Isotropic Manifolds of Indefinite Metric.

Isotropic Manifolds of Indefinite Metric[1])

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0. Introduction

In an earlier paper [17] we studied the pseudo-RIEMANNian manifolds of constant sectional curvature and classified the complete homogeneous ones except in the flat case; this paper is a more thorough study of a larger class of manifolds.

A pseudo-RIEMANNian (indefinite metric) manifold is (locally) isotropic if, given two nonzero tangentvectors of the same "length" at the same point, there is a (local) isometry carrying one vector to the other. The local version of this property is automatic in the case of constant curvature. Chapter I begins with the study of certain isotropic spaces, called model spaces, which are flat or are indefinite metric analogs of the real, complex and quaternionic elliptic and hyperbolic spaces and of the CAYLEY elliptic and hyperbolic planes. In § 4 it is shown that every locally isotropic manifold is locally isometric to a model space; an interesting consequence, extending a result of S. HELGASON ([8], § 4) in the LORENTZ case, is that the manifold must be of constant curvature if either the number of negative squares in the metric, or the number of positive squares in the metric, is odd. This local isometry extends to a metric covering in the case of complete manifolds (§ 5); as consequences we obtain a global classification of the complete locally isotropic manifolds of nonconstant curvature in about half the signatures of metric (§ 6), and we obtain a global classification of the isotropic manifolds (§ 7).

Chapter II is devoted to homogeneous locally isotropic manifolds. Using the techniques of [17] (§ 8), the complete ones are classified (§ 9). In § 12 it is shown that every isotropic pseudo-RIEMANNian manifold of strictly indefinite metric contains an (necessarily locally isotropic) open submanifold which is homogeneous but not complete. This is surprising in view of the RIEMANNian case where homogeneity implies completeness. It shows that one must add the hypothesis of completeness to Theorem 12 of [17] and Theorems 1 and 2 of [19].

I am indebted to Professor JACQUES TITS for a conversation on the indefinite CAYLEY plane. Professor TITS informs me that he has an unpublished classification of the isotropic pseudo-RIEMANNian manifolds obtained by means of a classification of subalgebras $\mathfrak{H}$ of small codimension in a simple LIE algebra $\mathfrak{G}$.

Note. It is possible to omit reading §§ 1.3.3–1.3.5, § 2.9, §§ 3.3–3.4 and § 10 while reading the rest of this paper. The main results are stated at the beginnings of §§ 4, 5, 6, 7, 9 and 12.

1. Preliminaries on pseudo-RIEMANNian manifolds, LIE groups and curvature

1. 1. Pseudo-RIEMANNian manifolds. A *pseudo-RIEMANNian manifold* is a pair $M_h^n = (M, Q)$ where M is an n-dimensional differentiable manifold, Q is a differentiable field of real nonsingular symmetric bilinear forms Q_x on the tangentspaces M_x of M, and each Q_x has signature $-\sum_1^h x_j^2 + \sum_{h+1}^n x_j^2$. We often say M_h^n when M is intended, and then the *pseudo-RIEMANNian metric* Q is understood. Given $X \in M_x$, we usually write $||X||^2$ for $Q_x(X, X)$. The *LEVI-CIVITA connection* is the unique torsionfree linear connection on the tangentbundle of M_h^n such that parallel translation is a linear isometry (preserves the inner products Q_x) on tangentspaces; M_h^n is *complete* if this connection is complete, i.e., if every geodesic can be extended to arbitrary values of the affine parameter.

An *isometry* of pseudo-RIEMANNian manifolds is a diffeomorphism $f: M_h^n \to N_h^n$ which induces linear isometries on tangentspaces. The collection of all isometries of M_h^n (onto itself) forms a LIE group $\mathbf{I}(M_h^n)$ called the *full group of isometries* of M_h^n; its identity component $\mathbf{I}_0(M_h^n)$ is the *connected group of isometries*. Given $x \in M_h^n$, we consider the collection of all local maps which are each an isometry (in induced pseudo-RIEMANNian structure) of an open neighborhood of x onto another open neighborhood and which leave x fixed, where two of these local maps are identified if they agree on a neighborhood of x; this collection of germs of maps forms a LIE group which is faithfully represented on the tangentspace $(M_h^n)_x$, and is called the *group of local isometries at* x; its LIE algebra is the subalgebra consisting of elements which vanish at x in the LIE algebra of germs of KILLING vectorfields at x.

M_h^n is *homogeneous* if $\mathbf{I}(M_h^n)$ is transitive on its points; M_h^n is *locally homogeneous* if, given x and y in M_h^n, there is an isometry of a neighborhood of x onto a neighborhood of y which carries x to y. M_h^n is *symmetric* (resp. *locally symmetric*) if, given $x \in M_h^n$, there is an $s_x \in \mathbf{I}(M_h^n)$ (resp. local isometry s_x at x) of order 2 and with x as isolated fixed point; s_x is then the *symmetry* (resp. *local symmetry*) at x. M_h^n is *isotropic* (resp. *locally isotropic*) if, given $x \in M_h^n$ and nonzero tangentvectors X, Y at x with $||X||^2 = ||Y||^2$, there exists $g \in \mathbf{I}(M_h^n)$ leaving x fixed (resp. a local isometry g at x) such that $g_* X = Y$.

If M_h^n is connected and either isotropic or symmetric, then it is both homogeneous and complete. For homogeneity one connects any two points x and y by a broken geodesic arc, chooses isometries fixing the midpoints (in affine parameter) and reversing the tangentvector there for each of the geodesic segments, and notes that the successive product of these isometries sends x

to y. For completeness, one extends a geodesic via its image under any isometry which fixes a point of the geodesic and reverses the tangentvector there. Combining these two arguments and applying the HEINE-BOREL Theorem, one can also see: If M_h^n is connected and is either locally isotropic or locally symmetric, then M_h^n is locally homogeneous. In fact we will soon see that M_h^n must be (locally) symmetric if it is (locally) isotropic.

A *pseudo-RIEMANNian covering* is a covering $\pi : N_h^n \to M_h^n$ of connected pseudo-RIEMANNian manifolds where π is locally an isometry. Then each deck transformation (homeomorphism $d : N_h^n \to N_h^n$ such that $\pi \cdot d = \pi$) is an isometry of N_h^n. If one of N_h^n and M_h^n is locally homogeneous (resp. locally symmetric, resp. locally isotropic, resp. complete) then π carries the same property over to the other. If M_h^n is homogeneous (resp. symmetric, resp. isotropic), then we can lift each isometry and N_h^n has the same property. But these global properties usually do not descend.

1. 2. Reductive groups. We assume familiarity with LIE groups. If G is a LIE group, then G_0 denotes the identity component, $\mathfrak{G}$ denotes the LIE algebra, ad and Ad denote the adjoint representations of G_0 and $\mathfrak{G}$ on $\mathfrak{G}$, and exp is the exponential map $\mathfrak{G} \to G_0$. The following conditions are equivalent, and G and $\mathfrak{G}$ are called *reductive* if one of them is satisfied: (1) ad (G_0) is fully reducible on $\mathfrak{G}$, (2) $\mathrm{Ad}(\mathfrak{G})$ is fully reducible on $\mathfrak{G}$, (3) $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{A}$ where $\mathfrak{A}$ is the center and $\mathfrak{G}'$ is a semisimple ideal (the *semisimple part*), (4) $\mathfrak{G}$ has a faithful fully reducible linear representation, (5) $\mathfrak{G}$ has a nondegenerate real symmetric $\mathrm{ad}(G_0)$-invariant bilinear form which is the trace form of a faithful linear representation. If H is a closed subgroup of G, it is called *reductive in* G if $\mathrm{ad}_G(H)$ is fully reducible on $\mathfrak{G}$; $\mathfrak{H}$ is reductive in $\mathfrak{G}$ if H_0 is reductive in G; an element $g \in G$ is *semisimple* if ad (g) is fully reducible.

The following well-known fact is immediate from SCHUR's Lemma because G is reductive; we will use it often.

1. 2. 1. Lemma. *Let G be an irreducible closed connected group of linear transformations of a real vectorspace V, let Z be the center of G, and let G' be the derived group of G. Then* (a) *G is semisimple, or* (b) *$G = G' \cdot Z$ where Z is a circle group which gives a complex structure to V, or* (c) *$G = G' \cdot Z$ where either Z consists of all nonzero real scalar transformations, or Z is generated by a circle group and the nonzero real scalars; case* (c) *does not occur if V has a nondegenerate G-invariant bilinear form.*

A *CARTAN involution* of a semisimple LIE algebra $\mathfrak{G}$ is an involutive automorphism of $\mathfrak{G}$ whose fixed point set is the LIE algebra of a maximal compact subgroup of the adjoint group of $\mathfrak{G}$; any two are conjugate by inner automorphisms of $\mathfrak{G}$. We will sometimes say CARTAN involution for an involutive automorphism of a reductive LIE algebra which induces a CARTAN involution on the semisimple part. If $\mathfrak{H}$ is an algebraic reductive subalgebra of a linear semisimple LIE algebra $\mathfrak{G}$, then it follows (see [4]) from G.D.MOSTOW's treatment of the case of semisimple $\mathfrak{H}$ [14] that there is a CARTAN involution of $\mathfrak{G}$ which preserves $\mathfrak{H}$ and induces a CARTAN involution of $\mathfrak{H}$.

1. 3. Curvature. Let M_h^n be a pseudo-RIEMANNian manifold. If S is a 2-dimensional subspace of a tangentspace $(M_h^n)_x$ on which Q_x is nondegenerate (*nonsingular subspace*), and if $\mathcal{R}$ is the curvature tensor of the LEVI-CIVITA connection, and is viewed as a transformation, then the *sectional curvature* of M_h^n along S is given by

$$K(S) = -\frac{Q_x(\mathcal{R}(X,Y)X,Y)}{Q_x(X,X)\cdot Q_x(Y,Y) - Q_x(X,Y)^2}\,.$$

M_h^n has *constant curvature* k (at x) if $K(S) = k$ for every nonsingular tangent 2-plane S to M_h^n (at x). If the group of local isometries at x contains the identity component $\mathbf{SO}^h(n)$ of the orthogonal group of Q_x, then M_h^n is easily seen to have constant curvature at x. This is particularly useful when M_h^n is connected and $n \geq 3$, for then F.SCHUR's Theorem says that M_h^n has constant curvature if it has constant curvature at every point. M_h^n is called *flat* if it has constant curvature zero. Finally, M_h^n is locally symmetric if $K(S)$ does not change as S undergoes parallel translation, i.e., if $\mathcal{R}$ is parallel.

Part 2 of the following result generalizes the theorem on sign of curvature of a RIEMANNian symmetric space.

1.3.1. Theorem. *Let $M = G/H$ where H is a closed subgroup of a semisimple* LIE *group G such that $\mathfrak{H}$ is invariant under a* CARTAN *involution σ of $\mathfrak{G}$. Let B be an* $\mathrm{ad}(G)$*-invariant σ-invariant nondegenerate symmetric bilinear form on $\mathfrak{G}$; $B = \Sigma a_i B_i$ where the B_i are the* KILLING *forms of the simple ideals of $\mathfrak{G}$; suppose that each $a_i > 0$. Then B is nondegenerate on $\mathfrak{H}$ and:*

1. *Let $\mathfrak{M} = \mathfrak{H}^\perp$ ($\perp$ relative to B). Then $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ is an* $\mathrm{ad}(H)$*-stable σ-stable vectorspace direct sum, $\mathfrak{M}$ is naturally identified with a tangentspace of M, and B induces a nondegenerate* $\mathrm{ad}(H)$*-invariant bilinear form on $\mathfrak{M}$ which defines a G-invariant pseudo-*RIEMANNian *metric Q on M.*

2. *Let S be a σ-invariant 2-dimensional linear subspace of $\mathfrak{M}$. Then S is nonsingular for B, so the sectional curvature $K(S)$ of (M, Q) along S is defined, and $K(S) \leq 0$.*

3. *The sectional curvatures of (M, Q) are bounded from above and below, if and only if (M, Q) has every sectional curvature non-positive.*

4. *If $\mathfrak{M}$ has a nonsingular 2-dimensional subspace S with orthonormal basis $\{X, Y\}$ and an element $Z \perp S$ with $[X, Z] = 0$ and $\|Y\|^2 \cdot \|Z\|^2 < 0$, then (M, Q) has a positive sectional curvature.*

Remark 1. If we had stipulated $a_i < 0$, then all signs of curvature would be reversed. If M is symmetric as a coset space, so $\mathfrak{H}$ is the fixed point set of an involutive automorphism τ of $\mathfrak{G}$, then $\mathfrak{H}$ is reductive because $\mathfrak{G}$ and τ are semisimple, and $\mathfrak{H}$ is algebraic as defined by τ, so it is automatic that $\mathfrak{H}$ is invariant under a CARTAN involution of $\mathfrak{G}$.

Remark 2. Parts 2 and 4 lead us to

Definition. In the notation of Theorem 1.3.1 a 2-plane in $\mathfrak{M}$ is *involutive* if it is invariant under a CARTAN involution of $\mathfrak{G}$ which preserves $\mathfrak{H}$, and the *involutive sectional curvatures* of (M, Q) are the sectional curvatures along involutive 2-planes.

Now the sign of involutive sectional curvature is constant for many manifolds, but it is unusual for the sign of sectional curvature to be constant. In fact, from Theorems 2.9 and 4.1 it will follow that a connected locally isotropic pseudo-RIEMANNian manifold M_h^n has sectional curvatures of one sign, if and only if either $h = 0$ or $h = n$ (definite metric), or M_h^n is of constant sectional curvature.

1.3.2. Proof. $\mathfrak{G} = \mathfrak{G}_+ + \mathfrak{G}_-$ where σ is $+1$ on $\mathfrak{G}_+$ and -1 on $\mathfrak{G}_-$ and where B is negative definite on $\mathfrak{G}_+$ and positive definite on $\mathfrak{G}_-$. $\sigma(\mathfrak{H}) = \mathfrak{H}$ implies $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{G}_+) + (\mathfrak{H} \cap \mathfrak{G}_-)$, so B is nondegenerate on $\mathfrak{H}$. Now the first statement is obvious. According to NOMIZU [15] the $\mathrm{ad}(G)$-invariance of B implies that the curvature tensor $\mathcal{R}$ of (M, Q) is given by

$$\mathcal{R}(X,Y)Z = \tfrac{1}{4}\{[X,[Y,Z]_{\mathfrak{M}}]_{\mathfrak{M}} - [Y,[X,Z]_{\mathfrak{M}}]_{\mathfrak{M}} - 2[[X,Y]_{\mathfrak{M}},Z]_{\mathfrak{M}} - 4[[X,Y]_{\mathfrak{H}},Z]_{\mathfrak{M}}\}$$

where $X, Y, Z \in \mathfrak{M}$ and subscripts denote B-orthogonal projections. For $X = Z$ the second term drops out and the first is minus half the third. If X and Y span a nonsingular plane S and if $B(X, Y) = 0$, then it follows that $||X||^2 \neq 0 \neq ||Y||^2$ and that

$$\begin{aligned}
||X||^2 \cdot ||Y||^2 \cdot K(S) &= -B(\mathcal{R}(X, Y)X, Y)\\
&= \tfrac{1}{4} B([[X, Y]_{\mathfrak{M}}, X]_{\mathfrak{M}}, Y) + B([[X, Y]_{\mathfrak{H}}, X]_{\mathfrak{M}}, Y)\\
&= \tfrac{1}{4} B([[X, Y]_{\mathfrak{M}}, X], Y) + B([[X, Y]_{\mathfrak{H}}, X], Y)\\
&= \tfrac{1}{4} B([X, Y]_{\mathfrak{M}}, [X, Y]) + B([X, Y]_{\mathfrak{H}}, [X, Y])\\
&= \tfrac{1}{4} B([X, Y]_{\mathfrak{M}}, [X, Y]_{\mathfrak{M}}) + B([X, Y]_{\mathfrak{H}}, [X, Y]_{\mathfrak{H}}).
\end{aligned}$$

Now let $Z = [X, Y]$. We have just seen that

$$||X||^2 \cdot ||Y||^2 \cdot K(S) = \tfrac{1}{4} B(Z_{\mathfrak{M}}, Z_{\mathfrak{M}}) + B(Z_{\mathfrak{H}}, Z_{\mathfrak{H}}).$$

Suppose further that $\sigma(S) = S$. We may then choose the basis $\{X, Y\}$ of S such that $\sigma(X) = \alpha X$ and $\sigma(Y) = \beta Y$ where $\alpha = \pm 1$ and $\beta = \pm 1$; in addition we may assume $||X||^2$ and $||Y||^2$ to have absolute value 1, and we may continue to assume $B(X, Y) = 0$. Then $\|X\|^2 = -\alpha$, $\|Y\|^2 = -\beta$ and $\sigma(Z) = \alpha\beta Z$. If $\alpha\beta = 1$, then $\|Z_{\mathfrak{M}}\|^2 \leq 0$ and $\|Z_{\mathfrak{H}}\|^2 \leq 0$, and it follows that $K(S) \leq 0$. If $\alpha\beta = -1$ then $\|Z_{\mathfrak{M}}\|^2 \geq 0$ and $\|Z_{\mathfrak{H}}\|^2 \geq 0$, and it follows that $K(S) \leq 0$. The second statement is now proved.

1. 3. 3. Suppose that the sectional curvatures of (M, Q) are bounded from above and below, and let S be a nonsingular 2-plane in $\mathfrak{M}$. We must prove $K(S) \leq 0$.

Given $Z \in \mathfrak{M}$, we write $Z = Z_+ + Z_-$ where $\sigma(Z_+) = Z_+$ and $\sigma(Z_-) = -Z_-$, and we write $Z_t = Z_+ + tZ_-$ for every real number t. Suppose that S has a basis $\{X, Y\}$ such that $X \perp Y_t$ for every t and $\{X, Y_t\}$ spans a 2-plane S_t. We then define

$$D_t = B(X, X) \cdot B(Y_t, Y_t) - B(X, Y_t)^2 = \|X\|^2(\|Y_+\|^2 + t^2\|Y_-\|^2) = D_0 + t^2 b$$

$$N_t = \tfrac{1}{4}\|[X, Y_t]_{\mathfrak{M}}\|^2 + \|[X, Y_t]_{\mathfrak{H}}\|^2 = N_0 + t^2(\tfrac{1}{4}\|[X, Y_-]_{\mathfrak{M}}\|^2 + \|[X, Y_-]_{\mathfrak{H}}\|^2)$$

$$+ t(\text{terms which vanish if } \sigma(X) = \pm X) = N_0 + t^2 u + tv$$

and have $K(S_t) = N_t/D_t$ if $D_t \neq 0$.

If $\sigma(S) = S$, we have seen that $K(S) \leq 0$. Now suppose $\sigma(S) \neq S$. If $\sigma(S) \cap S \neq 0$, then we have $0 \neq X \in S$ with $\sigma(X) = \pm X$. Suppose first that $\sigma(X) = X$. Then S has basis $\{X, Y\}$ with $X \perp Y_+$ and $Y_t \neq 0 \neq Y_-$, and each S_t is a plane. $D_t = D_0 + t^2 b$ with $D_0 > 0$ and $b < 0$, so D_t has precisely two zeros, $D_a = D_{-a} = 0$, $a > 0$. $N_t = N_0 + t^2 u$ with $N_0 \leq 0$ because $K(S_0) \leq 0$, and with $u \geq 0$. N_t must vanish if $D_t = 0$ by boundedness of $K(S_s)$ for s near t. If N_t is identically zero, then $K(S) = 0$. Otherwise N_t changes sign with D_t and $K(S) < 0$. A similar argument holds if $\sigma(X) = -X$.

Now suppose $\sigma(S) \cap S = 0$, and choose a basis $\{X, Y\}$ of S with $\|X\|^2 \neq 0$ and $X \perp Y_+$. Suppose first that $\|X\|^2 > 0$, and let T be the span of X and Y_-. We do not generally have $X \perp Y_t$, so D_t is of the form $D_t = D_0 + ta + t^2 b$. D_t has two zeros for S_0 is positive definite, and, for $|t|$ large, S_t is indefinite because $\|Y_t\|^2 < 0$. We may assume N_t not identically zero; then $N_t = N_0 + tv + t^2 u$ has at most two zeros, and boundedness of the defined $K(S_t)$ shows that a zero of D_t is a zero of N_t. Now we have $D_t = r(t - x)(t - y)$ and $N_t = s(t - x)(t - y)$ because the zeros x and y of D_t are distinct, and $K(S_t) = N_t/D_t = s/r$ when $D_t \neq 0$. As $K(S_0) \leq 0$ by the last paragraph, it follows that $K(S) \leq 0$. The necessity in the third statement is proved.

