SELF ADJOINT FUNCTION SPACES ON RIEMANNIAN SYMMETRIC MANIFOLDS

BY

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1. **Introduction and summary.** 1.1. K. de Leeuw and H. Mirkil [6] recently studied the closed rotation-invariant subalgebras A of the Banach algebra of continuous complex valued functions on the n-sphere, $n > 1$. Assuming that A contains the constants, they proved that there are only three possibilities: the constants, the whole algebra, and the algebra consisting of all functions f such that $f(x_1) = f(x_2)$ whenever x_1 is antipodal to x_2. Their proof depended on the high degree of transivity of the rotation group on the sphere, and this led I. Glicksberg to conjecture that their result should generalize to the compact two point homogeneous spaces (see [7] for these spaces; they are the spheres, the real, complex and quaternionic projective spaces, and the Cayley projective plane). Here their result is generalized to a much larger class of spaces, the class of all compact connected Riemannian symmetric spaces which are not locally isometric to a product in which one of the factors is a circle, a group manifold $\boldsymbol{SU}(n)$ $(n > 2)$, $\boldsymbol{SO}(4n+2)$ or $\boldsymbol{E}_6$, or a coset space $\boldsymbol{SU}(n)/\boldsymbol{SO}(n)$ $(n > 2)$, $\boldsymbol{SU}(2n)/\boldsymbol{Sp}(n)$ $(n > 2)$, $\boldsymbol{SO}(4n+2)/\boldsymbol{SO}(2n+1) \times \boldsymbol{SO}(2n+1)$, $\boldsymbol{E}_6/\boldsymbol{F}_4$, or $\boldsymbol{E}_6/(\boldsymbol{Sp}(4)/\{\pm I\})$.

I am indebted to I. Glicksberg both for the idea of working on this problem and for many valuable suggestions.

1.2. Let X be a compact connected Riemannian symmetric space. We study the action of the largest connected group G of isometries of X upon the Banach algebra $\boldsymbol{C}(X)$ of all continuous complex valued functions on X, looking for necessary and sufficient conditions on a closed G-invariant subspace V of $\boldsymbol{C}(X)$ that V be *self adjoint*[2], i.e., that V be closed under complex conjugation of functional values. If X is the sphere, then G is the rotation group, as in [6]. Some sufficiency conditions are obtained in §2 which ensure, for X in the class of symmetric spaces mentioned in §1.1, that every closed G-invariant subspace of $\boldsymbol{C}(X)$ is self adjoint. If X is not locally isometric to a product of lower dimensional spaces and the subspace is a subalgebra A properly containing the constants, then results of §7 (which do not depend on preceding results) show that it is $\boldsymbol{C}(Y)$ lifted to X where Y is a Riemannian symmetric space which admits a G-equivariant covering $X \to Y$; all possibilities are given in §7.4. The results of [6] are a special

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(2) We reserve the word *symmetric* for Riemannian manifolds, in order to avoid confusion.

case. Necessity conditions are much more difficult; tools are developed in §3, and the bulk of §§4–5 is devoted to the exibiting of a closed G-invariant subspace of $C(X)$ which is not self adjoint whenever X is not in the class mentioned in §1.1. These results are summarized as Theorem 6.1. §4 is of independent interest; some new results are proved relating the algebraic structure and the representation ring on a compact group.

1.3. We are forced to assume mild familiarity with Riemannian symmetric spaces and compact Lie groups, but recall the basic notions in order to establish notation.

Let X be a Riemannian manifold. The group of all isometries of X onto itself, endowed with the compact-open (uniform convergence on compact sets) topology, forms a Lie group. We will be concerned with the identity component G of that group. An isometry s of X of order two with $x \in X$ as isolated fixed point, is called the *symmetry* of X at x; if X is connected and s exists, then s is unique because it induces $-I$ (I = identity) on the tangentspace X_x. X is *symmetric* if it has a symmetry at every point. If X is symmetric and connected, then G is transitive on the points of X: by continuity it suffices to show that the group of all isometries is transitive, and this is done by joining two given points by a broken geodesic arc and applying the product of the symmetries at the midpoints of the smooth pieces.

Let X be connected and Riemannian symmetric. Then there is a Riemannian covering $\pi : \tilde{X} \to X$ (covering of Riemannian manifolds where the projection is a local isometry), $\tilde{X}$ simply connected and Riemannian symmetric. Then $\tilde{X}$ is a metric product $X_0 \times X_1 \times \cdots \times X_r$ where X_0 is a Euclidean space and the $X_i\,(i > 0)$ are irreducible (not Euclidean, and not locally a product). The $\pi(X_j)$ are the *local factors* of X. Now let X be compact. Then X_i is compact for $i > 0$. It follows that we can collapse X_0 and obtain a Riemannian covering $\pi' : X' \to X$ where $X' = X_0' \times X_1 \times \cdots \times X_r$ and X_0' is a flat torus.

Let X be a compact irreducible Riemannian symmetric space. Then there are only two possibilities. If G is not simple (in the sense of Lie groups), then X is itself a compact Lie group, and G consists of all transformations

$$(u, v) : x \to uxv^{-1}; \qquad u, v, x \in X.$$

Otherwise, G is simple.

The compact classical groups are the unitary groups $\boldsymbol{U}(n)$ in n complex variables, the special unitary groups $\boldsymbol{SU}(n)$ consisting of elements of determinant $+1$ in $\boldsymbol{U}(n)$, the orthogonal groups $\boldsymbol{O}(n)$ in n real variables, the special orthogonal (rotaticn) groups $\boldsymbol{SO}(n)$ consisting of elements of determinant $+1$ in $\boldsymbol{O}(n)$, the universal covering groups $\boldsymbol{Spin}(n)$ of $\boldsymbol{SO}(n)$, and the symplectic groups $\boldsymbol{Sp}(n)$ in n quaternionic variables, which are the quaternionic analogues of $\boldsymbol{U}(n)$. There are also the *compact simply connected* exceptional groups $\boldsymbol{G}_2$, $\boldsymbol{F}_4$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$ and $\boldsymbol{E}_8$ which are not easily described; only E_6 and E_7 have nontrivial centers, those centers being cyclic of respective orders 3 and 2, yielding quotient exceptional groups which are not simply connected.

We refer to Helgason [5] for details on Lie groups and symmetric spaces.

2. **Group theoretic criteria.** É. Cartan has shown [3, §16] that the symmetry to a compact Riemannian symmetric space sends every invariant function space to its adjoint. The following is a variation in which the identity transformation replaces the symmetry.

2.1. Theorem (joint work with I. Glicksberg). *Let X be a coset space G/K of a compact topological group by a closed subgroup. If $g^{-1} \in KgK$ for every $g \in G$, then every closed G-invariant subspace of $C(X)$ is self adjoint.*

Proof. Given $x \in X$ we have $g \in G$ with $x = g(w)$ where $w \in X$ is represented by the coset K. The hypothesis provides k and k' in K with $kg^{-1} = gk'$. Define $g_x = kg^{-1}$; now $g_x(w) = x$ and $g_x(x) = w$.

Let L be a closed G-invariant subspace of $C(X)$. As G is compact, L is the closed span of finite dimensional irreducible G-invariant subspaces, and L is self adjoint if these subspaces are. Thus we may assume that G acts irreducibly on L and that L has a basis $\{u_i\}$ on which G acts by unitary matrices. Evaluation at w being linear, we may also assume that $u_i(w) = \delta_{1i}$.

