HOMOGENEITY AND BOUNDED ISOMETRIES IN MANIFOLDS OF NEGATIVE CURVATURE

BY

JOSEPH A. WOLF[1]

1. Statement of results

The object of this paper is to prove Theorem 3 below, which gives a fairly complete analysis of the structure of Riemannian homogeneous manifolds of nonpositive sectional curvature. The main tool is

THEOREM 1. *Let M be a complete connected simply connected Riemannian manifold with every sectional curvature nonpositive. Let γ be an isometry of M; given $m \in M$, let X_m be the (unique by hypothesis on M) tangentvector to M at m such that* $\exp(X_m) = \gamma(m)$; *let X be the vectorfield on M defined by the X_m. Let M_0 be the Euclidean factor in the de Rham decomposition of M, so $M = M_0 \times M'$ where M' is the product of the irreducible factors. Then these are equivalent*:

(1) *There is an ordinary translation γ_0 of the Euclidean space M_0 such that the action of γ on $M = M_0 \times M'$ is given by $(m_0, m') \to (\gamma_0 m_0, m')$.*

(2) *X is a parallel vectorfield on M.*

(3) *γ is a Clifford translation[2] of M.*

(4) *γ is a bounded isometry[3] of M.*

In particular, if M_0 is trivial, then every bounded isometry of M is trivial.

As an immediate consequence of Theorem 1, we have

THEOREM 2. *Let M be a complete connected simply connected Riemannian manifold of nonpositive sectional curvature, and let Γ be a properly discontinuous group of fixed-point-free isometries of M. Then these are equivalent*:

(1) *M/Γ is isometric to the product of a flat torus with a complete simply connected Riemannian manifold of nonpositive sectional curvature.*

(2) *Every element of Γ is a Clifford translation of M.*

(3) *Every element of Γ is a bounded isometry of M.*

In particular, if one of these conditions holds, then M/Γ is diffeomorphic to the product of a torus and a Euclidean space.

To prove the following theorem, which is our goal, one notes that (1)

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[2] An isometry of a metric space is called a *Clifford translation* if the distance between a point and its image is the same for every point.

[3] A *bounded isometry* of a metric space is an isometry such that the distance between a point and its image is at most equal to some bound.

implies (2) trivially, that (2) is known to imply (3) [3, Théorème 2], that (3) implies (4) trivially, and that (4) implies (1) by Theorem 2.

THEOREM 3. *Let M be a connected simply connected Riemannian homogeneous manifold of nonpositive sectional curvature, and let Γ be a properly discontinuous group of fixed-point-free isometries of M. Then these are equivalent:*

(1) *M/Γ is isometric to the product of a flat torus with a simply connected Riemannian manifold (which is necessarily homogeneous and of nonpositive sectional curvature).*

(2) *M/Γ is a Riemannian homogeneous manifold.*

(3) *Every element of Γ is a Clifford translation of M.*

(4) *Every element of Γ is a bounded isometry of M.*

The fact that (2) implies (1) above has a number of consequences, most of which are immediate from Theorem 1:

COROLLARY 1. *Let N be a connected Riemannian homogeneous manifold with every sectional curvature nonpositive. Then*

(a) *N is isometric to the product of a flat torus with a simply connected manifold.*

(b) *N is diffeomorphic to the product of a torus and a Euclidean space.*

(c) *N admits a transitive solvable group of isometries.*

(d) *If there is no Euclidean factor in the de Rham decomposition of the universal Riemannian covering manifold of N, then* (d_1) *N is simply connected,* (d_2) *every bounded isometry of N is trivial,* (d_3) *every transitive group of isometries of N is centerless,* (d_4) *a transitive Lie group of isometries of N cannot admit a nontrivial bounded*[4] *inner automorphism, and* (d_5) *N is Riemannian symmetric if it admits a transitive semisimple Lie group of isometries.*

As our final result, Corollary 1 (d_4) will give us a result due to J. Tits:

COROLLARY 2. *Let β be a bounded automorphism of a connected semisimple Lie group G. Then β induces the identity transformation on every noncompact normal simple analytic subgroup of G.*

Remark. It is conceivable that Corollary 1 could be the basis for a classification of the Riemannian homogeneous manifolds of nonpositive sectional curvature.

Background. Corollary 1 (d_1) is an affirmative answer to "question (a)" raised by S. Kobayashi in his paper [2], and thus extends the main result of [2]. Kobayashi's paper led R. Hermann to conjecture Corollary 1 (b), and Hermann had already proved a slightly weaker statement when he learned of my result.

Acknowledgements. I am indebted to S. Kobayashi for showing me the

[4] An automorphism β of a topological group G is called *bounded* if G has a compact subset C such that $\beta(g)\cdot g^{-1} \in C$ for every $g \in G$.

proofs of [2], to R. Hermann for telling me his conjecture, and to L. W. Green for drawing my attention to a result in his thesis which is basic to the proof of Theorem 1.