1. 3. 4. Suppose that (M, Q) has every sectional curvature non-positive; we must prove the sectional curvatures bounded. If the sectional curvatures of (M, Q) are not bounded, then there is a sequence $\{P_i\}$ in the GRASSMANN manifold L of nonsingular 2-planes of some signature in $\mathfrak{M}$, such that $\{K(P_i)\} \to -\infty$. We may assume $\{P_i\} \to S$ for some necessarily singular 2-plane S in $\mathfrak{M}$ because the GRASSMANN manifold of all 2-planes in $\mathfrak{M}$ is compact.

Suppose first that S is not totally isotropic. Then S has a basis $\{X, Y\}$ such that either $\{X_+, Y_+\}$ or $\{X_-, Y_-\}$ is an orthonormal basis of a plane, and $||X||^2 \neq 0$. We may assume $\{X_+, Y_+\}$ orthonormal, and we have S_t, $D_t = D_0 + ta + t^2 b$ with $D_0 \neq 0$, and $N_t = N_0 + tv + t^2 u$, as in § 1.3.3, with $K(S_t) = N_t / D_t$ whenever $D_t \neq 0$. D_t changes sign at $t = 1$, and each P_i has an element Z_i with $||Z_i||^2 = ||X||^2$; this gives a sequence $\{t_i\} \to 1$ such that $(S_i = S_{t_i} \in L)$ $\{K(S_i)\} \to -\infty$, by choice of S. In particular N_t is not constant. Our hypothesis on sign of curvature implies that N_t and D_t change sign simultaneously. If D_t is linear, it follows that N_t is linear and both are constant multiples of $t - 1$; then $K(S_i)$ is constant, which is impossible. If D_t is not linear, then its zeros are distinct because it changes sign at $t = 1$, and again $K(S_i)$ is constant. This is a contradiction.

Now suppose S totally isotropic; then S has basis $\{X, Y\}$ such that both $\{X_+, Y_+\}$ and $\{X_-, Y_-\}$ are orthonormal. Let $Q_{s,t}$ be the plane spanned by X_s and Y_t.

$$D_{s,t} = B(X_s, X_s) \cdot B(Y_t, Y_t) - B(X_s, Y_t)^2 = (s^2 - 1)(t^2 - 1)$$

and the corresponding $N_{s,t}$ is quadratic in s and quadratic in t. For fixed $t \neq \pm 1$, $D_{s,t}$ changes sign as s crosses ± 1; thus the same is true of $N_{s,t}$ and so $s^2 - 1$ divides $N_{s,t}$. Similarly $t^2 - 1$ divides $N_{s,t}$, and it follows that $K(Q_{s,t})$ is constant for $|s| \neq 1 \neq |t|$. That is impossible because L contains a sequence of $Q_{s,t}$ converging to S. As before, this contradicts constancy of the $K(Q_{s,t})$.

The third statement is proven.

1.3.5. Let $X, Y, Z \in \mathfrak{M}$ be mutually orthogonal such that plane S spanned by X and Y has $K(S) \neq 0$, $[X, Z] = 0$ and $||Y||^2 \cdot ||Z||^2 < 0$. Then the plane T_t spanned by X and $Y + tZ$ is nonsingular for large t, and the formula for curvature then gives

$$||X||^2 \cdot ||Y||^2 \cdot K(S) = ||X||^2 \cdot ||Y + tZ||^2 \cdot K(T_t)$$

because $(Y + tZ) \perp X$ and $[X, Y + tZ] = [X, Y]$. For large t,

$$||Y + tZ||^2 \cdot ||Y||^2 < 0,$$

so $K(T_t)$ has sign opposite that of $K(S)$. Thus one of them must be positive.

This completes the proof of Theorem 1.3.1. *Q.E.D.*

Chapter I

Locally and Globally Isotropic Pseudo-RIEMANNian Manifolds

This Chapter develops the basic theory of isotropic pseudo-RIEMANNian manifolds. Important examples are described in § 2. In § 4 the local structure of locally isotropic manifolds is described in terms of these examples, and § 5 shows that these examples act as metric covering spaces for complete locally isotropic manifolds. § 6 extends that covering theorem to a partial classification, and § 7 gives the global classification of isotropic pseudo-RIEMANNian manifolds.

2. The model spaces

We shall describe spaces analogous to the isotropic RIEMANNian manifolds. Our later analyses of isotropic pseudo-RIEMANNian manifolds will depend on comparison with the spaces described below. The spaces $\mathbf{R}_h^n$ described below are flat complete connected simply connected isotropic pseudo-RIEMANNian manifolds in an obvious way ([17], § 3).

2. 1. Indefinite unitary groups over division algebras. Let $\mathbf{F}$ be a real division algebra $\mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{K}$ (quaternion). There is a natural conjugation $x \to \bar{x}$ of $\mathbf{F}$ over $\mathbf{R}$; this defines the multiplicative group $\mathbf{F}'$ of unmodular $(x\bar{x} = 1)$ elements of $\mathbf{F}$, and defines the notion of hermitian form on an $\mathbf{F}$-vectorspace. Given integers $0 \leq h \leq n$, $\mathbf{F}_h^n$ will denote the right $\mathbf{F}$-vectorspace of n-tuples together with the hermitian form

$$\mathfrak{b}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \Sigma\, x_i \bar{y}_i \,.$$

The "indefinite unitary" group over $\mathbf{F}$, $\mathbf{U}^h(n, \mathbf{F})$, is defined to be the group of automorphisms of $\mathbf{F}_h^n$. The usual notation for indefinite unitary groups over $\mathbf{F}$ is:

$$\mathbf{U}^h(n, \mathbf{R}) = \mathbf{O}^h(n), \text{ indefinite orthogonal group}$$

$$\mathbf{U}^h(n, \mathbf{C}) = \mathbf{U}^h(n), \text{ indefinite unitary group}$$

$$\mathbf{U}^h(n, \mathbf{K}) = \mathbf{Sp}^h(n), \text{ indefinite symplectic group}$$

If $h = 0$ or $h = n$, one omits writing it. Finally, $\mathbf{SO}^h(n)$ is defined to be the identity component of $\mathbf{O}^h(n)$, and $\mathbf{SU}^h(n)$ consists of the elements of $\mathbf{U}^h(n)$ of determinant 1.

2. 2. Indefinite GRASSMANN manifolds over division algebras. If $h' \leq h$ and $n' - h' \leq n - h$ then there is an inclusion $\mathbf{F}_{h'}^{n'} \subset \mathbf{F}_h^n$ consistent with the hermitian forms; it is unique up to action of an element of $\mathbf{U}^h(n, \mathbf{F})$, i.e., $\mathbf{U}^h(n, \mathbf{F})$ acts transitively on the collection of all subspaces $\mathbf{F}^{n'}_{h'}$ of $\mathbf{F}^n_h$. The subgroup of $\mathbf{U}^h(n, \mathbf{F})$ leaving invariant a particular $\mathbf{F}^{n'}_{h'}$ is isomorphic to $\mathbf{U}^{h-h'}(n - n', \mathbf{F}) \times \mathbf{U}^{h'}(n', \mathbf{F})$. Thus we may view the collection of all $\mathbf{F}^{n'}_{h'}$ in $\mathbf{F}^n_h$ as the coset space

$$\mathbf{G}_{h',h;n',n}(\mathbf{F}) = \mathbf{U}^h(n, \mathbf{F})/\{\mathbf{U}^{h-h'}(n - n', \mathbf{F}) \times \mathbf{U}^{h'}(n', \mathbf{F})\} . \qquad (2.2.1)$$

$\mathbf{G}_{h',h;n',n}(\mathbf{F})$ is the quotient of a LIE group by a closed subgroup, and thus carries the structure of a differentiable manifold; it has dimension $f(n - n')n'$ where f is the dimension of $\mathbf{F}$ over $\mathbf{R}$.

For the moment we write (2.2.1) above as $M = G/H$, where H is the isotropy group at $p \in M$. Then H has an element s, defined to be the identity I on p and $-I$ on $p^\perp$, such that $s^2 = I$ (on $\mathbf{F}^n_h$) and H is the full centralizer of s in G. In other words, M is a symmetric coset space and LIE algebras satisfy $\mathfrak{G} = \mathfrak{H} + \mathfrak{P}$ where $\mathfrak{H}$ is the eigenspace of $+1$ for $\mathrm{ad}(s)$ and $\mathfrak{P}$ is the eigenspace of -1; $\mathfrak{H} = \mathfrak{P}^\perp$ here with respect to the KILLING form B of $\mathfrak{G}$, and B is nondegenerate on $\mathfrak{P}$. The restriction of $-B$ to $\mathfrak{P}$ thus endows M with a G-invariant pseudo-RIEMANNian structure. This structure has signature (a, b) where $n'' = n - n'$, $h'' = h - h'$, $a = f(h'(n'' - h'') + h''(n' - h'))$ and $b = f(h'h'' + (n' - h')(n'' - h'')) = f(n - n')n'$; it has non-negative involutive sectional curvature. If we use B, rather than $-B$, to endow M with a metric, then the roles of a and b are interchanged and we obtain a pseudo-RIEMANNian structure of non-positive involutive sectional curvature. M is always viewed with one of these structures, and is called an indefinite GRASSMANN manifold over $\mathbf{F}$.

2. 3. The indefinite elliptic space $\mathbf{P}^n_h(\mathbf{F})$ is defined to be the GRASSMANN manifold of all $\mathbf{F}^1_0$ in $\mathbf{F}^{n+1}_h$ with the metric of non-negative involutive sectional curvature; it has dimension fn and signature $(fh, f(n - h))$; if $h = 0$ it is the ordinary elliptic space over $\mathbf{F}$, and if $h = n$ it is the usual hyperbolic space, with metric reversed, over $\mathbf{F}$. An obvious modification of the coset representation (2.2.1) yields

$$\mathbf{P}^n_h(\mathbf{R}) = \mathbf{SO}^h(n + 1)/\mathbf{O}^h(n) \qquad (2.3.1)$$

$$\mathbf{P}^n_h(\mathbf{C}) = \mathbf{SU}^h(n + 1)/\mathbf{U}^h(n) \qquad (2.3.2)$$

$$\mathbf{P}^n_h(\mathbf{K}) = \mathbf{Sp}^h(n + 1)/\{\mathbf{Sp}^h(n) \times \mathbf{Sp}(1)\} \qquad (2.3.3)$$

In the representation $\mathbf{P}^n_h(\mathbf{F}) = G/H$ immediately above, we still have the decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{P}$, and the negative of the KILLING form of $\mathfrak{G}$ gives $\mathfrak{P}$ the structure of $\mathbf{F}^n_h$. The transformations of $\mathfrak{P}$ generated by $\mathrm{ad}(H)$

are easily seen to be precisely the group generated by $\mathbf{U}^h(n, \mathbf{F})$ together with the scalar multiplications by elements of $\mathbf{F}'$. In particular, $\mathbf{P}^n_h(\mathbf{F})$ is an isotropic pseudo-RIEMANNian manifold M^{In}_{Ih}, the pseudo-RIEMANNian analog of the elliptic space $\mathbf{P}^n(\mathbf{F})$.

$\mathbf{P}^n_h(\mathbf{F})$ is complete because it is globally symmetric. If $\mathbf{F} \neq \mathbf{R}$, it is simply connected, as is seen by its representation (2.3.2) or (2.3.3) as coset space of a simply connected group by a closed connected subgroup. $\mathbf{P}^n_h(\mathbf{R})$ is the manifold $\mathbf{S}^n_h/\{\pm I\}$ of [17] with universal pseudo-RIEMANNian covering manifold $\tilde{\mathbf{S}}^n_h$ described in ([17], § 4.4 and § 11.2).

2. 4. The indefinite hyperbolic space $\mathbf{H}^n_h(\mathbf{F})$ is the GRASSMANN manifold of all $\mathbf{F}^1_0$ in $\mathbf{F}^{n+1}_{n-h}$ with the metric of non-positive involutive sectional curvature; one obtains it by replacing the metric with its negative on $\mathbf{P}^n_{n-h}(\mathbf{F})$. Thus we have

$$\mathbf{H}^n_h(R) = \mathbf{SO}^{h+1}(n+1)/\mathbf{O}^h(n) \tag{2.4.1}$$

$$\mathbf{H}^n_h(\mathbf{C}) = \mathbf{SU}^{h+1}(n+1)/\mathbf{U}^h(n) \tag{2.4.2}$$

$$\mathbf{H}^n_h(\mathbf{K}) = \mathbf{Sp}^{h+1}(n+1)/\{\mathbf{Sp}^h(n)\times \mathbf{Sp}(1)\} \tag{2.4.3}$$

from the obvious $\mathbf{U}^s(t, \mathbf{F}) = \mathbf{U}^{t-s}(t, \mathbf{F})$.

As with elliptic spaces, $\mathbf{H}^n_h(\mathbf{F})$ is a complete globally symmetric isotropic pseudo-RIEMANNian manifold M^{In}_{Ih}, which is the hyperbolic space over $\mathbf{F}$ in case $h = 0$; it is simply connected if $\mathbf{F} \neq \mathbf{R}$, and $\mathbf{H}^n_h(\mathbf{R})$ is the manifold $\mathbf{H}^n_h/\{\pm I\}$ of [17].

2. 5. Notation for some LIE groups. The compact simply connected LIE group of CARTAN classification type X will be denoted $\mathbf{X}$; thus $\mathbf{A}_n = \mathbf{SU}(n+1)$, $\mathbf{B}_n (n > 1)$ is the universal covering group $\mathbf{Spin}(2n+1)$ of $\mathbf{SO}(2n+1)$, $\mathbf{C}_n = \mathbf{Sp}(n)$, $D_n = \mathbf{Spin}(2n)$ for $n > 1$, and $\mathbf{G}_2$ is the group of sutomorphisms of the CAYLEY-DICKSON algebra **Cay**. In general, the CLIFFORD algebra construction gives a double covering group $\mathbf{Spin}^h(n)$ of $\mathbf{SO}^h(n)$. Finally, $\mathbf{F}^*_4$ will denote the connected LIE group of type F_4 whose maximal compact subgroup is $\mathbf{B}_4$. $\mathbf{F}^*_4$ is centerless and simply connected.

2. 6. The indefinite CAYLEY planes $\mathbf{P}^2_h(\mathbf{Cay})$ and $\mathbf{H}^2_h(\mathbf{Cay})$. Recall that the isotropic RIEMANNian manifolds are the EUCLIDean spaces, the spheres, the elliptic and hyperbolic spaces over a field $\mathbf{F}$, and the manifolds $\mathbf{F}_4/\mathbf{B}_4$ and $\mathbf{F}^*_4/\mathbf{B}_4$ in invariant RIEMANNian metric. The latter two, the CAYLEY elliptic and hyperbolic planes, have indefinite metric analogs; furthermore, each obviously gives a negative definite pseudo-RIEMANNian manifold on reversal of its metric.

Let σ and τ be the automorphisms of $\mathbf{F}_4$ defined by $\sigma = \mathrm{ad}(s)$ and $\tau = \mathrm{ad}(t)$

where $s \in \mathbf{B}_4 \subset \mathbf{F}_4$ is the unique central element of order 2 in $\mathbf{B}_4$, and where $t \in \mathbf{F}_4$ is the element of $\mathbf{B}_4$ with image $\begin{pmatrix} -I_8 & 0 \\ 0 & +1 \end{pmatrix}$ in $\mathbf{SO}(9)$. There is a direct sum vectorspace decomposition $\mathfrak{F}_4 = \mathfrak{B}_4 + \mathfrak{M}$ where $\mathfrak{B}_4$ is the eigenspace of $+1$ for σ and $\mathfrak{M}$ is the eigenspace of -1, and $\mathfrak{F}_4^* = \mathfrak{B}_4 + \sqrt{-1}\,\mathfrak{M}$. As $\sigma\tau = \tau\sigma$, we also have $\mathfrak{B}_4 = \mathfrak{H}_1 + \mathfrak{P}_1$ and $\mathfrak{M} = \mathfrak{H}_2 + \mathfrak{P}_2$ where $\mathfrak{H}_i$ is the eigenspace of $+1$ for τ and $\mathfrak{P}_i$ is the eigenspace of -1; $\mathfrak{F}_4' = \mathfrak{B}_4' + \mathfrak{M}'$ is the LIE algebra defined by $\mathfrak{B}_4' = \mathfrak{H}_1 + \sqrt{-1}\,\mathfrak{P}_1$ and $\mathfrak{M}' = \mathfrak{H}_2 + \sqrt{-1}\,\mathfrak{P}_2$ and σ induces an automorphism of $\mathfrak{F}_4'$ for which $\mathfrak{B}_4'$ is the eigenspace of $+1$ and $\mathfrak{M}'$ is the eigenspace of -1. Similarly, applying τ to the decomposition of $\mathfrak{F}_4^*$, we obtain a LIE algebra $\mathfrak{F}_4'' = \mathfrak{B}_4' + \sqrt{-1}\,\mathfrak{M}'$. To simplify the above, we first observe that $\mathfrak{B}_4' = \mathfrak{Spin}^1(9)$ from $\mathfrak{B}_4 = \mathfrak{Spin}(9)$ and the definition of τ. Let G' and G'' be the adjoint groups of $\mathfrak{F}_4'$ and $\mathfrak{F}_4''$, and let H' and H'' be the corresponding analytic subgroups for $\mathfrak{B}_4'$. G' and G'' are real forms of F_4, and are noncompact because H' and H'' are not compact. H' and H'' each has maximal compact subgroup of type D_4; thus the maximal compact subgroups of G' and G'' have dimensions ≥ 28; it follows that $G' = G'' = \mathbf{F}_4^*$ and $\mathfrak{F}_4' = \mathfrak{F}_4'' = \mathfrak{F}_4^*$. Now it is clear that we have only: $\mathfrak{F}_4 = \mathfrak{B}_4 + \mathfrak{M}$, $\mathfrak{F}_4^* = \mathfrak{B}_4 + \mathfrak{M}^*$ and $\mathfrak{F}_4^* = \mathfrak{B}_4' + \mathfrak{M}'$ where $\mathfrak{M}^* = \sqrt{-1}\,\mathfrak{M}$, $\mathfrak{M}'$ is given above, and $\mathfrak{B}_4' = \mathfrak{Spin}^1(9)$.

The decompositions above yield symmetric coset spaces $\mathbf{F}_4/\mathbf{B}_4$, $\mathbf{F}_4^*/\mathbf{B}_4$ and $\mathbf{F}_4^*/\mathbf{Spin}^1(9)$. With KILLING form for invariant metric, we obtain spaces of non-positive involutive curvature with metrics respectively of signature (16,0), (0,16), (8,8); the statement about (8,8) comes from the equivalence of $\mathfrak{M}'$ and $\sqrt{-1}\,\mathfrak{M}'$, which comes from the equivalence of $\mathbf{F}_4'$ and $\mathbf{F}_4''$ above.

The CAYLEY elliptic planes are the following symmetric coset spaces with pseudo-RIEMANNian structure of non-negative involutive sectional curvature induced by the negative of the KILLING form:

$\mathbf{P}_0^2(\mathbf{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$, positive definite CAYLEY elliptic plane. (2.6.1)

$\mathbf{P}_1^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}^1(9)$, indefinite CAYLEY elliptic plane. (2.6.2)

$\mathbf{P}_2^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}(9)$, negative definite CAYLEY elliptic plane. (2.6.3)

The CAYLEY hyperbolic planes are obtained the same way, with non-positive involutive sectional curvature from the KILLING form:

$\mathbf{H}_0^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}(9)$, positive definite CAYLEY hyperbolic plane. (2.6.4)

$\mathbf{H}_1^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}^1(9)$, indefinite CAYLEY hyperbolic plane. (2.6.5)

$\mathbf{H}_2^2(\mathbf{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$, negative definite CAYLEY hyperbolic plane. (2.6.6)

$\mathbf{P}^2_h(\mathbf{Cay})$ and $\mathbf{H}^2_h(\mathbf{Cay})$ are complete because they are globally symmetric; each is a pseudo-RIEMANNian manifold M^{16}_{8h}, and it is known from the theory of RIEMANNian symmetric spaces that they are isotropic for $h \neq 1$. We will now prove that $\mathbf{P}^2_1(\mathbf{Cay})$ is isotropic; isotropy then follows for $\mathbf{H}^2_1(\mathbf{Cay})$. To do this, we must show for every real number γ that $\mathbf{Spin}^1(9)$ is transitive on the nonzero vectors of square norm γ in $\mathfrak{M}' = \mathbf{R}^{16}_8$.