Let $x \in X$. $u_i \cdot g_x = \sum_j a_{ij} u_j$ with (a_{ij}) unitary; thus $u_i \cdot g_x^{-1} = \sum_j \overline{a_{ji}} u_j$. Now $a_{i1} = \sum_j a_{ij} u_j(w) = u_i(g_x(w)) = u_i(x) = u_i(g_x^{-1}(w)) = \sum_j \overline{a_{ji}} u_j(w) = \overline{a_{1i}}$. Thus $u_1(x)$ is real. Now u_1 is real valued, so $u_1 \in L \cap \bar{L}$, and $L = \bar{L}$ follows by irreducibility. Q.E.D.

2.2. *Cartan subalgebras and the Weyl group.* Let X be a compact connected Riemannian symmetric space; then X is a coset space G/K where G is the largest connected group of isometries and K is the isotropy subgroup of G at some point $x \in X$. The Lie algebra $\mathscr{G}$ of G is a vector space direct sum $\mathscr{K} + \mathscr{P}$ where the differential of the symmetry s at x is $+1$ on $\mathscr{K}$ and -1 on $\mathscr{P}$, and where $\mathscr{K}$ is the Lie algebra of K. $P = \exp(\mathscr{P})$ is the set of "transvections" through x, and every element $g \in G$ has expression $g = kp$ with $k \in K$ and $p \in P$. Every subalgebra of $\mathscr{G}$ contained in $\mathscr{P}$ is commutative, the maximal such subalgebras are called *Cartan subalgebras* of the symmetric pair $(\mathscr{G}, \mathscr{K})$, and any two Cartan subalgebras of $(\mathscr{G}, \mathscr{K})$ are conjugate by an element of K. Let $\mathscr{A}$ be a Cartan subalgebra of $(\mathscr{G}, \mathscr{K})$; we define $A = \exp(\mathscr{A})$, and it follows that every element of P is of the form kak^{-1} with $a \in A$ and $k \in K$; in particular, $G = KAK$. We refer to [5] for details on this and on the following.

Let $\mathscr{A}$ be a Cartan subalgebra of $(\mathscr{G}, \mathscr{K})$. Given $g \in G$, $\mathrm{ad}(g)$ will denote both the automorphism $h \to ghg^{-1}$ of G and the automorphism induced on $\mathscr{G}$. Consider the normalizer $N = \{k \in K : \mathrm{ad}(k)\mathscr{A} = \mathscr{A}\}$ of $\mathscr{A}$ in K. Conjugation by elements of N induces a finite group of linear transformations of $\mathscr{A}$; this group is called the *Weyl group* of X, and we will say that it "contains $-I$" if it contains the transformation $\theta \to -\theta$ of $\mathscr{A}$.

If X is a compact connected Lie group and T is a maximal toral subgroup, then the Weyl group of X (as Lie group) is the group of linear transformations of the Lie algebra $\mathscr{T}$ induced from conjugation by elements $x \in X$ with $\mathrm{ad}(x)\mathscr{T} = \mathscr{T}$. Endow X with a bi-invariant Riemannian metric; it becomes a Riemannian symmetric space where the symmetry at 1 is $x \to x^{-1}$ and the largest connected group G of isometries consists of the transformations

$$(u,v): x \to uxv^{-1}; \qquad u,\ v,\ x \in X.$$

Then $\mathscr{G} = \mathscr{X} \oplus \mathscr{X}, \mathscr{K}$ consists of all (θ,θ) with $\theta \in \mathscr{X}$, and $\mathscr{P}$ consists of the $(\theta, -\theta)$. Now $\theta \to (\theta, -\theta)$ maps $\mathscr{X}$ onto $\mathscr{P}$, mapping $\mathscr{T}$ isomorphically onto a Cartan subalgebra $\mathscr{A}$ of $(\mathscr{G}, \mathscr{K})$, and sending the Weyl group of X (as Lie group) isomorphically onto the Weyl group of X (as symmetic space). So there is no ambiguity in speaking of the Weyl group of X.

2.3. Theorem. *Let X be a compact connected Riemannian symmetric space. If the Weyl group of X contains $-I$, and if G denotes the largest connected group of isometries of X, then every G-invariant closed subspace of $C(X)$ is self adjoint.*

Proof. Let $\mathscr{A}$ be a Cartan subalgebra of $(\mathscr{G}, \mathscr{K})$, $A = \exp(\mathscr{A})$, and choose $k \in K$ inducing $-I$ in the Weyl group. Then $\mathrm{ad}(k)\theta = -\theta$ for $\theta \in \mathscr{A}$, so $\mathrm{ad}(k)a = a^{-1}$ for $a \in A$. Now let $g \in G$. As $G = KAK$ we have $g = k_1ak_2$ with $a \in A$ and $k_i \in K$. This gives

$$g^{-1} = k_2^{-1}a^{-1}k_1^{-1} \in Ka^{-1}K = Kk^{-1}a^{-1}kK = KaK = Kk_1ak_2K = KgK.$$

Now Theorem 2.1 shows that every closed G-invariant subspace of $C(X)$ is self adjoint. Q.E.D.

2.4. Remark. If K contains the symmetry s, then s induces the element $-I$ in the Weyl group of X, and we may apply Theorem 2.3. Here it is useful to know that *the following conditions are equivalent*: (i) *K contains the symmetry*, (ii) *the Lie groups G and K have the same rank* (= common dimension of maximal toral subgroups), (iii) *the Euler-Poincaré characteristic* $\chi(X) \neq 0$. In particular these conditions are satisfied if G admits no outer automorphism.

2.5. Remark. If X is a compact connected two point homogeneous space of dimension $\geqq 2$, then X is a Riemannian symmetric space whose Weyl group contains $-I$, so Theorem 2.3 applies.

2.6. Remark. The condition that the Weyl group contain $-I$ is preserved under formation of Riemannian products and passes down under Riemannian coverings. This is the main virtue of Theorem 2.3.

2.7. Proposition. *Let X be a compact connected hermitian symmetric or 2-point homogeneous space. If X is not a circle, and if G is the largest connected* *group of isometries of X, then every closed G-invariant subspace of $C(X)$ is self adjoint.*

This is immediate from Remarks 2.4 and 2.5.

3. **Representation theoretic criteria.** This section is based on the Peter-Weyl Theorem, a result of É. Cartan on induced representations, and

3.1. Lemma (I. Glicksberg). *Let H be a compact topological transformation group on a space Y, and let E be a self adjoint H-invariant finite dimensional subspace of $C(Y)$. Then E has a basis of real valued functions on which H acts by orthogonal matrices.*

Proof. As E is self adjoint, we have $E = E_R \otimes_R C = E_R + iE_R$ direct sum of real vector spaces where E_R consists of the real valued functions in E. H preserves each summand for, given $h \in H$ and $e \in E$, both e and $h(e)$ take the same set of values. E_R has a (real) basis $\{f_i\}$ on which H acts by orthogonal matrices; the lemma follows because $\{f_i\}$ is a (complex) basis of E. Q.E.D.

3.2. Recall that the *contragredient* of a linear representation π of a compact group H on V is the representation π^* induced on the dual space of V, and that π is called *self contragredient* if it is equivalent to π^*. If χ is the character of π, then the complex conjugate $\bar{\chi}$ is the character of π^*; thus π is self contragredient if and only if its character is real valued.

