{ad}(c)^{-1}\mathfrak{n}_1^- \subset \operatorname{ad}(c)^{-1}(\mathfrak{p}_2^C + \mathfrak{q}_2^-) \cap \mathfrak{g}^0 \\ &\quad + \operatorname{ad}(c)^{-1}\mathfrak{p}_1^- \cap \mathfrak{g}^0 \subset \operatorname{ad}(c)^{-1}\big(\tau(\mathfrak{q}_2^C + \mathfrak{p}_2^+) + \tau\mathfrak{p}_1^+\big) \cap \mathfrak{g}^0 \\ &= \operatorname{ad}(c)(\mathfrak{q}_2^C + \mathfrak{p}_2^+ + \mathfrak{p}_1^+) \cap \mathfrak{g}^0 \subset \mathfrak{b} .\end{aligned}$$

On the other hand, $\dim \mathfrak{b} = \dim \mathfrak{g}^0 - \dim \check{S} = \dim \mathfrak{g}^0 - (\dim \mathfrak{p}_2 + (1/2)\dim \mathfrak{p}_1) = \dim \mathfrak{k} + (1/2)\dim \mathfrak{p}_1 = \dim \mathfrak{k}' + \dim \mathfrak{q}_2 + (1/2)\dim \mathfrak{p}_1 = \dim \mathfrak{k}'^* + \dim \mathfrak{n}$. This proves $\mathfrak{b} = \operatorname{ad}(c)^{-1}\mathfrak{k}'^* + \operatorname{ad}(c)^{-1}\mathfrak{n}^-$, and $\operatorname{ad}(c)\mathfrak{b} = \mathfrak{k}'^* + \mathfrak{n}^-$ follows.

Let $\mathfrak{f}$ be the normalizer of $\operatorname{ad}(c)^{-1}\mathfrak{n}^-$ in $\mathfrak{g}^0$. By Lemma 5.3, $\mathfrak{b}$ is the sum of the non-positive eigenspaces of $\operatorname{ad}(-Y^0)$ on $\mathfrak{g}^0$, and $\operatorname{ad}(c)^{-1}\mathfrak{n}^-$ is the sum of the negative eigenspaces; thus

$$-Y^0 \in \mathfrak{b} \subset \mathfrak{f} ,$$

and so $\mathrm{ad}\,(-Y^0)\mathfrak{f} \subset \mathfrak{f}$. Now $\mathfrak{f} \neq \mathfrak{b}$ would imply the existence of an element $0 \neq F \in \mathfrak{f}$ with F contained in an eigenspace $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_1^+$ or $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_2^+$ of $\mathrm{ad}\,(-Y^0)$, and then $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_1^-$ or $\mathrm{ad}\,(c)^{-1}\mathfrak{n}_2^-$ would have a corresponding element F' with $[F, F'] \neq 0$. This would give

$$0 \neq [F, F'] \in \mathrm{ad}\,(c)^{-1}\mathfrak{k}'^* \cap \mathrm{ad}\,(c)^{-1}\mathfrak{n}^-$$

by addition of eigenvalues $F \in \mathfrak{f}$ and $F' \in \mathrm{ad}\,(c)^{-1}\mathfrak{n}^-$, which is impossible. We have proved that $\mathfrak{b}$ is the normalizer of $\mathrm{ad}\,(c)^{-1}\mathfrak{n}^-$ in $\mathfrak{g}^0$, and it follows that $\mathrm{ad}\,(c)\mathfrak{b}$ is the normalizer of $\mathfrak{n}^-$ in $\mathrm{ad}\,(c)\mathfrak{g}^0$, q.e.d.

5.6. The local structure of B is given by Proposition 5.5. The following definitions will be convenient in describing the global structure:

$$P_1^\pm = \exp(\mathfrak{p}_1^\pm) \subset G^C \quad \text{and} \quad N_1^\pm = P_1^\pm \cap \mathrm{ad}\,(c)G^0$$
$$R^\pm = \exp(\mathfrak{r}^\pm) \subset G^C \quad \text{and} \quad N^\pm = R^\pm \cap \mathrm{ad}\,(c)G^0 .$$

Lemma 5.4 shows that every $\mathrm{ad}\,(E)$, $E \in \mathfrak{r}^\pm$, is a nilpotent transformation of $\mathfrak{g}^C$. It follows that $\exp : \mathfrak{r}^\pm \to R^\pm$ is one-one onto, and that $R^\pm$ is a unipotent subgroup of G^C; in particular, $R^\pm$ is connected, simply connected and nilpotent. The same follows for $\exp : \mathfrak{p}_1^\pm \to P_1^\pm$ and for $P_1^\pm$.

5.7. LEMMA. *$N^\pm$ and $N_1^\pm$ are analytic subgroups of* $\mathrm{ad}\,(c)G^0$ *with Lie algebras $\mathfrak{n}^\pm$ and $\mathfrak{n}_1^\pm$.*

PROOF. Let η be the automorphism of G^C induced by the conjugation of $\mathfrak{g}^C$ over $\mathrm{ad}\,(c)\mathfrak{g}^0$. That conjugation preserves $\mathfrak{r}^\pm = \mathfrak{n}^\pm + i\mathfrak{n}^\pm$ because $\mathfrak{n}^\pm \subset \mathrm{ad}\,(c)\mathfrak{g}^0$, so $\eta(R^\pm) = R^\pm$. $N^\pm$ is the fixed point set of η in $R^\pm$ because $\mathrm{ad}\,(c)G^0$ is the fixed point set of η. As $R^\pm$ is a connected, simply connected nilpotent Lie group, it follows that $N^\pm = \exp(\mathfrak{n}^\pm)$. Similarly, $N_1^\pm = \exp(\mathfrak{n}_1^\pm)$, q.e.d.