2. Proof of Theorem 1

Let γ be a bounded isometry of M, and assume $\gamma \neq 1$. Given $x \in M$, let h_x be a geodesic on M containing both x and $\gamma(x)$. x cannot be a fixed point for γ because distinct geodesics through x diverge; thus h_x is uniquely determined. Now choose $m \in M$, and let g be a geodesic through m, $g \neq h_m$. Let S_g be the union of all h_x with $x \in g$. $\gamma(h_x) = h_x$ because they both contain $\gamma(x)$ and they do not diverge from each other; thus $\gamma(S_g) = S_g$. As the strip on S_g between g and $\gamma(g)$ is a regularly imbedded surface in M, for g and $\gamma(g)$ do not meet, because they are distinct and do not diverge apart, it follows that S_g is a regularly imbedded surface in M, and that any point of S_g is contained between some pair $(\gamma^{t-1}(g), \gamma^{t+1}(g))$ of nondivergent geodesics. Furthermore, any point z of S_g lies on some geodesic h_x of M which is contained in S_g, so the Gaussian curvature of S_g at z is bounded above by the sectional curvature in M of the tangentplane to S_g at z, and they are equal if and only if this plane is parallel along h_x at z. This shows that S_g is of nonpositive Gaussian curvature, and that S_g is totally geodesic in M if it is flat. By using the ruling of S_g by the h_x and the striping of S_g by the $\gamma^t(g)$, it is easy to see that S_g is complete and simply connected; as S_g is of nonpositive Gaussian curvature and any point lies between two nondivergent geodesics (of the form $\gamma^t(g)$), a deep result of L. W. Green [1, Corollary 4.2] shows that S_g is flat, and it follows that S_g is totally geodesic in M. Now γ is a bounded isometry on the Euclidean plane S_g; it follows that γ is an ordinary translation of S_g [3, proof of Théorème 4], and thus that γ is of constant displacement on S_g for the induced metric. As S_g is totally geodesic, γ is of constant displacement on S_g for the metric of M. M is the union of the various S_g, and each S_g contains m; thus γ is of constant displacement on M. This shows that (4) implies (3).

Let γ be a Clifford translation of M. In particular, γ is a bounded isometry of M, and we can apply the constructions above. Given $m \in M$, let $\{X_m ; Y_2, \cdots, Y_n\}$ be a basis of the tangentspace M_m. Let g_i be the geodesic through m tangent to Y_i, and let $S_i = S_{g_i}$. Each S_i is totally geodesic in M, is isometric to the Euclidean plane, and is invariant under γ, and γ induces an ordinary translation in each S_i. Thus the restriction of X to S_i is a parallel field of tangentvectors to S_i; in particular, X is parallel along each g_i, and along h_m. Thus every covariant derivative of X vanishes at m. As m was arbitrary, it follows that X is parallel over M. This shows that (3) implies (2).

Let X be parallel on M. If we look at the de Rham decomposition, it is clear that, at every point of M, X is tangent to the Euclidean factor. This shows that the action of γ on $M = M_0 \times M'$ is of the form

$$(m_0, m') \rightarrow (\gamma_0 m_0, m')$$

where γ_0 is an isometry of M_0. γ_0 is a Clifford translation of M_0 because X has constant length; it follows [3, Théorème 4] that γ_0 is an ordinary translation of M_0. This shows that (2) implies (1).

It is trivial that (1) implies (4). Theorem 1 is now proved, Q.E.D.

3. Proof of the corollaries

Theorem 1 has been proved, and Theorems 2 and 3 follow; only the corollaries remain to be proved.

Proof of Corollary 1. Statements (a), (b), (d_1), and (d_2) are immediate from Theorems 1 and 3, (d_3) follows from (d_2) because [3, Théorème 2] a central element of a transitive group of isometries would be a Clifford translation, and (d_4) follows from (d_2) because an element inducing a bounded inner automorphism of a transitive Lie group of isometries would have to be a bounded isometry. Symmetry provides a transitive semisimple group of isometries in (d_5) because N has no Euclidean factor. Conversely, if G is a transitive semisimple group of isometries in (d_5), then we may assume G connected, and G is centerless by (d_3); as $N = G/K$ with K maximal compact in G because N is diffeomorphic to a Euclidean space, N is symmetric.

Only (c) remains, and Theorem 3 shows that (c) need only be proved in case N is simply connected and without Euclidean factor. Let G be a connected transitive Lie group of isometries of N, let R and S be the connected radical and a Levi-Whitehead complement, and decompose $S = S_N \cdot S_C$, where S_N is the product of the noncompact normal simple subgroups, and S_C is the product of the compact ones. An isotropy subgroup K of G is maximal compact because N is diffeomorphic to Euclidean space; thus we may assume S chosen so that $K = K_N \cdot S_C \cdot K_R$ with K_N maximal compact in S_N and K_R maximal compact in R. The Iwasawa decomposition of S_N provides a solvable subgroup B such that $S_N = B \cdot K_N$; now

$$(B \cdot R) \cdot K = B \cdot K \cdot R = G,$$

so $B \cdot R$ is a solvable transitive group of isometries of N. Corollary 1 is now proved, Q.E.D.

Proof of Corollary 2. By passing to the adjoint group of G and then dividing out by the maximal compact normal subgroup, it is easily seen that it suffices to prove that $\beta = 1$ provided that G is a product of noncompact centerless simple groups. If β is inner, now, it must be trivial by Corollary 1 (d_4) because G can be realized as a transitive group of isometries of a Riemannian symmetric space of nonpositive curvature. It follows that β is of finite order, and thus semisimple; now it is easy to see that β is inner, and thus trivial, Q.E.D.

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University of California
Berkeley, California