DISCRETE GROUPS, SYMMETRIC SPACES, AND GLOBAL HOLONOMY.*

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1. Introduction. Let M be a connected simply connected Riemannian manifold and let Γ be a properly discontinuous group of isometries such that M/Γ is compact. If every sectional curvature of M is negative, in particular if M is a noncompact irreducible symmetric space of rank 1, then a method of É. Cartan shows that every abelian subgroup of Γ is either finite or the product of a finite group with an infinite cyclic group. If M is the Euclidean space R^n, then a calculation shows that every abelian subgroup of Γ is the product of a finite group with a free abelian group on $\leqq n$ generators. These phenomena are unified by one of the conclusions of our Theorem 6.2: *If M is Riemannian symmetric and v is the maximum of the dimensions of those totally geodesic submanifolds of M which are isometric to Euclidean spaces, then every abelian subgroup of Γ is the product of a finite group with a free abelian group on $\leqq v$ generators, and Γ has a subgroup which is free abelian on v generators.* We also prove that an abelian subgroup of Γ must preserve a flat connected totally geodesic submanifold of M; if M/Γ is a manifold, it follows that M/Γ contains a maximal connected flat totally geodesic submanifold which is closed, and every abelian subgroup of $\pi_1(M/\Gamma)$ can be represented by closed geodesic arcs lying in a connected flat totally geodesic submanifold (Corollary 6.6). In addition, we analyze the group of components of the homogeneous holonomy group of a locally symmetric Riemannian manifold N (Theorem 7.1), prove that N has compact homogeneous holonomy group if N is compact[2] (Corollary 7.2), and give conditions for every manifold locally isometric to N to have compact homogeneous holonomy group (Corollary 7.3).

Our bounds are obtained by estimating the "size" of abelian subgroups of discrete uniform subgroups of Lie groups $L = E \times G$ where E is a semi-direct product of a compact group and a vector group, such as the Euclidean group, and G is a reductive Lie group with only finitely many components.

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[2] If N is flat, this is just the classical Bieberbach Theorem [3]. This does not give a new proof of the Bieberbach Theorem because that result is used in our arguments.

The estimates are first made for reductive groups (Theorem 4.2), and then extended by a generalization of Bieberbach's Theorem (Theorem 5.1).

Let Γ be a discrete uniform subgroup of a reductive Lie group G, where G has only finitely many components, and let Δ be an abelian subgroup of Γ. Our main idea is that the size of Δ can be estimated by finding a Cartan subgroup H of G which is normalized by Δ, and observing that $\Delta \cap H$ has finite index in Δ. In order to find H, we first prove that every element of Γ is a semisimple element of G (Theorem 3.2), and then apply a result of A. Borel and G. D. Mostow (Corollary 3.7). The main tool in our proof of Theorem 3.2 is a geometric characterization of the semisimple elements of G (Lemma 3.6).

2. Preliminaries.

2.1 *Lie groups.* Given a Lie group G, G_0 will denote the identity component, $\mathfrak{G}$ will denote the Lie algebra, and $\exp: \mathfrak{G} \to G$ will denote the exponential map. G and $\mathfrak{G}$ are called *reductive* if the adjoint representation of $\mathfrak{G}$ (or, equivalently, of G_0) is fully reducible, i.e., if $\mathfrak{G}$ is the direct sum of an abelian ideal $\mathfrak{A}$ and a semisimple ideal $\mathfrak{G}'$; then $\mathfrak{A}$ is the center of $\mathfrak{G}$, $A = \exp(\mathfrak{A})$ is the identity component of the center of G_0 and is called the *connected center* of G_0, $\mathfrak{G}'$ is the derived algebra of $\mathfrak{G}$ and is called the *semisimple part* of $\mathfrak{G}$, $G' = \exp(\mathfrak{G}')$ is called the *semisimple part* of G_0, and there is a natural homomorphism $(a, g) \to ag$ of $A \times G'$ onto G_0.

If $\mathfrak{G}$ is reductive, then the Cartan subalgebras of $\mathfrak{G}$ are the subalgebras of the form $\mathfrak{A} \oplus \mathfrak{H}'$, where $\oplus$ denotes direct sum of ideals and $\mathfrak{H}'$ is a Cartan subalgebra of $\mathfrak{G}'$; thus the Cartan subalgebras of $\mathfrak{G}$ are abelian. By *Cartan subgroup* of a Lie group G, we mean a (necessarily connected) group of the form $\exp(\mathfrak{H})$ where $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{G}$.

Under the adjoint representation of a Lie group G, an element $g \in G$ induces an automorphism $\mathrm{ad}(g)$ of $\mathfrak{G}$; we will call g *semisimple* if $\mathrm{ad}(g)$ is a fully reducible linear transformation of $\mathfrak{G}$. If g is a semisimple element of a reductive Lie group G, and if Z is the centralizer of g in G, then not only is Z reductive but the adjoint representation of G induces a fully reducible representation of $\mathfrak{Z}$ on $\mathfrak{G}$.

If S and T are subsets of a group G, then the commutator $[S, T]$ denotes the set of all elements $[s, t] = sts^{-1}t^{-1}$ where $s \in S$, $t \in T$.