5.8. Next we give an explicit description of B, the isotropy subgroup of G^0 at $c(x)$. B and its identity component B_0 have the important property of being transitive on M. In fact, $G^0 = BK$, for $G^0 = KB$ because K is transitive on $\check{S} = G^0/K$. Also $M = B/L$ since $L = B \cap K$, by definition of L; as M is connected and acyclic this shows that $B = L \cdot B_0$ and that L is a maximal compact subgroup of B.

5.9. THEOREM. *B is the normalizer of* $\mathrm{ad}\,(c)^{-1}N^-$ *in G^0, $B = L \cdot B_0$, $B_0 =$* $\mathrm{ad}\,(c)^{-1}K'^* \cdot \mathrm{ad}\,(c)^{-1}N^-$ *Chevalley semi-direct product decomposition into reductive and unipotent parts, and L is the centralizer of X^0 in K.*

PROOF. B and $\mathrm{ad}\,(c)^{-1}K'^* \cdot \mathrm{ad}\,(c)^{-1}N^-$ have the same Lie algebra by Proposition 5.5. As the latter is connected by Lemma 5.7, and because K'^* is the analytic group for $\mathfrak{k}'^*$, it must be B_0.

Let $\widetilde{B}$ be the normalizer of $\mathrm{ad}\,(c)^{-1}N^-$ in G^0, and define $\widetilde{L}$ to be the centralizer of X^0 in K. $B \subset \widetilde{B}$, because $\mathrm{ad}\,(c)^{-1}\mathfrak{n}^-$ is the nilradical of $\mathfrak{b}$. $\widetilde{L} \subset L$

because, given $k \in \tilde{L}$, $\operatorname{ad}(c)k = k$ and so k is contained in the isotropy subgroup $\operatorname{ad}(c)K$ of G at $c(x)$. $\tilde{B}_0 = B_0$ because $\mathfrak{b}$ is the normalizer of $\operatorname{ad}(c)^{-1}\mathfrak{n}^-$ in $\mathfrak{g}^0$, $\tilde{L}_0 = L_0$ because $\mathfrak{l}$ is the centralizer of $-Y^0$ (and thus of $X^0 = [Y^0, Z]$) in $\mathfrak{k}$, and $\tilde{L} \subset L \subset B \subset \tilde{B}$. Thus we need only prove that $\tilde{L}$ meets every component of $\tilde{B}$.

Let $g \in \tilde{B}$. $\operatorname{ad}(g)$ preserves $\mathfrak{b}$ and $\tilde{B}$ is a linear algebraic group by definition, so there exists $g_1 \in g\tilde{B}_0$ such that $\operatorname{ad}(g_1)$ preserves the reductive part $\operatorname{ad}(c)^{-1}\mathfrak{k}'^*$ of $\mathfrak{b}$ and $g_1 \in K$. Now there exists $g_2 \in K \cap g\tilde{B}_0$ such that $\operatorname{ad}(g_2)$ preserves both $\operatorname{ad}(c)^{-1}\mathfrak{k}'^*$ and the Cartan subalgebra $J\mathfrak{a}^0$ of the symmetric pair $(\operatorname{ad}(c)^{-1}\mathfrak{k}'^*, \mathfrak{l})$. As in Theorem 3.6, $\operatorname{ad}(g_2)$ acts by a signed permutation on the Y^0_α, $\operatorname{ad}(g_2)^{-1}Y^0 = \sum a_\alpha Y^0_\alpha$ with $a_\alpha = \pm 1$. We have $W_\alpha \in \mathfrak{g}^0_\alpha$ with $[Y^0_\alpha, W_\alpha] = +W_\alpha$. $W_\alpha \in \operatorname{ad}(c)^{-1}\mathfrak{n}^-$ because $\operatorname{ad}(-Y^0)W_\alpha = -W_\alpha$, so $\operatorname{ad}(g_2)W_\alpha \in \operatorname{ad}(c)^{-1}\mathfrak{n}^-$ because $g_2 \in \tilde{B}$.

$$\operatorname{ad}(-Y^0)\cdot(\operatorname{ad}(g_2)W_\alpha) = \operatorname{ad}(g_2)\{\operatorname{ad}(-\textstyle\sum a_\beta Y^0_\beta)\cdot W_\alpha\} = -a_\alpha \operatorname{ad}(g_2)W_\alpha\ .$$

Thus $a_\alpha = +1$ and $\operatorname{ad}(g_2)Y^0 = Y^0$. Now $\operatorname{ad}(g_2)X^0 = X^0$ so $g_2 \in \tilde{L} \cap g\tilde{B}_0$, q.e.d.

REMARK. Theorem 5.9 shows that B is a parabolic subgroup of G^0. This means that $\mathfrak{b}$ is the sum of the non-positive *restricted root* spaces for the Lie algebra of a split algebraic torus in G^0 (over the reals), and B is the largest subgroup of G^0 with Lie algebra $\mathfrak{b}$. As B is parabolic in G^0, one can prove, using a rationality theorem of Borel, that $S = G^0/B$ is a real projective variety defined over the rational number field.