Let σ be a CARTAN involution of $\mathfrak{F}^*_4$ which preserves $\mathfrak{Spin}^1(9)$; $\mathfrak{M}' = \mathfrak{A} + \mathfrak{B}$ under σ where $\mathfrak{A}$ is positive definite and $\mathfrak{B}$ is negative definite. $\mathbf{Spin}(8)$, the subgroup of $\mathbf{Spin}^1(9)$ which preserves this decomposition, maximal compact subgroup of $\mathbf{Spin}^1(9)$, acts by $u: a + b \to f(u)\,(a) + f(t(u))\,(b)$ where $f: \mathbf{Spin}(8) \to \mathbf{SO}(8)$ is the projection and t is triality. If $a \neq 0 \neq b$, it follows that the isotropy subgroup of $\mathbf{Spin}(8)$ at $a + b$ lies in the fixed point set $\mathbf{G}_2$ of t. Given $\alpha > 0 > \beta$ let $S_\alpha = \{a \in \mathfrak{A} : ||a||^2 = a\}$ and $S_\beta = \{b \in \mathfrak{B} : ||b||^2 = \beta\}$; it follows by counting dimensions that $\mathbf{Spin}(8)$ is transitive on $S_\alpha \times S_\beta$ and $\mathbf{G}_2$ is its isotropy subgroup[2]. Finally, let $\{g_t\}$ be a 1-parameter group in $\mathbf{Spin}^1(9)$ such that $\sigma(g_t) = g_t^{-1}$. As each g_t is a semisimple linear transformation of $\mathfrak{M}'$ with all eigenvalues real, and each g_t preserves the inner product on $\mathfrak{M}' = \mathbf{R}^{16}_8$, we can replace the parameter by a multiple and find $r \in \mathfrak{A}$ and $s \in \mathfrak{B}$ such that $||r||^2 = 1 = -\,||s||^2$, $g_t(r) = \cosh(t) r + \sinh(t) s$, and $g_t(s) = \sinh(t) r + \cosh(t) s$.

Let $x \in \mathfrak{M}'$, $x \neq 0$, $||x||^2 = \gamma$; $x = a + b$ with $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$; $||a||^2 = \alpha$ and $||b||^2 = \beta$. Applying an element of $\mathbf{Spin}(8)$, we may assume

$$a = \sqrt{\alpha}\, r \quad \text{and} \quad b = \sqrt{-\beta}\, s\,.$$

Now a close look at $\mathbf{R}^2_1$ and matrices $\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$ shows:

(1) If $\gamma > 0$, then some g_t carries x to $\sqrt{\gamma}\, r$.

(2) If $\gamma = 0$, then some g_t carries x to $r + s$.

(3) If $\gamma < 0$, then some g_t carries x to $\sqrt{-\gamma}\, s$.

This completes the proof that $\mathbf{P}^2_1(\mathbf{Cay})$ is isotropic.

We have proved that each of the CAYLEY planes is an isotropic pseudo-RIEMANNian manifold. Observe also that each CAYLEY plane is complete, symmetric, and simply connected.

2. 7. CARTAN's technique for the full group of isometries of a RIEMANNian symmetric space can be extended to indefinite metric. We will make that extension now, and then calculate the full group of isometries for each of our model spaces in § 2.8.

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[2]) I am indebted to Professor JACQUES TITS for bringing to my attention this fact that **Spin** (8) is transitive on $\mathbf{S}^7 \times \mathbf{S}^7$.

Let M_h^n be an irreducible complete connected simply connected pseudo-RIEMANNian symmetric manifold, let s be the symmetry at $x \in M_h^n$; let $G = \mathbf{I}_0(M_h^n)$ and $G' = \mathbf{I}(M_h^n)$, and let H and H' be their respective isotropy subgroups at x. The problem is to find G' and H' from G, H and s. $G' = G \cdot H'$ because M_h^n is connected, so it suffices to find H'; knowing the subgroup $H'' = H \cup s \cdot H$ of H', then, it suffices to find $\{k_1 = 1, k_2, \ldots\} \subset H'$ such that H' is the disjoint union of the $k_i H''$. The whole point is that the following gives enough information about the k_i so that, in any particular case, one can find them:

Theorem. *Each k_i, $i > 1$, induces an outer automorphism of H which is induced by an automorphism of G. The automorphisms of H induced by k_i and k_j differ by an inner automorphism of H, if and only if $k_i = k_j$.*

Let $\sigma = \mathrm{ad}(s)$; then $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ where $\mathfrak{H}$ is the σ-eigenspace of $+1$ and $\mathfrak{M}$ is the eigenspace of -1. As M_h^n is irreducible, i.e., as $\mathrm{ad}(H)$ is irreducible on $\mathfrak{M}$, the centralizer of $\mathrm{ad}(H)$ in the algebra of linear endomorphisms of $\mathfrak{M}$ is a real division algebra $\mathbf{F}$ (by SCHUR's Lemma). Now identify H, H'' and H' with the linear groups they induce on $\mathfrak{M}$. I claim that we need only prove $\mathbf{F}' \subset H''$. For then if $k' \in H'$ induces an inner automorphism of H, so $k'k \in \mathbf{F}$ for some $k \in H$, we have $k'k \in \mathbf{F}'$ because (since it preserves the metric) $\det.(k'k) = \pm 1$, whence $k'k \in H''$ and so $k' \in H''$; the theorem would follow.

The fact that $\mathbf{F}' \subset H''$ is obvious if $\mathbf{F} = \mathbf{R}$ (in which case $1 \in H$ is $1 \in \mathbf{F}'$ and $s \in H''$ is $-1 \in \mathbf{F}'$); if $\mathbf{F} \neq \mathbf{R}$ it is a consequence of:

Lemma. (1) $\mathbf{F} = \mathbf{R}$ *if and only if H is semisimple.* (2) $\mathbf{F} \neq \mathbf{K}$. (3) *These are equivalent*: (a) *the center of H is a circle group,* (b) *H is not semisimple,* (c) *M_h^n has a G-invariant indefinite metric KÄHLER structure which induces the original pseudo-RIEMANNian structure,* (d) $\mathbf{F} = \mathbf{C}$.

Remark. This is the indefinite-metric version of ([20], Lemma 2.4.3), which is essentially due to CARTAN [5, 6] with part of the proof taken from CARTAN [5] and BOREL [3].

Proof of Lemma. Let Z be the center of H; then $Z \subset \mathbf{F}'$ because $Z \subset \mathbf{F}$ and $\det.z = \pm 1$ for every $z \in Z$. Thus $\mathbf{F} \neq \mathbf{C}$ implies that Z is finite. H is reductive because it is an irreducible linear group. It follows that H is semisimple if $\mathbf{F} \neq \mathbf{C}$.

Suppose that $\mathbf{F} \neq \mathbf{R}$, i.e., that $\mathbf{F}'$ has an element J with $J^2 = -I$. Then we have real subalgebras $\mathfrak{L} = \mathfrak{H}^C + (I + \sqrt{-1}\, J)\, \mathfrak{M}$ and $\overline{\mathfrak{L}} = \mathfrak{H}^C + (I - \sqrt{-1}\, J)\, \mathfrak{M}$ of the complexified algebra $\mathfrak{G}^C$ which are complex conjugate over $\mathfrak{G}$, which span $\mathfrak{G}^C$ over $\mathbf{R}$, and which have intersection $\mathfrak{H}^C$. As H is connected (because M_h^n is simply connected), this is A. FROLICHER's criterion ([7], § 20) that J define a G-invariant complex structure on $M_h^n = G/H$. From our pseudo-RIEMANNian metric we now have a G-invariant indefinite hermitian metric on M_h^n; the complex structure is parallel because H contains the holonomy group, so the indefinite hermitian metric is KÄHLERian.

Continue to assume $\mathbf{F} \neq \mathbf{R}$. Let T be a maximal compact subgroup of H, so $T'' = T \cup s \cdot T$ is maximal compact in H''; extend T to a maximal compact subgroup S of G, so $S'' = S \cup s \cdot S$ is maximal compact in $G'' = G \cup s \cdot G$. The group G is semisimple ([15], p. 56), because M_h^n is not flat as a consequence of $\mathbf{F} \neq \mathbf{R}$. Thus $\mathfrak{G}''$ has a CARTAN involution τ which induces a CARTAN involution of $\mathfrak{H}''$, and we may assume τ trivial on S'' and T''. In particular, τ commutes with $\sigma = \mathrm{ad}(s)$ and preserves the summands in $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$. The transformation t of $\mathfrak{M}$ induced by τ normalizes the linear group H, so $t\mathbf{F}'t^{-1} = \mathbf{F}'$. If $\mathbf{F} = \mathbf{K}$, then t centralizes $\mathbf{F}$ because $\mathbf{F}'$ has no outer automorphism and its element of order 2 is central. If $\mathbf{F} = \mathbf{C}$ and t does not centralize $\mathbf{F}$, then $tft^{-1} = \bar{f}$ for every $f \in \mathbf{F}$. Now let $\mathfrak{H} = \mathfrak{H}_+ + \mathfrak{H}_-$ and $\mathfrak{M} = \mathfrak{M}_+ + \mathfrak{M}_-$ where subscripts denote the eigenspace of $+1$ or -1 for τ, and we define $\mathfrak{H}^* = \mathfrak{H}_+ + \sqrt{-1}\,\mathfrak{H}_-$, $\mathfrak{M}^* = \mathfrak{M}_+ + \sqrt{-1}\,\mathfrak{M}_-$ and $\mathfrak{G}^* = \mathfrak{H}^* + \mathfrak{M}^*$. If G^* is the adjoint group of $\mathfrak{G}^*$ and H^* is the analytic subgroup for $\mathfrak{H}^*$, then G^*/H^* is a RIEMANNian symmetric space. If t centralizes $\mathbf{F}$, then J preserves both $\mathfrak{M}_+$ and $\mathfrak{M}_-$, and thus induces a transformation J^* of square $-I$ on $\mathfrak{M}^*$ which commutes with the irreducible linear group $H^*(= \mathrm{ad}(H^*))$ on $\mathfrak{M}^*$. Otherwise, J interchanges $\mathfrak{M}_+$ and $\mathfrak{M}_-$. Then J induces a transformation J' of $\mathfrak{M}^*$ by $J'(X) = \sqrt{-1}\,J(X)$ if $X \in \mathfrak{M}_+$ or if $X \in \sqrt{-1}\,\mathfrak{M}_-$. J' centralizes H^* and has square $+I$. This is impossible because H^* is irreducible on $\mathfrak{M}^*$. Thus J preserves both $\mathfrak{M}_+$ and $\mathfrak{M}_-$, and induces J^*. It follows, as with J on G/H, that J^* induces a G^*-invariant KÄHLER metric on the *compact* RIEMANNian symmetric space G^*/H^*. Now the cohomology group $\mathbf{H}^2(G^*/H^*; \mathbf{R}) \neq 0$ so the homotopy group $\pi_2(G^*/H^*)$ is infinite, and the homotopy sequence

$$0 = \pi_2(G^*) \to \pi_2(G^*/H^*) \to \pi_1(H^*) \to \pi_1(G^*) = \text{finite}$$

shows that $\pi_1(H^*)$ is infinite. Thus H^*, and consequently H, is not semisimple.

We have proven that H is not semisimple when $\mathbf{F} \neq \mathbf{R}$, and that M_h^n has a G-invariant indefinite metric KÄHLER structure then which induces its original pseudo-RIEMANNian structure. As H is reductive, it also follows that the center Z of H is infinite; as $Z \subset \mathbf{F}'$, it follows that $\mathbf{F} = \mathbf{C}$. The Lemma now follows, and the Theorem is proved. *Q. E. D.*

2. 8. Enumeration of the full groups of isometries of the model spaces.

It is known ([17], § 3) and easily seen that the full group of isometries $\mathbf{I}(\mathbf{R}_h^n)$ is the semidirect product $\mathbf{O}^h(n) \cdot \mathbf{R}^n$ acting by

$$(A, \vec{a}) : \vec{x} \to A(\vec{x}) + \vec{a}$$

where $A \in \mathbf{O}^h(n)$, $\vec{a} \in \mathbf{R}^n$ = underlying vector group of $\mathbf{R}_h^n$, and $\vec{x} \in \mathbf{R}_h^n$; the action of A is the usual linear action.

If $h < n - 1$, then the universal pseudo-RIEMANNian covering manifold $\tilde{\mathbf{S}}_h^n$ of $\mathbf{P}_h^n(\mathbf{R})$ is the quadric $\mathbf{S}_h^n = \{\vec{x} \in \mathbf{R}_h^{n+1} : \|\vec{x}\|^2 = 1\}$ with induced metric; if $h = n$, then $\tilde{\mathbf{S}}_h^n$ is one of the two components of $\mathbf{S}_h^n$; if $h = n - 1$, then the situation is rather complicated ([17], § 11) and $\tilde{\mathbf{S}}_h^n$ is an infinite covering manifold of $\mathbf{S}_h^n$. Similarly, the universal pseudo-RIEMANNian covering manifold $\tilde{\mathbf{H}}_h^n$ of $\mathbf{H}_h^n(\mathbf{R})$ is described by the quadric $\mathbf{H}_h^n = \{\vec{x} \in \mathbf{R}_{h+1}^{n+1} : \|\vec{x}\|^2 = -1\}$ with induced metric. It is known ([17], § 4.5) that $\mathbf{I}(\mathbf{S}_h^n) = \mathbf{O}^h(n+1)$ and $\mathbf{I}(H_h^n) = \mathbf{O}^{h+1}(n+1)$. Together with ([17], § 11), this will suffice for our purposes.

We will apply the technique of § 2.7 to the other model spaces. The starting point is a result of K. NOMIZU ([15], Th. 16.1) which assures us that the representations $M = G/H$ of model spaces M of nonconstant curvature, given by (2.3.2–3), (2.4.2–3) and (2.6.1–6), have the property: $\mathbf{I}_0(M)$ *consists of the isometries induced by* G. In each case, this means $\mathbf{I}_0(M) = G/Z$ where Z is the center of G; Z is trivial if G is of type F_4, $Z = \{\pm I\}$ if $G = \mathbf{Sp}^k(n+1)$, and Z consists of the scalar matrices $\exp(2\pi\sqrt{-1}\,m/(n+1))I$ if $G = \mathbf{SU}^k(n+1)$.

Let M be a CAYLEY plane G/H as in (2.6.1–6). Then $G = \mathbf{I}_0(M)$, G contains the symmetry, and $H = \mathbf{Spin}^k(9)$ has no outer automorphism. By Theorem 2.7, $G = \mathbf{I}(M)$:

$$\mathbf{I}(\mathbf{P}_0^2(\mathbf{Cay})) = \mathbf{I}(\mathbf{H}_2^2(\mathbf{Cay})) = \mathbf{F}_4;\ \mathbf{Spin}\ (9) \text{ is isotropy subgroup.} \tag{2.8.1}$$

$$\mathbf{I}(\mathbf{P}_1^2(\mathbf{Cay})) = \mathbf{I}(\mathbf{H}_1^2(\mathbf{Cay})) = \mathbf{F}_4^*;\ \mathbf{Spin}^1\ (9) \text{ is isotropy subgroup.} \tag{2.8.2}$$

$$\mathbf{I}(\mathbf{P}_2^2(\mathbf{Cay})) = \mathbf{I}(\mathbf{H}_0^2(\mathbf{Cay})) = \mathbf{F}_4^*;\ \mathbf{Spin}\ (9) \text{ is isotropy subgroup.} \tag{2.8.3}$$

Let M be an indefinite quaternionic elliptic space G/H; $G = \mathbf{Sp}^h(n+1)$ and $\mathbf{I}_0(M) = G/\{\pm I\}$. If $n > 1$, then $\mathbf{Sp}^h(n)$ is not isomorphic to $\mathbf{Sp}(1)$, and it follows that $H/\{\pm I\}$ has no outer automorphism. If $n = 1$, then G is of the type C_2; CARTAN type C_2 is the same as type B_2 so one would expect that M have constant curvature; indeed, it is standard that $\mathbf{P}_0^1(\mathbf{K})$ is isometric to the sphere $\mathbf{S}^4$ and easily seen that $\mathbf{P}_1^1(\mathbf{K})$ is real hyperbolic space $\mathbf{H}^4$ with metric reversed. We set aside these spaces of constant curvature. Now G contains the symmetry, so Theorem 2.7 yields:

$$\mathbf{I}(\mathbf{P}_h^n(\mathbf{K})) = \mathbf{I}(\mathbf{H}_{n-h}^n(\mathbf{K})) = \mathbf{Sp}^h(n+1)/\{\pm I\} \text{ when } n > 1; \tag{2.8.4}$$
$$\{\mathbf{Sp}^h(n) \times \mathbf{Sp}(1)\}/\{\pm I\} \text{ is isotropy subgroup.}$$

Let M be an indefinite complex elliptic space G/H; $G = \mathbf{SU}^h(n+1)$ and $\mathbf{I}_0(M)$ is $\mathbf{SU}^h(n+1)/\{\text{scalars}\}$. The image of H in $\mathbf{I}_0(M)$ is isomorphic to $\mathbf{U}^h(n)$; its only outer automorphism, the usual conjugate-transpose, is induced by the automorphism $\mathrm{ad}(\alpha)$ of G where α is the isometry of M induced by conjugation of $\mathbf{C}$ over $\mathbf{R}$. As G contains the symmetry, Theorem 2.7 yields:

$$\mathbf{I}(\mathbf{P}^n_h(\mathbf{C})) = \mathbf{I}_0(\mathbf{P}^n_h(\mathbf{C})) \cup \alpha \cdot \mathbf{I}_0(\mathbf{P}^n_h(\mathbf{C})) = \mathbf{I}(\mathbf{H}^n_{n-h}(\mathbf{C})) , \qquad (2.8.5)$$

where α is induced by conjugation of $\mathbf{C}$ over $\mathbf{R}$; $\mathbf{U}^h(n) \cup \alpha \cdot \mathbf{U}^h(n)$ is isotropy subgroup.

We have now determined the full groups of isometries of the model spaces.

2. 9. Sectional curvatures of the model spaces are not well behaved, except in the cases of definite metric and constant curvature:

Theorem. *Let (M, Q) be a pseudo-Riemannian manifold $\mathbf{P}^n_h(\mathbf{F})$ or $\mathbf{H}^n_h(\mathbf{F})$ with $\mathbf{F} = \mathbf{C}$ or $\mathbf{K}$ and $0 < h < n$, or $\mathbf{P}^2_1(\mathbf{Cay})$ or $\mathbf{H}^2_1(\mathbf{Cay})$. Then the sectional curvatures of (M, Q) are not bounded and do not keep one sign.*

Proof. Let $G = \mathbf{I}_0(M, Q)$ and let H be the isotropy subgroup of G at $x \in M$. Then G is semisimple, $\mathfrak{H}$ is the eigenspace of $+1$ for the involutive isometry τ of $\mathfrak{G}$ induced by the symmetry to (M, Q) at x, $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ where $\mathfrak{M}$ is the eigenspace of -1 for τ and both summands are stable under a Cartan involution σ of $\mathfrak{G}$, and a nonzero real multiple of the Killing form B of $\mathfrak{G}$ induces Q through its restriction to $\mathfrak{M}$. Thus we are in the situation of Theorem 1.3.1. If one changes Q to $a \cdot Q$, he changes a sectional curvature $K(S)$ to $a^{-1}K(S)$; thus the assertion on $\mathbf{P}^n_h(\mathbf{F})$ or $\mathbf{P}^2_1(\mathbf{Cay})$ will follow from the corresponding assertion for $\mathbf{H}^n_{n-h}(\mathbf{F})$ or $\mathbf{H}^2_1(\mathbf{Cay})$, and we may assume that Q is induced by B. As $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{H}$, we have seen in the proof of Theorem 1.3.1 that a nonsingular 2-plane $S \subset \mathfrak{M}$ with basis $\{X, Y\}$ has sectional curvature

$$K(S) = \frac{||[X, Y]||^2}{||X||^2 \cdot ||Y||^2 - B(X, Y)^2}$$

where $||Z||^2$ denotes $B(Z, Z)$. We also write $\mathfrak{G} = \mathfrak{G}_+ + \mathfrak{G}_-$, $\mathfrak{M} = \mathfrak{M}_+ + \mathfrak{M}_-$ and $\mathfrak{H} = \mathfrak{H}_+ + \mathfrak{H}_-$ where subscripts denote the eigenspace of $+1$ or -1 for the Cartan involution σ, and we write $Z = Z_+ + Z_-$ for every $Z \in \mathfrak{G}$ where $Z_+ \in \mathfrak{G}_+ = \mathfrak{H}_+ + \mathfrak{M}_+$ and $Z_- \in \mathfrak{G}_- = \mathfrak{H}_- + \mathfrak{M}_-$. B is positive definite on $\mathfrak{G}_-$ and negative definite on $\mathfrak{G}_+$.