3.3. Theorem. *Let G be a compact topological group; let $X = G/K$ where K is a closed subgroup; let D be the set of equivalence classes of irreducible representations of G; given $\pi \in D$, let n_π denote the multiplicity of the trivial representation of K in the restriction $\pi|_K$. Then every closed G-invariant subspace of $C(X)$ is self adjoint if and only if $\pi \in D$ implies* (i) *$n_\pi \leqq 1$ and* (ii) *π is self contragredient if $n_\pi = 1$.*

3.4. Let V be an irreducible nonzero G-invariant subspace of $C(X)$, let π be the representation of G on V, and suppose that π satisfies conditions (i) and (ii) of the theorem; we must show that V is self adjoint. If $\beta \in D$, then β^* denotes the contragredient, W^β is the representation space, $E^\beta = W^\beta \otimes W^{\beta^*}$ is the space of matrix coefficients, and $(E^\beta)^K$ denotes the eigenspace of $+1$ for $1 \otimes \beta^*(K)$. $(E^\beta)^K$ consists of the functions in E^β invariant under right translation by elements of K, and we have Fourier-Peter-Weyl developments

$$(*) \qquad C(G) \sim \sum_{\beta \in D} E^\beta, \quad C(X) \sim \sum_{\beta \in D} (E^\beta)^K.$$

Thus conditions (i) and (ii) on π, together with the fact that $V \neq \{0\}$ implies $n_\pi \neq 0$ by (*), say that $V = (E^\pi)^K$ and π is self contragredient. Let V^* be the adjoint of V; then a glance at functional values show that G acts by π^* on V^*. As $\pi = \pi^*$ this implies $V^* \subset (E^\pi)^K = V$; thus V is self adjoint.

3.5. Let every closed G-invariant subspace of $C(X)$ be self adjoint and let $\pi \in D$; we must prove (i) and (ii). There is nothing to prove if $n_\pi = 0$, so we may assume $n_\pi > 0$. Then $U = (E^\pi)^K \neq 0$. U is self adjoint by hypothesis, and finite dimensional and G-invariant by construction; thus the representation α of G on U is self contragredient by Lemma 3.1. As $\alpha = n_\pi \cdot \pi$, π is self contragredient. This proves (ii).

Now suppose $n_\pi > 1$ and let E be the real vector space consisting of the real valued functions in E^π. Self contragredience of the representation π of G implies that the character χ of π is real valued, so $0 \neq \chi \in E$, and thus $E \neq \{0\}$. As E^π is an irreducible $\pi(G) \otimes \pi^*(G)$-invariant subspace of $C(G)$, this implies that E^π is self adjoint, so $E^\pi = E \otimes_R C$ by Lemma 3.1. Define $F = (E^\pi)^K \cap E$; as E is $\pi(G) \otimes \pi^*(G)$-invariant by construction, it follows that $U = (E^\pi)^K$ satisfies $U = F \otimes_R C$. Now $F = F_1 \oplus \cdots \oplus F_n$, $n = n_\pi$, where F_i is G-invariant and G acts on $U_i = F_i \otimes_R C$ by π. Let $\{f_1^i, \cdots, f_r^i\}$ be a basis of F_i over R (and thus of U_i over C) such that, given $g \in G$, g acts on the various U_i with the same matrix in these bases. Define $f = f_1^1 + i f_1^2$ and let W be the subspace of U spanned by the G- translates of f; W is self adjoint by hypothesis. Thus W contains $f_1^1 = \mathrm{Re}.f$ and $f_1^2 = \mathrm{Im}.f$, so $W = U_1 \oplus U_2$. But the action of G on W is π, thus irreducible, by construction of W. This contradicts $n_\pi > 1$.

Theorem 3.3 is proved. Q.E.D.

As we worked with only one representation at a time in the proof of Theorem 3.3, we have in fact proved

3.6. Corollary. *Let G be a compact topological group, let $X = G/K$ where K is a closed subgroup, and suppose for every irreducible representation π of G that the trivial representation of K has multiplicity at most* 1 *in $\pi|_K$. Let V be a closed G-invariant subspace of $C(X)$. Then V is self adjoint if and only if the representation of G on V is self contragredient.*

In order to apply Theorem 3.3 and Corollary 3.6 to symmetric spaces, we need the following lemma which is a mild extension of a result of É. Cartan [3, §17].

3.7. Lemma. *Let G be the largest connected group of isometries of a compact connected Riemannian symmetric space X. Let F be the representation of G on $C(X)$ and let f be an irreducible representation of G. Then the multiplicity of f in F is at most* 1.

Proof. Let K be the isotropy subgroup of G at $x \in X$, let s be the symmetry to X at x, and define

$$G' = G \cup s \cdot G \text{ and } K' = K \cup s \cdot K.$$

Then $X = G'/K'$. Let the Lie algebra $\mathscr{G} = \mathscr{K} + \mathscr{P}$ and define $P = \exp(\mathscr{P})$ as in §2.2. Then $G' = K' \cdot P$ because $G = K \cdot P$, and $sps^{-1} = p^{-1}$ for every $p \in P$. Let $g \in G$; $g = kp$ with $k \in K'$ and $p \in P$. Now

$$g^{-1} = p^{-1}k^{-1} \in K'p^{-1}K' = K'sp^{-1}s^{-1}K' = K'pK' = K'gK'.$$

Thus every closed G'-invariant subspace of $C(X)$ is self adjoint by Theorem 2.3. Let F' be the representation of G' on $C(X)$ and let f' be an irreducible representation of G'. Then Theorem 3.3 says that the multiplicity of f' in F' is at most 1.

Consider the Fourier-Peter-Weyl development $C(X) \sim \sum(E^{\beta})^K$ under G. If G is reducible on $(E^f)^K$, then we choose a subspace $V_1 \oplus V_2$ of $(E^f)^K$ where G acts irreducibly on each V_i. Define $W_i = V_i + s(V_i)$. Then G' acts on both W_i by the same irreducible representation, so $W_1 = W_2$. Now we may assume $V_2 = s(V_1)$, and it follows from Schur's Lemma that s commutes with every element of G. This implies $s \in G$, so $V_1 = V_2$ in contradiction to reducibility of G on $(E^f)^K$. Now G acts irreducibly on $(E^f)^K$. Our assertion follows. Q.E.D

3.8. THEOREM. *Let X be a compact connected Riemannian symmetric space. Let G be the largest connected group of isometries of X and let V be a closed G-invariant subspace of $C(X)$. Then V is self adjoint if and only if the representation of G on V is self contragredient.*

Proof. Let π be an irreducible representation of G. Comparing the Fourier-Peter-Weyl developments of $C(G)$ and $C(X)$, we see that the multiplicity of π in the representation of G on $C(X)$ is equal to the multiplicity of the trivial representation of K (= isotropy subgroup of G on X) in $\pi|_K$. The former multiplicity being at most 1 by Lemma 3.7, the latter multiplicity is at most 1. The theorem now follows from Corollary 3.6. Q.E.D.

4. **Function spaces on group manifolds.** The goal of this section is:

4.1. THEOREM. *Let X be a compact topological group and define G to be the group of transformations of X generated by the right and left translations. Then the following three conditions are equivalent.*

(1) *Every closed G-invariant subspace of $C(X)$ is self adjoint.*

(2) *Every linear representation of X is equivalent to its contragredient.*

(3) *Every element of X is conjugate to its inverse.*

Suppose further that X is a compact connected Lie group in bi-invariant Riemannian metric. Then G is the largest connected group of isometries of X, and the list of equivalent conditions extends to include the following three conditions.