6. The Cayley transform of D

In this section we shall construct the Cayley transform of the domain D embedded in $\mathfrak{p}^-$. It will turn out that this is always a tube domain over a homogeneous self-dual cone or a Siegel domain of type II.

The space M is embedded in M^* as $G^0(x)$; the map $\xi : \mathfrak{p}^- \to M^*$ defined by $\xi(E) = \exp(E)\cdot K^{\mathbf{C}}P^+ \in G^{\mathbf{C}}/K^{\mathbf{C}}P^+ = M^*$ is a holomorphic homeomorphism onto an open subset containing $G^0(x)$. M is embedded in $\mathfrak{p}^- = \mathfrak{p}^-_1 + \mathfrak{p}^-_2$ as the bounded domain $D = \xi^{-1}(G^0(x))$. We shall see that $\xi^{-1}c\xi$ is a holomorphic homeomorphism of D onto a domain $D^c \subset \mathfrak{p}^-$, which will be called the Cayley transform of D.

6.1. We denote by ν and ν^0 the involutions of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$ and $\mathfrak{g}^0$, respectively. We have $\nu^0 = \nu\sigma = \sigma\nu$, where σ denotes the symmetry. It is easy to see that ν maps $\mathfrak{p}^+_2$ and $\mathfrak{p}^-_2$ onto each other. Let $\langle\ ,\ \rangle$ denote the Killing form; then $\langle U, V\rangle_\nu = -\langle U, \nu V\rangle$ defines a positive definite hermitian form on $\mathfrak{g}^{\mathbf{C}}$. If f is a linear transformation on $\mathfrak{g}^{\mathbf{C}}$, we denote by f^* its adjoint with respect to this hermitian form. A trivial computation shows that $\mathrm{ad}(V)^* = -\mathrm{ad}(\nu V)$ for any $V \in \mathfrak{g}^C$. It also follows that $\mathrm{ad}(K)$ acts on $\mathfrak{p}^-$ by unitary transformations with respect to $\langle\ ,\ \rangle_\nu$.

By Lemma 5.3, $\mathfrak{n}_1^-$ is a real form of the complex vector space $\mathfrak{p}_1^-$. An element of $\mathfrak{p}_1^-$ will be called *real* (resp. *imaginary*) if it is contained in $\mathfrak{n}_1^-$ (resp. $i\mathfrak{n}_1^-$). A complex-linear transformation of $\mathfrak{p}_1^-$ will be called *real* if it preserves $\mathfrak{n}_1^-$. The restriction of $\langle\ ,\ \rangle_\nu$ to $\mathfrak{n}_1^-$ is a real positive definite bilinear form.

Given any element $E \in \mathfrak{p}^-$ we denote by E_1, E_2 its projections onto $\mathfrak{p}_1^-$ and $\mathfrak{p}_2^-$, respectively. So $E = E_1 + E_2$, $E_1 \in \mathfrak{p}_1^-$, $E_2 \in \mathfrak{p}_2^-$.

In this section we denote the zero element of $\mathfrak{p}^-$, which is the base point of D, by $\mathfrak{o}$, and its image under $\xi^{-1}c\xi$ by $\mathfrak{o}^c$. By Lemma 4.2 we have

$$\mathfrak{o}^c = \xi^{-1}\big(c(x)\big) = i \textstyle\sum_{\alpha\in\Delta} E_{-\alpha}\ .$$

6.2. PROPOSITION. *We define $\mathfrak{c}$ by $i\mathfrak{c} = K_1^*(\mathfrak{o}^c)$. Then $\mathfrak{c}$ is a cone in $\mathfrak{n}_1^-$, self-dual with respect to the restriction to $\mathfrak{n}_1^-$ of the positive definite form $\langle\ ,\ \rangle_\nu$.*

PROOF. First we note that $\mathfrak{o}^c \in i\mathfrak{n}_1^-$. In fact,

$$-\, i\mathfrak{o}^c = \mathrm{ad}\,(c) \textstyle\sum_{\alpha\in\Delta} E_{-\alpha} = \dfrac{1}{2} \textstyle\sum_{\alpha\in\Delta} (X_\alpha^0 - iH_\alpha) \in \mathfrak{g}^0\ ,$$

as one verifies readily by a computation in the three-dimensional simple Lie algebra. Hence $-i\mathfrak{o}^c \in \mathfrak{p}_1^- \cap \mathrm{ad}\,(c)\mathfrak{g}^0 = \mathfrak{n}_1^-$.

$\mathrm{ad}\,(K_1^*)$ acts on $\mathfrak{p}_1^-$ by real linear transformations, since K_1^* normalizes N_1^-. It follows that $i\mathfrak{c} \subset i\mathfrak{n}_1^-$, $\mathfrak{c} \subset \mathfrak{n}_1^-$. The isotropy group of K_1^* at $\mathfrak{o}^c$ is $K_1^* \cap B = L_1$; thus $i\mathfrak{c} = K_1^*/L_1$. This implies that $\dim \mathfrak{c} = \dim \mathfrak{k}_1^* - \dim \mathfrak{l}_1 = \dim \mathfrak{q}_1 = \dim \mathfrak{n}_1^-$; hence $\mathfrak{c}$ is open in $\mathfrak{n}_1^-$.