Choose $X \in \mathfrak{M}_-$ with $||X||^2 = 1$ and let U be the subgroup of H consisting of all elements which preserve $\mathfrak{M}_+$ and $\mathfrak{M}_-$. U is a maximal compact subgroup of H and $\mathfrak{U} = \mathfrak{H}_+$. Let V be the isotropy subgroup of U at X. Transitivity of H on the light cone $||W||^2 = 0 \neq W$ in $\mathfrak{M}$ implies transitivity of V on the unit sphere $\mathbf{S}_+ = \{W \in \mathfrak{M}_+ : ||W||^2 = -1\}$ in $\mathfrak{M}_+$. If $(M, Q) = \mathbf{H}^n_h(\mathbf{F})$, then $\mathfrak{M}$ carries the structure of $\mathbf{F}^n_h$ and H acts as $\mathbf{U}^h(n, \mathbf{F}) \cdot \mathbf{F}'$. We then define a subspace $0 \neq \mathfrak{N} = X^\perp \cap X \cdot \mathbf{F} \subset \mathfrak{M}_-$ which is nonzero because $\mathbf{F} \neq \mathbf{R}$. Let $\mathbf{S}_-$ be the unit sphere $\{W \in \mathfrak{N} : ||W||^2 = 1\}$ in $\mathfrak{N}$; if $\mathbf{F} = \mathbf{C}$ then $\mathbf{S}_-$ is two points; if $\mathbf{F} = \mathbf{K}$ then V is transitive on $\mathbf{S}_-$. If $(M, Q) = \mathbf{H}_1^2(\mathbf{Cay})$, we define $\mathfrak{N} = X^\perp \cap \mathfrak{M}_-$. Then $U = \mathbf{Spin}(8)$; let t be its triality automorphism and let $f\colon \mathbf{Spin}(8) \to \mathbf{SO}(8)$ be the projection; if $\mathbf{SO}(8)$ acts on $\mathfrak{M}_+$ and $\mathfrak{M}_-$ by its usual action on $\mathbf{R}^8$, then U acts on $\mathfrak{M} = \mathfrak{M}_+ \oplus \mathfrak{M}_-$ by $u \to f(u) \oplus f(t(u))$; then the action of V on $\mathfrak{N}$ is the usual action of $\mathbf{SO}(7)$ on $\mathbf{R}^7$, so V is transitive on the unit sphere $\mathbf{S}_- = \{W \in \mathfrak{N} : ||W||^2 = 1\}$ in $\mathfrak{N}$.

From transitivity of V on $\mathbf{S}_+$, and on $\mathbf{S}_-$ when the latter is connected, we see that there are real numbers α and β such that $K(S) = \alpha$ and $K(T) = \beta$ if S is spanned by X and some $Y \in \mathbf{S}_-$, and if T is spanned by X and some $Z \in \mathbf{S}_+$. Now $\alpha \neq \beta$ because (M, Q) is not of constant sectional curvature.

Choose $Y \in \mathbf{S}_-$ and $Z \in \mathbf{S}_+$; for every real number b, let $b' = \sqrt{b^2 + 1}$, define $W_b = bY + b'Z$, and let S_b be the plane spanned by X and W_b. We have $X \perp W_b$ and $||W_b||^2 = -1$. Thus $K(S_b) = -||[X, W_b]||^2 = -\{||[X, bY]||^2 + ||[X, b'Z]||^2\} = -\{b^2\alpha - (b^2+1)\beta\} = b^2(\beta - \alpha) + \beta$, which is unbounded for $|b|$ large. To see that sectional curvature does not keep one sign, we can either apply Part 3 of Theorem 1.3.1 or observe that the plane T_b spanned by X and $b'Y + bZ$ has $K(T_b) = \alpha + b^2(\alpha - \beta)$. Either way, Theorem 2.9 is proved. *Q. E. D.*

Remark. One can show $\alpha = 4\beta < 0$ above.

3. Groups transitive on cones and quadrics

The first step in our study of isotropic spaces is:

3.1. Theorem. *Let G be a closed connected subgroup of $\mathbf{SO}^h(n)$ which is transitive both on a component L of the light cone $\{x \in \mathbf{R}_h^n : ||x||^2 = 0,\ x \neq 0\}$ and a component Q of a non-empty quadric $\{x \in \mathbf{R}_h^n : ||x||^2 = a\}$ $(a \neq 0)$. Then*

(a) $G = \mathbf{SO}^h(n)$;

or

(b) $G = \mathbf{SU}^{h/2}(n/2)$ *or* $\mathbf{U}^{h/2}(n/2)$;

or

(c) $G = \mathbf{Sp}^{h/4}(n/4)$ *or* $\mathbf{Sp}^{h/4}(n/4) \cdot \mathbf{T}$ *or* $\mathbf{Sp}^{h/4}(n/4) \cdot \mathbf{Sp}(1)$;

or

(d) $G = \mathbf{Spin}(9)$ *with* $n = 16$ *and* $h = 0$ *or* 16; *or* $G = \mathbf{Spin}^1(9)$ *with* $n = 16 = 2h$;

or

(e) $G = \mathbf{Spin}(7)$ *with* $n = 8$ *and* $h = 0$ *or* 8; *or* $G = \mathbf{Spin}^3(7)$ *with* $n = 8 = 2h$;

or

(f) $G = \mathbf{G}_2$ *with* $n = 7$ *and* $h = 0$ *or* 7; *or* $G = \mathbf{G}_2^*$ *with* $n = 7$ *and* $h = 3$ *or* 4.

Here $\mathbf{G}_2^*$ *denotes the noncompact centerless real group of the type* G_2.

Conversely, every group listed above is transitive on the nonzero elements of any given square norm in $\mathbf{R}_h^n$.

Remark. Our proof is based on the case $h = 0$, which is the celebrated MONTGOMERY-SAMELSON-BOREL classification of groups transitive on spheres ([12], [1], [2]; see § 27 of [16]).

3.2. *Proof.* If $n = 2$, then $\dim. Q = 1 = \dim. \mathbf{SO}^h(n)$ and necessarily $G = \mathbf{SO}^h(n)$. Now assume $n > 2$; then G is irreducible on $\mathbf{R}_h^n$ ([17], Lemma 8.2). Now G is reductive and its connected center either is trivial or is a circle group which gives a complex structure to $\mathbf{R}_h^n$; it follows ([14], Th. 6) that $\mathfrak{G}$ is invariant under a CARTAN involution σ of $\mathfrak{SO}^h(n)$ which induces a CARTAN involution of $\mathfrak{G}$. $\sigma = 1$ if $h = 0$ or $h = n$.

Let $x \in Q$. The isotropy subgroup H of $\mathbf{SO}^h(n)$ at x is reductive, algebraic and connected, and thus ([14], Th. 6) $\mathfrak{H}$ is invariant under a CARTAN involution τ; σ and τ are conjugate in the group of inner automorphisms of $\mathbf{SO}^h(n)$, so we may change x and assume $\sigma = \tau$. Now $\mathfrak{SO}^h(n) = \mathfrak{K} + \mathfrak{P}$ where $\mathfrak{K}$ is the eigenspace of $+1$ for σ and $\mathfrak{P}$ is the eigenspace of -1; $\mathfrak{G} = \mathfrak{G}_K + \mathfrak{G}_P$ and $\mathfrak{H} = \mathfrak{H}_K + \mathfrak{H}_P$ where subscripts denote intersection with $\mathfrak{K}$ or $\mathfrak{P}$. Define $\mathfrak{SO}^h(n)^* = \mathfrak{K} + \sqrt{-1}\,\mathfrak{P}$, $\mathfrak{G}^* = \mathfrak{G}_K + \sqrt{-1}\,\mathfrak{G}_P$ and $\mathfrak{H}^* = \mathfrak{H}_K + \sqrt{-1}\,\mathfrak{H}_P$; then $\mathfrak{SO}^h(n)^* = \mathfrak{SO}(n)$ and $\mathfrak{H}^* = \mathfrak{SO}(n-1)$. Let G^* be the analytic subgroup of $\mathbf{SO}(n)$ corresponding to $\mathfrak{G}^* \subset \mathfrak{SO}(n)$. Transitivity of G on Q implies $\mathfrak{G} + \mathfrak{H} = \mathfrak{SO}^h(n)$; thus $\mathfrak{G}^* + \mathfrak{H}^* = (\mathfrak{G} + \mathfrak{H})^* = \mathfrak{SO}(n)$ and it follows that G^* is transitive on the sphere $\mathbf{S}^{n-1} = \mathbf{SO}(n)/\mathbf{SO}(n-1)$.

According to MONTGOMERY, SAMELSON and BOREL, transitivity of G^* on $\mathbf{S}^{n-1}$ implies

(a) $G^* = \mathbf{SO}(n)$;

or

(b) $G^* = \mathbf{SU}(n/2)$ or $\mathbf{U}(n/2)$;

or

(c) $G^* = \mathbf{Sp}(n/4)$ or $\mathbf{Sp}(n/4) \cdot \mathbf{T}$ or $\mathbf{Sp}(n/4) \cdot \mathbf{Sp}(1)$;

or

(d) $G^* = \mathbf{Spin}(9)$ with $n = 16$;

or

(e) $G^* = \mathbf{Spin}(7)$ with $n = 8$;

or

(f) $G^* = \mathbf{G}_2$ with $n = 7$.

Because $G \subset \mathbf{SO}^h(n)$, cases (a), (b) and (c) above yield the corresponding possibilities for G in the Theorem; they have the required transitivity. Now we need only examine (d), (e) and (f) above when $0 < h < n$. We will need the fact [13] that, by transitivity of G on L, a maximal compact subgroup $M \subset G$ is transitive on a compact deformation retract of L, and that this retract is a component of $\mathbf{S}^{h-1} \times \mathbf{S}^{n-h-1}$ because the full light cone is homeomorphic to $\mathbf{S}^{h-1} \times \mathbf{R} \times \mathbf{S}^{n-h-1}$. As M preserves complementary positive definite and negative definite subspaces of $\mathbf{R}^n_h$ ([18], p. 79), we may take $\mathbf{S}^{n-h-1}$ and $\mathbf{S}^{h-1}$ to be the respective unit spheres in these subspaces. We may assume $h \leq n-h$.

Let $G^* = \mathbf{Spin}(9)$ with G noncompact and $0 < h \leq n-h$. Then $n = 16$, $G = \mathbf{Spin}^k(9)$ with $1 \leq k \leq 4$, and M is a quotient of $\mathbf{Spin}(k) \times \mathbf{Spin}(9-k)$. Now $k = 1$ because a simple factor of M is transitive on $\mathbf{S}^{n-h-1}$ and $n-h-$ $-1 \geq 7$; thus $h = 8$ and we are in case (d) of the Theorem. The required transitivity of G was seen in our proof that $\mathbf{P}^2_1(\mathbf{Cay})$ be isotropic.

Let $G^* = \mathbf{Spin}(7)$ with G noncompact and $0 < h \leq n-h$. Then $n = 8$, $G = \mathbf{Spin}^k(7)$ with $1 \leq k \leq 3$, and M is a quotient of $\mathbf{Spin}(k) \times \mathbf{Spin}(7-k)$. If $k = 1$, then $\mathbf{Spin}(6)$ is transitive on $\mathbf{S}^{n-h-1}$; $n-h-1 \geq 3$ then implies $n-h-1 = 5$ and $h-1 = 1$, so the isotropy subgroup $\mathbf{Spin}(5)$ of $\mathbf{Spin}(6)$ on $\mathbf{S}^5$ is transitive on $\mathbf{S}^1$; that is impossible. If $k = 2$, then $\mathbf{Spin}(5)$ is transitive on $\mathbf{S}^{n-h-1}$ because $\mathbf{Spin}(2)$ is not; thus $n-h-1 = 4$, so $h-1 = 2$ and $M' \times \mathbf{Spin}(2)$ is transitive on $\mathbf{S}^2$ where M' is isotropy of $\mathbf{Spin}(5)$ on $\mathbf{S}^4$; then $M' = \mathbf{Spin}(4)$ implies that M has a 2-dimensional subgroup which acts trivially on all of $\mathbf{R}^n_h$; that is impossible. Now $k = 3$, so M is locally $(\mathbf{Spin}(3))^3$; thus $h = 4$, two factors of M acting as $\mathbf{SO}(4)$ on one $\mathbf{S}^3$ and the third acting simply transitively on the other $\mathbf{S}^3$, as in case (e) of the Theorem.

Let $G^* = \mathbf{G}_2$ with G noncompact and $0 < h \leq n-h$. Then $n = 7$ and M is a quotient of $\mathbf{SU}(2) \times \mathbf{SU}(2)$. It follows that $h = 3$, $M \cong \mathbf{SO}(4)$, G is the noncompact centerless group $\mathbf{G}^*_2$ of type G_2, and we are in case (f) of the Theorem.

3. 3. In order to complete the proof of the Theorem, we must find actions of $\mathbf{Spin}^3(7)$ on $\mathbf{R}^8_4$ and $\mathbf{G}^*_2$ on $\mathbf{R}^7_4$ with the desired transitivity properties.

We have $\mathbf{Spin}(7) \subset \mathbf{SO}(8)$ transitive on the unit sphere $\mathbf{S}^7 \subset \mathbf{R}^8$ and with isotropy subgroup $\mathbf{G}_2$ at $x \in \mathbf{S}^7$. The theory of RIEMANNian symmetric spaces provides an element $t \in \mathbf{G}_2$ of order 2 whose centralizer K_1 in $\mathbf{G}_2$ is isomorphic to $\mathbf{SO}(4)$. Let A and B be the eigenspaces of $+1$ and of -1 for t on $\mathbf{R}^8$.

We first prove $\dim. A = 4 = \dim. B$. B has even dimension $2b$, $1 \leq b \leq 3$, and we must prove $b = 2$. Let y be an element of the unit sphere $\mathbf{S}^6$ in the tangentspace $(\mathbf{S}^7)_x$. $\mathbf{G}_2$ is transitive on $\mathbf{S}^6$ with isotropy subgroup $J \cong \mathbf{SU}(3)$ at y. As rank. $J = 2$, we may assume y chosen so that $t \in J$. We identify $(\mathbf{S}^7)_x$ with $x^\perp$ by a $\mathbf{G}_2$-equivariant parallel translation; now x and y span a plane T in A, and J acts on $T^\perp$. Thus if $b \neq 2$, t would be $-I$ on $T^\perp$ and thus would be central in J, or would represent an element of $\mathbf{U}(3)$ not in $\mathbf{SU}(3)$ and thus not be in J. This shows $b = 2$.

We will produce groups acting on $\mathbf{R}^8_4 = A + \sqrt{-1}\, B$ and $\mathbf{R}^7_4 = A' + {}$ $+ \sqrt{-1}\, B$, where $A' = A \cap x^\perp$. Let τ be the automorphism $\operatorname{ad}(t)$ of $\mathfrak{Spin}(7)$; then $\mathfrak{Spin}(7) = \mathfrak{K} + \mathfrak{P}$ where $\mathfrak{K}$ is the eigenspace of $+1$ for τ and $\mathfrak{P}$ is the eigenspace of -1, and $\mathfrak{G}_2 = \mathfrak{K}' + \mathfrak{P}'$ where $\mathfrak{P}' = \mathfrak{G}_2 \cap \mathfrak{P}$, and $\mathfrak{K}' = \mathfrak{G}_2 \cap \mathfrak{K}$ is the algebra of K_1. Now $\mathfrak{G}_2^* = \mathfrak{K}' + \sqrt{-1}\, \mathfrak{P}'$ is the Lie algebra of $\mathbf{G}_2^*$. $\mathfrak{K} + \sqrt{-1}\, \mathfrak{P}$ is $\mathfrak{Spin}^3(7)$ because it is contained in $\mathfrak{SO}^4(8)$ where $\mathbf{SO}^4(8)$ acts on $\mathbf{R}^8_4$, and because $\mathbf{SO}^4(8)$ cannot contain $\mathbf{Spin}(5)$. Thus $\mathbf{G}_2^*$ acts on $\mathbf{R}^7_4$ and $\mathbf{Spin}^3(7)$ acts on $\mathbf{R}^8_4$.

Given $\alpha > 0 > \beta$, we define 3-spheres $S_\alpha = \{w \in A : ||w||^2 = \alpha\}$ and $S_\beta = \{w \in \sqrt{-1}\, B : ||w||^2 = \beta\}$ and a 2-sphere $S'_\alpha = S_\alpha \cap A'$. K_1 acts effectively on $A' \oplus B$ and is isomorphic to $\mathbf{SO}(4)$; thus we view K_1 as the image of a representation $f \oplus g$ of $\mathbf{SO}(4)$ on $A' \oplus B$. g is faithful because $f \oplus g$ is faithful. If f is not trivial, then $f\colon \mathbf{SO}(4) \to \mathbf{SO}(3)$ is onto and it follows from the local product structure of $\mathbf{SO}(4)$ that K_1 is transitive on $S'_\alpha \times S_\beta$. We will prove that f is not trivial. Choose nonzero $z \in B$, let $\mathbf{G}'_2$ be the isotropy subgroup of $\mathbf{Spin}(7)$ at z, let e be the central element of order 2 in $\mathbf{Spin}(7)$, and define $t' = te$. Now e acts as $-I$ on $\mathbf{R}^8$ by irreducibility of $\mathbf{Spin}(7)$, so $t' = -t$ on $\mathbf{R}^8$ and $t' \in \mathbf{G}'_2$. Any two elements of order 2 in $\mathbf{G}_2$ being conjugate, $\mathbf{Spin}(7)$ has an element h such that $h\mathbf{G}_2h^{-1} = \mathbf{G}'_2$ and $hth^{-1} = t'$. Now $K_2 = hK_1h^{-1}$ is the centralizer of t' in $\mathbf{G}'_2$. h interchanges A and B because $t' = -t$; it follows that the group K generated by K_1 and K_2 preserves both A and B. If f is trivial, then K is the product $\mathbf{SO}(4) \times$ $\times\ \mathbf{SO}(4)$ of the rotation groups of A and B; then rank. $K = 4 > 3 =$ $=$ rank. $\mathbf{Spin}(7)$, which is impossible. This proves f nontrivial.

We have proved that K_1 is transitive on $S'_\alpha \times S_\beta$. The group K above is centralized by t and thus is contained in $\mathbf{Spin}^3(7)$. It is locally a product of groups $\mathbf{SO}(3)$, the number of factor being at least 3 because it contains K_1 properly and the number of factors being at most 3 because rank. $\mathbf{Spin}(7) = 3$. Thus K is the group with Lie algebra $\mathfrak{K}$ and is maximal compact in $\mathbf{Spin}^3(7)$. A close look at f, g and h above shows that K is transitive on $S_\alpha \times S_\beta$. Now, as in the proof that $\mathbf{P}^2_1(\mathbf{Cay})$ is isotropic, it follows that $\mathbf{G}_2^*$ (resp. $\mathbf{Spin}^3(7)$) is transitive on the nonzero elements of any given square norm in $\mathbf{R}^7_4$ (resp. in $\mathbf{R}^8_4$). *Q.E.D.*

3.4. Some coset representations for quadrics. In $\mathbf{R}_h^n$ we define quadrics

$$\mathbf{S} = \{x \in \mathbf{R}_h^n : ||x||^2 = 1\} \text{ and } \mathbf{H} = \{x \in \mathbf{R}_h^n : ||x||^2 = -1\}.$$

Besides the usual representations of $\mathbf{S}$ and $\mathbf{H}$ as coset spaces of $\mathbf{SO}^h(n)$, $\mathbf{U}^{h/2}(n/2)$ and $\mathbf{SU}^{h/2}(n/2)$, and $\mathbf{Sp}^{h/4}(n/4)\cdot\mathbf{Sp}(1)$, $\mathbf{Sp}^{h/4}(n/4)\cdot\mathbf{T}$ and $\mathbf{Sp}^{h/4}(n/4)$, we have:

1. $\mathbf{Spin}(9)/\mathbf{Spin}(7) = \mathbf{S} \subset \mathbf{R}_0^{16}$
2. $\mathbf{Spin}^1(9)/\mathbf{Spin}(7) = \mathbf{S} \subset \mathbf{R}_8^{16}$
3. $\mathbf{Spin}(7)/\mathbf{G}_2 = \mathbf{S} \subset \mathbf{R}_0^8$
4. $\mathbf{Spin}^3(7)/\mathbf{G}_2^* = \mathbf{S} \subset \mathbf{R}_4^8$
5. $\mathbf{G}_2/\mathbf{SU}(3) = \mathbf{S} \subset \mathbf{R}_0^7$
6. $\mathbf{G}_2^*/\mathbf{SU}^1(3) = \mathbf{S} \subset \mathbf{R}_4^7$
7. $\mathbf{G}_2^*/\mathbf{SL}(3;\mathbf{R}) = \mathbf{S} \subset \mathbf{R}_3^7$

Replacing the inner product by its negative on $\mathbf{R}_h^n$, these representations yield representations for $\mathbf{H}$ in $\mathbf{R}_{16}^{16}$, $\mathbf{R}_8^{16}$, $\mathbf{R}_8^8$, $\mathbf{R}_4^8$, $\mathbf{R}_7^7$, $\mathbf{R}_4^7$, and $\mathbf{R}_3^7$.