(4) *X is semisimple, and none of its simple components is locally isomorphic to $SU(n)$ $(n > 2)$, to $SO(4n+2)$, nor to E_6.*

(5) *The Weyl group of X contains $-I$.*

(6) *Every central element of the universal covering group of X has square* 1.

4.2. COROLLARY. *Let $\mathscr{L}$* (resp. *L*) *be a real or complex semisimple Lie algebra* (resp. *connected Lie group*). *Then every finite dimensional representation of*

$\mathcal{L}$ (resp. L) is equivalent to its contragredient, if and only if, no simple ideal of $\mathcal{L}$ (resp. no simple component of L) is of Cartan classification type A_n $(n > 1)$, nor D_{2n+1}, nor E_6.

REMARK. Corollary 4.2 is due to E. B. Dynkin [4] for complex Lie algebras. We find it easier to work directly than to reduce to Dynkin's result.

4.3. (1) *implies* (2). Let D denote the set of equivalence classes of irreducible representations of X. Given $\pi \in D$, d_π is its degree, π^* is its contragredient, W^π is the representation space, and $E^\pi = W^\pi \otimes W^{\pi^*}$ is the space of matrix coefficients. We have the Peter-Weyl development

$$C(X) \sim \sum_{\pi \in D} E^\pi$$

and the E^π are the irredicible G-invariant subspaces of $C(X)$. Now assume (1), i.e., suppose that each E^π is self adjoint. Lemma 3.1 provides a basis $\{f_i\}$ of E^π on which G acts by orthogonal transformations; in particular the subgroup $X \times 1$ of G acts on $\{f_i\}$ by real matrices, so the character χ of this representation of X is real. But that reprerentation is $\pi \oplus \cdots \oplus \pi$ (d_π times), so the character χ_π of π is real, and $\chi_{\pi^*} = \overline{\chi_\pi}$ now implies $\chi_\pi = \chi_{\pi^*}$. This proves that π is equivalent to π^*, completing the derivation of (2) from (1).

4.4. (2) *implies* (3). Let every linear representation of X be self contragredient. Then every character on X is real. If $x \in X$ and χ is the character of a representation π, one sees that $\chi(x^{-1}) = \overline{\chi(x)}$ by diagonalizing $\pi(x)$; thus $\chi(x^{-1}) = \chi(x)$. As the characters separate conjugacy classes it follows that x is conjugate to x^{-1}, completing the derivation of (3) from (2).

4.5. (3) *implies* (1). Let every element of X be conjugate to its inverse and let K be the isotropy subgroup of G at $1 \in X$. K consists of all $(a,a): x \to axa^{-1}$ for $a \in X$. Given $g = (u,v) \in G$, $g: x \to uxv^{-1}$, our hypothesis provides $b \in X$ such that $b(v^{-1}u)b^{-1} = u^{-1}v$. Thus $g^{-1} = (u^{-1},v^{-1}) \in K(u^{-1},v^{-1})K$ $= K(1, v^{-1}u)(u^{-1},u^{-1})K = K(1, v^{-1}u)K = K(b,b)(1,v^{-1}u)(b^{-1}, b^{-1})K$ $= K(1,u^{-1}v)K = K(u,u)(1,u^{-1}v)K = K(u,v)K = KgK$. Now every G-invariant subspace of $C(X)$ is self adjoint by Theorem 2.1, so (1) has been derived from (3).

This completes the proof of equivalence of the first three statements of Theorem 4.1. We now assume that X is a compact connected Lie group.

4.6. (5) *implies* (1) by Theorem 2.3.

4.7. (1) *implies* (6). Assume (1); then (2) follows, so every representation of X is self contragredient. X is semisimple because it cannot have a circle group for homomorphic image. Now let $X' \to X$ be the universal covering group, and let Z' and Z be the respective centers of X' and X; $X'/Z' = X/Z$ and Z' is finite of some order n.

We first check that every representation of X' is self contragredient. If X' has a representation which is not self contragredient, then some irreducible summand

β is not self contragredient, so the character χ_β is not real. Let $\gamma = \beta \otimes \cdots \otimes \beta$ (n times), representation of X' with character $\chi_\gamma = (\chi_\beta)^n$. If $z \in Z'$, then $z^n = 1$ so $\beta(z) = e^{2\pi ik/n}I$ for some integer k by Schur's Lemma; now $\gamma(z)(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$; thus Z' is in the kernel of γ. Now γ induces a representation α of $X'/Z' = X/Z$; composition of α with $X \to X/Z$ is a representation π of X whose character χ_π assumes the same values as χ_γ. χ_π is real by our hypothesis that π is self contragredient; thus $\chi_\beta(x)^n$ is real for every $x \in X'$. Choose $x \in X'$ with $\chi_\beta(x)$ not real and let $\{x_t\}$ be a one parameter subgroup of X' with $x_1 = x$. Then $t \to \chi_\beta(x_t)$ is a smooth curve in the complex plane. As $\chi_\beta(x_t)^n$ is real, the curve is confined to the n or $n/2$ lines through 0 and $e^{2\pi ik/n}$, and thus lies completely in one of those lines because it is smooth. This is impossible because $\chi_\beta(x_1)$ and $\chi_\beta(x_0)$ cannot lie on the same line. We remark that there is an alternate argument: the highest and lowest weights of the complex Lie algebra representation induced by β are not symmetric about zero, and one can prove that it follows that the same is true for the greatest irreducible summand of γ.

Now every representation of X' is self contragredient. Let β be a faithful finite dimensional representation of X'; $\beta = \beta_1 \oplus \cdots \oplus \beta_r$ where the β_i are irredicible, and the character of β_i is real. If $x \in Z'$, then $\beta_i(z)$ is scalar by Schur's Lemma and has real trace, so $\beta_i(z) = \pm I$; thus $\beta(z)^2 = I$, and it follows that $z^2 = 1$. We have derived (6) from (1).

4.8. (6) *implies* (4). Assume (6) and let $X' \to X$ be the universal covering group. X is semisimple because X' cannot have a vector group as direct factor. If a simple component of X is locally isomorphic to $\boldsymbol{SU}(n)$ $(n > 2)$ (resp. $\boldsymbol{SO}(4n + 2)$, resp. $\boldsymbol{E}_6$), then X' has a direct factor $\boldsymbol{SU}(n)$ $(n > 2)$ (resp. $\boldsymbol{Spin}(4n + 2)$, resp. $\boldsymbol{E}_6$) which is impossible because the center of that direct factor is cyclic of order $n > 2$ (resp. of order 4, resp. of order 3). This completes the derivation of (4) from (6).

4.9. (4) *implies* (5). Let W be the Weyl group of X with respect to a maximal torus T; W is the group of transformations of its Lie algebra $\mathcal{T}$ induced from conjugation by elements $g \in X$ such that $gTg^{-1} = T$. Assume (4); we wish to find $w \in W$ inducing the transformation $\theta \to -\theta$ on $\mathcal{T}$. The existence of w is equivalent to the existence of a corresponding element in the Weyl group of each simple factor of X, so we may assume X simple. Locally isomorphic Lie groups have isomorphic Weyl groups, so we may check any group locally isomorphic to X.

$\boldsymbol{SO}(2m$ or $2m + 1)$. In 2×2 blocks let

$$u_i = \mathrm{diag.}\left\{0, \cdots, 0, \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}, 0, \cdots, 0, (1)\right\}.$$

Then the u_i form a basis of $\mathcal{T}$, W acting by all signed permutations in the $2m + 1$ case, and W acting by all permutations with an even number of sign changes in the $2m$ case. Thus W contains $-I$ except in the $2m$ case with m odd. That case was excluded.