By Theorem 4.9 (3), K_1/L_1 is a compact symmetric space; now $i\mathfrak{c}$ (or $\mathfrak{c}$) is its non-compact dual. With respect to the restriction of $\langle\ ,\ \rangle_\nu$ to $\mathfrak{n}_1^-$, $\mathrm{ad}\,(\mathfrak{l}_1)$ acts by skew-symmetric, $\mathrm{ad}\,(i\mathfrak{q}_1)$ by symmetric, real linear transformations. We have $Z^0 \in \mathfrak{q}_1$, and $\mathrm{ad}\,(Z^0)E = iE$ for all $E \in \mathfrak{p}_1^-$. It follows that

$$\{\mathrm{ad}\,(\exp\,(itZ^0)) : t \text{ real}\} \subset \mathrm{ad}\,(K_1^*)$$

acts on $\mathfrak{p}_1^-$ as the group of positive real multiplications; thus $\mathfrak{c}$ is a cone,

$\mathfrak{c} = -i\,\mathrm{ad}(K_1^*)(\mathfrak{o}^c)$ is just $-i\exp(\mathrm{ad}\,(i\mathfrak{q}_1))(\mathfrak{o}^c)$, and for every $Q \in i\mathfrak{q}_1$, we have $\langle -i\exp(\mathrm{ad}(Q))(\mathfrak{o}^c),\ -i\mathfrak{o}^c\rangle_\nu > 0$ since $\mathrm{ad}\,(Q)$ is symmetric. Thus $\langle y,\ -i\mathfrak{o}^c\rangle_\nu > 0$ for all $y \in \mathfrak{c}$. This means that $-i\mathfrak{o}^c$ is in the dual cone $\mathfrak{c}^*$ of $\mathfrak{c}$; in particular, $\mathfrak{c}^*$ is non-empty, and $\mathfrak{c}$ contains no entire straight line.

Now Vinberg's argument [10, Theorem 3] can be applied, and shows that $\mathfrak{n}_1^-$ can be given the structure of a formally real Jordan algebra in such a way that $\mathfrak{c}$ coincides with the interior of the set of squares. Again by Vinberg's result, this shows that $\mathfrak{c}$ is a self-dual cone, q. e. d.

REMARK 1. Using the results of Vinberg [10] or Koecher [5], it is easy to see that the Jordan algebra structure on $\mathfrak{n}_1^-$ is given as follows. For any $U \in \mathfrak{n}_1^-$ there is a unique $Q \in i\mathfrak{q}_1$ such that $U = \mathrm{ad}\,(Q)(-i\mathfrak{o}^c)$. Given another element $V = \mathrm{ad}\,(R)(-i\mathfrak{o}^c) \in \mathfrak{n}_1^-$ ($R \in i\mathfrak{q}_1$) the Jordan product is defined as $U_{\mathrm{j}} V = \mathrm{ad}\,(Q)\,\mathrm{ad}\,(R)(-i\mathfrak{o}^c)$.

REMARK 2. In the case where $\mathfrak{g}_1 = \mathfrak{g}$, we have $i\mathfrak{c} = K^*/L$. Thus $i\mathfrak{c}$ (or $\mathfrak{c}$) is just the non-compact dual of the Bergman-Šilov boundary $\check{S}$, which, by Theorem 4.9 (4), is a symmetric space in this case.

6.3. LEMMA. *For all* $U \in \mathfrak{p}_2^+$, *we have* $\tau(U) = -[U, \mathfrak{o}^c]$.

PROOF. First we show that, restricted to $\mathfrak{p}_2^C + \mathfrak{q}_2^C$, τ and $\mathrm{ad}\,(X)$ coincide. By Lemma 5.3, $\mathfrak{p}_2^C + \mathfrak{q}_2^C$ is the sum of the (± 1)-eigenspaces of $\mathrm{ad}\,(-Y^0)$ on $\mathfrak{g}^C$. We have $X = iX^0 = i\,\mathrm{ad}\,(Z^0)(-Y^0)$, and $\mathfrak{p}_2^C + \mathfrak{q}_2^C$ is invariant under $\mathrm{ad}\,(Z^0)$. It follows that $\mathfrak{p}_2^C + \mathfrak{q}_2^C$ is the sum of the $(\pm i)$-eigenspaces of $\mathrm{ad}\,(X)$. Now, if $\mathrm{ad}\,(X)\,U = \pm\, iU$, then $\tau(U) = \exp\{(\pi/2)\,\mathrm{ad}\,(X)\}\,U = e^{\pm i\pi/2}U = \pm\, iU$, proving the assertion.

To finish the proof of the Lemma, let $U \in \mathfrak{p}_2^+$. Then we have

$$\tau(U) = -[U, X] = -[U, i\textstyle\sum_{\alpha\in\Delta} E_\alpha] - [U, i\textstyle\sum_{\alpha\in\Delta} E_{-\alpha}] = -[U, \mathfrak{o}^c]\,,$$

since $[U, E_\alpha] = 0$ for all $\alpha \in \Delta$, $\mathfrak{p}^+$ being abelian, q.e.d.