If S and T are groups and ϕ is a homomorphism of S into the group of automorphisms of T, then the *semidirect product* $S \cdot_\phi T$ is the set $S \times T$ with the group structure $(s_1, t_1)(s_2, t_2) = (s_1 s_2, (\phi(s_2^{-1})(t_1)) t_2)$. If S is given as a group of automorphisms of T, then the semidirect product is denoted $S \cdot T$.

2.2. *Discrete groups.* A subgroup Γ of a topological group G is *discrete* if G has an open set U such that $\Gamma \cap U$ is just the identity element $1 \in G$. A subgroup H of G is *uniform* if the coset space $G/\bar{H}$ ($\bar{H}$ is the closure of H in G) is compact.

Let Γ be a discrete uniform subgroup of G. C. L. Siegel [9] has shown that G is locally compact, and, if every covering of G by open sets has a countable refinement,[3] then G has a compact subset F such that $\Gamma \cdot F = G$, every $g \in G$ has a neighborhood contained in a finite union of the γF, $\Gamma_F = \{\gamma \in \Gamma : \gamma F \text{ meets } F\}$ is finite, and Γ_F generates Γ if G is connected. It follows that Γ finitely generated if G/G_0 is finitely generated, but this is better seen directly [7]. F will be called a *fundamental domain* for the action of Γ on G by right translations.

2.3. *Symmetric spaces.* It is well known that a connected simply connected Riemannian symmetric space M is isometric to a product $M_0 \times M_1 \times \cdots \times M_t$ where M_0 is a Euclidean space and each M_i $(i > 0)$ is an irreducible Riemannian (non-Euclidean and not isometric to a product of lower dimensional Riemannian manifolds) symmetric space; M_0 is the *Euclidean part* of M, $M' = M_1 \times \cdots \times M_t$ is the *non-Euclidean part* of M, and the M_i $(i > 0)$ are the *irreducible factors* of M. We will say that M is *strictly non-Euclidean* if $M = M'$, i.e., if $\dim . M_0 = 0$, and will say that M is *strictly noncompact* if every irreducible factor of M is noncompact. If M is strictly noncompact, then every sectional curvature on M is $\leqq 0$.

Full groups of isometries are related by $I(M) = I(M_0) \times I(M')$, and $I(M')$ is generated by $I(M_1) \times \cdots \times I(M_t)$ together with all permutations on mutually isometric sets of M_i. Connected groups of isometries are related by $I_0(M) = I_0(M_0) \times I_0(M_1) \times \cdots \times I_0(M_t)$, and $I(M)/I_0(M)$ is finite. $I(M_0)$ is the Euclidean group $E(\dim . M_0)$, $I(M_i)$ is a compact semisimple Lie group if M_i is compact, and $I_0(M_i)$ is a noncompact centerless real or complex simple Lie group if M_i is noncompact $(i > 0)$.

Let S be a maximal connected flat (all sectional curvatures zero) totally geodesic submanifold of M. $I(M)$ acts transitively on the set of all such submanifolds, and the *rank* of M (denoted $\operatorname{rank}. M$) is defined to be their common dimension. S is isometric to a product $M_0 \times S_1 \times \cdots \times S_t$ where S_i is a maximal connected flat totally geodesic submanifold of M_i, whence it

[3] Siegel requires that G have a countable basis for open sets, but uses only this weaker property.

is easily seen that S is a closed submanifold of M. The symmetry to M_i at a point of S_i induces a symmetry of S_i; it follows that each S_i (and thus S) is a connected Riemannian symmetric manifold of constant curvature zero. Thus S_i is a flat torus if M_i is compact and S_i is a Euclidean space if M_i $(i > 0)$ is noncompact; to see this, we use [10, Théorème 4] or [11, § 14] together with the fact that S_i is the orbit of some Cartan subgroup of maximal vector rank in $I(M_i)$. If M_C is the product of the compact irreducible factors of M and M_N is the product of M_0 with the noncompact irreducible factors of M, it follows that S is isometric to the product of a flat torus of dimension rank. M_C with a Euclidean space of dimension rank. M_N. For this reason, we define the *vector rank* of M (denoted v-rank. M) to be rank. M_N. Observe that $v\text{-rank.}\, M = \dim.\, M_0 + v\text{-rank.}\, M'$.

Let Γ be a subgroup of $I(M)$. The action of Γ on M is *properly discontinuous* if every element of M has a neighborhood which meets its transforms by only a finite number of elements of Γ; this is equivalent to Γ being a discrete subgroup of $I(M)$. The action of Γ on M is *free* if $1 \neq \gamma \in \Gamma$ and $x \in M$ implies $\gamma(x) \neq x$. $M \to M/\Gamma$ is a covering space if and only if Γ acts freely and properly discontinuously on M. If M is strictly noncompact, then the isotropy subgroups of $I(M)$ are the maximal compact subgroups, and, if Γ acts properly discontinuously on M, it follows that Γ acts freely if and only if every element $\neq 1$ of Γ has infinite order. If Γ acts properly discontinuously on M, then M/Γ is a Hausdorff topological space (although it need not be a manifold), and M/Γ is compact if and only if Γ is a uniform subgroup of $I(M)$.