4. Structure of locally isotropic manifolds

Theorem 3.1 enumerated the candidates for group of local isometries in a locally isotropic manifold M_h^n. This is used in the structure theorem, Theorem 4 below. It turns out that not all the groups listed in Theorem 3.1 occur here, however; this happens in such a way that we are able to extend, in Theorem 4 (4.), the theorem that an isotropic LORENTZ manifold has constant curvature.

Theorem. *Let M_h^n be a connected locally isotropic pseudo-RIEMANNian manifold. Given $x \in M_h^n$, let G_x be the group of local isometries at x, let $\mathfrak{L}_x$ denote the LIE algebra of germs of KILLING vectorfields at x, and view $\mathfrak{G}_x$ as the subalgebra of $\mathfrak{L}_x$ consisting of the elements which vanish at x. Then:*

(1) *M_h^n is locally symmetric.*

(2) *M_h^n is locally homogeneous: given $x, y \in M_h^n$, there is an isometry f of a neighborhood of x onto a neighborhood of y with $f(x) = y$. In particular, f induces isomorphisms $\mathfrak{L}_x \cong \mathfrak{L}_y$, $\mathfrak{G}_x \cong \mathfrak{G}_y$ and $G_x \cong G_y$.*

(3) *M_h^n is characterized locally by n, h, G_x, $\mathfrak{L}_x$, and (only necessary if $\mathfrak{L}_x = \mathfrak{F}_4^*$ and $n = 2h = 16$) the sign of involutive sectional curvature, in the sense that the only possibilities are:*

G_x	$\mathfrak{L}_x$	M^n_h locally isometric to
$\mathbf{SO}^h(n)$, case of constant curvature	not semisimple	$\mathbf{R}^n_h$
	$\mathfrak{SO}^h(n+1)$	$\mathbf{P}^n_h(\mathbf{R})$
	$\mathfrak{SO}^{h+1}(n+1)$	$\mathbf{H}^n_h(\mathbf{R})$
$\mathbf{U}^{h'}(n')$, $2h' = h$, $2n' = n$	$\mathfrak{SU}^{h'}(n'+1)$	$\mathbf{P}^{n'}_{h'}(\mathbf{C})$
	$\mathfrak{SU}^{h'+1}(n'+1)$	$\mathbf{H}^{n'}_{h'}(\mathbf{C})$
$\mathbf{Sp}^{h''}(n'') \cdot \mathbf{Sp}(1)$, $4h'' = h$, $4n'' = n$	$\mathfrak{Sp}^{h''}(n''+1)$	$\mathbf{P}^{n''}_{h''}(\mathbf{K})$
	$\mathfrak{Sp}^{h''+1}(n''+1)$	$\mathbf{H}^{n''}_{h''}(\mathbf{K})$
$\mathbf{Spin}(9)$, $n = 16$, $h = 0$ or $h = 16$	$\mathfrak{F}_4$	$\mathbf{P}^2_0(\mathbf{Cay})$ or $\mathbf{H}^2_2(\mathbf{Cay})$
	$\mathfrak{F}_4^*$	$\mathbf{H}^2_0(\mathbf{Cay})$ or $\mathbf{P}^2_2(\mathbf{Cay})$
$\mathbf{Spin}^1(9)$, $n = 2h = 16$	$\mathfrak{F}_4^*$	$\mathbf{P}^2_1(\mathbf{Cay})$ or $\mathbf{H}^2_1(\mathbf{Cay})$

(4) *If h is odd, or if $n - h$ is odd, then M^n_h is of constant sectional curvature.*

Proof. The case $n = 2$ is trivial because there each $G_x = \mathbf{SO}^h(n)$ and M^n_h has constant curvature. Now let $n > 2$. As seen in the proof of Theorem 3.1, G_x acts irreducibly on the tangentspace T_x and every representation of G_x is fully reducible.

Let $\pi : \mathfrak{L}_x \to T_x$ be evaluation at x. π is a linear map with kernel $\mathfrak{G}_x$. We have a natural representation $g \to \mathrm{ad}(g)$ of G_x on $\mathfrak{L}_x$ by automorphisms, and $\pi(\mathrm{ad}(g)X) = g(\pi(X))$ for every $g \in G_x$ and every $X \in \mathfrak{L}_x$. Given $z \in M^n_h$, $\mathfrak{G}_z$ extends to a Lie algebra of Killing vectorfields on a normal coordinate neighborhood U of z. We may choose z with $x \in U$. By irreducibility of G_z on T_z, now, $\mathfrak{L}_x$ has an element which does not vanish at x. By irreducibility of G_x and equivariance of π, it follows that $\pi(\mathfrak{L}_x) = T_x$ and in particular $\mathfrak{L}_x$ has dimension $n + \dim. G_x$.

G_x is represented faithfully on T_x because M^n_h is connected. Identifying T_x with $\mathbf{R}^n_h$ and G_x with its linear action, the identity component G of G_x is listed in Theorem 3.1. By irreducibility of $\mathfrak{G}_x$ on T_x, and by the Levi-Whitehead-Malcev' Theorem, $\mathfrak{G}_x$ is a maximal subalgebra of $\mathfrak{L}_x$; if $\mathfrak{G}_x$ is semisimple it follows either that $\mathfrak{L}_x$ is semisimple or that $\mathfrak{L}_x = \mathfrak{G}_x + \mathrm{Rad}(\mathfrak{L}_x)$; in the latter case $\mathrm{Rad}(\mathfrak{L}_x)$ is commutative by irreducibility of $\mathfrak{G}_x$, whence M^n_h is flat in a neighborhood of x, and $G_x = \mathbf{O}^h(n)$.

If G is listed under (e) (resp. (f)) in Theorem 3.1, so G is $\mathbf{Spin}(7)$ on $\mathbf{R}^8_0$ or $\mathbf{R}^8_8$ or $\mathbf{Spin}^3(7)$ on $\mathbf{R}^8_4$ (resp. $\mathbf{G}_2$ on $\mathbf{R}^7_0$ or $\mathbf{R}^7_7$ or $\mathbf{G}^*_2$ on $\mathbf{R}^7_4$ or $\mathbf{R}^7_3$), then we choose $X \in T_x = \mathbf{R}^n_h$ with $||X||^2 \neq 0$ and observe that the isotropy subgroup of G at X is $\mathfrak{G}_2$ or $\mathfrak{G}_2^*$ (resp. $\mathbf{SU}(3)$ or $\mathbf{SU}^1(3)$ or the self-contragredient group of type $\mathbf{SL}(3, \mathbf{R}) \subset \mathbf{SO}^3(6)$). From the combined transitivity properties of G and this subgroup (see Lemma 8.3 of [17] for $\mathbf{SL}(3, \mathbf{R})$), it follows that G is transitive on positive definite or on negative definite 2-planes in T_x, so all such planes give the same sectional curvature; these planes form an open subset in the GRASSMANN manifold of 2-planes in T_x, and thus their sectional curvatures determine the curvature tensor of M^n_h at x; it follows that M^n_h has constant curvature in a neighborhood of x and so $G_x = \mathbf{O}^h(n)$. This is a contradiction. We conclude that G is listed under (a), (b), (c) or (d) in Theorem 3.1.

If G is listed under (a), then $-I \in G_x = \mathbf{O}^h(n)$. If G is listed under (b), (c) or (d), then $-I \in G \subset G_x$. In any case, M^n_h is locally symmetric at x. This proves the first statement of the Theorem.

Define two points of M^n_h to be equivalent if a neighborhood of one is isometric to a neighborhood of the other as in the second statement of the Theorem. The equivalence classes are open subsets of M^n_h because $\pi(\mathfrak{L}_x) = T_x$ for every x. As the equivalence classes are disjoint, it follows that each is a union of components of M^n_h. The second statement of the Theorem follows from connectedness of M^n_h.

We have seen that G_x has an element t which is $-I$ on T_x; let $\tau = \mathrm{ad}(t)$. Then under τ, $\mathfrak{L}_x = \mathfrak{G}_x + \mathfrak{M}_x$ where $\mathfrak{M}_x$ is the eigenspace for -1 in $\mathfrak{L}_x$; $\mathfrak{G}_x$ is the eigenspace for $+1$. We have also seen that M^n_h is flat and $G_x = \mathbf{O}^h(n)$ if $\mathfrak{L}_x$ is not semisimple; now assume $\mathfrak{L}_x$ semisimple. Then ([14], Theorem 6) there is a CARTAN involution σ of $\mathfrak{L}_x$ which induces a CARTAN involution of $\mathfrak{G}_x$. t lies in a compact subgroup of G_x because $t^2 = 1$; replacing t by a conjugate (this does nothing because t is central), we have $\sigma\tau = \tau\sigma$. $\mathfrak{L}_x = \mathfrak{L}_K + \mathfrak{L}_P$ and $\mathfrak{G}_x = \mathfrak{G}_K + \mathfrak{G}_P$ where K denotes σ-eigenspace of $+1$ and P denotes σ-eigenspace of -1. Let $\mathfrak{L}^* = \mathfrak{L}_K + \sqrt{-1}\ \mathfrak{L}_P$ and $\mathfrak{G}^* = \mathfrak{G}_K + \sqrt{-1}\ \mathfrak{G}_P$; then τ induces an involutive automorphism of $\mathfrak{L}^*$ with $\mathfrak{G}^*$ as fixed point set. Let L^* be a (necessarily compact) connected group with LIE algebra $\mathfrak{L}^*$, and let G^* be an analytic subgroup corresponding to $\mathfrak{G}^*$.

Recall $\mathfrak{L}_x = \mathfrak{G}_x + \mathfrak{M}_x$ under τ; $\pi : \mathfrak{M}_x \to T_x$ is an equivariant isomorphism, so $\mathrm{ad}(G_x)$ has an $(n-1)$-dimensional orbit on $\mathfrak{M}_x$. Let $\mathfrak{M}^* = (\mathfrak{M}_x \cap \mathfrak{L}_K) + \sqrt{-1}(\mathfrak{M}_x \cap \mathfrak{L}_P)$; then $\mathfrak{L}^* = \mathfrak{G}^* + \mathfrak{M}^*$ under τ, and it follows that $\mathrm{ad}(G^*)$ has an $(n-1)$-dimensional orbit on $\mathfrak{M}^*$. In other words, the compact RIEMANNian symmetric space L^*/G^* has rank 1. By Theorem 3 and our above elimination of (e) and (f), it follows that the identity component of G_x is (a) $\mathbf{SO}^h(n)$, (b) $\mathbf{U}^{h/2}(n/2)$, (c) $\mathbf{Sp}^{h/4}(n/4) \cdot \mathbf{Sp}(1)$, or (d)

Spin (9) or $\mathbf{Spin}^1(9)$, and that $\mathfrak{L}_x$ is given respectively as an algebra of type (a) $B_{n/2}$ if n is even, $D_{(n+1)/2}$ if n is odd, (b) $A_{n/2}$, (c) $C_{(n/4)+1}$, or (d) F_4.

If $G_x = \mathbf{SO}^h(n)$, then M_h^n is of constant curvature, and thus is locally isometric to $\mathbf{R}_h^n, \mathbf{P}_h^n(\mathbf{R})$ or $\mathbf{H}_h^n(\mathbf{R})$. If $G_x = \mathbf{U}^{h'}(n')$ where $2h' = h$ and $2n' = n$, then $\mathfrak{L}_x$ is either $\mathfrak{SU}^{h'}(n'+1)$ or $\mathfrak{SU}^{h'+1}(n'+1)$ because it contains $\mathfrak{U}^{h'}(n')$ and is of type $A_{n'}$; thus M_h^n is locally isometric to $\mathbf{P}_{h'}^{n'}(\mathbf{C})$ or to $\mathbf{H}_{h'}^{n'}(\mathbf{C})$. If $G_x = \mathbf{Sp}^{h''}(n'') \cdot \mathbf{Sp}(1)$ where $4h'' = h$ and $4n'' = n$, then $\mathfrak{L}_x$ is either $\mathfrak{Sp}^{h''}(n''+1)$ or $\mathfrak{Sp}^{h''+1}(n''+1)$ because it is of type $C_{n''+1}$ and $\mathfrak{Sp}^{h''}(n'') \oplus \mathfrak{Sp}(1)$ is a maximal subalgebra; thus M_h^n is locally isometric either to $\mathbf{P}_{h''}^{n''}(\mathbf{K})$ or to $\mathbf{H}_{h''}^{n''}(\mathbf{K})$. If G_x is **Spin** (9) or $\mathbf{Spin}^1(9)$, then $\mathfrak{L}_x$ is $\mathfrak{F}_4$ (only when $G_x = \mathbf{Spin}(9)$) or $\mathfrak{F}_4^*$, and M_h^n is locally isometric to a CAYLEY plane.

We have just completed the proof of the third statement of the Theorem. The fourth statement follows at a glance. *Q.E.D.*

Theorem 4 allows us to formulate:

Definition. Let M_h^n be a locally isotropic pseudo-RIEMANNian manifold and let S be a nonsingular tangent 2-plane at $x \in M_h^n$, so S is sent to a nonsingular tangent 2-plane $f_*(S)$ at $f(x) \in N_h^n$ under an isometry f of a neighborhood of x onto a neighborhood in

$$N_h^n = \mathbf{R}_h^n, \mathbf{P}_k^m(\mathbf{R}, \mathbf{C}, \mathbf{K} \text{ or } \mathbf{Cay}) \text{ or } \mathbf{H}_k^m(\mathbf{R}, \mathbf{C}, \mathbf{K} \text{ or } \mathbf{Cay}).$$

Then S is *involutive*, and the sectional curvature $K(S)$ is an *involutive sectional curvature*, if and only if either $N_h^n = \mathbf{R}_h^n$ or $f_*(S)$ is involutive on N_h^n.

5. The universal covering theorem

Our global classifications will depend on the following consequence of Theorem 4:

Theorem. *Let M_h^n be a connected pseudo-RIEMANNian manifold. Then M_h^n is complete and locally isotropic, if and only if its universal pseudo-RIEMANNian covering manifold is* (a) $\mathbf{R}_h^n, \tilde{\mathbf{S}}_h^n$ *or* $\tilde{\mathbf{H}}_h^n$; *or, if $h' = h/2$ and $n' = n/2$ are integers,* (b) $\mathbf{P}_{h'}^{n'}(\mathbf{C})$ *or* $\mathbf{H}_{h'}^{n'}(\mathbf{C})$; *or, if $h'' = h/4$ and $n'' = n/4$ are integers,* (c) $\mathbf{P}_{h''}^{n''}(\mathbf{K})$ *or* $\mathbf{H}_{h''}^{n''}(\mathbf{K})$; *or, if $n = 16$ and $h''' = h/8$ is an integer,* (d) $\mathbf{P}_{h'''}^{2}(\mathbf{Cay})$ *or* $\mathbf{H}_{h'''}^{2}(\mathbf{Cay})$.

Remark. $\tilde{\mathbf{S}}_h^n$ and $\tilde{\mathbf{H}}_h^n$ are the universal pseudo-RIEMANNian covering manifolds of $\mathbf{P}_h^n(\mathbf{R})$ and $\mathbf{H}_h^n(\mathbf{R})$.

Proof. It suffices to prove the Theorem when M_h^n is simply connected, and completeness and local isotropy are immediate from the existence of a pseudo-

RIEMANNian covering by one of the manifolds listed. Now assume M_h^n complete, simply connected and locally isotropic; we must show that it is isometric to one of the manifolds listed.

Choose $x \in M_h^n$. By Theorem 4, there is an isometry f of a neighborhood U of x onto an open subset V of a manifold N_h^n in our list. The curvature tensors are parallel on both M_h^n and N_h^n, for those manifolds are locally symmetric. The torsion forms vanish because we are dealing with LEVI-CIVITA connections. The proof of Theorem 5 is now identical to the proof of the case of constant curvature ([17], Theorem 5): one uses N. HICKS' extension ([9], Theorem 1) of AMBROSE's Theorem to extend f to a global isometry by extending it along broken geodesics. *Q. E. D.*

6. A partial classification of complete locally isotropic manifolds

As in the case of constant nonzero curvature [18], one can classify the non-flat complete locally isotropic manifolds in certain signatures of metric:

6. 1. Theorem. *Let M_s^n be a complete connected locally isotropic pseudo-RIEMANNian manifold which is not flat.*

1. *If M_s^n is of non-negative involutive sectional curvature with $2s \leq n$ and $s \neq n-1$, or if M_s^n is of non-positive involutive sectional curvature with $2s \geq n$ and $s \neq 1$, then the fundamental group $\pi_1(M_s^n)$ is finite.*

2. *If $\pi_1(M_s^n)$ is finite, then*

(a) *M_s^n is of constant sectional curvature. In [18], the global classification of such spaces is reduced to the classical CLIFFORD-KLEIN spherical space form problem in dimension $n-s$ for positive curvature, and thus in dimension s for negative curvature;*

or (b) *M_s^n is simply connected and thus isometric to one of the model spaces;*

or (c) *M_s^n is isometric to $\mathbf{P}_{s/2}^{n/2}(\mathbf{C})/\{1, \alpha J\}$ where $r = s/4$ and $t = ((n/2)+1)/2$ are integers, α is the isometry induced by conjugation of $\mathbf{C}$ over $\mathbf{R}$, and J is the isometry induced by*

$$\begin{pmatrix} 0 & I_r & & \\ -I_r & 0 & & \\ & & 0 & I_{t-r} \\ & & -I_{t-r} & 0 \end{pmatrix} \in \mathbf{SU}(2r) \times \mathbf{SU}(2t-2r) \subset \mathbf{SU}^{s/2}((n/2)+1)\,;$$

or (d) *M_s^n is isometric to $\mathbf{H}_{s/2}^{n/2}(\mathbf{C})/\{1, \alpha J\}$ where $r = ((s/2)+1)/2$ and $t = ((n/2)+1)/2$ are integers, and α and J are as above.*

Remark. It suffices to prove Theorem 6.1 in the case where M_s^n is of non-negative involutive sectional curvature; the case of non-positive involutive sectional curvature will then follow by reversing (replacing with its negative) the metric.

6.2. Proof of Part 1. M_s^n is non-negative involutive curvature with $2s \leq n$ and $s \neq n-1$. We may suppose that M_s^n is not of constant curvature, for the result is known in the case of constant positive curvature ([18], Theorem 1).

Now suppose that M_s^n is not covered by a CAYLEY plane; then $\pi : \mathbf{P}_h^m(\mathbf{F}) \to M_s^n$ is the universal pseudo-RIEMANNian covering where $\mathbf{F} = \mathbf{C}$ or $\mathbf{K}$ has degree f over $\mathbf{R}$, $fm = n$ and $fh = s$. Let Γ be the group of deck transformations of π; we must prove Γ finite; as $\mathbf{I}(\mathbf{P}_h^m(\mathbf{F}))$ has only finitely many components, it suffices to prove $\Gamma \cap \mathbf{I}_0(\mathbf{P}_h^m(\mathbf{F}))$ finite; thus we may assume $\Gamma \subset \mathbf{I}_0(\mathbf{P}_h^m(\mathbf{F}))$. The principle fibring $\psi : \mathbf{S}_s^{n+f-1} \to \mathbf{P}_h^m(\mathbf{F})$ given by

$$\mathbf{U}^h(m+1, \mathbf{F})/\mathbf{U}^h(m, F) \to \mathbf{U}^h(m+1, \mathbf{F})/\mathbf{U}^h(m, \mathbf{F}) \times \mathbf{U}(1, \mathbf{F})$$

induces a homomorphism Φ of $\mathbf{SU}^h(m+1)$ or $\mathbf{Sp}^h(m+1)$ onto $\mathbf{I}_0(\mathbf{P}_h^m(\mathbf{F}))$; now it suffices to prove $\Phi^{-1}(\Gamma)$ finite. $\Phi^{-1}(\Gamma)$ is properly discontinuous because Φ is ψ-equivariant, Γ is properly discontinuous, and the kernel Ker. Φ is finite. $\Phi^{-1}(\Gamma)$ acts freely because both Γ and Ker. Φ act freely. $2s \leq n$ implies $2s \leq n + f - 1$, and $f > 1$ then gives $s \neq (n + f - 1) - 1$. Now our result ([18], Theorem 1) for constant curvature shows that $\Phi^{-1}(\Gamma)$ is finite, and so $\pi_1(M_s^n)$ is finite.

