On the Classification of Hermitian Symmetric Spaces

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1. Introduction. The hermitian symmetric spaces were first studied by É. Cartan [3], who classified them by means of his classification [2] of the Riemannian symmetric spaces. The work of Borel and de Siebenthal on subgroups of maximal rank in compact Lie groups [1] gives a simpler proof of Cartan's classification result. Other proofs result from H. C. Wang's work on C-spaces [7] and J. Tits' work on parabolic groups [6]. We will give a direct proof which appears to be the simplest available.

2. Preliminaries. In order to establish notation and terminology and to reduce our problem to a problem on compact simple Lie groups, we recall some basic notions on symmetric spaces. The reader is referred to Helgason's book [4] for details.

2.1. A *Riemannian symmetric space* is a connected Riemannian manifold M such that, given $x \in M$, there is a (globally defined) isometry s_x which preserves x and has differential $-I$ on the tangent space M_x; s_x is the *symmetry to M at x*. Given a complex manifold with hermitian metric, one obtains a Riemannian manifold by taking the underlying real manifold and the real part of the hermitian metric. If this Riemannian manifold is symmetric, and if the symmetries are hermitian isometries, one says that the original complex manifold with hermitian metric is a *hermitian symmetric space*. A hermitian symmetric space is always Kählerian.

Let $\tilde{M}$ be a complete simply connected Riemannian manifold. Then [5] $\tilde{M}$ is isometric to a product $M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a Euclidean space and the other M_i are irreducible (not Euclidean, and not locally isometric to a product of lower dimensional Riemannian manifolds.) $\tilde{M}$ is symmetric if and only if each of the M_i is symmetric. $\tilde{M}$ is the real structure of a Kähler manifold if and only if each of the M_i is the real structure of a Kahler manifold,

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and then $M \cong M_0 \times M_1 \times \cdots \times M_r$ may be viewed as a hermitian isometry. In particular, $\tilde{M}$ is hermitian symmetric if and only if each M_i is hermitian symmetric.

Let M be a hermitian symmetric space and let $\pi : \tilde{M} \to M$ be the universal covering. Then there is a unique complex structure and a unique hermitian metric on $\tilde{M}$, such that π is locally a hermitian isometry. As the symmetries of M lift, $\tilde{M}$ is a hermitian symmetric space in these structures. Now $M = \tilde{M}/\Gamma$ where Γ is a discontinuous group of hermitian isometries, and $\tilde{M} \cong M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a complex Euclidean space and the other M_i are irreducible simply connected hermitian symmetric spaces. Now [8, §3] Γ preserves each M_i, acts on M_0 as a group of pure translations, and acts trivially on the other M_i. We have proved:

Lemma 1. *Every hermitian symmetric space admits a hermitian isometry with a space $M_0' \times M_1 \times \cdots \times M_r$ where M_0' is the quotient of a complex Euclidean space by a discrete group of pure translations and the other M_i are irreducible simply connected hermitian symmetric spaces.*

2.2. Let M be a simply connected Riemannian symmetric space. We obtain a triple $(\mathfrak{G}, \sigma, B)$ and a decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ as follows. G is the largest connected group of isometries of M and K in the subgroup which leaves fixed a point $x \in M$, so the coset space G/K is diffeomorphic to M under $gK \to g(x)$. $\mathfrak{G}$ is the Lie algebra of G, σ is the automorphism of $\mathfrak{G}$ induced from conjugation by the symmetry s_x, and $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ is the decomposition into $(+1)$- and (-1)-eigenspaces; $\mathfrak{K}$ is the Lie algebra of K and

$$(2.2.1) \qquad [\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}, \quad [\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P} \quad \text{and} \quad [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}.$$

In particular the image $ad_G(K)$ in the adjoint representation of G preserves $\mathfrak{P}$. Let π_* be the differential at $1 \in G$ of the map $\pi : g \to g(x)$ of G onto M; if $k \in K$, let k_* be its differential at x. Then π_* restricts to a linear isomorphism of $\mathfrak{P}$ onto the tangent space M_x, and this isomorphism is K-equivariant in the sense that

$$(2.2.2) \qquad \pi_*(ad(k)X) = k_*(\pi_* X) \quad \text{for} \quad k \in K \quad \text{and} \quad X \in \mathfrak{P}.$$

The Riemannian metric on M is determined by its G-invariance and its value at x. The latter is a positive definite K_*-invariant inner product on M_x; by (2.2.2) it is specified by a positive definite $ad_G(K)$-invariant inner product B on $\mathfrak{P}$.