6.4. LEMMA. *The function* $\Phi : \mathfrak{p}_2^- \times \mathfrak{p}_2^- \to \mathfrak{p}_1^-$ *defined by*

$$\Phi(U, V) = -\frac{i}{2}\,\mathrm{ad}\,(U)\,\mathrm{ad}\,(V)^*\mathfrak{o}^c = -\frac{i}{2}\,[U, \tau\nu(V)]$$

has the following properties:

(i) Φ *is complex linear in its first argument,*

(ii) Φ *is hermitian with respect to the real form* $\mathfrak{n}_1^-$ *of* $\mathfrak{p}_1^-$, i. e., *if* μ *denotes conjugation in* $\mathfrak{p}_1^-$ *with respect to* $\mathfrak{n}_1^-$, *then*

$$\Phi(U, V) = \mu\Phi(V, U)\,,$$

(iii) *for any* $k \in K'^*$,

$$\Phi(\mathrm{ad}\,(k)\,U, \mathrm{ad}\,(k)\,V) = \mathrm{ad}\,(k)\,\Phi(U, V)\,,$$

(iv) *for any* $U \in \mathfrak{p}_2^-$

$$\Phi(U, U) \in \bar{\mathfrak{c}},$$

(v) $\Phi(U, U) = 0$ *only if* $U = 0$.

PROOF. First we show that the two forms of the definition of $\Phi(U, V)$ are equal. In fact, since ν maps $\mathfrak{p}_2^-$ onto $\mathfrak{p}_2^+$, we have, by Lemma 6.3,

$$\tau\nu(V) = -[\nu(V), \mathfrak{o}^c] = -\mathrm{ad}\,(\nu V)\,\mathfrak{o}^c = \mathrm{ad}\,(V)^*\mathfrak{o}^c$$

for all $V \in \mathfrak{p}_2^-$. Hence $-(i/2)[U, \tau\nu(V)] = -(i/2)\,\mathrm{ad}\,(U)\,\mathrm{ad}\,(V)^*\mathfrak{o}^c$, as asserted. It is also clear that $\Phi(U, V) \in \mathfrak{p}_1^-$ for all $U, V \in \mathfrak{p}_2^-$.

Property (i) is trivial. To prove (ii), it suffices to show that $\Phi(U, U) \in \mathfrak{n}_1^-$ for all $U \in \mathfrak{p}_2^-$. Since $\mathfrak{n}_1^- = \mathfrak{p}_1^- \cap \mathrm{ad}\,(c)\mathfrak{g}^0$, it suffices to show that the involution of $\mathfrak{g}^C$ with respect to $\mathrm{ad}\,(c)\mathfrak{g}^0$ keeps $\Phi(U, U)$ fixed. This involution is

$$\mathrm{ad}\,(c)\nu^0\,\mathrm{ad}\,(c)^{-1} = \mathrm{ad}\,(c)\sigma\nu\,\mathrm{ad}\,(c^{-1}) = \sigma\,\mathrm{ad}\,(c^{-1})\nu\,\mathrm{ad}\,(c^{-1}) = \sigma\nu\tau^{-1}\,.$$

Now

$$\sigma\nu\tau^{-1}(U) = -\sigma\nu\tau(U) = -\sigma\tau\nu(U) = -\tau\nu(U)\,,$$

and

$$\sigma\nu\tau^{-1}\big(\tau\nu(U)\big) = \sigma(U) = -U\,;$$

thus $\sigma\nu\tau^{-1}\Phi(U, U) = (i/2)[-\tau\nu(U), -U] = -(i/2)\,[U, \tau\nu(U)] = \Phi(U, U)$ which was to be shown.

To prove (iii), we note that τ and ν are both trivial on $\mathfrak{l}$ and equal to $-I$ on $i\mathfrak{q}_1$. Hence $\tau\nu$ is trivial on $\mathfrak{k}'^*$, and thus $\tau\nu$ commutes with every $\mathrm{ad}\,(k)$ $(k \in K'^*)$. It follows that

$$\begin{aligned}\Phi\big(\mathrm{ad}\,(k)U, \mathrm{ad}\,(k)V\big) &= -\frac{i}{2}\,[\mathrm{ad}\,(k)U, \tau\nu\,\mathrm{ad}\,(k)V] \\ &= -\frac{i}{2}\,\mathrm{ad}\,(k)[U, \tau\nu(V)] = \mathrm{ad}\,(k)\Phi(U, V)\,.\end{aligned}$$

To prove (iv), we note that $\mathrm{ad}\,(K)$ is unitary with respect to the form $\langle\ ,\ \rangle_\nu$; hence by Proposition 6.2, it is enough to show that, for all $U \in \mathfrak{p}_2^-$ and $V \in \mathfrak{c}$, $\langle\Phi(U, U), V\rangle_\nu \geqq 0$.

Given such U and V, by Proposition 6.2 there exists an element $k \in K_1^*$ such that $\mathrm{ad}\,(k)V = -i\mathfrak{o}^c$. Denoting $U' = \mathrm{ad}\,(k)U$, using (iii), and using that $\mathrm{ad}\,(k)$ is unitary, we have

$$\begin{aligned}\langle\Phi(U, U), V\rangle_\nu &= \langle\Phi(U', U'), -i\mathfrak{o}^c\rangle_\nu \\ &= -\left\langle \frac{i}{2}\,\mathrm{ad}\,(U')\,\mathrm{ad}\,(U')^*\mathfrak{o}^c, -i\mathfrak{o}^c\right\rangle_\nu \\ &= \frac{1}{2}\langle\mathrm{ad}\,(U')^*\mathfrak{o}^c, \mathrm{ad}\,(U')^*\mathfrak{o}^c\rangle_\nu \geqq 0\end{aligned}$$

by the positive definiteness of $\langle\ ,\ \rangle_\nu$.