Now suppose that M_s^n is covered by a CAYLEY plane $\mathbf{P}_h^2(\mathbf{Cay})$; $h = 0$ or 1 by hypothesis. If $h = 0$, then the CAYLEY plane is compact and our assertion is trivial; now assume $h = 1$ and let Γ be the group of deck transformations of the universal pseudo-RIEMANNian covering $\pi : \mathbf{P}_1^2(\mathbf{Cay}) \to M_s^n$. $\mathbf{P}_1^2(\mathbf{Cay})$ is a coset space G/H where $G = \mathbf{F}_4^*$ is the full group of isometries and H is the isotropy subgroup at some point x. Let σ be a CARTAN involution of $\mathfrak{G}$ which preserves $\mathfrak{H}$ and induces a CARTAN involution of $\mathfrak{H}$; $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ where $\mathfrak{K}$ is the eigenspace of $+1$ for σ, where $\mathfrak{P}$ is the eigenspace for -1, and where $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{K}) + (\mathfrak{H} \cap \mathfrak{P})$. Let $\mathfrak{A}$ be a 1-dimensional subspace of $\mathfrak{P}$, and define $A = \exp(\mathfrak{A})$, $K = \exp(\mathfrak{K})$, and $P = \exp(\mathfrak{P})$. As G/K is an irreducible RIEMANNian symmetric space of rank 1 and of noncompact type we have $G = K \cdot P$ and $P = \mathrm{ad}(K)A$; thus $G = KAK$. Now dim. $\mathfrak{H} = 36$ and dim. $(\mathfrak{H} \cap \mathfrak{K}) = 28$ because $\mathfrak{H}$ is of type B_4 and $\mathfrak{H} \cap \mathfrak{K}$ is of type D_4; thus $\mathfrak{H} \cap \mathfrak{P} \neq 0$ and we may choose $\mathfrak{A} \subset \mathfrak{H} \cap \mathfrak{P}$; it follows that $G = KHK$. Define $X = K(x) \subset \mathbf{P}_1^2(\mathbf{Cay})$ and let $g \in G$; then $g = k_1 h k_2$ with $k_i \in K$ and $h \in H$, $k_2^{-1}(x) \in X$, $k_1(x) \in X$, and $g(k_2^{-1}(x)) = k_1(x)$; it follows that

$g(X)$ meets X. Now X is a compact subset of $\mathbf{P}_1^2(\mathbf{Cay})$ such that $\gamma(X)$ meets X whenever $\gamma \in \Gamma$. As in [18], proper discontinuity of Γ now implies that Γ in finite. Thus $\pi_1(M_s^n)$ is finite.

We have now proved the first part of Theorem 6.1.

6.3. Proof of Part 2. Let Γ be a finite group of isometries acting freely on a model space N of non-constant and nonnegative involutive sectional curvature. Γ lies in a maximal compact subgroup K of $\mathbf{I}(N)$, and there is a point $y \in N$ such that the symmetry to N at y normalizes K in $\mathbf{I}(N)$. Let H be the isotropy subgroup of $\mathbf{I}(N)$ at y; then $L = H \cap K$ is a maximal compact subgroup of H, and Γ acts freely on K/L. The possibilities for N are $\mathbf{P}_h^2(\mathbf{Cay})$, $\mathbf{P}_h^m(\mathbf{K})$ and $\mathbf{P}_h^m(\mathbf{C})$; they all have the property that the compact LIE groups K and L have the same rank; thus every element of K_0 is conjugate to an element of a maximal torus of L and consequently has a fixed point on K/L; it follows that $\Gamma \cap K_0 = \{1\}$. As $K_0 = K \cap \mathbf{I}_0(N)$, this proves $\Gamma \cap \mathbf{I}_0(N) = \{1\}$.

Suppose that $\Gamma \neq \{1\}$. Then the curvature conditions on N and the results of § 2.8 show that $N = \mathbf{P}_h^m(\mathbf{C})$ and Γ is a 2-element group $\{1, \alpha g\}$ where α is the isometry induced by conjugation of $\mathbf{C}$ over $\mathbf{R}$ and $g \in \mathbf{I}_0(\mathbf{P}_h^m(\mathbf{C}))$ is induced by some element (also denoted g) of the matrix group $\mathbf{SU}^h(m+1)$. $(\alpha g)^2 = 1 \in \Gamma$ implies that ${}^t g^{-1} g$ is a scalar matrix $c\,I \in \mathbf{SU}^h(m+1)$, so $g = c\,{}^t g$. As ${}^t({}^t g) = g$ we have $c = \pm 1$, so ${}^t g = \pm g$. Let Z be the group of all matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where $A \in \mathbf{U}(h)$, $B \in \mathbf{U}(m+1-h)$ and $(\det. A)(\det. B) = 1$; Z is the pre-image of K_0 in $\mathbf{SU}^h(m+1)$. Now $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with those conditions, for we may assume $K = K_0 \cup \alpha K_0$. If $g = {}^t g$, then $a = {}^t a$ and $b = {}^t b$, and there are unitary matrices u and v with $u a {}^t u = I_h$ and $v b {}^t v = I_{m+1-h}$. It follows that $(\det. u)^2 (\det. v)^2 = 1$, so we may replace u or v by a scalar multiple, if necessary, and assume $h = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in Z$. Now $h g {}^t h = I$, and this gives $\mathrm{ad}({}^t h^{-1})(\alpha g) = {}^t h^{-1} \alpha g {}^t h = {}^t h^{-1} \alpha h^{-1} h g {}^t h = {}^t h^{-1} \alpha h^{-1} =$ $= {}^t h^{-1} \cdot {}^t h \alpha = \alpha$, so αg has a fixed point on $K(y) \subset \mathbf{P}_h^m(\mathbf{C})$. This being impossible, we have $g = -{}^t g$. Now $a = -{}^t a$ and $b = -{}^t b$, so $N = N_s^n$ gives that $r = h/2 = s/4$ and $t = (m+1)/2 = ((n/2)+1)/2$ are integers. Define $J_w = \begin{pmatrix} 0 & I_w \\ -I_w & 0 \end{pmatrix}$ for every integer $w > 0$, and then define

$$J = \begin{pmatrix} J_r & 0 \\ 0 & J_{t-r} \end{pmatrix} \in \mathbf{SU}(2r) \times \mathbf{SU}(2t-2r) \subset \mathbf{SU}^h(m+1).$$

Now we have unitary matrices u and v such that $u a {}^t u = J_r$ and $v b {}^t v = J_{t-r}$. As before we may assume $h = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in Z$. Now Γ is conjugate to $\{1, \mathrm{ad}({}^t h^{-1})(\alpha g)\}$, and the second element is given by

$$ {}^t h^{-1} \alpha g {}^t h = {}^t h^{-1} \alpha h^{-1} h g {}^t h = {}^t h^{-1} \alpha h^{-1} J = {}^t h^{-1} \cdot {}^t h \alpha J = \alpha J . $$

As αJ has no fixed point on $\mathbf{P}_h^m(\mathbf{C})$, the Theorem follows. *Q. E. D.*

6.4. A finiteness criterion. The latter part of § 6.2 consisted of a technique which could be useful in a variety of situations. For this reason we state it separately.

Theorem. *Let G be a connected semisimple* Lie *group with finite center, let H be a closed subgroup of G, and let Γ be a subgroup of G which acts properly discontinuously on G/H. Suppose that $\mathfrak{H}$ is preserved by a* Cartan *involution σ of $\mathfrak{G}$, that $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ is the decomposition under σ, and that $\mathfrak{H} \cap \mathfrak{P}$ contains a* Cartan *subalgebra of the symmetric pair $(\mathfrak{G}, \mathfrak{K})$. Then Γ is finite.*

Corollary. *Let G be a group $\mathbf{F}_4^*$, $\mathbf{Sp}^1(n)$, $\mathbf{SU}^1(n)$ or $\mathbf{SO}^1(n)$. Let H be a closed noncompact subgroup of G whose* Lie *algebra is preserved by a* Cartan *involution of $\mathfrak{G}$, and let Γ be a subgroup of G which acts properly discontinuously on G/H. Then Γ is finite.*

For if G is semisimple, then, in the notation of Theorem 6.4, $(\mathfrak{G}, \mathfrak{K})$ has rank 1 and $\mathfrak{H} \cap \mathfrak{P} \neq 0$. If G is not semisimple, then n is so small that $G = H$.

Remark. Part 1 of Theorem 6.1 could be proved from Theorem 6.4; the proof then would be more uniform and would avoid appeal to [18], but the method used in § 6.2 is more elementary.

7. The classification theorem for isotropic manifolds

Extending the results ([17], §§ 16—17) for constant curvature, we will prove:

7.1. Theorem. *Let M_h^n be a connected locally isotropic pseudo-*Riemannian *manifold which is not flat. Then these are equivalent:*

1. *M_h^n is isotropic.*
2. *M_h^n is symmetric.*
3. *Let Γ be the group of deck transformations of the universal pseudo-*Riemannian *covering $\pi : N_h^n \to M_h^n$. Then M_h^n is complete and Γ is a normal subgroup of $\mathbf{I}(N_h^n)$.*
4. *There is a pseudo-*Riemannian *covering $M_h^n \to \mathbf{P}_k^m$(*$\mathbf{R}$, $\mathbf{C}$, $\mathbf{K}$ *or* **Cay**) *or $M_h^n \to \mathbf{H}_k^m$(*$\mathbf{R}$, $\mathbf{C}$, $\mathbf{K}$ *or* **Cay**).
5. *Either M_h^n is isometric to a model space $\mathbf{P}_k^m$(*$\mathbf{C}$, $\mathbf{K}$ *or* **Cay**) *or $\mathbf{H}_k^m$(*$\mathbf{C}$, $\mathbf{K}$ *or* **Cay**), *or there is a pseudo-*Remannian *covering $M_h^n \to \mathbf{P}_h^n(\mathbf{R})$ or $\mathbf{H}_h^n(\mathbf{R})$.*

Corollary. *Let M_h^n be a connected isotropic pseudo-*RIEMANN*ian manifold. If M_h^n is of constant nonzero curvature, then it is explicitly described in* ([17], Theorem 16.1). *Otherwise, M_h^n is simply connected and is isometric to*

$$\mathbf{R}_h^n,\ \mathbf{P}_k^m(\mathbf{C}, \mathbf{K} \text{ or } \mathbf{Cay}) \text{ or } \mathbf{H}_k^m(\mathbf{C}, \mathbf{K} \text{ or } \mathbf{Cay}).$$

7.2. Proof of Theorem. If M_h^n is symmetric or isotropic, then it is complete. In the notation of statement (3.), if M_h^n is symmetric, then every symmetry lifts to a symmetry of N_h^n, and consequently every symmetry of N_h^n is a π-fibre map. Thus every symmetry of N_h^n normalizes Γ. Now $\mathbf{I}_0(N_h^n)$ lies in the closure of the group generated by the symmetries and thus normalizes Γ, and thus centralizes Γ because Γ is discrete. Now Γ lies in the centralizer Z of $\mathbf{I}_0(N_h^n)$ in $\mathbf{I}(N_h^n)$. Given $g \in \mathbf{I}(N_h^n)$ and $z \in Z$, one can check the various cases and see that gzg^{-1} is z or z^{-1}; in fact this is trivial in nonconstant curvature and is contained in ([17], § 16) for constant curvature. Thus Γ is normal in $\mathbf{I}(N_h^n)$. On the other hand, if we assume M_h^n to be isotropic instead of symmetric, we see that every isotropy subgroup of $\mathbf{I}_0(N_h^n)$ consists of π-fibre maps, and it follows as above that Γ is normal in $\mathbf{I}(N_h^n)$. Thus (1) implies (3) and (2) implies (3).

Assume (3). If M_h^n is of constant curvature, then we know ([17], Th. 16.1) that (5) follows. If M_h^n is of nonconstant curvature, then $\mathbf{I}_0(N_h^n)$ is centerless, and it is easily checked that the centralizer Z of $\mathbf{I}_0(N_h^n)$ in $\mathbf{I}(N_h^n)$ is trivial; as $\Gamma \subset Z$, it follows that π is an isometry. Thus (3) implies (5). And it is clear that (5) implies (4).

Assume (4) and let $\psi: M_h^n \to D_h^n$ be the covering of (4). Given $d \in D_h^n$, $g \in \mathbf{I}(D_h^n)$ such that $g(d) = d$, and $x \in \psi^{-1}(d)$, there is a lift $\tilde{g} \in \mathbf{I}(M_h^n)$ of g such that $\tilde{g}(x) = x$. Now both (1) and (2) follow.

This completes the proof of Theorem 7.1.

If M_h^n is not flat, then Corollary 7.1 is the statement that (1) implies (5) in Theorem 7.1; if M_h^n is flat, it is the statement that (1) implies (3) in Theorem 15 of [17]. The Corollary follows.

Chapter II

Homogeneous Locally Isotropic Pseudo-RIEMANNian Manifolds

The goal of this Chapter is the global classification of complete homogeneous locally isotropic pseudo-RIEMANNian manifolds which are not flat. That classification was accomplished in an earlier paper [17] in the case of constant curvature. Our main technique is to reduce to the case of constant curvature and use the results of [17]. After the classification, we give some examples which show that the hypothesis of completeness is essential.

8. Application of the classification in constant curvature

8.1. Theorem. *Let M_s^r be a complete connected homogeneous locally isotropic pseudo-*Riemannian *manifold of nonconstant sectional curvature. Let Γ be the group of deck transformations of the universal pseudo-*Riemannian *covering $\pi : N_s^r \to M_s^r$. Then $\Gamma \cap \mathbf{I}_0(N_s^r) = \{1\}$.*

Remark. If N_s^r is not a complex elliptic or hyperbolic space, it immediately follows from Theorems 6.1 and 8.1 that M_s^r is simply connected and π is a global isometry; if M_s^r is not simply connected, it follows that M_s^r is isometric to one of the two manifolds of (2c) and (2d) of Theorem 6.1. This does not complete the classification, however, as those manifolds must be checked for homogeneity.

Proof. The possibility of reversing the metrics of M_s^r and N_s^r shows that we need only consider the case of non-negative involutive sectional curvature. If $N_s^r = \mathbf{P}_k^2(\mathbf{Cay})$ with $k < 2$, then Theorem 6.1 proves $\Gamma = \{1\}$. If $N_s^r = \mathbf{P}_2^2(\mathbf{Cay})$, then we reverse metrics and have several theorems (see [20], [21] or [11], for example) which imply triviality of Γ. Thus we need only consider the case $N_s^r = \mathbf{P}_k^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{C}$ or $\mathbf{F} = \mathbf{K}$.

Let $\Delta = \Gamma \cap \mathbf{I}_0(N_s^r)$. According to Theorem 6.1, we need only prove that Δ is finite. Thus we need only prove Δ finite under the assumption $N_s^r = \mathbf{P}_k^m(\mathbf{F} = \mathbf{C}$ or $\mathbf{K})$.

8.2. Let f be the degree of $\mathbf{F}$ over $\mathbf{R}$, and define h and n by $h = s = fk$ and $n + 1 = f(m + 1) = r + f$. We have $\psi : \mathbf{S}_h^n \to \mathbf{P}_k^m(\mathbf{F})$ given by

$$\mathbf{U}^k(m + 1;\ \mathbf{F})/\mathbf{U}^k(m, \mathbf{F}) \to \mathbf{U}^k(m + 1;\ \mathbf{F})/\mathbf{U}^k(m, \mathbf{F}) \times \mathbf{U}(1, \mathbf{F})$$

and a ψ-equivariant epimorphism $\Phi : H \to \mathbf{I}_0(\mathbf{P}_k^m(\mathbf{F}))$ where $H = \mathbf{SU}^k(m + 1)$ or $\mathbf{Sp}^k(m + 1)$. Define $D = \Phi^{-1}(\Delta)$.

$\mathbf{P}_k^m(\mathbf{F})$ consists of all positive definite lines in $\mathbf{F}_k^{m+1}$, and

$$\mathbf{S}_h^n = \{x \in \mathbf{F}_k^{m+1} : ||x||^2 = 1\}$$

where the norm is taken with respect to the hermitian form on $\mathbf{F}_k^{m+1}$. View $\mathbf{F}_k^{m+1}$ as a real vectorspace V; $\mathbf{F}$ acts on V. Let Q be the real part of the hermitian form on $\mathbf{F}_k^{m+1}$; Q is a symmetric bilinear form giving V the structure of $\mathbf{R}_h^{n+1}$. Let $\mathbf{F}'$ denote the multiplicative group of unimodular element of $\mathbf{F}$; now $\mathbf{F}' \subset \mathbf{O}^h(n + 1)$ and $H \subset \mathbf{O}^h(n + 1)$ where everything acts on V and H is the group mapped by Φ.

8.3. Lemma. *Let G' be the centralizer of D in H and define $G = G' \cdot \mathbf{F}'$. Then G is transitive on the points of $\mathbf{S}_h^n$. In particular, $\mathbf{S}_h^n/D$ is a homogeneous pseudo-*Riemannian *manifold of constant positive curvature.*

Proof of Lemma. Let $x, y \in \mathbf{S}_h^n$. Homogeneity of M_s^r implies ([17], Th. 2.5) that the centralizer of Δ in $\mathbf{I}_0(N_s^r)$ is transitive on N_s^r; it follows that G' is transitive on N_s^r; thus G' has an element g_1 mapping $x\mathbf{F}$ onto $y\mathbf{F}$. As $||x||^2 = ||y||^2$, $\mathbf{F}'$ has an element g_2 sending g_1x to y. This proves transitivity of G.

Transitivity of G shows that D acts freely on $\mathbf{S}_h^n$, for every element of G commutes with every element of D. The action of Δ on $\mathbf{P}_k^m(\mathbf{F})$ is properly discontinuous and $\mathrm{Ker}.\,\Phi$ is finite; thus the action of D on $\mathbf{S}_h^n$ is properly discontinuous. Now $\mathbf{S}_h^n \to \mathbf{S}_h^n/D$ is a normal covering, and homogeneity follows ([17], Th. 2.5).

Lemma 8.3 is now proved.

8.4. Lemma. *If D is not finite, then the* $\mathbf{R}$*-linear action of D on V is not fully reducible.*

Proof of Lemma. Lemma 8.4 is based on ([17], Lemma 8.4). Suppose that D is infinite and fully reducible. Then Lemma 8.3 and ([17], Th. 10.1) show that V is the direct sum $U \oplus W$ of two totally Q-isotropic $\mathbf{R}$-subspaces and that there is a nonzero real number $t \neq \pm 1$ such that D has an element d whose restriction to U is tI, whose restriction to W is $t^{-1}I$, and such that d and possibly also $-I$ generate D. U and W are $\mathbf{F}$-subspaces of V because D is $\mathbf{F}$-linear and $t \neq t^{-1}$. Let Q_F be the $\mathbf{F}$-hermitian form on $V = \mathbf{F}_k^{m+1}$; $m + 1 = 2k$, k is the common dimension of U and W over $\mathbf{F}$, U and W are totally Q_F-isotropic ([17], § 8.4), and there are $\mathbf{F}$-bases $\{e_i\}$ of U and $\{f_i\}$ of W with $Q_F(e_i, f_j) = 2\delta_{ij}$. Now let σ represent $\mathbf{GL}(k, \mathbf{F})$, the general linear group of $\mathbf{F}^k$, on V by: $\sigma(\alpha)$ has matrix $\begin{pmatrix} \alpha & 0 \\ 0 & {}^t\overline{\alpha}^{-1} \end{pmatrix}$ in the basis $\{e_i, f_j\}$ of V over $\mathbf{F}$. The image of σ is precisely our group G', consisting of all $\mathbf{F}$-linear transformations of V which preserve Q_F and commute with each element of D.