Conversely $(\mathfrak{G}, \sigma, B)$ determines M up to isometry. For $\mathfrak{G}$ and σ give $\mathfrak{K}$ and $\mathfrak{P}$. If G' is the simply connected group for $\mathfrak{G}$ and K' is the analytic subgroup for $\mathfrak{K}$, then we have diffeomorphisms

$$G'/K' \approx G/K \approx M$$

by rendering G' effective, and B determines the Riemannian metric of M.

Define $\mathfrak{P}^* = (-1)^{1/2}\, \mathfrak{P}$ and $\mathfrak{G}^* = \mathfrak{K} + \mathfrak{P}^* \subset \mathfrak{G}^{\mathbf{C}}$. Extend σ to $\mathfrak{G}^{\mathbf{C}}$ and then let

σ^* be the restriction to $\mathfrak{G}^*$. Define $B^*(X, Y) = B((-1)^{1/2}X, (-1)^{1/2}Y)$ for $X, Y \in \mathfrak{P}^*$. Now the *dual* of M is the simply connected Riemannian symmetric space M^* determined by $(\mathfrak{G}^*, \sigma^*, B^*)$. M^* is well defined by (2.2.1) and (2.2.2), and $(M^*)^*$ is isometric to M.

Let M be irreducible. This is equivalent to irreducibility of $ad_G(K)$ on $\mathfrak{P}$. If β and β^* are the Killing forms of $\mathfrak{G}$ and $\mathfrak{G}^*$, it follows that $\beta|_{\mathfrak{P}} = rB$ and $\beta^*|_{\mathfrak{P}^*} = (-r)B^*$ for some nonzero real number r. $\beta(\mathfrak{K}, \mathfrak{P}) = 0 = \beta^*(\mathfrak{K}, \mathfrak{P}^*)$, and β and β^* are negative definite on $\mathfrak{K}$, so just one of them is negative definite. Thus the metric on M is determined up to a multiple by $(\mathfrak{G}, \mathfrak{K})$, and just one of M and M^* is compact. Furthermore M is the real structure of a hermitian symmetric space if and only if K is not semisimple. In that case the center of K is a circle group by irreducibility of $ad_G(K)$ on $\mathfrak{P}$ and by Schur's Lemma, the almost-complex structure of M being one of the two complex-vectorspace structures on $\mathfrak{P}$ in which this circle group acts as multiplication by unimodular complex numbers. The hermitian metric is determined by the Riemannian metric and the complex structure. Note that s_x must be the nontrivial element of square 1 in the center of K, so σ preserves every ideal of $\mathfrak{G}$; irreducibility now implies that $\mathfrak{G}$ is simple.

Duality and the remarks just above prove:

Lemma 2. *In order to classify the irreducible hermitian symmetric spaces, it suffices to classify the pairs* $(\mathfrak{G}, \mathfrak{K})$ *where*

(i) $\mathfrak{G}$ *is a compact simple Lie algebra,*

(ii) $\mathfrak{K}$ *is the fixed point set of some involutive automorphism* σ *of* $\mathfrak{G}$, *and*

(iii) $\mathfrak{K}$ *is not semisimple.*

3. Reduction to a problem on roots. Let $\mathfrak{H}$ be a Cartan subalgebra of a real semisimple Lie algebra $\mathfrak{G}$. Then $\mathfrak{H}^{\mathfrak{c}}$ is a Cartan subalgebra of $\mathfrak{G}^{\mathfrak{c}}$, and we will refer to the roots of $\mathfrak{G}^{\mathfrak{c}}$ relative to $\mathfrak{H}^{\mathfrak{c}}$ as the $\mathfrak{H}$-roots of $\mathfrak{G}$. These roots are linear functionals on $\mathfrak{H}^{\mathfrak{c}}$. if λ is an $\mathfrak{H}$-root of $\mathfrak{G}$, then $\mathfrak{G}_\lambda$ will denote the *root space for* λ, complex subspace of dimension 1 in $\mathfrak{G}^{\mathfrak{c}}$ characterized by

$$[H, X] = \lambda(H)\cdot X \quad \text{for} \quad H \in \mathfrak{H}^{\mathfrak{c}} \quad \text{and} \quad X \in \mathfrak{G}_\lambda .$$