$\boldsymbol{Sp}(n)$. $\mathscr{T}$ has a basis on which W acts by all signed permutations; thus $-I \in W$.

$\boldsymbol{G}_2$. $\boldsymbol{G}_2$ has a subgroup isomorphic to $\boldsymbol{SO}(4)$, and we choose $T \subset \boldsymbol{SO}(4)$. Now W contains the Weyl group of $\boldsymbol{SO}(4)$, which contains $-I$.

$\boldsymbol{F}_4$. $\boldsymbol{F}_4$ has a subgroup locally isomorphic to $\boldsymbol{SO}(9)$. Apply the trick used for $\boldsymbol{G}_2$.

$\boldsymbol{E}_7$. $\boldsymbol{E}_7$ has a subgroup H locally isomorphic to $\boldsymbol{SU}(8)$ whose normalizer $N = H \cup aH$ where conjugation of H by a is induced by complex conjugation of unitary matrices [9, §5.7]. Choose $\mathscr{T} \subset \mathscr{H}$ corresponding to the diagonal pure imaginary 8×8 matrices; now a induces $-I \in W$.

$\boldsymbol{E}_8$. $\boldsymbol{E}_8$ has a subgroup locally isomorphic to $\boldsymbol{SO}(3) \times \boldsymbol{E}_7$. Apply the trick used for $\boldsymbol{G}_2$.

We have derived (5) from (4). This completes the proof of Theorem 4.1. Q.E.D.

4.10. **Proof of Corollary 4.2.** Let β represent an object A on a finite dimensional hermitian vector space V. If it can be defined, the contragredient β^* is the representation induced on the dual space V^*. If the superscript ^{t} denotes transpose of matrices, the matrices representing linear transformations via an orthonormal basis B of V, then V is identified to V^* by the hermitian form and

(i) if A is a group, then $\beta^*(a) = {}^t\beta(a)^{-1}$;

(ii) if A is a Lie algebra, then $\beta^*(a) = -{}^t\beta(a)$.

Now suppose that A is a connected Lie group with Lie algebra $\mathscr{A}$, and let β_* denote the representation of $\mathscr{A}$ induced by β; as β_* is the differential of β, it is easily verified that β is self contragredient if and only if β_* is self contragredient. Every finite dimensional representation of a Lie algebra being induced by one of the corresponding connected simply connected Lie group, the proof of Corollary 4.2 is reduced to the proof of the statement on groups.

Let L be a connected semisimple Lie group, let $\mathscr{L}$ be its Lie algebra, and let π represent L on a finite dimensional vector space V. If L is a real Lie group, let $\mathscr{G}$ be the complexification of $\pi_*(\mathscr{L})$ and let G be the corresponding analytic matrix group. If L is a complex Lie group, let $\mathscr{G} = \pi_*(\mathscr{L})$ and $G = \pi(L)$. Let $\alpha : x \to x$ (resp. $\beta : z \to z$) denote the identity representation of $\pi(L)$ (resp. of G); π is self contragredient if and only if α is self contragredient, i.e., if and only if the identity representation α_* of $\pi_*(\mathscr{L})$ is self contragredient. If α_* is self contragredient, then there is a nonsingular matrix b such that $-b^t x b^{-1} = x$ for every $x \in \pi_*(\mathscr{L})$. As every element of $\mathscr{G}$ has form $x + iy$, $i^2 = -1$, x and y in $\pi_*(\mathscr{L})$, this gives $-b^t z b = z$ for every $z \in \mathscr{G}$, so β_* is self contragredient. On the other hand, if β_* is self contragredient, so there is a nonsingular matrix c such that $-c^t z c^{-1} = z$ for every $z \in \mathscr{G}$, then $-c^t z c^{-1} = z$ if $z \in \pi_*(\mathscr{L}) \subset \mathscr{G}$, so α_* is self contragredient. This proves that π is self contragredient if and only if β_* is self contragredient.

Let K be a maximal compact subgroup of G, let $\mathscr{K}$ be the corresponding real subalgebra of $\mathscr{G}$, and let γ be the restriction of β to K. It is standard from Lie theory that $\mathscr{G}$ is the complexification of $\mathscr{K}$; thus the argument above shows that γ is self contragredient if and only if β_* is self contragredient. This proves that π if self contragredient if and only if γ is self contragredient.

If no simple component of L is of type A_n $(n > 1)$, D_{2n+1} nor E_6, then the same is true of $\mathcal{L}$, thus of $\pi_*(\mathcal{L})$, thus of $\mathcal{G}$, thus of $\mathcal{K}$, and thus of K; it then follows that K has every representation self contragredient by Theorem 4.1, and this now implies that π is self contragredient.

Finally, let L have a simple component S of type A_n $(n > 1)$, D_{2n+1} or E_6, and let M be the compact connected centerless Lie group of that type. By Theorem 4.1 there is a representation δ of M which is not self contragredient. Now δ complexifies to a representation σ of the centerless connected complex group N of the same type as S, and N has a subgroup isomorphic to the quotient S/Y of S by its center. Let Z be the center of L; then S/Y is a direct factor of L/Z; let p be the projection. Define τ to be the representation of L obtained by $L \to L/Z$ followed by p followed by $S/Y \to N$. We know from above that self contragredience of τ is equivalent to that of δ; thus τ is not self contragredient.

This completes the proof of Corollary 4.2. Q.E.D.

5. **Function spaces on symmetric quotient manifolds of simple groups.** We will prove:

5.1. THEOREM. *Let X be a compact conected Riemannian symmetric space, let G be the largest connected group of isometries of X, and suppose that G is simple. Then the following conditions are equivalent*:

(1) *Every closed G-invariant subspace of $C(X)$ is self adjoint.*

(2) *The Weyl group of X contains $-I$.*

(3) *X is not locally isometric to one of the spaces*

(3a) $\boldsymbol{SU}(n)/\boldsymbol{SO}(n)$, $n > 2$, *or*

(3b) $\boldsymbol{SU}(2n)/\boldsymbol{Sp}(n)$, $n > 2$, *or*

(3c) $\boldsymbol{SO}(4n+2)/\boldsymbol{SO}(2n+1) \times \boldsymbol{SO}(2n+1)$, $n > 0$, *or*

(3d) $\boldsymbol{E}_6/\boldsymbol{F}_4$, *or*

(3e) $\boldsymbol{E}_6/(\boldsymbol{Sp}(4)/\{\pm I\})$.

The proof makes essential use of É. Cartan's classification [1].

5.2. (3) *implies* (2). If the Euler-Poincaré characteristic $\chi(X) \neq 0$, i.e., if the isotropy subgroup K of G at $x \in X$ contains the symmetry s to X at x, then s induces $-I$ in the Weyl group. Now suppose $\chi(X) = 0$. According to É. Cartan's classification [1], X must be locally isometric to (a) $\boldsymbol{SU}(n)/\boldsymbol{SO}(n)$, $n > 2$, or (b) $\boldsymbol{SU}(2n)/\boldsymbol{Sp}(n)$, $n > 1$, or (c) $\boldsymbol{SO}(p+q)/\boldsymbol{SO}(p) \times \boldsymbol{SO}(q)$, p and q odd, or (d) $\boldsymbol{E}_6/\boldsymbol{F}_4$, or (e) $\boldsymbol{E}_6/(\boldsymbol{Sp}(4)/\{\pm I\})$. The hypothesis (3) and the fact that $\boldsymbol{SU}(4)/\boldsymbol{Sp}(2)$ is the same as the 5-sphere $\boldsymbol{SO}(6)/\boldsymbol{SO}(5) \times \boldsymbol{SO}(1)$ eliminates all of these except $\boldsymbol{SO}(p+q)/\boldsymbol{SO}(p) \times \boldsymbol{SO}(q)$, p and q odd, $p \neq q$. We must show that the Weyl group contains $-I$ in that case.