We have $k > 1$ because M_s^r is not of constant curvature. Define $w_i = (1/2)(e_i - f_i)$ and $w_{k+i} = (1/2)(e_i + f_i)$ for $1 \leq i \leq k$; now $\{w_j\}$ is a Q_F-orthonormal basis of V. As $w_{k+1} \in \mathbf{S}_h^n$ and $G = G' \cdot \mathbf{F}'$ is transitive on $\mathbf{S}_h^n$, whenever we have $x \in \mathbf{S}_h^n$ we must have $b \in \mathbf{F}'$ and $\alpha \in \mathbf{GL}(k, \mathbf{F})$ such that $\sigma(\alpha): w_{k+1} \to xb$. Let $x = \Sigma w_j x_j$ with $x_j \in \mathbf{F}$; then it follows as in the proof of ([17], Lemma 8.4) that $\sum_{i=1}^{k} (x_i b)(\overline{x_{k+i} b}) \in \mathbf{R} \subset \mathbf{F}$, i.e., that $\sum_{i=1}^{k} x_i \overline{x_{k+i}} \in \mathbf{R}$. As $\mathbf{F} \neq \mathbf{R}$, $\mathbf{S}_h^n$ has many elements $x = \Sigma w_j x_j$ for which $\sum_{i=1}^{k} x_i \overline{x_{k+i}}$ is not real. Thus D cannot be both infinite and fully reducible.

Lemma 8.4 is now proved.

8.5. Lemma. *If D is not finite, then* $\mathbf{F} \neq \mathbf{C}$.

Proof of Lemma. Suppose that D is infinite and $\mathbf{F} = \mathbf{C}$. Nonconstancy of sectional curvature of M_s^r shows $m + 1 > 2$, so the kernel $\mathrm{Ker}.\,\Phi$, which is isomorphic to cyclic group $\mathbf{Z}_{m+1}$, is a cyclic subgroup of D of order $q > 2$. D has an element d of infinite order; let D_1 be the subgroup generated by d and Ker. Φ. Ker. Φ is central in D so $D_1 \cong \mathbf{Z} \times \mathbf{Z}_q$, $q > 2$. Lemmas 8.3 and 8.4 hold for D_1. Now D_1 is abelian but not fully reducible on V; Theorem 10.1 of [17] says that D_1 is isomorphic to $\mathbf{Z}, \mathbf{Z} \times \mathbf{Z}_2$, $\mathbf{Z} \times \mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_2$. Thus $\mathbf{F} \neq \mathbf{C}$ if D is infinite.

Lemma 8.5 is now proved.

8.6. The following is the last and most delicate step in the proof of Theorem 8.1.

Lemma. *If D is not finite, then* $\mathbf{F} \neq \mathbf{K}$.

Proof of Lemma. Suppose that D is infinite and $\mathbf{F} = \mathbf{K}$. D is not fully reducible on V and we apply the method of ([17], § 10.4). Lemmas 8.2 and 8.6 of [17] provide a G-invariant D-invariant maximal totally Q-isotropic subspace U of V and a Q-orthonormal $\mathbf{R}$-basis $\gamma = \{v_1, \ldots, v_{n+1=h+p}\}$ such that ($e_i = v_{h+i} + v_i$ and $f_i = v_{h+i} - v_i$ for $1 \leq i \leq p$) $\{e_i\}$ is a basis of U and $\{v_{p+1}, \ldots, v_h; e_1, \ldots, e_p\}$ is a basis of $U^{\perp}$. Let β be the basis $\{f_1, \ldots, f_p; v_{p+1}, \ldots, v_h; e_1, \ldots, e_p\}$ of V and let $\mathbf{B}$ be the algebra of linear transformations of V generated by D. If $\underline{a}$ is an element of $\mathbf{B}$, D or G, then $\underline{a}$ preserves both U and $U^{\perp}$, so $\underline{a}$ has matrix relative to β given in block form by

$$\underline{a} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix}.$$

If $\underline{a} \in G$ or $\underline{a} \in D$, then $\underline{a} \in \mathbf{O}^h(n+1)$ and it follows that $a_6 = {}^t a_1^{-1}$ and $a_4 \in \mathbf{O}(h-p)$.

Let r_1, r_4 and r_6 be the matric representations of respective degrees p, $h-p$ and p of G given by $\underline{a} \to a_1$, $\underline{a} \to a_4$ and $\underline{a} \to a_6$. r_6 is irreducible ([17], Lemma 8.2) and r_1 is contragredient to r_6. If $\underline{a} \in G$ sends v_{h+1} to $x = \Sigma x_j v_j$, then one checks that the first row of a_1 is $(x_{h+1} - x_1, \ldots, x_{h+p} - x_p) = q(x)$. As x ranges over $\mathbf{S}_h^n$, $q(x)$ ranges over a subset of $\mathbf{R}^p$ on which no nondegenerate quadratic form is bounded. In [17] this was observed to imply that r_1 has no nonzero symmetric bilinear invariant. As $G = G' \cdot \mathbf{K}'$ and $\mathbf{K}'$ is compact, it also shows that the restriction r_1' of r_1 to G' is without a nonzero symmetric bilinear invariant. It follows that $r_6' = r_6|_{G'}$ has no nonzero symmetric bilinear invariant.

After considering bilinear invariants of r_6, § 10.4 of [17] goes on to show that every element $d \in D$ has matrix of the form

$$d = \begin{pmatrix} d_1 & 0 & d_3 \\ 0 & d_4 & 0 \\ 0 & 0 & d_1 \end{pmatrix} \text{ relative to } \beta.$$

It is further shown that an element d of D can be chosen with $d_3 \neq 0$ and that, this choice made, d_3 is a bilinear invariant of r_6, i.e., ${}^t g_6 d_3 g_6 = d_3$ for every $g \in G$. Now let $g \in G'$ and $u \in \mathbf{K}'$. Then ${}^t g_6 (d_3 u_6) g_6 = {}^t g_6 d_3 g_6 u_6 = d_3 u_6$, whence $d_3 u_6$ is a bilinear invariant of r_6'. It follows that $d_3 u_6$ is antisymmetric for every $u \in \mathbf{K}'$. Now as in [17] we take $u, v \in \mathbf{K}'$, observe that $w \in \mathbf{K}'$ gives $d_3 w_6 = -{}^t(d_3 w_6) = -{}^t w_6 \cdot {}^t d_3 = {}^t w_6 d_3$, and conclude that $u_6 v_6 = (uv)_6 = d_3^{-1} \cdot {}^t(uv)_6 \cdot d_3 = d_3^{-1} \cdot {}^t v_6 \cdot d_3 \cdot d_3^{-1} \cdot {}^t u_6 \cdot d_3 = v_6 u_6$. As the restriction of r_6 to $\mathbf{K}'$ is faithful, it follows that $\mathbf{K}'$ is commutative. That is absurd.

Lemma 8.6 is now proved.

8.7. We saw in § 8.1 that, in order to prove Theorem 8.1, it was sufficient to assume $N_s^r = \mathbf{P}_k^m(\mathbf{F} = \mathbf{C}$ or $\mathbf{K})$ and prove Δ finite. For this, it is sufficient to prove D finite. If D is infinite, then Lemma 8.5 shows $\mathbf{F} \neq \mathbf{C}$ and Lemma 8.6 shows $\mathbf{F} \neq \mathbf{K}$. This completes the proof of Theorem 8.1. *Q. E. D.*

9. The classification theorem for complete homogeneous locally isotropic manifolds of nonconstant curvature

The following theorem gives the classification mentioned above. Combined with Theorem 12 of [17], it gives the classification up to global isometry of the complete homogeneous locally isotropic manifolds which are not flat. A partial classification is available [19] in the flat case.

9.1. Theorem. *Let M_h^n be a complete connected locally isotropic pseudo-RIEMANNian manifold of nonconstant sectional curvature. Then these are equivalent:*

1. *M_h^n is homogeneous.*
2. *$\pi_1(M_h^n)$ is finite.*
3. *$\pi_1(M_h^n)$ has only 1 or 2 elements.*
4. *M_h^n is isometric to one of the spaces:*

(a) $\mathbf{P}_k^2(\mathrm{Cay})$ or $\mathbf{H}_k^2(\mathrm{Cay})$ *where* $n = 16$ *and* $h = 8k$.

(b) $\mathbf{P}_k^m(\mathbf{K})$ *or* $\mathbf{H}_k^m(\mathbf{K})$ *where* $n = 4m$ *and* $h = 4k$.

(c) $\mathbf{P}_k^m(\mathbf{C})$ *or* $\mathbf{H}_k^m(\mathbf{C})$ *where* $n = 2m$ *and* $h = 2k$.

(d) $\mathbf{P}_{2s}^{2t-1}(\mathbf{C})/\{1, \alpha J\}$ *where* $n = 4t-2$, $h = 4s$, α *is the isometry induced by conjugation of* $\mathbf{C}$ *over* $\mathbf{R}$, *and* J *is the isometry induced by*

$$\begin{pmatrix} 0 & I_s & & \\ -I_s & 0 & & \\ & & 0 & I_{t-s} \\ & & -I_{t-s} & 0 \end{pmatrix} \in \mathbf{SU}^{2s}(2t) .$$

(e) $\mathbf{H}_{2r-1}^{2t-1}(\mathbf{C})/\{1, \alpha J\}$ *where* $n = 4t-2$, $h = 4r-2$, *and* α *and* J *are given as above with* $s = t - r$.

9. 2. *Proof*. Theorem 6.1 is the equivalence of (2), (3) and (4), Theorems 8.1 and 6.1 show that (1) implies (4), and the manifolds (4a), (4b) and (4c) are homogeneous. As one passes from (4e) to (4d) by reversing the metric, the Theorem will now follow as soon as we prove that $\mathbf{P}^{2t-1}_{2s}(\mathbf{C})/\{1, \alpha J\}$ is homogeneous.

Let $V = \mathbf{C}^{2t}_{2s}$ and identify J with its matrix as given in (4d). Let $\mathbf{A}$ be the algebra of $\mathbf{R}$-linear endomorphisms of V generated by αJ and $\sqrt{-1}\, I_{2t}$. As both αJ and $\sqrt{-1}\, I_{2t}$ have square $-I$, and as they anticommute, $\mathbf{A}$ is a quaternion algebra. Let Q_C be the $\mathbf{C}$-hermitian form on V and let Q be its real part. There is a unique $\mathbf{A}$-hermitian form Q_A on V whose real part is Q. We now have $(V, Q) = \mathbf{R}^{4t}_{4s}$, $(V, Q_C) = \mathbf{C}^{2t}_{2s}$ and $(V, Q_A) = \mathbf{K}^{t}_{s}$. One can describe $\mathbf{S}^{4t-1}_{4s}$ by $Q(x, x) = 1$, by $Q_C(x, x) = 1$, or by $Q_A(x, x) = 1$. The unitary group $\mathbf{Sp}^s(t)$ of (V, Q_A) is a subgroup of the unitary group $\mathbf{U}^{2s}(2t)$ of (V, Q_C); by construction, $\mathbf{Sp}^s(t)$ is the centralizer of αJ in $\mathbf{U}^{2s}(2t)$. As $\mathbf{Sp}^s(t)$ acts transitively on $\mathbf{S}^{4t-1}_{4s}$ and $\mathbf{C}$-linearly on V, it induces a transitive group of motions of $\mathbf{P}^{2t-1}_{2s}(\mathbf{C})$. It follows that $\mathbf{P}^{2t-1}_{2s}(\mathbf{C})/\{1, \alpha J\}$ has a transitive group of isometries. *Q.E.D.*

9. 3. Combining Theorem 9.1 with Theorem 12 of [17], we obtain the rather strange result:

Theorem. *Let M^n_h be a complete connected locally isotropic pseudo-*Riemannian* manifold which is homogeneous but not isotropic. Then M^n_h is of constant sectional curvature, if and only if $\pi_1(M^n_h)$ is not of order 2. M^n_h is of nonconstant sectional curvature, if and only if $n = 4t - 2$, $h = 4s$ (resp. $h = 4r - 2$) and M^n_h is isometric to $\mathbf{P}^{2t-1}_{2s}(\mathbf{C})/\{1, \alpha J\}$ (resp. $\mathbf{H}^{2t-1}_{2r-1}(\mathbf{C})/\{1, \alpha J\}$).*

For if N^n_h is complete, connected, of constant sectional curvature, and with fundamental group of order 2, then N^n_h is isometric to $\mathbf{P}^n_h(\mathbf{R})$ with $h < n - 1$ or to $\mathbf{H}^n_h(\mathbf{R})$ with $h > 1$. Those manifolds are isotropic. Theorem 9.3 now follows from Theorems 7.1 and 9.1.

10. A characterization and completeness criterion for locally symmetric affine homogeneous spaces

This section provides a tool for the study of incomplete homogeneous spaces.

10. 1. An *affine manifold* is a differentiable manifold with an affine connection on its tangentbundle. If M is an affine manifold, then $\mathbf{A}(M)$ denotes the group of all connection preserving diffeomorphisms of M, and M is *affine homogeneous* if $\mathbf{A}(M)$ is transitive on the points of M. Given $x \in M$, there is an open neighborhood U of x on which the geodesic symmetry at x is a diffeomorphism; M is *locally symmetric* if, given $x \in M$, U can be chosen such that the geodesic symmetry preserves the induced connection on U; this is equivalent to vanishing of the torsion tensor and parallelism of the curvature tensor for the affine connection on M. M is symmetric if it is locally symmetric and the geodesic symmetries extend to elements of $\mathbf{A}(M)$. If the affine connection of M is the LEVI-CIVITA connection of a pseudo-RIEMANNian metric, then M is (locally) symmetric in the affine sense if and only if it is (locally) symmetric in the pseudo-RIEMANNian sense, for an element of $\mathbf{A}(M)$ or $\mathbf{A}(U)$ is an isometry if it is an isometry at some fixed point; on the other hand, M can be affine homogeneous but not pseudo-RIEMANNian homogeneous.

10. 2. Theorem. *Let M be a locally symmetric connected affine homogeneous manifold. Then there is a connected simply connected affine symmetric manifold N, a point $y \in N$, and a closed connected subgroup $B \subset \mathbf{A}(N)$, such that*

1. *$B(y)$ is an open submanifold of N.*
2. *M and $B(y)$ have the same universal affine covering manifold.*
3. *M is complete if and only if $B(y) = N$.*

10. 3. Corollary. *Let M be a compact connected locally symmetric affine homogeneous manifold. If $\pi_1(M)$ is finite, then M is complete.*

For $B(y)$ must be closed in N, as it is a continuous image of the compact manifold which is the universal covering of M.

10. 4. Corollary. *Let M be a connected locally symmetric affine manifold, and suppose that there is a connected solvable subgroup $G \subset \mathbf{A}(M)$ which is transitive on M. If the space N of Theorem 10.2 is not contractible, then M is not complete.*

For the proof of Theorem 10.2 will show that B can be chosen to be solvable when $\mathbf{A}(M)$ has a solvable transitive subgroup. If M is complete, then $B(y) = N$, so $N = B/F$ where F is the isotropy subgroup of B at y. F is closed in B, and is connected because $\pi_1(N) = \{1\}$; it follows that N has a deformation retraction onto a torus. As $\pi_1(N) = \{1\}$, this says that N is contractible. Thus M cannot be complete if N is not contractible.

10. 5. Proof of Theorem. Choose a connected transitive subgroup $G \subset \mathbf{A}(M)$, close it if it is not closed, let $x \in M$, and let H be the isotropy subgroup of G at x. The universal covering $\beta: G' \to G$ gives an action of the universal covering group G' on M by $g': m \to \beta(g')(m)$. A result of K. NOMIZU ([15], Th. 17.1) provides a connected simply connected affine symmetric manifold N, a point $y \in N$, open neighborhoods U of x and V of y, and an affine diffeomorphism f of U onto V carrying x to y. Let W be an open neighborhood of $1 \in G'$ such that $g'(x) \in U$ for every $g' \in W$. Given $g' \in W$, now, we have a small open neighborhood X of x such that $X \cup g'(X) \subset U$; thus g' induces an affine equivalence g'' of $f(X)$ onto $f(g'(X))$.

N is complete because one can extend geodesics by the symmetries. As N is simply connected, and as its affine connection is real analytic ([15], Th. 15.3), it follows that g'' extends to a unique element (also denoted g'') of $\mathbf{A}(N)$. This map $g' \to g''$ gives a homomorphism of G' onto a subgroup L of $\mathbf{A}(N)$. $L(y)$ contains V and thus is an open submanifold of N. Let B be the closure of L in $\mathbf{A}(N)$. B preserves the closed set $N - L(y)$ because so does L, and thus $B(y) = L(y)$. The first statement of the Theorem is proved. For Corollary 10.4, one observes that B is solvable if G is solvable.

In the universal covering $\beta : G' \to G$, let H' be the identity component of $\beta^{-1}(H)$. β induces a universal covering $\pi : G'/H' \to M$; pull back the connection of M so that π is the universal affine covering. Now let $\alpha : G' \to B(y)$ be defined by $\alpha(g') = g''(y)$. $\alpha(H') = y$ because every vectorfield on U corresponding to a 1-parameter subgroup of H vanishes at x. Thus α induces a map Φ of G'/H' onto $B(y)$, and Φ is a covering. If we take a small π-admissible open set in M, lift to G'/H' and then map by Φ, we have an affine equivalence, for everything can be translated back into U and V; thus Φ is an affine covering. The second statement of the Theorem is proved.

The second statement shows that M is complete if and only if $B(y)$ is complete. As N is complete, M is complete if $B(y) = N$. Now suppose $B(y) \neq N$; we must prove that $B(y)$ is not complete. As $B(y) \neq N$ but $B(y)$ is open in N, we have a boundary point v of $B(y)$ with $v \notin B(y)$. Some geodesic of N through v must have a point in $B(y)$; otherwise v would have a geodesic coordinate neighborhood disjoint from $B(y)$ and thus would not be a boundary point. Let $\{v_t\}$ be a geodesic of N, $v_0 = v$ and $v_1 \in B(y)$. $v_{1+\varepsilon} \in B(y)$ for $|\varepsilon|$ small; thus $B(y)$ contains an arc of $\{v_t\}$. If $B(y)$ were complete it would contain all of $\{v_t\}$. As $v \notin B(y)$, $B(y)$ is not complete. *Q.E.D.*

10. 6. The following can occasionally be useful in applying Theorem 10. 2.

Theorem. *Let N be a complete connected affine homogeneous manifold; let $B \subset \mathbf{A}(N)$ be a closed connected subgroup with an open orbit $B(y)$; let H be the isotropy subgroup of the identity component $\mathbf{A}_0(N)$; and let K_B, K_H and K_A be respective maximal compact subgroups of B, H and $\mathbf{A}_0(N)$ such that $K_B \subset K_A$ and $K_H \subset K_A$. If K_B is not transitive on K_A/K_H, then $B(y) \neq N$, and consequently $B(y)$ is not complete.*

For we can assume $L = K_H \cap K_B$ to be maximal compact in $H \cap B$. If K_B is not transitive on K_A/K_H, then an orbit K_B/L has lower dimension and thus (these manifolds are compact) has different homotopy type. Then $B(y)$ has different homotopy type than N, for $B(y) = B/(B \cap H)$ retracts onto K_B/L and $N = \mathbf{A}_0(N)/H$ retracts onto K_A/K_H. It follows that $B(y) \neq N$.

11. A tool for the construction of incomplete manifolds

We will develop tools for the construction of incomplete homogeneous pseudo-RIEMANNian manifolds as orbits of groups which preserve the exponential image of a set obtained from a light cone. The basic idea is illustrated by:

11. 1. Theorem. *Let U be a nonzero totally isotropic linear subspace of $\mathbf{R}^n_h$, and define M^n_h to be the open set $\mathbf{R}^n_h - U^\perp$ with the induced pseudo-RIEMANNian structure. Then M^n_h is a flat homogeneous incomplete pseudo-RIEMANNian manifold.*

Remark. There are min. $\{h, n-h\}$ non-isometric manifolds M^n_h above, corresponding to the possibilities for the dimension of U.

Remark. Given a group H of linear transformations of U which is transitive on $U - \{0\}$, define G to be the subgroup of $\mathbf{I}(\mathbf{R}^n_h)$ consisting of elements which preserve M^n_h, whose homogeneous part preserves U, and whose homogeneous part restricts to an element of H. We will in fact prove that G is transitive on M^n_h.