We have $\mathfrak{G}^{\mathfrak{c}} = \mathfrak{H}^{\mathfrak{c}} + \sum \mathfrak{G}_\lambda$ and $\mathfrak{G} = \mathfrak{H} + \sum \{\mathfrak{G} \cap (\mathfrak{G}_\lambda + \mathfrak{G}_{-\lambda})\}$. H_λ will denote the unique element of $\mathfrak{H}^{\mathfrak{c}}$ such that

$$\beta(H_\lambda , H) = \lambda(H) \quad \text{for every} \quad H \in \mathfrak{H}^{\mathfrak{c}} ,$$

where β is the Killing form of $\mathfrak{G}^{\mathfrak{c}}$.

A lexicographic ordering of the dual space of $\mathfrak{H}^{\mathfrak{c}}$ induces an ordering of the roots and the notions of positive and negative root. Then a *simple root* is a positive root which is not a sum of positive roots, and every positive (resp. negative) root is a linear combination of simple roots with non-negative (resp. nonpositive) integral coefficients. The set of all simple roots depends on the ordering and is called a *system of simple roots.*

Let $\Psi = \{\varphi, \psi_1, \cdots, \psi_r\}$ be a system of simple $\mathfrak{H}$-roots of $\mathfrak{G}$. We will say that φ *is of noncompact type* if, for every $\mathfrak{H}$-root λ of $\mathfrak{G}$, either λ is of the form

$$\lambda = \pm \sum_{i=1}^{r} a_i \psi_i, \qquad a_i \geqq 0, \tag{3.1}$$

or λ is of the form

$$\lambda = \pm\left(\varphi + \sum_{i=1}^{r} a_i \psi_i\right), \qquad a_i \geqq 0. \tag{3.2}$$

In that case roots of the form (3.1) will be called *φ-compact* and roots of the form (3.2) will be called *φ-noncompact.*

Lemma 3. *Let $\mathfrak{G}$ be a semisimple Lie algebra, let $\mathfrak{H}$ be a Cartan subalgebra, and let $\Psi = \{\varphi, \psi_1, \cdots, \psi_r\}$ be a system of simple $\mathfrak{H}$-roots of $\mathfrak{G}$ such that φ is of noncompact type. Define*

$$\mathfrak{K}_\varphi = \mathfrak{H} + \sum \{\mathfrak{G} \cap (\mathfrak{G}_\lambda + \mathfrak{G}_{-\lambda})\}, \qquad \lambda \ \varphi\text{-compact},$$

and

$$\mathfrak{P}_\varphi = \sum \{\mathfrak{G} \cap (\mathfrak{G}_\lambda + \mathfrak{G}_{-\lambda})\}, \qquad \lambda \ \varphi\text{-noncompact},$$

and let σ_φ be the linear transformation of $\mathfrak{G}$ which is $+1$ on $\mathfrak{K}_\varphi$ and -1 on $\mathfrak{P}_\varphi$. Then σ_φ is an involutive automorphism of $\mathfrak{G}$, $\mathfrak{K}_\varphi$ is its fixed point set, and $\mathfrak{K}_\varphi$ is a subalgebra of $\mathfrak{G}$ whose center has dimension one.

Conversely, if $\mathfrak{G}$ is a simple Lie algebra and $\mathfrak{K}$ is a non semisimple subalgebra which is the fixed point set of an involutive automorphism σ of $\mathfrak{G}$, then $\mathfrak{H}$, Ψ and φ exist as above such that $\mathfrak{K} = \mathfrak{K}_\varphi$ and $\sigma = \sigma_\varphi$.