To prove (v), suppose that $\Phi(U, U) = 0$. Then

$$\langle\Phi(U, U), \mathfrak{o}^c\rangle_\nu = \frac{1}{2}\langle\mathrm{ad}\,(U)^*\mathfrak{o}^c, \mathrm{ad}\,(U)^*\mathfrak{o}^c\rangle_\nu = 0\,,$$

whence $\mathrm{ad}\,(U)^*\mathfrak{o}^c = 0$. This means $\mathrm{ad}\,(\nu(U))\mathfrak{o}^c = 0$, i.e., $[\nu(U), \mathfrak{o}^c] = 0$. By definition of Y^0, and by the fact that $\nu(U) \in \mathfrak{p}_2^+$, it follows that $[\nu(U), Y^0] = 0$, i. e., $\mathrm{ad}(-Y^0)\nu(U) = 0$. By Lemma 5.3 and by $\nu(U) \in \mathfrak{p}_2^+$, it follows that $\nu(U) = 0$, and finally, since ν is a real linear isomorphism, $U = 0$, q.e.d.

6.5. LEMMA. *The map $I - \tau\nu$ is a real linear isomorphism of $\mathfrak{p}_2^-$ onto $\mathfrak{n}_2^-$; so every element of $\mathfrak{n}_2^-$ can uniquely be written as $V - \tau\nu(V)$, with $V \in \mathfrak{p}_2^-$.*

PROOF. If $V \in \mathfrak{p}_2^-$, then $\nu(V) \in \mathfrak{p}_2^+$ and $\tau\nu(V) \in \mathfrak{q}_2^-$. $\mathfrak{n}_2^- = (\mathfrak{p}_2^- + \mathfrak{q}_2^-) \cap \mathrm{ad}(c)\mathfrak{g}^0$ is a real form of $\mathfrak{p}_2^- + \mathfrak{q}_2^-$, so $\dim \mathfrak{n}_2^- = \dim \mathfrak{p}_2^- = \dim \mathfrak{q}_2^-$. It suffices to prove that $V - \tau\nu(V) \in \mathrm{ad}(c)\mathfrak{g}^0$ for all $V \in \mathfrak{p}_2^-$. The involution of $\mathfrak{g}^C$ with respect to $\mathrm{ad}(c)\mathfrak{g}^0$ was computed to be $\sigma\nu\tau^{-1}$ in the proof of Lemma 6.4, property (ii). A simple computation, similar to that in Lemma 6.4, shows now that $V - \tau\nu(V)$ is invariant under $\sigma\nu\tau^{-1}$, q.e.d.

6.6. PROPOSITION. *N^- acts on $\mathfrak{p}^-$ by*

$$g(E) = E + U + V + 2i\Phi(E_2, V) + i\Phi(V, V) ,$$

where $g = \exp(U + (I - \tau\nu)(V))$, $U \in \mathfrak{n}_1^-$, $V \in \mathfrak{p}_2^-$. K'^* acts on $\mathfrak{p}^-$ by the adjoint representation; it preserves $\mathfrak{p}_1^-$ and $\mathfrak{p}_2^-$. On $\mathfrak{p}_1^-$, K_1^* is real, L_2 is trivial. These actions are ξ-equivariant; in particular, $K'^* \cdot N^-$ preserves $\xi(\mathfrak{p}^-)$.

PROOF. First we note that the adjoint action of K^C on $\mathfrak{p}^-$ is ξ-equivariant because

$$\begin{aligned}\xi\big(\mathrm{ad}(k)E\big) &= \exp\big(\mathrm{ad}(k)E\big)K^C P^+ = k \cdot \exp(E) k^{-1} K^C P^+ \\ &= k \cdot \exp(E) K^C P^+ = k\big(\xi(E)\big) .\end{aligned}$$

K'^* commutes with τ^2, hence $\mathrm{ad}(K'^*)$ preserves the eigenspaces $\mathfrak{p}_1^-, \mathfrak{p}_2^-$ of τ^2 in $\mathfrak{p}^-$. $\mathrm{ad}(K_1^*)$ is real on $\mathfrak{p}_1^-$, and L_2 is trivial on $\mathfrak{p}_1^-$, since $\mathfrak{l}_2$ centralizes $\mathfrak{p}_1$. By Lemma 6.5, every element $g \in N^-$ is of the given form. Applying the Campbell-Hausdorff formula, we have

$$g = \exp\big(U + V - \tau\nu(V)\big) \exp(U) \cdot \exp\left(\frac{1}{2}[V, \tau\nu(V)]\right) \cdot \exp(V) \cdot \exp\big(-\tau\nu(V)\big) ,$$

since by Lemma 5.4 all other brackets vanish. We have $-\tau\nu(V) \in \mathfrak{q}_2^- \subset \mathfrak{k}^C$, so $\exp(-\tau\nu(V))$ acts on $\mathfrak{p}^-$ by the adjoint representation;

$$\exp\big(-\tau\nu(V)\big)(E) = E - [\tau\nu(V), E] = E + [E_2, \tau\nu(V)] ,$$

since all other brackets vanish again by Lemma 5.4. The other three factors in the expression of g are exponentials of elements in $\mathfrak{p}^-$; hence they act on $\mathfrak{p}^-$ by translations. Using the definition of Φ, we obtain the desired formula for $g(E)$, q.e.d.

6.7. We define the the Cayley transform of $D = \xi^{-1}(G^0(x))$ by

$$D^c = \xi^{-1}\big(c(G^0(x))\big) \subset \mathfrak{p}^- .$$

This definition makes sense, since

$$c\big(G^0(x)\big) = c\big(B(x)\big) = \mathrm{ad}\,(c)B\cdot c(x) = K'^*\cdot N^-\cdot c(x) \subset \xi(\mathfrak{p}^-) ,$$

by Proposition 6.6 and by transitivity of B on M (§ 5.8). We also define $\check{S}^c = c(\check{S})$, the Cayley transform of the Bergman-Šilov boundary of M in M^*.