Proof. Let β be a skew basis of $\mathbf{R}^n_h$ with respect to U. This means that we choose an orthonormal basis $\{v_1, \ldots, v_h;\ v_{h+1}, \ldots, v_n\}$ of $\mathbf{R}^n_h$ such that ($t = \dim. U$; $f_i = -v_i + v_{n-t+i}$ and $e_i = v_i + v_{n-t+i}$ for $1 \leq i \leq t$) $\{e_1, \ldots, e_t\}$ is a basis of U, and we define β to be the basis $\{f_1, \ldots f_t;\ v_{t+1}, \ldots, v_{n-t};\ e_1, \ldots, e_t\}$ of $\mathbf{R}^n_h$. $U^\perp$ is spanned by the e_i and the v_j in β; thus $G_U = \{g \in \mathbf{O}^h(n) : g(U) = U\}$ consists of the elements $g \in \mathbf{O}^h(n)$ whose matrix relative to β has block form

$$g_\beta = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix}.$$

Let H be a subgroup of $\mathbf{GL}(U)$ which is transitive on $U - \{0\}$. There is a monomorphism $r \to r'$ of H onto a subgroup H' of G given by:

$$(r')_\beta = \begin{pmatrix} {}^t r^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & r \end{pmatrix}$$

where r denotes the matrix of r with respect to $\{e_i\}$. Let G be the subgroup of $\mathbf{I}(\mathbf{R}^n_h)$ consisting of all isometries whose homogeneous part lies in H' and whose translation part is in $U^\perp$; G preserves M^n_h. Given $x \in M^n_h$, $x = f + v$ where $v \in U^\perp$ and f is a nonzero element of the span of the f_i; choose $r' \in H'$ with $r'(f_1) = f$; G contains the transformation $y \to r'(y) + v$, and this transformation sends f_1 to x. Thus G is transitive on M^n_h. *Q. E. D.*

Corollary. *The connected noncommutative 2-dimensional LIE group admits an incomplete flat left-invariant LORENTZ metric.*

Proof. Suppose $n = 2$ and $t = h = 1$ above. Then H consists of nonzero scalars and G is simply transitive on M_1^2. The diffeomorphism $g \to g(f_1)$ of G onto M_1^2 endows G with an incomplete flat left-invariant LORENTZ metric, and the group of the Corollary is isomorphic to the identity component of G. *Q.E.D.*

11.2. In order to apply the idea of Theorem 11.1 to a manifold which is symmetric but not flat, one replaces H' by a corresponding subgroup of the isotropy group at a point, and one replaces translations along $U^\perp$ by transvections. It is difficult to see that a group analogous to G will then result, and it is very difficult to see whether the orbits are proper submanifolds. But the construction of most of our model spaces allows us to replace those considerations of LIE algebra by considerations of linear associative algebra while preserving the idea of Theorem 11.1. The associative algebra consists of:

11.3. Theorem. *Let Q_F be the hermitian form on $\mathbf{F}_k^m$, define quadrics $\mathbf{S} = \{x \in \mathbf{F}_k^m : Q_F(x, x) = 1\}$ and $\mathbf{H} = \{x \in \mathbf{F}_k^m : Q_F(x, x) = -1\}$, and let U be a nonzero maximal totally Q_F-isotropic $\mathbf{F}$-subspace of $\mathbf{F}_k^m$. Let P be any group of $\mathbf{F}$-linear transformations of U which contains the real scalars and is transitive on $U - \{0\}$, define $G = \{g \in \mathbf{U}^k(m, \mathbf{F}) : g(U) = U$ and $g|_U \in P\}$, let $\mathbf{F}'$ be the group of transformations of $\mathbf{F}_k^m$ which are scalar multiplication by unimodular elements of $\mathbf{F}$, and let G' be the group of $\mathbf{R}$-linear transformations of $\mathbf{F}_k^m$ generated by G and $\mathbf{F}'$. Let α be a Q_F-orthonormal $\mathbf{F}$-basis $\{v_1, \ldots, v_k;\ v_{k+1}, \ldots, v_m\}$ of $\mathbf{F}_k^m$ which gives a skew $\mathbf{F}$-basis β of $\mathbf{F}_k^m$ with respect to U. If $2k > m$, then $G'(v_1)$ is a proper open submanifold of $\mathbf{H}$. If $2k < m$, then $G'(v_m)$ is a proper open submanifold of $\mathbf{S}$.*

Proof. Let $t = \dim._F U$ and $f = \dim._R \mathbf{F}$ so $tf = \dim._R U$, let $\beta = \{f_1, \ldots, f_t;\ v_{t+1}, \ldots, v_{m-t};\ e_1, \ldots, e_t\}$ be the skew $\mathbf{F}$-basis of $\mathbf{F}_k^m$ derived from α so that $\varepsilon = \{e_1, \ldots, e_t\}$ spans U over $\mathbf{F}$, and let S^{tf-1} be the unit sphere $\{\overset{t}{\underset{1}{\Sigma}} e_i a_i : \overset{t}{\underset{1}{\Sigma}} a_i \bar{a}_i = 1\}$ in U. P acts transitively on S^{tf-1} by $p : u \to p(u) \cdot \{\overset{t}{\underset{1}{\Sigma}} \bar{b}_i b_i\}^{-1/2}$ where $u \in S^{tf-1}$ and $p(u) = \Sigma e_i b^i$; it follows that a maximal compact subgroup $K \subset P$ is already transitive on S^{tf-1}, by a check if $tf \leq 2$, and because S^{tf-1} is simply connected if $tf > 2$. We may now alter ε, changing β and α in consequence, so that the $\mathbf{F}$-linear action of K preserves $\mathbf{S}^{tf-1}$; this change will not effect the Theorem.

We wish to prove certain orbits $G'(v)$ proper and open in quadrics Q, i.e., properly contained in Q and of the same dimension as Q. We may cut G' to a subgroup in proving $G'(v)$ open, and we may do the same in proving $G'(v) \neq Q$ because our reason for that inequality (see § 11.5) will depend only on m, k and the fact that $G'(U) = U$. Thus we may replace P by its subgroup generated by K and the real scalars. This done, the matrix p_ε of an element $p \in P$ relative to ε satisfies $c \cdot p_\varepsilon = {}^t\bar{p}_\varepsilon^{-1}$ where $0 \neq c \in \mathbf{R}$.

Let $g \in G$. Its matrix relative to β has block form

$$g_\beta = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix},$$

and g_6 is a real scalar multiple of g_1 because $g_6 = p_\varepsilon$ for some $p \in P$ by definition of G, $g_1 = {}^t\bar{g}_6^{-1}$ because $g \in \mathbf{U}^k(m, \mathbf{F})$, and ${}^t\bar{p}_\varepsilon^{-1}$ is a real multiple of p_ε. Let $(g_i)_u$ and $(g_i)_{uv}$ denote the u row and the (u, v) element of g_i. Then we have

$$g(e_1 + f_1) = \sum_1^t \{e_i(g_6)_{1i} + f_i(g_1)_{1i} + e_i(g_3)_{1i}\} + \sum_{t+1}^{m-t} v_i(g_2)_{1i}$$

$$g(e_1 - f_1) = \sum_1^t \{e_i(g_6)_{1i} - f_i(g_1)_{1i} - e_i(g_3)_{1i}\} - \sum_{t+1}^{m-t} v_i(g_2)_{1i}$$

thus $2v_1 = e_1 - f_1$ tells us that $g(v_1) = v_1$ if and only if

(a) $(g_2)_1 = (0, \ldots, 0)$

(b) $(g_1)_1 = (1, 0, \ldots, 0) = (g_6)_1 - (g_3)_1$.

Similarly $2v_{m-t+1} = e_1 + f_1$ says that $g(v_{m-t+1}) = v_{m-t+1}$ if and only if

(a′) $(g_2)_1 = (0, \ldots, 0)$

(b′) $(g_1)_1 = (1, 0, \ldots, 0) = (g_6)_1 + (g_3)_1$.

Now (a) coincides with (a′), and the first equality of both (b) and (b′) imply $(g_1)_1 = (g_6)_1$, so (b) is equivalent to (b′). It follows that $g(v_1) = v_1$ if and only if $g(v_{m-t+1}) = v_{m-t+1}$. In particular, as $v_m \in G(v_{m-t+1})$ so that $G(v_{m-t+1}) = G(v_m)$ we have dim. $G(v_1) =$ dim. $G(v_m)$.

To calculate these dimensions we need:

11. 4. Lemma. *Retain the notation of Theorem* 11.3. *If* $2k \geq m$, *then* G *is transitive on* $\mathbf{S}$. *If* $2k \leq m$, *then* G *is transitive on* $\mathbf{H}$.

Replacement of Q_F by $-Q_F$ shows the two statements to be equivalent, so only the first need be proved. Assume $2k \geq m$; then $t = m - k$ because U is *maximal* totally isotropic, whence the span of $\{v_{t+1}, \ldots, v_{m-t-k}\}$ is negative definite under Q_F. One now repeats the transitivity argument of ([17], p. 127) replacing EUCLIDean pairings by $\mathbf{F}$-hermitian pairings and writing scalars on the right.

11. 5. Combining Lemma 11.4 with dim. $G(v_1) =$ dim. $G(v_m)$, we see that $G(v_1)$ and $G'(v_1)$ are open submanifolds of $\mathbf{H}$, and that $G(v_m)$ and $G'(v_m)$ are open submanifolds of $\mathbf{S}$. As G and G' preserve U, we know ([17], Lemma 8.2) that transitivity of G or G' on $\mathbf{S}$ would imply $2k \geq m$, and thus, by reversal of Q_F, transitivity of G or G' on $\mathbf{H}$ would imply $2k \leq m$. The Theorem follows. *Q. E. D.*

11. 6. A conjecture. Theorem 11.3 will allow us to deal with all of the isotropic manifolds of strictly indefinite metric except $\mathbf{P}_1^2(\mathbf{Cay})$ and $\mathbf{H}_1^2(\mathbf{Cay})$. In those spaces, however, we will see that the normalizer of a BOREL subgroup of the full group of isometries has an open proper orbit. I believe that this is the case for every symmetric pseudo-RIEMANNian manifold G/H of strictly indefinite metric and of rank one where G is simple and has the same rank as H. Unfortunately I have been unable to give a general proof of this conjecture, even for the spaces covered by Theorem 11.3.

11. 7. Theorem. *Let B be a BOREL subgroup of $\mathbf{F}_4^*$ and let P be the normalizer of B in $\mathbf{F}_4^*$. Then P has a proper open orbit on $\mathbf{F}_4^*/\mathbf{Spin}(9)$.*

Proof. Let $G = \mathbf{F}_4^*$ acting on $M = \mathbf{F}_4^*/\mathbf{Spin}^1(9)$. Choose $x \in M$, let H be the isotropy subgroup of G at x, and choose a maximal compact subgroup K of G which is normalized by the symmetry s of M at x. We have involutive automorphisms τ and $\sigma = \mathrm{ad}(s)$ of $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{K} + \mathfrak{L} = \mathfrak{H} + \mathfrak{M}$ where σ is $+1$ on $\mathfrak{H}$ and is -1 on $\mathfrak{M}$, and where τ is $+1$ on $\mathfrak{K}$ and is -1 on $\mathfrak{L}$.

Metrize M as $\mathbf{H}_1^2(\mathbf{Cay})$ with the KILLING form of $\mathfrak{G}$; let $\mathfrak{U}$ be an isotropic (= light like) line in $M_x = \mathfrak{M}$, let γ be the light like geodesic $\exp(\mathfrak{U}) \cdot x$ in M, and choose $y \in \gamma$ within a geodesically convex normal coordinate neighborhood of x. The point y cannot be fixed under $\exp(\mathfrak{L})$, for $\exp(\mathfrak{L})$ generates G. Thus we have an one-dimensional subspace $\mathfrak{A} \subset \mathfrak{L}$ such that $\mathfrak{A} \not\subset \mathfrak{H}'$, where H' is the isotropy subgroup of G at y.

$A = \exp(\mathfrak{A})$ is maximal among the algebraic tori of G which split over $\mathbf{R}$, for the RIEMANNian symmetric space G/K is of rank 1; thus there is a unipotent subgroup $N \subset G$ such that $G = KAN$ is an Iwasawa decomposition. We may replace B by a conjugate, assuming $B = AN$. Now $P = ZB$ where Z is the centralizer of A in G. $\tau(\mathfrak{Z}) = \mathfrak{Z}$ because $\tau(\mathfrak{A}) = \mathfrak{A}$; thus $\mathfrak{Z} = \mathfrak{Q} + \mathfrak{R}$ where $\mathfrak{Q} = \mathfrak{Z} \cap \mathfrak{K}$ and $\mathfrak{R} = \mathfrak{Z} \cap \mathfrak{L}$. $\mathfrak{R} = \mathfrak{A}$ as rank $(G/K) = 1$, and $Q = \exp(\mathfrak{Q})$ is compact; $P = QAN$.

Let $J = P \cap H'$. J is algebraic because P and H' are algebraic; thus $\mathfrak{J} = \mathfrak{S} + \mathfrak{C} + \mathfrak{V}$ where $\mathfrak{V}$ is a unipotent ideal, $\mathfrak{S} + \mathfrak{C}$ is reductive, $\mathfrak{S}$ is semisimple, and $\mathfrak{C}$ is the center of $\mathfrak{S} + \mathfrak{C}$. $S = \exp(\mathfrak{S})$ is a compact subgroup of H which preserves the geodesic γ: compactness follows because S is a semisimple subgroup of the compact group Q, and γ is preserved because S fixes both x and y. In proving $\mathbf{H}_1^2(\mathbf{Cay})$ to be isotropic, we saw that the maxima among the compact subgroups of H preserving a nonzero element of the light cone of M_x were of type G_2. It follows that S is contained in a subgroup $\mathbf{G}_2$ of H; in particular, $\dim. \mathfrak{S} \leq 14$. $\mathfrak{C} + \mathfrak{V}$ lies in a BOREL subalgebra $\mathfrak{B}'$ of $\mathfrak{H}'$. $\mathfrak{B}'$ is of dimension 8 because $H'/(\text{maximalcompact}) = \mathbf{Spin}^1(9)/\mathbf{Spin}(8)$ is real hyperbolic 8-space; thus $\dim. (\mathfrak{C} + \mathfrak{V}) \leq 8$. If $\dim. (\mathfrak{C} + \mathfrak{V}) = 8$, then $\mathfrak{C} + \mathfrak{B} = \mathfrak{B}'$; it would follow that $\mathfrak{C} + \mathfrak{B}$ contains the LIE algebra of an algebraic torus split over $\mathbf{R}$; $\mathfrak{A}$ being the only such algebra in $\mathfrak{P}$ because Q is compact and $\mathfrak{A}$ is the only such algebra in $\mathfrak{B}$, it would follow that $\mathfrak{A} \subset \mathfrak{H}'$; by our choice of $\mathfrak{A}$, this is not the case. Thus $\dim.(\mathfrak{C} + \mathfrak{B}) \leq 7$. This proves $\dim. \mathfrak{J} \leq 21$.

$\mathfrak{P} = \mathfrak{Q} + \mathfrak{B}$ direct sum as $P = QAN$. $\mathfrak{B}$ is of dimension 16, 16 being the dimension of $\mathbf{H}_0^2(\mathbf{Cay}) = G/K$. Q is the centralizer of $\mathfrak{A}$ in K, which is isomorphic to $\mathbf{Spin}(7)$. Thus $\mathfrak{P}$ has dimension 37. It follows that $\dim. P(y) = \dim. P\text{-}\dim. J \geq 16$, which proves that $P(y)$ is open in M. On the other hand if P were transitive on M, then we could retract M onto the orbit of a maximal compact subgroup K' of P, and it would follow that K' is transitive on a sphere $\mathbf{S}^8 = K/(K \cap H)$; then K' would have to be of type B_4 and would thus be a maximal subgroup of G; as P is not compact it would follow that $P = G$, which is impossible because B is normal in P and G is simple. We have proved that $P(y)$ is a proper open subset of M. *Q. E. D.*

12. Existence of incomplete homogeneous spaces

The following shows that the hypothesis of completeness is essential in Theorems 9.1 and 9.3, in Theorem 12 of [17], and in Theorems 1 and 2 of [19].

12.1. Theorem. *Let N_h^n be an isotropic pseudo-RIEMANNian manifold. Suppose either that N_h^n is of non-negative involutive sectional curvature with $0 < h \leq \frac{n}{2}$ or that N_h^n is of non-positive involutive sectional curvature with $\frac{n}{2} \leq h < n$. Then there is a subgroup $G \subset \mathbf{I}(N_h^n)$ and a point $y \in N_h^n$ such that $G(y)$ is a proper open subset of N_h^n; in the induced pseudo-RIEMANNian structure, $G(y)$ is an incomplete homogeneous pseudo-RIEMANNian manifold locally isometric to N_h^n.*

Remark. Let M_h^n be a non-flat manifold $G(y)$ constructed in the proof of Theorem 12.1. The construction of G and the arguments of ([17], § 10) imply: *If Γ is a properly discontinuous group of isometries acting freely on M_h^n, and if M_h^n/Γ is homogeneous, then Γ is finite.* The signatures of metric involved are about the same as those involved in the finiteness statement (part 1) of Theorem 6.1. This suggests that Theorem 6.1 (part 1) might be valid without assuming completeness.

12. 2. Theorem 12.1 is contained in Theorem 11.1 if N_h^n is flat, and is contained in Theorem 11.7 if N_h^n is a CAYLEY plane. Reversing metrics if necessary, we may assume N_h^n to be of non-negative involutive sectional curvature. Now by Theorem 7.1 we have reduced to the case where there is a pseudo-RIEMANNian covering $\pi: N_h^n \to \mathbf{P}_s^r(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{K}$.

Our hypothesis $0 < h \leq \frac{n}{2}$ translates to $0 < 2s \leq r$.

Recall the quadric $\mathbf{S} = \{x \in \mathbf{F}_s^{r+1} : ||x||^2 = 1\}$ where norm is in the hermitian form on $\mathbf{F}_s^{r+1}$. Let $\mathbf{F}'$ be the group of scalar multiplications of $\mathbf{F}_s^{r+1}$ by unimodular elements of $\mathbf{F}$. As $0 < 2s < r + 1$, Theorem 11.3 yields a subgroup $G' \subset \mathbf{U}^s(r + 1, \mathbf{F})$ and a point $b \in \mathbf{S}$ such that $(H = G' \cdot \mathbf{F}')\, H(b)$ is a proper open subset of $\mathbf{S}$. The projection $\beta: \mathbf{S} \to \mathbf{S}/\mathbf{F}' = \mathbf{P}_s^r(\mathbf{F})$ induces a homomorphism $\alpha: H \to H/\mathbf{F}' = H'' \subset \mathbf{I}(\mathbf{P}_s^r(\mathbf{F}))$; $H(b) = \beta^{-1}(H''(\beta b))$ because $\mathbf{F}' \subset H$. Thus $H''(\beta b)$ is a proper open subset of $\mathbf{P}_s^r(\mathbf{F})$.

Choose $y \in \pi^{-1}(\beta b)$ and lift H'' to a group G of isometries of N_h^n; $G(y) = \pi^{-1}(H''(\beta b))$ and so $G(y)$ is a proper open subset of N_h^n, necessarily incomplete in the induced metric. *Q. E. D.*

12.3. The group G' in the proof of Theorem 12.1 was constructed, in Theorem 11.3, from any subgroup P of the group of all $\mathbf{F}$-linear transformations of a maximal totally isotropic subspace U of $\mathbf{F}_s^{r+1}$ such that (1) P is transitive on $U - \{0\}$ and (2) P contains the real scalars. Often there are several such groups, but I do not know whether the resulting incomplete homogeneous spaces are distinct. Condition (1) cannot be avoided, but condition (2) was used only to ensure that (3) P has a subgroup which satisfies (1) and whose isotropy subgroup at any $u \in U - \{0\}$ preserves a subspace of U complementary to $u\mathbf{F}$. Condition (2) implies that P has no $\mathbf{R}$-bilinear invariant. But if a group P could be found satisfying (1) and (3), then the resulting group $G' \cdot \mathbf{F}'$ would have in its center one of the hyperbolic translation groups $\mathfrak{T}(+)$ described in ([17], § 9.2). If this could be shown to act discontinuously on the resulting orbit $M_h^n \subsetneqq \mathbf{P}_s^r(\mathbf{F})$, then $M_h^n/\mathfrak{T}(+)$ would be homogeneous. This would be of interest in comparison to Theorems 6.1, 8.1, 9.1 and 9.3.

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