Proof. We have $[\mathfrak{K}_\varphi, \mathfrak{K}_\varphi] \subset \mathfrak{K}_\varphi$, $[\mathfrak{K}_\varphi, \mathfrak{P}_\varphi] \subset \mathfrak{P}_\varphi$ and $[\mathfrak{P}_\varphi, \mathfrak{P}_\varphi] \subset \mathfrak{K}_\varphi$ by (3.1) and (3.2). This proves that σ_φ is an involutive automorphism of $\mathfrak{G}$. $\mathfrak{K}_\varphi$ is its fixed point set by construction, and $\{\sum_{i=1}^{r} H_{\psi_i} \cdot \mathfrak{C}\}^\perp \cap \mathfrak{H}^{\mathfrak{C}}$ ($\perp$ relative to the Killing form) is the center of $\mathfrak{K}_\varphi^{\mathfrak{C}}$. As the $\{H_\varphi, H_{\psi_1}, \cdots, H_{\psi_r}\}$ is a basis of $\mathfrak{H}^{\mathfrak{C}}$ and the $(H_\varphi \cdot \mathfrak{C}) \cap \mathfrak{G}$ and $(H_{\psi_i} \cdot \mathfrak{C}) \cap \mathfrak{G}$ span $\mathfrak{H}$, the center of $\mathfrak{K}_\varphi$ has dimension one.

For the converse we may suppose $\mathfrak{G}$ compact because σ commutes with a Cartan involution of $\mathfrak{G}$. Now let G be the centerless group with Lie algebra $\mathfrak{G}$, let K be the analytic subgroup with Lie algebra $\mathfrak{K}$, and let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be the decomposition into $(+1)$- and (-1)-eigenspaces of σ. $\mathfrak{K}$ is the direct sum of a semisimple ideal and an abelian ideal because G is compact, so $K = K' \cdot U$ where K' is the derived group and U is the identity component of the center. $ad_G(K)|_{\mathfrak{P}}$ is a faithful irreducible representation of K on $\mathfrak{P}$, and the restriction of the Killing form to $\mathfrak{P}$ is nondegenerate and $ad_G(K)$-invariant; now U is a circle group by Schur's Lemma. Let $\mathfrak{U}$ be the Lie algebra of U; now there is an element $Z \in \mathfrak{U}$ such that $ad(Z)\mathfrak{K} = 0$ and $ad(Z)|_{\mathfrak{P}}$ has only $\pm(-1)^{1/2}$ as eigenvalues. In particular $\mathfrak{K}$ is the full centralizer of $\mathfrak{U}$ in $\mathfrak{G}$. Thus $\mathfrak{U} \subset \mathfrak{H} \subset \mathfrak{K}$ for some Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$.

$\mathfrak{H}$ normalizes both $\mathfrak{K}$ and $\mathfrak{P}$; thus the set of $\mathfrak{H}$-roots of $\mathfrak{G}$ is partitioned into classes A and B such that

$$\mathfrak{K}^{\mathfrak{C}} = \mathfrak{H}^{\mathfrak{C}} + \sum_{\lambda \varepsilon A} \mathfrak{G}_\lambda \quad \text{and} \quad \mathfrak{P}^{\mathfrak{C}} = \sum_{\lambda \varepsilon B} \mathfrak{G}_\lambda .$$

Now $\lambda(Z) = 0$ if $\lambda \varepsilon A$, and $\lambda(Z) = \pm(-1)^{1/2}$ if $\lambda \varepsilon B$. Choose an ordering of the $\mathfrak{H}$-roots of $\mathfrak{G}$ such that $\lambda(Z) = (-1)^{1/2}$ for $\lambda > 0$ and $\lambda \varepsilon B$, and let Ψ be the corresponding system of simple roots. Then $\Psi = \{\varphi_1 , \cdots , \varphi_t ; \psi_1 , \cdots , \psi_r\}$ with $\varphi_i \varepsilon B$ and $\psi_i \varepsilon A$. Let μ be the greatest root;

$$\mu = \sum_{i=1}^{t} m_i \varphi_i + \sum_{i=1}^{r} n_i \psi_i$$

with $m_i > 0$ and $n_i > 0$. Now

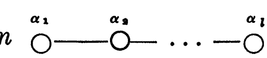
$$\mu(Z) = \sum_{i=1}^{t} m_i \varphi_i(Z) + \sum_{i=1}^{r} n_i \psi_i(Z) = \left(\sum_{i=1}^{t} m_i\right)(-1)^{1/2}.$$

Thus $t = 1 = m_1$. As every positive root has coefficients at most those of μ, it follows that φ_1 is of noncompact type, $\mathfrak{K} = \mathfrak{K}_{\varphi_1}$, $\mathfrak{P} = \mathfrak{P}_{\varphi_1}$, and consequently $\sigma = \sigma_{\varphi_1}$. *Q.E.D.*

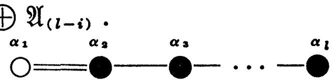
4. The simple roots of noncompact type. Lemmas 1, 2 and 3 reduce the classification of hermitian symmetric spaces to the classification of simple roots of noncompact type in compact simple Lie algebras.