Let $X = G/K$ be locally isometric to $\boldsymbol{SO}(p+q)/\boldsymbol{SO}(p) \times \boldsymbol{SO}(q)$, p and q odd, $p \neq q$; we may assume $p < q$. Now the space $\mathcal{P}$ of §2.2 consists of all

$$b^* = \begin{bmatrix} 0 & b \\ -{}^t b & 0 \end{bmatrix}$$

where b is a real $p \times q$ matrix, and it is straightforward to verify that the Cartan subalgebra $\mathscr{A}$ may be chosen to be subset of $\mathscr{P}$ consisting of all b^* for which b is of the form

$$b = \begin{bmatrix} b_1 & & & & 0 & \cdots & 0 \\ & \cdot & & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot \\ & & & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & & \cdot \\ & & & b_p & 0 & \cdots & 0 \end{bmatrix}.$$

We now define, using $p < q$,

$$\kappa = \begin{bmatrix} I_p & & \\ & -I_{p+1} & \\ & & I_{q-p-1} \end{bmatrix},$$

observe that $\kappa \in K$ because $p + 1$ is even, and check that $\mathrm{ad}(\kappa)b^* = -b^*$ for every $b^* \in \mathscr{A}$. Thus the Weyl group of $X' = \mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ contains $-I$; it follows that the Weyl group of X contains $-I$. This completes our checking that (3) implies (2).

5.3. (2) *implies* (1) by Theorem 2.3.

5.4. (1) *implies* (3). The proof is by contradiction. Let $X = G/K$ be one of the spaces (3a) – (3e). We need a closed G-invariant subspace of $\mathbf{C}(X)$ which is not self adjoint.

Suppose first that $X = G/K$ is simply connected. By Theorem 3.8 we need only construct a subspace V of $\mathbf{C}(X)$ on which G acts by an irreducible representation π which is not self contragredient. This is easy in cases (3d) and (3e), for there the center $\mathbf{Z}$ of G is cyclic of order 3, and, the representation of G on $\mathbf{C}(X)$ being faithful, we have a subspace V on which G acts by an irreducible representation π which does not annihilate Z; then π is not self contragredient because its character takes the value $(\dim V)e^{2\pi i/3}$ which is not real.

In case (3a), we let α denote the usual representation of $\mathbf{SU}(n)$ and define $\pi = s^2(\alpha)$, second symmetrization, action on homogeneous polynomials of degree 2; π is not self contragredient because $n > 2$, and its restriction to $\mathbf{SO}(n)$ contains the trivial representation because $\pi(\mathbf{SO}(n))$ preserves $\sum z_j^2$; if n is even (so G is $\mathbf{SU}(n)/\{\pm I\}$ rather than $\mathbf{SU}(n)$), π induces a representation of G because it annihilates $-I$; now for any $n > 2$ we have a non-self contragredient representation π of G on a subspace of $\mathbf{C}(X)$ by comparing the Fourier-Peter-Weyl developments of $\mathbf{C}(G)$ and $\mathbf{C}(X)$. Case (3b) is similar; π is defined to be $a^2(\alpha)$, second alternation, action on antisymmetric tensors of degree 2; $\pi(\boldsymbol{Sp}(n))$ preserves $\sum_1^n(z_jz'_{j+n}-z_{j+n}z'_j)$; non-self contragredience of π follows from $n > 2$.

Finally suppose that we are in case (3c). If $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the usual real representation space of $G = \boldsymbol{SO}(4n+2)$, $m = 4n+2$, then the complex Clifford algebra M is the complex algebra generated by $\{1, e_1, \cdots, e_m\}$ with $e_ie_j + e_je_i = \delta_{ij}$. The universal covering group $\boldsymbol{Spin}(m)$ of G is a multiplicative subgroup of M; it projects to G by sending $(\sum a_ie_i)(\sum b_ie_i)$ to the product of reflections in the corresponding hyperplanes. The spin representation σ of $\boldsymbol{Spin}(m)$ is induced by the minimal faithful representation of M; it is a sum $\sigma_1 \oplus \sigma_2$ with σ_i irreducible and not self contragredient (half spin representations) but $\sigma_1^* = \sigma_2$. Let H be the subgroup of $\boldsymbol{Spin}(m)$ over K, $H = \boldsymbol{Spin}(2n+1) \times \boldsymbol{Spin}(2n+1)$. If τ is the spin representation of $\boldsymbol{Spin}(2n+1)$, one checks that $\sigma_i|H = \tau \otimes \tau$. σ_i has no bilinear invariant, not being self contragredient, but τ (thus $\tau \otimes \tau$) has one because it is self contragredient. Let N be the normalizer of H in $\boldsymbol{Spin}(m)$; it follows that $\sigma_i(N)$ has a bilinear invariant η. Now either $s^2(\sigma_1)$ or $a^2(\sigma_2)$ (in fact [4, Theorem 1.4] shows it is $s^2(\sigma_1)$) has an irreducible summand π whose restriction to N contains the trivial representation, and which is not self contragredient. By construction π annihilates the central element of order 2 in $\boldsymbol{Spin}(m)$ and thus induces a representation π of G; on G, π is not self contragredient, π is irreducible, and $\pi|_{K^*}$ contains the trivial representation of K^* where

$$K^* = K \cup bK\,,\ b = \begin{pmatrix} 0 & I_{2n+1} \\ -I_{2n+1} & 0 \end{pmatrix}.$$

Ignoring b for the moment, we are done as in case (3a) by comparing Fourier-Peter-Weyl developments. This finishes the case where X is simply connected.

We now treat the general case. Let G' be the largest connected group of isometries of the universal Riemannian covering manifold X' of X; $G' \to G$ is a covering group, and $X' = G'/K'$ where K'' is the inverse image of K in G' and K' is the identity component of K''. In cases (3a), (3b), (3d) and (3e) one sees from [8, §§5.5.3, 5.5.4, 5.5.14] that K is the image of K' in G; in case (3d) one sees from [8, §5.5.11] that K is the image either of K' or of $K^* = K' \cup bK$, in the notation of the preceding paragraph. Thus K'' is generated by K', the kernel of $G' \to G$, and (case (3d) only) possibly b.

We have an irreducible representation π of G' on a subspace V of $\boldsymbol{C}(X')$ which is not self contragredient; in case (3d) we may view $V \subset \boldsymbol{C}(G'/K^*)$. The center of G' has some finite order r and thus acts trivially on $U = \{v^r\colon v \in V\}$; here $v^r(x') = v(x')^r$. U is invariant under G' and thus spans an invariant (and finite dimensional) subspace W of $\boldsymbol{C}(X')$; we may view $W \subset \boldsymbol{C}(G'/K^*)$ in case (3d). The argument of §4.7 shows that the action of G' on W is not self contragredient, so W is not self adjoint. As the center of G' is trivial on W, our analysis of K''

above shows that we may view W as a non-selfadjoint G-invariant subspace of $\boldsymbol{C}(X)$.

Theorem 5.1 is proved. Q.E.D.