2.4. *Holonomy groups.* The *homogeneous holonomy group* $H(M, x)$ of a Riemannian manifold M at a point $x \in M$ is the group of linear transformations of the tangentspace M_x obtained by parallel translation of tangentvectors along sectionally smooth closed arcs based at x. The Riemannian metric gives M_x a positive definite inner product; $H(M, x)$ is a subgroup of the corresponding orthogonal group and carries the induced topology. The *restricted homogeneous holonomy group* is the identity component $H_0(M, x)$, consists of those elements of $H(M, x)$ obtained from nullhomotopic closed arcs, and is a closed subgroup of the orthogonal group of M_x; in particular, $H_0(M, x)$ is compact, and now $H(M, x)$ is compact if and only if it has only finitely many components. Thus we have a natural homomorphism of the fundamental group $\pi_1(M, x)$ onto the quotient $H(M, x)/H_0(M, x)$. If M is connected, then we speak of $H(M)$ and $H_0(M)$ in the same sense as $\pi_1(M)$.

Suppose that M is a connected simply connected Riemannian symmetric space, and $M = M_0 \times M_1 \times \cdots \times M_t$ is the decomposition into Euclidean and irreducible non-Euclidean parts. Then $H(M_0) = 1$ and $H(M_i, x)$ is the group of linear transformations of $(M_i)_x$ induced by the isotropy subgroup of $I_0(M_i)$ at x.

If M and N are Riemannian manifolds, $x \in M$ and $y \in N$, then

$$H(M \times N, (x,y)) = H(M,x) \times H(N,y).$$

3. Semisimplicity of discrete uniform subgroups.

3.1. If a Cartan subgroup of a semisimple Lie group G is normalized by an element $g \in G$, then it is known [5, Proposition 7.7] that g is a semisimple element of G. In order to make the estimates described in § 1, then, we need:

3.2. THEOREM.[4] *If Γ is a discrete uniform subgroup of a reductive Lie group G such that G/G_0 has no element of infinite order, then every element of Γ is a semisimple element of G.*

The essential part of the reduction to the semisimple case is given by:

3.3. LEMMA. *Let Γ be a discrete uniform subgroup of a connected reductive Lie group G, let A be the connected center of G, let G' be the semisimple part of G, and let $\Gamma' = \{g' \in G' \colon g' = a\gamma$ for some $a \in A$, $\gamma \in \Gamma\}$. Then Γ' is a discrete uniform subgroup of G' and $\Gamma \cap A$ is a discrete uniform subgroup of A.*

Proof. It is sufficient to consider the case where G' has no compact factor, for replacing G' by G'/K, where K is the maximal compact normal subgroup of G', affects neither hypotheses nor conclusions of the Lemma. Similarly, we may assume $G = A \times G'$.

Let $\{\gamma_i'\} \to 1$ be a sequence in Γ'; this gives us a sequence $\{\gamma_i\}$ in Γ with $\gamma_i = a_i\gamma_i'$, $a_i \in A$. A being central in G, $\{[\gamma_i,\gamma]\} = \{[\gamma_i',\gamma]\} \to 1$ for every $\gamma \in \Gamma$; thus γ_i commutes with γ for large i because Γ is discrete. Γ being finitely generated, it follows that γ_i is central in Γ for large i. Now let $\pi\colon G \to G'$ be the projection; $\Gamma' = \pi(\Gamma)$. Being uniform in G, Γ has the Selberg density property (S) in G ([8, Lemma 1] or [4, Lemma 1.4]); thus Γ' has the property (S) in G' [4, § 1.2]; it follows that the centralizer of Γ' in G' is just the center of G' [4, Corollary 4.4], whence γ_i' is central in G'

[4] As the proof will show, the essential case is that of a semisimple linear group, for which the result is known; see Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups," *Annals of Mathematics*, vol. 75 (1962), pp. 485-535, esp. § 11.2.

for large i. As G' has discrete center, this contradicts $\{\gamma_i'\} \to 1$. Thus Γ' is a discrete subgroup of G'.

Let F be a compact fundamental domain for the action of Γ on G by right translations. Γ' is a uniform subgroup of G' because $\pi(F)$ is compact and $G' = \Gamma' \cdot \pi(F)$.

Our proof that $\Gamma \cap A$ be uniform in A is a modification of an argument of A. Weil.[5] Let $a \in A$ and retain the notation above. Then $a = \gamma f$ for some $f \in F$ and some $\gamma \in \Gamma$ such that $\pi(\gamma) \in \pi(F)^{-1} \cap \pi(\Gamma)$. $\pi(F)^{-1}$ is compact and $\pi(\Gamma)$ was just seen discrete; this gives $\{\gamma_1, \cdots, \gamma_t\} \subset \Gamma$ such that the $\pi(\gamma_i)$ exhaust $\pi(F)^{-1} \cap \pi(\Gamma)$; it follows that $\gamma = \delta\gamma_i$ for some i and some $\delta \in \Gamma \cap A$. In other words, $a \in (\Gamma \cap A) \cdot \bigcup_{i=1}^{t} \gamma_i F$ for every $a \in A$. As $\bigcup_{i=1}^{t} \gamma_i F$ is compact, it follows that $A/(\Gamma \cap A)$ is compact. *q. e. d.*

The semisimple case is reduced to the linear semisimple case by means of:

3.4. LEMMA. *Let Γ be a discrete uniform subgroup of a connected semisimple Lie group G, let Z be the center of G, and let $\pi\colon G \to G/Z$ be the projection. Then $\pi(\Gamma)$ is a discrete uniform subgroup of G/Z and Γ has finite index in $\Gamma \cdot Z$.*