We adopt the notation of the Cartan classification for the compact simple Lie algebras. Thus $\mathfrak{A}_l$ is the Lie algebra of the special unitary group $\mathbf{SU}(l+1)$ in $l+1$ complex variables, $\mathfrak{B}_l$ is the Lie algebra of the rotation group $\mathbf{SO}(2l+1)$ in $2l+1$ real variables, $\mathfrak{C}_l$ is the Lie algebra of the quaternionic unitary group $\mathbf{Sp}(l)$ in l quaternion variables, $\mathfrak{D}_l$ is the Lie algebra of $\mathbf{SO}(2l)$, and $\mathfrak{G}_2$, $\mathfrak{F}_4$, $\mathfrak{E}_6$, $\mathfrak{E}_7$ and $\mathfrak{E}_8$ denote the exceptional structures. $\mathfrak{T}$ denotes the (commutative) Lie algebra of dimension 1.

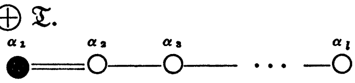
Lemma 4. *A complete list of the simple roots of noncompact type in compact simple Lie algebras is given as follows.*

$\mathfrak{A}_l$ *has diagram* 

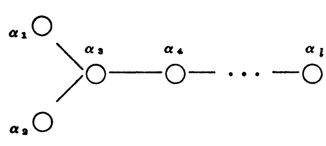
and each α_i *is of noncompact type.* $\mathfrak{K}_{\alpha_i} \cong \mathfrak{A}_{i-1} \oplus \mathfrak{T} \oplus \mathfrak{A}_{(l-i)}$.

$\mathfrak{B}_l$ *has diagram* 

and only α_l *is of noncompact type.* $\mathfrak{K}_{\alpha_l} \cong \mathfrak{B}_{l-1} \oplus \mathfrak{T}$.

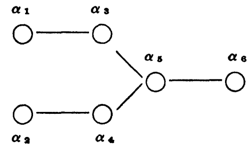
$\mathfrak{C}_l$ *has diagram* 

and only α_1 *is of noncompact type.* $\mathfrak{K}_{\alpha_1} \cong \mathfrak{A}_{l-1} \oplus \mathfrak{T}$.

$\mathfrak{D}_l$ *has diagram* 

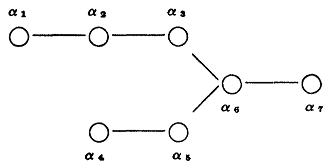
and only α_1, α_2 *and* α_l *are of* *noncompact type.* $\mathfrak{K}_{\alpha_1} \cong \mathfrak{K}_{\alpha_2} \cong \mathfrak{A}_{l-1} \oplus \mathfrak{T}$ *and* $\mathfrak{K}_{\alpha} \cong \mathfrak{D}_{l-1} \oplus \mathfrak{T}$.

$\mathfrak{E}_6$ *has diagram*



and only α_1 *and* α_2 *are of noncompact type.* $\mathfrak{K}_{\alpha_1} \cong \mathfrak{K}_{\alpha_2} \cong \mathfrak{D}_5 \oplus \mathfrak{T}$.

$\mathfrak{E}_7$ *has diagram*



and only α_1 *is of noncompact type.* $\mathfrak{K}_{\alpha_1} \cong E_6 \oplus \mathfrak{T}$.

Proof. Let $\mathfrak{G}$ be a simple compact Lie algebra, choose a Cartan subalgebra $\mathfrak{H}$, and let $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ be a system of simple $\mathfrak{H}$-roots of $\mathfrak{G}$, numbered in conformity to the diagrams above where this condition is applicable. Let $\mu = \sum m_i\alpha_i$ be the greatest root. If λ is any positive root, then $\lambda = \sum n_i\alpha_i$ with $0 \leqq n_i \leqq m_i$. Thus α_i is of noncompact type if and only if $m_i = 1$.