6. **Function spaces on symmetric spaces.** Combining the results of §§2–5, we will prove:

6.1. THEOREM. *Let X be a compact connected Riemannian symmetric space. If G is the largest connected group of isometries of X, then the following conditions are equivalent.*

(1) *Every closed G-invariant subspace of $\boldsymbol{C}(X)$ is self adjoint.*

(2) *The Weyl group of X contains $-I$.*

(3) *The universal Riemannian covering manifold of X is a product of compact irreducible symmetric spaces, none of which is isometric to $\boldsymbol{SU}(n)$ $(n > 2)$, $\boldsymbol{SU}(n)/\boldsymbol{SO}(n)$ $(n > 2)$, $\boldsymbol{SU}(2n)/\boldsymbol{Sp}(n)$ $(n > 2)$, $\boldsymbol{Spin}(4n+2)$ $(n > 0)$, $\boldsymbol{SO}(4n+2)/\boldsymbol{SO}(2n+1) \times \boldsymbol{SO}(2n+1) (n > 0)$, $\boldsymbol{E}_6$, $\boldsymbol{E}_6/\boldsymbol{F}_4$ nor $\boldsymbol{E}_6/(\boldsymbol{Sp}(4)/\{\pm I\})$.*

(4) *X is locally isometric to a product $X_1 \times \cdots \times X_r$ where each X_i is one of* (i) *complex Grassmann manifold,* (ii) *real Grassmann manifold except* $(2n+1)$-*planes in* $(4n+2)$-*space,* (iii) $\boldsymbol{SO}(2n)/\boldsymbol{U}(n)$, (iv) $\boldsymbol{Sp}(n)/\boldsymbol{U}(n)$, (v) *quaternionic Grassmann manifold,* (vi) $\boldsymbol{G}_2/\boldsymbol{SO}(4)$, (vii) *Cayley plane,* (viii) $\boldsymbol{F}_4/\boldsymbol{Sp}(3)\cdot\boldsymbol{Sp}(1)$, (ix) $\boldsymbol{E}_6/\boldsymbol{SU}(6)\cdot\boldsymbol{SU}(2)$, (x) $\boldsymbol{E}_6/\boldsymbol{SO}(10)\cdot\boldsymbol{SO}(2)$, (xi) $\boldsymbol{E}_7/(\boldsymbol{SU}(8)/\{\pm I, \pm iI\})$, (xii) $\boldsymbol{E}_7/\boldsymbol{SO}(12)\cdot\boldsymbol{SU}(2)$, (xiii) $\boldsymbol{E}_7/\boldsymbol{E}_6\cdot\boldsymbol{SO}(2)$, (xiv) $\boldsymbol{E}_8/\boldsymbol{SO}(16)$, (xv) $\boldsymbol{E}_8/\boldsymbol{E}_7\cdot\boldsymbol{SU}(2)$, (xvi) $\boldsymbol{Spin}(n)$ *for* $n \not\equiv 2 \pmod 4$, (xvii) $\boldsymbol{Sp}(n)$, (xviii) $\boldsymbol{G}_2$, (xix) $\boldsymbol{F}_4$, (xx) $\boldsymbol{E}_7$, (xxi) $\boldsymbol{E}_8$.

REMARK. Equivalence of (3) and (4) is immediate from É. Cartan's classification [1], his classification of the compact simple Lie groups, and the existence of a Riemannian covering $X_0 \times X_1 \times \cdots \times X_r \to X$ where X_0 is a flat torus and $X_i (i > 0)$ is a compact simply connected symmetric space.

6.2. In the decomposition $\mathscr{G} = \mathscr{K} + \mathscr{P}$, all three objects split as X is decomposed locally into a direct product; thus the Cartan subalgebra $\mathscr{A}$ of $(\mathscr{G}, \mathscr{K})$ splits. Now (2) and (3) are equivalent by Theorem 4.1 and 5.1. (2) implies (1) by Theorem 2.3. (1) implies (3) by Theorems 3.8 and 5.1. Q.E.D.

7. **Application to function algebras.** We will prove:

7.1. THEOREM. *Let X be a compact connected irreducible Riemannian symmetric space, let G be the largest connected group of isometries of X, and let Δ be the centralizer of G in the group of all isometries of X.*

1. *If A is a G-invariant subset of $\boldsymbol{C}(X)$, then either A is a set of constant function, or the identification space*(3) *X/A admits the structure of Riemannian symmetric space in such a manner that the projection $\pi: X \to X/A$ is a G-equivariant Riemannian covering.*

(3) Let two elements $x_i \in X$ be *equivalent* if $a(x_1) = a(x_2)$ for every $a \in A$. Then X/A denotes the set of equivalence classes with the strongest topology for which $X \to X/A$ is continuous.

2. *There are one-one correspondences between the subgroups $\Gamma \subset \Delta$, the Riemannian coverings $X \to Y$ with Y Riemannian symmetric, and the closed self adjoint G-invariant subalgebras $A \subset C(X)$ which properly contain the constant functions, given as follows*:

(i) Γ *yields the covering $X \to X/\Gamma$, and the algebra of all functions which are invariant under every element of* Γ.

(ii) $X \to Y$ *yields the group of deck transformations of the covering, and the algebra $C(Y)$ viewed as a subalgebra of $C(X)$.*

(iii) A *yields the covering $X \to X/A$, and the group of all isometries of X which preserve every element of A.*

REMARK. For simply connected X, the groups Δ will be listed in §7.4. Then the group corresponding to Δ for X/Σ is Δ/Σ. The groups Δ listed have very simple structure, so it is easy to find all subgroups of all quotient groups, resulting in a classification for compact irreducible symmetric X of all closed self adjoint G-invariant subalgebras of $C(X)$. Then Theorem 6.1 tells us in most cases that we have classified all the closed G-invariant subalgebras of $C(X)$.

7.2. **Proof of 1.** Let $x \in X$; then $X = G/K$ where K is the isotropy subgroup of G at x. Let $y = \pi(x)$, image of x in X/A. G acts on X/A because $G(A) = A$; now $X/A = G/H$ where H is the isotropy subgroup of G at y; $K \subset H \subset G$. H is a closed subgroup of G because $X/A = X/\bar{A}$ where $\bar{A}$ is the closure of A in $C(X)$; thus H is a Lie subgroup, and the Lie algebras $\mathscr{K} \subset \mathscr{H} \subset \mathscr{G}$. Irreducibility of X is equivalent to maximality of $\mathscr{K}$ among proper subalgebras of $\mathscr{G}$; thus $\mathscr{K} = \mathscr{H}$ or $\mathscr{H} = \mathscr{G}$.

If $\mathscr{H} = \mathscr{G}$, then $G = \exp(\mathscr{G}) = \exp(\mathscr{H}) \subset H$ so $G = H$; this means that X/A is a single point and A is a set of constant functions.

Now suppose $\mathscr{K} = \mathscr{H}$. Then K is a subgroup of finite index in H, whence $\pi : X \to X/A$ is given by $G/K \to G/K\backslash H = G/H$, covering with finite fibre $K\backslash H$. G being compact, X/A admits a G-invariant Riemannian metric ds^2. The lift π^*ds^2 to X is a G-invariant Riemannian metric; by irreducibility of X it must be a constant multiple of the original metric on X. Thus we may choose ds^2 so that π^*ds^2 is the metric on X, i.e., so that π is a local isometry, i.e., so that π is a Riemannian covering. This done, $(X/A, ds^2)$ is locally symmetric; but the local transvections are globally defined because they are induced by the elements of G which are transvections on X; it follows that $(X/A, ds^2)$ is globally symmetric.