Proof. It suffices to show $\Gamma \cdot Z$ discrete in G, and for this we may assume that G has no compact factor. Let $\{\gamma_i z_i\} \to 1$ be a sequence in $\Gamma \cdot Z$ with $\gamma_i \in \Gamma$ and $z_i \in Z$. Given $\gamma \in \Gamma$, $\{[\gamma_i, \gamma]\} = \{[\gamma_i z_i, \gamma]\} \to 1$; thus γ_i is central in Γ for large i because Γ is discrete and finitely generated. As in the previous lemma, it follows that $\gamma_i \in Z$ for large i, whence $\gamma_i z_i \in Z$ for i large. Z being discrete, this contradicts $\{\gamma_i z_i\} \to 1$. *q. e. d.*

3.5. *Proof of Theorem* 3.2. $\Gamma \cap G_0$ is a discrete uniform subgroup of G_0, G_0 satisfies the hypotheses of the Theorem, every element of Γ has a finite power in $\Gamma \cap G_0$, and γ^m cannot be semisimple unless γ is semisimple. Thus we may assume G connected. Semisimplicity of γ depending only on the automorphism $\mathrm{ad}(\gamma)$ of $\mathfrak{G}$, Lemmas 3.3 and 3.4 now allow us to assume that G is a centerless semisimple Lie group. Finally, every automorphism of a compact simple Lie algebra being semisimple (because it preserves the Killing form, which is negative definite), we may factor G by its maximal compact normal subgroup and assume that G has no compact factor. In summary, we need only consider the case where G is a product of noncompact centerless connected simple Lie groups.

[5] See his paper "Discrete subgroups of Lie groups II," *Annals of Mathematics*, vol. 75. Observer that the argument proves: If Δ is a discrete uniform subgroup of a connected group S and T is a closed normal subgroup of S, then $\Delta \cap T$ is uniform in T if and only if the image of Δ in S/T is discrete.

Let K be a maximal compact subgroup of G and let M be the Riemannian symmetric space G/K; G is the identity component of the full group of isometries of M, whence Γ is represented faithfully on M by isometries; the action of Γ on M is properly discontinuous because Γ is discrete in G. The adjoint representation of G represents Γ faithfully as a definitely generated real matrix group, so Γ has a subgroup of finite index with no element $\neq 1$ of finite order [8, Lemma 8]. We cut Γ down to this subgroup, and may thus assume that no element $\neq 1$ of Γ has a fixed point on M, so $M \to M/\Gamma$ is a covering of Riemannian manifolds which is a local isometry (a Riemannian covering). M/Γ is compact because G/Γ is compact. M being a complete connected simply connected Riemannian manifold with every sectional curvature $\leqq 0$, it follows [6] that every $\gamma \in \Gamma$ preserves some geodesic on M. Theorem 3.2 follows from:

3.6. Lemma. *Let M be a strictly non-Euclidean Riemannian symmetric space, let $G = I(M)$, the full group of isometries of M, and let $g \in G$. Then g is a semisimple element of G if and only if some power g^m, $m \neq 0$, preserves a geodesic on M. If g preserves a geodesic σ through $x \in M$ and has no fixed point on σ, then $g = kp = pk$ where $k \in G$, $k(x) = x$, and p is a transvection along σ.*

Proof. Let g be semisimple. Then g normalizes a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ [5, Theorem 7.6]. Choose $m \geqq 1$ such that $g^m \in H = \exp(\mathfrak{H})$; this is possible because G is semisimple and G/G_0 is finite. There is an element $x \in M$ and a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ where $\mathfrak{K}$ is the Lie algebra of the isotropy subgroup K of G at x, such that $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{K}) + (\mathfrak{H} \cap \mathfrak{P})$. This holds for compact G_0 by conjugacy of maximal tori and because an involutive automorphism must conserve a maximal torus; it is known for noncompact linear simple G_0 ([12], p. 107); it now follows in our case. Now $g^m = kp$ with $k \in H \cap K$ and $p = \exp(X)$ for some $X \in \mathfrak{H} \cap \mathfrak{P}$; thus g^m preserves the geodesic $\sigma = \{\exp(tX)x\}$ on M.

Let g^m preserve a geodesic σ on M; we wish to show g semisimple, and it suffices to show g^m semisimple. Thus we may assume g to preserve σ. As g^2 preserves σ, we now replace g by g^2 if g has a fixed point on σ. σ being a totally geodesic submanifold of M, g induces an isometry of σ onto itself, and the possible replacement of g by g^2 shows that $g : \sigma_t \to \sigma_{t+a}$ for some real number a, where t is arc length. Let $x \in \sigma$, say $x = \sigma_0$, and take the Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ at x; this gives $X \in \mathfrak{P}$ with $\sigma_t = \exp(tX)x$. $k = \exp(-aX)g \in K$, and k commutes with X because it preserves every σ_t, whence k commutes with $p = \exp(aX)$. Now $g = kp = pk$, k is semisimple because it lies in a compact group K, p is a transvection along σ, and p is semisimple because it is represented on $\mathfrak{G}$ by a positive definite matrix. Being the product of two commuting semisimple elements, g is semisimple. *q.e.d.*

3.7. COROLLARY. *If Γ is a discrete uniform subgroup of a reductive Lie group G such that G/G_0 has no element of infinite order, and if Δ is an abelian subgroup of Γ, then Δ normalizes a Cartan subgroup of G.*