6.8. THEOREM. *The Cayley transform D^c is equal to the domain*

$$\{E : \mathrm{Im}\, E_1 - \Phi(E_2, E_2) \in \mathfrak{c}\} .$$

PROOF. By a simple computation, Proposition 6.6 shows that the domain $\{E : \mathrm{Im}\, E_1 - \Phi(E_2, E_2) \in \mathfrak{c}\}$ is transformed into itself by the groups K'^* and N^-. Hence it is transformed into itself by $\mathrm{ad}\,(c)B = K'^*\cdot N^-$. On the other hand, given any point $E = E_1 + E_2$ in this domain, it can successively be transformed by $\exp\,(E_2 - \tau\nu(E_2)) \in N^-$ into a point $E' = E_1' + 0$, then by $\exp\,(-\mathrm{Re}\, E_1') \in N^-$ into a point $E'' = E_1'' + 0$ such that $E_1'' \in i\mathfrak{c}$, and finally by an element $k \in K_1^*$ into the point $\mathfrak{o}^c$. This shows that our domain is just the orbit of $\mathfrak{o}^c$ under $\mathrm{ad}\,(c)\cdot B$, q.e.d.

REMARK 1. If $\mathfrak{g} = \mathfrak{g}_1$, then $\mathfrak{p}_2 = 0$ and $D^c = \{E : \mathrm{Im}\, E \in \mathfrak{c}\}$, the tube domain over $\mathfrak{c}$. Now Proposition 3.12 shows that M is of tube type if and only if conditions (i) through (v) of Theorem 4.9 hold.

REMARK 2. We have $D^c \cap \mathfrak{n}_1^- = \xi^{-1}(K'^*(c(x))) = \xi^{-1}(K_1^*(c(x))) = \mathfrak{c}$ and $D^c \cap \mathfrak{p}_1^- = \xi^{-1}(\mathrm{ad}\,(c)\cdot G_1^0\cdot(c(x)))$ equal to the tube domain over $\mathfrak{c}$, by Proposition 6.6 and Remark 1.

6.9. THEOREM. $\check{S}^c \cap \xi(\mathfrak{p}^-) = \xi\big(\{E : \mathrm{Im}\, E_1 - \Phi(E_2, E_2) = 0\}\big)$, *and this is a dense open subset of* $\check{S}^c$.

PROOF. We denote $T = \{E : \mathrm{Im}\, E_1 - \Phi(E_2, E_2) = 0\}$. By Proposition 6.6, it follows that $T = N^-(\mathfrak{o}) = K'^*\cdot N^-(\mathfrak{o})$, and that for any $E \in \mathfrak{p}^-$,

$$\dim N^-(E) = \dim T = \dim \check{S} .$$

We have $\dim K'^*N^-(E) \geqq \dim T$, with equality if and only if $E \in T$, since K'^* contains the positive multiples of the identity transformation on $\mathfrak{p}_1^-$. Now, $\check{S}^c$ and $\xi(\mathfrak{p}^-)$ are both stable under $\mathrm{ad}\,(c)B = K'^*\cdot N^-$, hence $\xi^{-1}(\check{S}^c \cap \xi(\mathfrak{p}^-))$ is a union of $\mathrm{ad}\,(c)B$-orbits. Each such orbit has dimension bounded by $\dim \check{S} = \dim T$, and thus must be equal to T by the above discussion. That proves that $\xi(T) = \check{S}^c \cap \xi(\mathfrak{p}^-)$, and that $\xi(T)$ is open in $\check{S}^c$.

Let $F = M^* - \xi(\mathfrak{p}^-)$, and define the real analytic map f of $K \times \mathfrak{a}$ onto M^* by $f(k, V) = (k\cdot\exp V)(x)$. Observe that

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = \begin{pmatrix}1 & 0\\ \tan t & 1\end{pmatrix}\begin{pmatrix}\cos t & 0\\ 0 & \dfrac{1}{\cos t}\end{pmatrix}\begin{pmatrix}1 & -\tan t\\ 0 & 1\end{pmatrix};$$

with strong orthogonality of Δ, this implies that $\exp \sum t_i X_i \in P^- K^{\mathbf{C}} P^+$ if and only if each $\tan t_i \neq 0$. Now $f^{-1}(\xi(\mathfrak{p}^-))$ is the complement of a proper real analytic subvariety of $K \times \mathfrak{a}$. Finiteness of the Weyl group and the exponential description of $\mathfrak{a} \to A$ now show that F is a proper real analytic subvariety of M^*.

$\check{S}^c$ is a real analytic submanifold of M^*. It follows that $F' = F \cap \check{S}^c$ is a real analytic subvariety of $\check{S}^c$. As $\check{S}^c$ has a non-empty open subset $\xi(T)$ disjoint from F', and as $\check{S}^c$ is connected, it follows that $\dim F' < \dim \check{S}^c$. Now $\xi(T)$ is a dense open subset of $\check{S}^c$, q.e.d.

Appendix

We prove that in the tube case the mapping p from the tube domain D^c to a bounded domain defined by Koecher [5, p. 118] is up to a factor i equal to the inverse of the Cayley transformation $\xi^{-1}c\xi$.

