THE AFFINE GROUP OF A LIE GROUP
JOSEPH A. WOLF

1. If $G$ is a Lie group, then the group $\text{Aut}(G)$ of all continuous automorphisms of $G$ has a natural Lie group structure. This gives the semidirect product $A(G) = G \cdot \text{Aut}(G)$ the structure of a Lie group. When $G$ is a vector group $\mathbb{R}^n$, $A(G)$ is the ordinary affine group $A(n)$. Following L. Auslander [1] we will refer to $A(G)$ as the affine group of $G$, and regard it as a transformation group on $G$ by $(g, \alpha): h \rightarrow g \cdot \alpha(h)$ where $g, h \in G$ and $\alpha \in \text{Aut}(G)$; in the case of a vector group, this is the usual action on $A(n)$ on $\mathbb{R}^n$.

If $B$ is a compact subgroup of $A(n)$, then it is well known that $B$ has a fixed point on $\mathbb{R}^n$, i.e., that there is a point $x \in \mathbb{R}^n$ such that $b(x) = x$ for every $b \in B$. For $A(n)$ is contained in the general linear group $\text{GL}(n+1, \mathbb{R})$ in the usual fashion, and $B$ (being compact) must be conjugate to a subgroup of the orthogonal group $\text{O}(n+1)$. This conjugation can be done leaving fixed the $(n+1, n+1)$-place matrix entries, and is thus possible by an element of $A(n)$. This done, the translation-parts of elements of $B$ must be zero, proving the assertion.

L. Auslander [1] has extended this theorem to compact abelian subgroups of $A(G)$ when $G$ is connected, simply connected and nilpotent. We will give a further extension.

Theorem. Let $G$ be a connected Lie group and let $S$ be the identity component of the radical of $G$. Then the following conditions are equivalent:

1. If $B$ is a compact subgroup of the affine group $A(G)$, then $G$ has an element $x$ such that $b(x) = x$ for every $b \in B$.
2. Every compact subgroup of $A(G)$ is conjugate to a subgroup of $\text{Aut}(G)$.
3. $G$ has no nontrivial compact subgroup.
4. $G$ is homeomorphic to Euclidean space.
5. $S$ is simply connected and $G/S$ is a direct product of copies of the universal covering group of the real special linear group $\text{SL}(2, \mathbb{R})$.
6. $G$ is simply connected, and every simple analytic subgroup of $G$ is a 3-dimensional noncompact group.

Equivalence of (3) and (4) is contained in the Cartan-Iwasawa theorem [3, Theorem 13]. Equivalence of (4) and (5) follows from
Chevalley’s theorem on the topology of solvable groups [2], the fact that the universal covering of \( \text{SL}(2, \mathbb{R}) \) is the only simple Lie group homeomorphic to Euclidean space, and the global Levi-Whitehead decomposition of \( G \). It is not difficult to see that (5) is equivalent to (6) and it is clear that (1) is equivalent to (2). Finally, a nontrivial compact subgroup of \( G \) is a compact subgroup of \( \text{Aut}(G) \) which is not conjugate to a subgroup of \( \text{Aut}(G) \); thus (2) implies (3). The proof of the Theorem is now reduced to the proof that (3) implies (2).

2. Suppose that \( G \) has no nontrivial compact subgroup. Then \( G \) is simply connected, and it follows that \( \text{Aut}(G) \) has only finitely many connected components because \( \text{Aut}(G) \) is isomorphic to the group \( \text{Aut}(\mathfrak{g}) \) of automorphisms of the Lie algebra \( \mathfrak{g} \) of \( G \), and \( \text{Aut}(\mathfrak{g}) \) is a real algebraic matrix group. Thus \( A(G) \) has only finitely many connected components. The Cartan-Iwasawa theorem [3, Theorem 13] is valid for Lie groups with only finitely many components; thus \( A(G) \) has maximal compact subgroups and, if \( K \) is one of them, every compact subgroup of \( A(G) \) is conjugate to a subgroup of \( K \). The proof that (3) implies (2) is now reduced to the proof that \( \text{Aut}(G) \) contains a maximal compact subgroup of \( A(G) \).

Let \( K \subseteq \text{Aut}(G) \subseteq A(G) \) be a maximal compact subgroup of \( \text{Aut}(G) \); we will prove that \( K \) is a maximal compact subgroup of \( A(G) \). Let \( K' \) be a maximal compact subgroup of \( A(G) \) with \( K \subseteq K' \); we must prove \( K = K' \). It is easily seen that \( K \) meets every component of \( A(G) \); it follows that we need only prove that \( K \) and \( K' \) have the same identity component. Again because \( K \subseteq K' \), it suffices to show that \( \dim K = \dim K' \). Let \( f: A(G) \to \text{Aut}(G) \) be the canonical homomorphism \( (g, \alpha) \to \alpha \) with kernel \( G \). \( K \cap G \) and \( K' \cap G \) are compact subgroups of \( G \) and thus are trivial by hypothesis. Furthermore \( K = f(K) = f(K') \) because \( K \) is a maximal compact subgroup of \( \text{Aut}(G) \), and because \( f(K) \) is contained in the compact subgroup \( f(K') \). This gives \( \dim K = \dim f(K') = \dim K' \) which proves the Theorem.

3. It is worth remarking that the main part of the Theorem—the equivalence of (1), (2) and (3)—can be proved in the same way when \( G \) is assumed to have only finitely many connected components.

References