Proof. Theorem 3.2 says that $\mathrm{ad}(\Delta)$ is an abelian group of semisimple automorphisms of $\mathfrak{G}$; thus $\mathrm{ad}(\Delta)$ is diagonable on $\mathfrak{G}$ over the complex numbers, and it follows that $\mathrm{ad}(\Delta)$ has a finitely generated subgroup D such that every D-invariant subspace of $\mathfrak{G}$ is $\mathrm{ad}(\Delta)$-invariant. Finitely generated and abelian, D is of type $(MP)^*$ (see [5, p. 404]); thus D leaves invariant a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ [5, Theorem 7.6]. $\mathfrak{H}$ is $\mathrm{ad}(\Delta)$-invariant by choice of D, whence Δ normalizes the Cartan subgroup $\exp(\mathfrak{H})$ of G. *q.e.d.*

3.8. *Remark.* If Ψ is a subgroup of Γ which has normal subgroups Ψ_i such that $\Psi = \Psi_k \supset \Psi_{k-1} \supset \cdots \supset \Psi_1 \supset \Psi_0 = 1$ with Ψ_i/Ψ_{i-1} cyclic, then $\mathrm{ad}(\Psi)$ is a group of semisimple automorphisms of type $(MP)^*$ of $\mathfrak{G}$, so Ψ normalizes a Cartan subgroup of G by [5, Theorem 7.6].

4. Bounds for reductive groups.

4.1. Let G be a reductive Lie group. G has only a finite number of conjugacy classes of Cartan subgroups; let $\{H_1, \cdots, H_m\}$ be a maximal collection of mutually nonconjugate Cartan subgroups of G. H_i being a connected abelian Lie group of dimension $r = \mathrm{rank}.\, G$, it is isomorphic to the product of a vector group R^{u_i} with a torus T^{r-u_i}. The *vector rank* of G, denoted v-rank. G, is defined to be the maximum of the u_i If G/G_0 is finite, then each H_i has finite index in its normalizer in G; in this case there is a smallest integer, which we define to be the *torsion rank* of G and denote t-rank, G such that every finite abelian subgroup of G (which will automatically normalize a Cartan subgroup by [5, Theorem 7.6]) can be expressed as the product of $\leqq t$-rank. G cyclic groups.

4.2. THEOREM. *Let Γ be a discrete uniform subgroup of a reductive Lie group G such that G/G_0 is finite, and let Δ be an abelian subgroup of Γ. Then Δ can be expressed as the product of $\leqq t$-rank. G finite cyclic groups with a free abelian group on $\leqq v$-rank. G generators; in particular, Δ is finitely generated. Furthermore, the second bound is best possible in the sense that Γ has a subgroup which is free abelian on v-rank. G generators. Finally, if Σ is a subgroup of Γ which is the product of a finite abelian group and a* *free abelian group on m generators, if H is a Cartan subgroup of G normalized by Σ, and if Γ does not have an abelian subgroup Ψ such that $\Sigma \cap \Psi$ be of finite index in Σ but of infinite index in Ψ, then $\Sigma \cap H$ is uniform in H.*

Proof. By Corollary 3.7, Δ normalizes a Cartan subgroup A of G. $\Delta \cap A$ is finitely generated because it is a discrete subgroup of the connected abelian Lie group A, and $\Delta \cap A$ has finite index in Δ because A has finite index in its normalizer in G; thus Δ is finitely generated. The first statement now follows from the structure theorem for finitely generated abelian groups and the definitions of t-rank. G and v-rank. G. We need some lemmas for the other statements.

4.3. LEMMA.[6] *Let Γ be a discrete uniform subgroup of a Hausdorff topological group G, let D be a finitely generated subgroup of Γ, and let G_D and Γ_D be the respective centralizers of D in G and Γ. Then Γ_D is a discrete uniform subgroup of G_D.*

Proof. Let $\pi\colon G \to G/\Gamma$ be the projection. As $\Gamma_D = \Gamma \cap G_D$ and G/Γ is a compact Hausdorff space, Γ_D is uniform in G_D if and only if $\pi(G_D)$ is closed in G/Γ. If $\pi(G_D)$ is not closed in G/Γ, then $\pi^{-1}\pi(G_D)$ is not closed in G, so there is a sequence $\{\gamma_i\}$ in Γ of elements distinct $\operatorname{mod} G_D$ and a sequence $\{g_i\}$ in G_D with $\{\gamma_i g_i\} \to x$ for some $x \in G$. Given $d \in D$, $\{[\gamma_i, d]\} = \{[\gamma_i g_i, d]\} \to [x, d]$, whence $[\gamma_i, d] = [x, d]$ for large i because Γ is discrete, implying that $\gamma_i^{-1}\gamma_j$ commute with d for large i and j. As D is finitely generated, $\gamma_i^{-1}\gamma_j \in G_D$ for large i and j, contradicting our choice of the sequence $\{\gamma_i\}$. *q. e. d.*

