THE AFFINE GROUP OF A LIE GROUP

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1. If G is a Lie group, then the group $\mathbf{Aut}(G)$ of all continuous automorphisms of G has a natural Lie group structure. This gives the semidirect product $\mathbf{A}(G)=G\cdot\mathbf{Aut}(G)$ the structure of a Lie group. When G is a vector group $\mathbf{R}^n$, $\mathbf{A}(G)$ is the ordinary affine group $\mathbf{A}(n)$. Following L. Auslander [1] we will refer to $\mathbf{A}(G)$ as the *affine group of* G, and regard it as a transformation group on G by $(g, \alpha): h \to g\cdot\alpha(h)$ where $g, h \in G$ and $\alpha \in \mathbf{Aut}(G)$; in the case of a vector group, this is the usual action on $\mathbf{A}(n)$ on $\mathbf{R}^n$.

If B is a compact subgroup of $\mathbf{A}(n)$, then it is well known that B has a fixed point on $\mathbf{R}^n$, i.e., that there is a point $x \in \mathbf{R}^n$ such that $b(x)=x$ for every $b \in B$. For $\mathbf{A}(n)$ is contained in the general linear group $\mathbf{GL}(n+1, \mathbf{R})$ in the usual fashion, and B (being compact) must be conjugate to a subgroup of the orthogonal group $\mathbf{O}(n+1)$. This conjugation can be done leaving fixed the $(n+1, n+1)$-place matrix entries, and is thus possible by an element of $\mathbf{A}(n)$. This done, the translation-parts of elements of B must be zero, proving the assertion.

L. Auslander [1] has extended this theorem to compact abelian subgroups of $\mathbf{A}(G)$ when G is connected, simply connected and nilpotent. We will give a further extension.

THEOREM. *Let G be a connected Lie group and let S be the identity component of the radical of G. Then the following conditions are equivalent*:

1. *If B is a compact subgroup of the affine group* $\mathbf{A}(G)$, *then G has an element x such that $b(x)=x$ for every $b \in B$.*
2. *Every compact subgroup of* $\mathbf{A}(G)$ *is conjugate to a subgroup of* $\mathbf{Aut}(G)$.
3. *G has no nontrivial compact subgroup.*
4. *G is homeomorphic to Euclidean space.*
5. *S is simply connected and G/S is a direct product of copies of the universal covering group of the real special linear group* $\mathbf{SL}(2, \mathbf{R})$.
6. *G is simply connected, and every simple analytic subgroup of G is a 3-dimensional noncompact group.*

Equivalence of (3) and (4) is contained in the Cartan-Iwasawa theorem [3, Theorem 13]. Equivalence of (4) and (5) follows from

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Chevalley's theorem on the topology of solvable groups [2], the fact that the universal covering of $\mathbf{SL}(2, \mathbf{R})$ is the only simple Lie group homeomorphic to Euclidean space, and the global Levi-Whitehead decomposition of G. It is not difficult to see that (5) is equivalent to (6) and it is clear that (1) is equivalent to (2). Finally, a nontrivial compact subgroup of G is a compact subgroup of $\mathbf{A}(G)$ which is not conjugate to a subgroup of $\mathbf{Aut}(G)$; thus (2) implies (3). The proof of the Theorem is now reduced to the proof that (3) implies (2).

2. Suppose that G has no nontrivial compact subgroup. Then G is simply connected, and it follows that $\mathbf{Aut}(G)$ has only finitely many connected components because $\mathbf{Aut}(G)$ is isomorphic to the group $\mathbf{Aut}(\mathfrak{G})$ of automorphisms of the Lie algebra $\mathfrak{G}$ of G, and $\mathbf{Aut}(\mathfrak{G})$ is a real algebraic matrix group. Thus $\mathbf{A}(G)$ has only finitely many connected components. The Cartan-Iwasawa theorem [3, Theorem 13] is valid for Lie groups with only finitely many components; thus $\mathbf{A}(G)$ has maximal compact subgroups and, if K is one of them, every compact subgroup of $\mathbf{A}(G)$ is conjugate to a subgroup of K. The proof that (3) implies (2) is now reduced to the proof that $\mathbf{Aut}(G)$ contains a maximal compact subgroup of $\mathbf{A}(G)$.

Let $K\subset\mathbf{Aut}(G)\subset\mathbf{A}(G)$ be a maximal compact subgroup of $\mathbf{Aut}(G)$; we will prove that K is a maximal compact subgroup of $\mathbf{A}(G)$. Let K' be a maximal compact subgroup of $\mathbf{A}(G)$ with $K\subset K'$; we must prove $K=K'$. It is easily seen that K meets every component of $\mathbf{A}(G)$; it follows that we need only prove that K and K' have the same identity component. Again because $K\subset K'$, it suffices to show that $\dim K=\dim K'$. Let $f\colon \mathbf{A}(G)\to\mathbf{Aut}(G)$ be the canonical homomorphism $(g, \alpha)\to\alpha$ with kernel G. $K\cap G$ and $K'\cap G$ are compact subgroups of G and thus are trivial by hypothesis. Furthermore $K=f(K)=f(K')$ because K is a maximal compact subgroup of $\mathbf{Aut}(G)$, and because $f(K)$ is contained in the compact subgroup $f(K')$. This gives $\dim K = \dim f(K') = \dim K'$ which proves the Theorem.

3. It is worth remarking that the main part of the Theorem—the equivalence of (1), (2) and (3)—can be proved in the same way when G is assumed to have only finitely many connected components.

References

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3. K. Iwasawa, *On some types of topological groups*, Ann. of Math. (2) **50** (1949), 507–558.

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