A calculation (or see [1], p. 219) shows that the greatest root of $\mathfrak{G}$ is given by

$$
\begin{aligned}
\mathfrak{A}_l &: \quad \alpha_1 + \alpha_2 + \cdots + \alpha_l \\
\mathfrak{B}_l &: \quad 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}) + \alpha_l \\
\mathfrak{C}_l &: \quad \alpha_1 + 2(\alpha_2 + \alpha_3 + \cdots + \alpha_l) \\
\mathfrak{D}_l &: \quad \alpha_1 + \alpha_2 + 2(\alpha_3 + \alpha_4 + \cdots + \alpha_{l-1}) + \alpha_l \\
\mathfrak{G}_2 &: \quad 2\alpha_1 + 3\alpha_2 \\
\mathfrak{F}_4 &: \quad 2(\alpha_1 + \alpha_2) + 3\alpha_3 + 4\alpha_4 \\
\mathfrak{E}_6 &: \quad \alpha_1 + \alpha_2 + 2(\alpha_3 + \alpha_4 + \alpha_6) + 3\alpha_5 \\
\mathfrak{E}_7 &: \quad \alpha_1 + 2(\alpha_2 + \alpha_4 + \alpha_7) + 3(\alpha_3 + \alpha_5) + 4\alpha_6 \\
\mathfrak{E}_8 &: \quad 2(\alpha_1 + \alpha_2) + 3(\alpha_3 + \alpha_4) + 4(\alpha_5 + \alpha_6) + 5\alpha_7 + 6\alpha_8
\end{aligned}
$$

where Ψ is indexed appropriately for $\mathfrak{G}_2$, $\mathfrak{F}_4$ and $\mathfrak{E}_8$. The assertion on roots of noncompact type follows.

Let $\alpha \in \Psi$ be of noncompact type. Then the diagram of the semisimple part of $\mathfrak{K}_\alpha$ is obtained by deleting α from the diagram of $\mathfrak{G}$, by definition of $\mathfrak{K}_\alpha$ and and the standard construction of all positive roots from the simple roots. As $\mathfrak{K}_\alpha$ is the direct sum of its semisimple part and a one-dimensional ideal, our assertions on $\mathfrak{K}_\alpha$ are proved. Q.E.D.

5. The classification. Let G be the largest connected group of isometries of an irreducible hermitian symmetric space G/K. Then the center of K is a circle group $\mathbf{T}$; now $K = K' \cdot \mathbf{T}$ where K' is the derived group, C is the center of

K', $\delta : C \to \mathbf{T}$ is a homomorphism, and $K' \cdot \mathbf{T} = (K' \times \mathbf{T})/\{(c^{-1}, \delta(c)\}$. When it is convenient we write G_1/K_1 for G/K, where G_1 is a covering group of G and K_1 is the full inverse image of K. Finally, $\mathbf{A}_l, \cdots, \mathbf{E}_8$ will denote the compact simply connected group of Cartan classification type $A_l, \cdots, E_8$; and $ad(\mathbf{A}_l), \cdots, ad(\mathbf{E}_8)$ will denote the centerless version. We can now combine our four lemmas.

Theorem. *Let M be a complex manifold with hermitian metric. Then M is a hermitian symmetric space if and only if it is hermitian-isometric to a product*

$$M_0 \times M_1 \times \cdots \times M_r$$

where M_0 is the quotient of a complex Euclidean space by a discrete group of pure translations, and where each M_i $(i > 0)$ is one of the Riemannian symmetric spaces

$$\mathbf{SU}(p+q)/[\mathbf{SU}(p+q) \cap \{\mathbf{U}(p) \times \mathbf{U}(q)\}]$$

$$\mathbf{SO}(n+2)/\mathbf{SO}(n) \times \mathbf{SO}(2)$$

$$\mathbf{Sp}(n)/\mathbf{U}(n)$$

$$\mathbf{SO}(2n)/\mathbf{U}(n)$$

$$ad(\mathbf{E}_6)/\mathbf{SO}(10) \cdot \mathbf{SO}(2)$$

$$ad(\mathbf{E}_7)/\mathbf{E}_6 \cdot \mathbf{T}$$

or is the dual of one of those spaces, and where each M_i has the hermitian metric derived from the Riemannian metric and one of the two invariant complex structures.

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