4.4. We will prove the last statement of the theorem, retaining the notation of Lemma 4.3. Theorem 3.2 and an induction on the number of generators of Σ show that G_Σ is a reductive Lie group, so its Lie algebra is a direct sum $\mathfrak{A} \oplus \mathfrak{C} \oplus \mathfrak{N}$ with $\mathfrak{A}$ abelian, $\mathfrak{C}$ a sum of compact simple Lie algebras, and $\mathfrak{N}$ a sum of noncompact simple Lie algebras. By Lemma 3.3 and assumption on Σ, we see that $\mathfrak{N} = 0$; thus $(G_\Sigma)_0 = V \times K$ where V is a vector group in $\exp(\mathfrak{A})$ and K is a compact group containing $\exp(\mathfrak{C})$. $\Sigma \cap H$ having finite index in Σ, and $H \subset (G_\Sigma)_0$, gives $m \leqq v\text{-rank}. H \leqq \dim. V$. On the other hand, $V \cap \Gamma_\Sigma$ is free abelian on $\dim. V$ generators by Lemmas 4.3 and 3.3; thus $\dim. V \leqq m$ by assumption on Σ. It follows that $m = \text{v-rank}. H$, proving the last statement of the Theorem.

4.5. LEMMA. *Let H be a Cartan subgroup of a reductive Lie group G,*

[6] This is essentially the same as A. Selberg's result [8, Lemma 2].

and suppose that H has an element h such that, given $g \in G_0$, the number of distinct absolute values among the eigenvalues of $\mathrm{ad}(h)$ *(acting on* $\mathfrak{G}$*) is at least as large as the number of distinct absolute values among the eigenvalues of* $\mathrm{ad}(g)$. *Then* v-rank. $H = v$-rank. G.

For $\mathfrak{H}$ has the maximal number of linearly independent real-valued roots among all Cartan subalgebras of $\mathfrak{G}$.

4.6.[7] We will finish the proof of Theorem 4.2 by proving the second statement. Let $g \in G_0$, the number of whose absolute values of eigenvalues in the adjoint representation is maximal among the elements of G_0. Taking powers only separates further the absolute values of the eigenvalues, so we have a neighborhood U of 1 in G_0 and an integer m such that:

1. If μ is the Haar measure on G/Γ and $\pi \colon G \to G/\Gamma$ is the projection, then $m > \mu(G/\Gamma)/\mu(\pi(U))$.
2. If $1 \leqq a \leqq m$ and $u_i \in U$, then the number of distinct absolute values of eigenvalues of $\mathrm{ad}(u_1 g^a u_2^{-1})$ is not less than the number for $\mathrm{ad}(g)$.

We choose [8, Lemma 1] the integer a and the $u_i \in U$ such that $u_1 g^a u_2^{-1} = \gamma \in \Gamma$, replace γ by a power which lies in a Cartan subgroup H of G, and observe that v-rank. $H = v$-rank. G by Lemma 4.5. We may take γ to be a regular element of G, whence $H/(H \cap \Gamma)$ is compact by Lemma 4.3 with $D = \{\gamma\}$. Thus $H \cap \Gamma$ has a subgroup which is free abelian on v rank. G generators. The Theorem is proved. *q. e. d.*

5. Bounds for groups with Euclidean factor. Our tool for the treatment of groups with Euclidean factor is the following generalization of theorems of L. Bieberbach [3] and L. Auslander [1]:

5.1. THEOREM. *Let K be a compact group of automorphisms of a connected simply connected nilpotent Lie group N, let E be the semidirect product $K \cdot N$, let $L = E \times G$ where G is a reductive Lie group with G/G_0 finite, and let Γ be a discrete uniform subgroup of L. Then $\Gamma \cap (N \times G)$ is a normal subgroup of finite index in Γ which is a discrete uniform subgroup of $N \times G$.*

Proof.[8] As in § 3.5, we may assume G to be connected and without

[7] Compare with paragraph 3, p. 151, of [8], where A. Selberg considers the case $G = SL(n; R)$.

[8] This Theorem can be proved as a consequence of [2, Theorem 1], to which it is similar. We find it more convenient, however, to give an argument which is a variation on the proof of that result.

compact normal subgroup; then $G = A \cdot G'$ where A, the connected center of G, is a vector group and G', the semisimple part of G, is without compact factor. Let $p\colon A \times G' \to G$ be the projection; then $(1 \times p)^{-1}(\Gamma)$ is a discrete uniform subgroup of $E \times A \times G'$ with the same projection on K as has Γ. Replacing N by $N \times A$, we see that it suffices to prove the Theorem when G is semisimple and without compact factor.

Let J be the closure of $\Gamma \cdot N$ in L, and let $\alpha\colon L \to G$ and $\beta\colon L \to K$ be be the projections. J_0 is solvable by the generalized Zassenhaus Lemma [2, Proposition 2], and J is clearly normalized by Γ; as $\alpha(\Gamma)$ has the Selberg density property (S) in G (for Γ has it in L), $\alpha(\Gamma)$normalizes $\alpha(J)$ and G is semisimple without compact factor, it follows [4, Theorem 4.1] that $\alpha(J)$ has discrete closure in G. Thus $\alpha(\Gamma)$ is discrete, being contained in $\alpha(J)$ The proof of the last part of Lemma 3.3 now shows $\Gamma \cap E$ uniform in E.