Let M be of tube type. Now $\mathfrak{n}^- = \mathfrak{n}_1^-$, $\mathfrak{p}^- = \mathfrak{p}_1^-$, $K = K' = K_1$, etc. In the Remark following Proposition 6.2, we described the Jordan algebra structure on $\mathfrak{n}^- = \operatorname{Re} \mathfrak{p}^-$. By complexification, this gives a Jordan structure on $\mathfrak{p}^-$. The unit of the Jordan algebra $\mathfrak{p}^-$ is $U = \sum_{\alpha \in \Delta} E_{-\alpha}$. Koecher's map $p \colon D^c \to \mathfrak{p}^-$ is defined by $p(E) = (E - iU)_{\mathfrak{j}}(E + iU)^{-1}$. We have the inverse Cayley transformation $q = \xi^{-1}c^{-1}\xi$ defined on D^c.

THEOREM. *$q(E) = ip(E)$ for all $E \in D^c$.*

PROOF. Let $\mathfrak{h}^* = i\mathfrak{h}^-$. This is a Cartan subalgebra of the symmetric pair $(\mathfrak{k}^*, \mathfrak{l})$, and we have $K^* = LH^*L$. So $i\mathfrak{c} = K^*(\mathfrak{o}^c) = LH^*L(\mathfrak{o}^c) = LH^*(\mathfrak{o}^c)$. $\mathfrak{h}^*$ is spanned by the $H_\alpha (\alpha \in \Delta)$; so $H^*(\mathfrak{o}^c)$ can be computed, using the strong orthogonality of Δ and $\operatorname{ad}(\exp tH_\alpha)E_{-\alpha} = e^{2t}E_{-\alpha}$. We have

$$H^*(\mathfrak{o}^c) = \{i \textstyle\sum_{\alpha \in \Delta} a_\alpha E_{-\alpha} \mid a_\alpha > 0\} .$$

Now let $E = i \sum_{\alpha \in \Delta} a_\alpha E_{-\alpha}$ be an element in $H^*(\mathfrak{o}^c)$. The $E_{-\alpha} (\alpha \in \Delta)$ are orthogonal idempotents in the Jordan algebra, since $E_{-\alpha} = \operatorname{ad}(-H_\alpha/2)U$, and thus $(E_{-\alpha})_{\mathfrak{j}}(E_{-\beta}) = \operatorname{ad}(-H_\alpha/2)E_{-\beta} = \delta_{\alpha\beta}E_{-\beta}$ (where $\delta_{\alpha\beta}$ is the Kronecker symbol). Since $U = \sum_{\alpha \in} E_{-\alpha}$, the computation of $p(E)$ can be performed in the direct sum of the Jordan subalgebras generated by the $E_{-\alpha}$, $\alpha \in \Delta$, each of which is isomorphic to $\mathbf{C}$. So we obtain

$$p(E) = \textstyle\sum_{\alpha \in \Delta} \dfrac{a_\alpha - 1}{a_\alpha + 1} E_{-\alpha} .$$

Now we compute $q(E)$. We have

$$\xi q(E) = c^{-1}\xi(E) = \exp\left(\textstyle\sum_{\alpha \in \Delta} X_\alpha\right) \exp\left(i \textstyle\sum_{\alpha \in \Delta} a_\alpha E_{-\alpha}\right) .$$

By strong orthogonality of Δ, we can calculate in the simple algebras $\mathfrak{g}_\alpha^{\mathbf{C}}$;

using the easily verified relation

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ia_\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\dfrac{a_\alpha - 1}{a_\alpha + 1} & 1 \end{pmatrix}\begin{pmatrix} \dfrac{a_\alpha + 1}{\sqrt{2}} & 0 \\ 0 & \dfrac{\sqrt{2}}{a_\alpha + 1} \end{pmatrix}\begin{pmatrix} 1 & \dfrac{-i}{a_\alpha + 1} \\ 0 & 1 \end{pmatrix},$$

we get

$$\xi q(E) = \exp\left(i \sum_{\alpha \in \Delta} \frac{a_\alpha - 1}{a_\alpha + 1} E_{-\alpha}\right) \cdot u$$

where $u \in K^C P^+$. It follows that

$$q(E) = i \sum_{\alpha \in \Delta} \frac{a_\alpha - 1}{a_\alpha + 1} E_{-\alpha} = ip(E) .$$

Now let $g \in L$. It is known [5, pp. 76–78] that $\mathrm{ad}\,(L)$ is just the automorphism group of the Jordan algebra $\mathfrak{p}^-$. Hence, since p is given purely in terms of the Jordan structure, $p(\mathrm{ad}\,(g)E) = \mathrm{ad}\,(g)p(E)$. Also we know that g centralizes c; hence $q(\mathrm{ad}\,(g)E) = \mathrm{ad}\,(g)q(E)$. Finally, as $\mathrm{ad}\,(L)$ acts by real linear transformations, $\mathrm{ad}\,(g)(iE) = i\,\mathrm{ad}\,(g)(E)$. It follows that

$$q\big(\mathrm{ad}\,(g)E\big) = ip\,\big(\mathrm{ad}\,(g)E\big)$$

for all $E \in H^*(\mathfrak{o}^c)$ and $g \in L$. This means that q and ip are equal on $LH^*(\mathfrak{o}^c) = ic$. But they both are holomorphic maps on $D^c = \mathrm{Re}\,\mathfrak{p}^- + ic$, and hence they have to be equal on the whole domain D^c, q.e.d.

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