7.3. **Proof of 2.** Let Γ be a subgroup of Δ. Δ is finite because G is semisimple and G has finite index in the group of all isometries of X; thus Γ is finite. If an element $\gamma \in \Gamma$ has a fixed point, $\gamma(x) = x$, then $\gamma = 1$; for given $x' \in X$ we choose $g \in G$ with $x' = g(x)$, and then $\gamma(x') = \gamma g x = g \gamma x = g(x) = x'$. This shows that $X \to X/\Gamma$ is a covering; it is a Riemannian covering because Γ acts by isometries, and Γ is the group of deck transformations of the covering. G preserves Γ-orbits on X, thus acting on X/Γ. As with X/A above, it follows that X/Γ is Riemannian symmetric. Let A be the set $\{f \in C(X) : \gamma(f) = f$ for every $\gamma \in \Gamma\}$; it is trivial to check that A is the algebra of all functions constant on Γ-orbits, and thus G-invariant. In other words, A is $C(X/\Gamma)$ lifted to X, and in particular A is closed, self adjoint, and properly contains the constants.

Let $X \to Y$ be a Riemannian covering with Y Riemannian symmetric. The fundamental group $\pi_1(Y)$ is commutative [8, Theorem 6.4] so $\pi_1(X)$ is injected as a normal subgroup of $\pi_1(Y)$. This implies that the covering is of the form $X \to X/\Gamma$ where Γ, the group of deck transformations of the covering, is a group of homeomorphisms of X isomorphic to $\pi_1(Y)/\pi_1(X)$. As the covering is Riemannian, one easily checks that Γ is a group of isometries of X. The symmetries of Y lift to symmetries of X, so the symmetries of X send Γ-orbits to Γ-orbits, whence transvections of X (being products of two symmetries) send Γ-orbits to Γ-orbits. G is generated by the transvections to X; thus G sends Γ-orbits to Γ-orbits. In other words, G normalizes Γ in the group of all isometries of X. As G is connected and Γ is discrete, it follows that $\Gamma \subset \Delta$. Construction of $C(Y) \subset C(X)$ from $X \to Y$ was explained in the preceding paragraph.

Let A be a closed self adjoint G-invariant subalgebra of $C(X)$ which properly contains the constants. Then $X \to X/A = Y$ is a Riemannian covering with Y Riemannian symmetric, by §7.2. The Stone-Weierstrass Theorem ensures that A is $C(Y)$ lifted to X. The correspondence between A and the group of deck transformations of $X \to Y$ has already been analyzed. Q.E.D.

7.4. Let X be simply connected; we will list Δ for the various possibilities. If X is a group manifold, then Δ is the center acting by left translation. The list is direct verification from É. Cartan's work [2], using simplifications from [8, §2.4, §5].

a. The following is a complete list of the cases where $\Delta = \{1\}$, i.e., where $C(X)$ and the constants are the only closed G-invariant self adjoint subalgebras of $C(X)$ which contain the constants. (i) complex Grassmann manifolds, except linear p-planes in $2p$-space, (ii) $SO(4n+2)/U(2n+1)$, (iii) quaternionic Grassmann manifolds, except linear p-planes in $2p$-space, (iv) $G_2/SO(4)$, (v) Cayley projective plane, (vi) $F_4/Sp(3)\cdot Sp(1)$, (vii) $E_6/SU(6)\cdot SU(2)$, (viii) $E_6/SO(10)\cdot SO(2)$, (ix) $E_7/SO(12)\cdot SU(2)$, (x) $E_8/SO(16)$, (xi) $E_8/E_7\cdot SO(2)$, (xii) G_2, (xiii) F_4, and (xiv) E_8. In all of these cases, $C(X)$ and the constants are the only closed G-invariant subalgebras of $C\ (X)$ containing the constants.

b. The following is a complete list of the cases where Δ is cyclic of prime order $r > 1$, i.e., where $C(X)$ and the constants and just one other are the only closed G-invariant self adjoint subalgebras of $C(X)$ containing the constants. (xv) $E_7/E_6\cdot SO(2)$, $r = 2$, $\Delta \not\subset G$; (xvi) $E_7/(SU(8)/\{\pm I, \pm iI\})$, $r = 2$, $\Delta \not\subset G$; (xvii) $E_6/(Sp(4)/\{\pm I\})$, $r = 3$, $\Delta =$ center of G; (xviii) E_6/F_4, $r = 3$, $\Delta =$ center of G; (xix) $Sp(n)/U(n)$, $r = 2$, $\Delta \not\subset G$; (xx) quaternionic Grassmann manifold of p-planes in $2p$-space, $\Delta = \{1, \beta\}$ where $\beta : x \to x^{\perp}$ for every $x \in X$; (xxi) $SO(4n)/U(2n)$, $r = 2$, $\Delta \not\subset G$; (xxii) real oriented Grassmann manifold lexcept p-panes in $2p$-space, $\Delta = \{1, \alpha\}$ where α reverses orientation of every

p-plane; (xxiii) complex Grassmann manifold of p-planes in $2p$-space, $\Delta = \{1,\beta\}$ as in (xx); (xxiv) $\boldsymbol{E}_7$, $r = 2$, (xxv) $\boldsymbol{E}_6$, $r = 3$, (xxvi) $\boldsymbol{Sp}(n)$, $r = 2$; (xxvii) $\boldsymbol{Spin}(2n + 1)$, $r = 2$; (xxviii) $\boldsymbol{SU}/(\boldsymbol{SO}$ or $\boldsymbol{Sp})$, see below. In cases (xv), (xix), (xxi), (xxii) for 2-planes in $(n + 2)$-space, and (xxiii), X is a hermitian symmetric space and the nontrivial element of Δ is antiholomorphic. Except in cases (xvii), (xviii), (xxv) and (xxviii), every closed G-invariant subalgebra of $\boldsymbol{C}(X)$ containing the constants is $\boldsymbol{C}(X)$, the constants, or the other one.

c. Let $\boldsymbol{Z}_r$ denote the cyclic group of order r. The remaining cases are: (xxix) $\boldsymbol{SU}(2n)/\boldsymbol{SO}(2n)$, $G = \boldsymbol{SU}(2n)/\{\pm I\}$, $\Delta = {}_n\boldsymbol{Z} \times \boldsymbol{Z}_2$ where $\boldsymbol{Z}_n$ is the center of G; (xxx) $\boldsymbol{SU}(2n + 1)/\boldsymbol{SO}(2n + 1)$, $\Delta = \boldsymbol{Z}_{2n+1}$ = center of G; (xxxi) $\boldsymbol{SU}(2n)/\boldsymbol{Sp}(n)$, $G = \boldsymbol{SU}(2n)/\{\pm I\}$, $\Delta = \boldsymbol{Z}_n$ = center of G; (xxxii) $\boldsymbol{SU}(n)$, $\Delta = \boldsymbol{Z}_n$; (xxxiii) real Grassmann manifold of oriented p-planes in $2p$-space $\Delta = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \{1,\alpha,\beta,\alpha\beta\}$ with β as in (xx) and α as in (xxii); (xxxiv) $\boldsymbol{Spin}(4n)$, $\Delta = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$; (xxxv) $\boldsymbol{Spin}(4n+2)$, $\Delta = \boldsymbol{Z}_4$. In both case (xxxiii) for p even and case (xxxiv), the corresponding five self adjoint subalgebras of $\boldsymbol{C}(X)$ exhaust all closed G-invariant subalgebras of $\boldsymbol{C}(X)$ containing the scalars.

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