$\Gamma \cap E$ being a discrete uniform subgroup of E, the generalized Bieberbach Theorem [1, Theorem 1] shows that $(\Gamma \cap E) \cap N = \Gamma \cap N$ is a discrete uniform subgroup of N. $\Gamma \cap N$ is normal in Γ, and is thus normalized by $\beta(\Gamma)$. On the other hand, interpreting K as a group of automorphisms of N, uniformity of $\Gamma \cap N$ in N implies that an element of K is determined by its action on $\Gamma \cap N$ [1, Theorem 2]. It follows that $\beta(\Gamma)$ is finite. *q. e. d.*

5.2. Corollary. *Let Γ be a discrete uniform subgroup of $L = E \times G$ where E is a semidirect product $K \cdot V$, K is a compact group of automorphisms of the vector group V, and G is a reductive Lie group with G/G_0 finite. If Δ is an abelian subgroup of Γ, then Δ normalizes some Cartan subgroup H of G, and Δ can be expressed as the product of $\leqq t$-rank.$(K \times G)$ finite cyclic groups with a free abelian group on $\leqq$ dim. $V + v$-rank. G generators. If Γ has no abelian subgroup Σ with the property that $\Delta \cap \Sigma$ has finite index in Δ and infinite index in Σ, then $\Delta \cap (V \times H)$ is uniform in $V \times H$. Finally, Γ has a subgroup which is free abelian on* dim. $V + v$-rank. G *generators.*

Proof. By Theorems 3.2 and 5.1, every element of Γ is a semisimple element of the reductive Lie group $G'' = \Gamma \cdot (V \times G)$, and G''/G_0'' is finite. The Cartan subgroups of G'' being of the form $V \times$(Cartan subgroup of G), we have v-rank. $G'' =$ dim. $V + v$-rank. G. Every finite subgroup of E being conjugate to a subgroup of K, we have t-rank. $G'' \leqq t$-rank. $(K \times G)$. The Corollary now follows from Theorem 4.2. *q. e. d.*

6. Application to symmetric spaces.

6.1. Let M be a connected simply connected Riemannian symmetric space, and let $L = I(M)$, the full group of isometries of M. If $M = M_0 \times M'$

is the decomposition of M into the product of its Euclidean and non-Euclidean parts, then $L = E \times G$ where $G = I(M')$ is a semisimple (and thus reductive) Lie group with G/G_0 finite and $E = I(M_0)$ is the Euclidean group $E(n)$, $n = \dim. M_0$, semidirect product $O(n) \cdot V$ where $O(n)$ is the orthogonal group of a vectorspace V which can be identified with M_0. As every flat maximal connected totally geodesic submanifold of M' is an orbit of a Cartan subgroup of maximal vector rank in G, we have v-rank. $M' = v$-rank. G. Thus v-rank. $M = \dim. M_0 + v$-rank. $I(M')$.

6.2. Theorem. *Let M be a connected simply connected Riemannian symmetric space, let M_0 and M' be the Euclidean and non-Euclidean parts of M, let Γ be a properly discontinuous group of isometries of M with M/Γ compact, and let Δ be an abelian subgroup of Γ. Then M has a closed connected Δ-invariant flat totally geodesic submanifold S_Δ whose image in M/Γ is compact, and Δ can be expressed as the product of $\leqq t$-rank. $(O(\dim. M_0) \times I(M'))$ finite cyclic groups with a free abelian group on $\leqq v$-rank. M generators. If Γ has no abelian subgroup Σ with the property that $\Delta \cap \Sigma$ has finite index in Δ but infinite index in Σ, then S_Δ/Δ is compact. Finally, Γ has a subgroup Ψ which is free abelian on v-rank. M generators, and S_Ψ can be taken to be a maximal connected flat totally geodesic submanifold of M.*

Proof. Γ acts by isometries, M/Γ is compact, and $I(M)$ acts transitively on M with compact isotropy subgroups; it follows that Γ is a discrete uniform subgroup of $I(M)$.

6.3. Retaining the notation of § 6.1, we will define S_Δ to be an orbit of one of the groups $V \times H$ where H is a Cartan subgroup of $I(M')$ normalized by Δ; such groups exist by Corollary 5.2. When the choice of H is made, we will choose $x \in M_0$ to be the origin of V and choose $y \in M'$ such that $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{K}) + (\mathfrak{H} \cap \mathfrak{P})$ where K is the isotropy subgroup of $I(M')$ at y and $\mathfrak{P}$ is the orthogonal complement of $\mathfrak{K}$ in $\mathfrak{I}(M')$ under the Killing form of $\mathfrak{I}(M')$. Then $S_\Delta = (V \times H)(x, y) = M_0 \times H(y)$ is clearly closed, connected, flat and totally geodesic in M. Let $\pi: I(M) \to I(M')$ be the projection and let $\delta \in \Delta$. $\pi(\Delta)$ normalizes H, and is thus contained in $K \cdot H$; now $\pi(\delta) = kh$. This gives $\pi(\delta) H(y) = khH(y) = kHk^{-1}(y) = H(y)$, proving S_Δ to be Δ-invariant.

We choose H to be a Cartan subgroup of $I(M')$ which is normalized by some maximal abelian subgroup Φ of Γ which contains Δ, and set $\Phi' = \Phi \cap (V \times H)$; Φ' has finite index in Φ by Theorem 5.1 and finiteness of G/G_0. Φ is finitely generated, and thus, by Lemma 4.3, is uniform in its centralizer A in $I(M)$; it follows that Φ' is uniform in $V \times H$. Thus the image of S_Δ in M/Γ is compact. As the bounds on the size of Δ follow from Corollary 5.2, this proves the first statement of the Theorem.

6.4. Observe that not only is the image of S_Δ in M/Γ compact, but S_Δ/Φ is compact. Thus S_Δ/Δ is compact if Δ has finite index in Φ. The structure of abelian subgroups of Γ is such that Δ has finite index in Φ if Γ has no abelian subgroup Σ with the property that $\Delta \cap \Sigma$ has finite index in Δ but infinite index in Σ. This proves the second statement of the Theorem.

Corollary 5.2 shows that Γ has a subgroup which is free abelian on v-rank. M generators. If Δ is such a subgroup, then H is a Cartan subgroup of maximal vector rank in $I(M')$. Examining the compact and noncompact factors of M' separately, we see that, for proper choice of $y \in M'$, S_Δ is a maximal connected flat totally geodesic submanifold of M. *q. e. d.*

6.5. The complete connected locally symmetric Riemannian manifolds are precisely those manifolds whose universal Riemannian covering manifold is symmetric. Thus Theorem 6.2 gives us:

6.6. COROLLARY. *Let N be a compact connected locally symmetric Riemannian manifold, let D be an abelian subgroup of the fundamental group $\pi_1(N)$, and let M_c be the product of the compact irreducible factors of the universal Riemannian covering manifold M of N. Then N has a closed connected flat totally geodesic submanifold S_D and an element $x \in S_D$ such that D is represented by closed geodesic arcs in S_D based at x, and D can be expressed as the product of $\leqq t$-rank.$I(M_c)$ finite cyclic groups with a free abelian group on $\leqq v$-rank. M generators; thus D is free abelian on $\leqq v$-rank. M generators if M is strictly noncompact. $\pi_1(N)$ has a subgroup P which is free abelian on v-rank. M generators, and S_P can be taken to be a maximal connected flat totally geodesic submanifold of N; thus N has a maximal connected flat totally geodesic submanifold which is closed in N.*

7. Holonomy groups of locally symmetric spaces. Corollary 7.2 is the only part of § 7 which uses the results of preceding sections; in fact, it uses only Theorem 5.1, which does not depend on preceding results.

7.1. THEOREM. *Let Γ be the group of deck transformations of the universal Riemannian covering $\pi: M \rightarrow N$ of a complete locally symmetric Riemannian manifold N, let $M = M_0 \times M'$ be the decomposition into Euclidean and non-Euclidean parts, and let V be the group of pure translations of M_0. Then there is a canonical isomorphism between*

$$\Gamma \cdot (V \times I_0(M'))/(V \times I_0(M'))$$

and the group $H(N)/H_0(N)$ *of components of the homogeneous holonomy group of* N.

Proof. Recall the homomorphism β of Γ onto $H(N,\pi(x))/H_0(N,\pi(x))$ defined by $\beta(\gamma)=t_\gamma\cdot H_0(N,\pi(x))$, where t_γ is the operation defined by $\pi(\tau_\gamma)$ and τ_γ is any sectionally smooth arc in M from x to $\gamma(x)$. We can represent t_γ on the tangentspace M_x as the differential $\gamma_*: M_x\to M_{\gamma(x)}$ followed by parallel translation of tangentvectors backwards along τ_γ.

Let $x\in M=M_0\times M'$ have repesentation $x=(x_0,x')$, let K' be the isotropy subgroup of $I(M')$ at x', and let $P'=\exp(\mathfrak{P}')$ where $\mathfrak{P}'$ is the orthogonal complement of $\mathfrak{K}'$ in $\mathfrak{I}(M')$ under the Killing form of $\mathfrak{I}(M')$. The every element of $I(M')$ has expression $p'k'$ with $k'\in K'$ and $p'\in P'$. Observe that the identity component K_0' is the isotropy subgroup of $I_0(M')$ at x', and its action on the tangentspace M_x is that of $H(M,x)=H_0(N,\pi(x))$; also, P' is the set of transvections along geodesics in M' which pass through x', and $p_*':M_{x'}'\to M_{p'(x')}'$ is parallel translation along the geodesic arc from x' to $p'(x')$ on which p' is a transvection.

Let $\gamma\in\Gamma$; $\gamma=\gamma_0\gamma'$ with $\gamma_0\in I(M_0)$ and $\gamma'\in I(M')$. $\gamma'=p'k'$ as above and $\gamma_0=p_0k_0$ with $k_0(x_0)=x_0$ and $p_0\in V$; thus $\gamma=(p_0p')(k_0k')$. Let $y=(y_0,y')$ be the image of x; then p_0 is transvection along a geodesic arc τ_0 in M_0 from x_0 to y_0, p' is transvection along a geodesic arc τ' in M' from x' to y', we define τ_γ to be the geodesic arc in M from x to y with projections τ' and τ_0 and it is then clear that t_γ is represented by the differential of k_0k' on M_x. Thus the canonical homomorphism β of Γ onto $H(N)/H_0(N)$ induces the isomorphism of the Theorem. *q. e. d.*

7.2. Corollary. *Let* N *be a compact locally symmetric Riemannian manifold. Then the homogeneous holonomy group* $H(N)$ *is compact, i. e.,* $H(N)/H_0(N)$ *is finite.*

This follows easily from Theorems 5.1 and 7.1.

7.3. Corollary. *Let* M *be a connected simply connected Riemannian symmetric space with Euclidean part* M_0 *and non-Euclidean part* M'. *If* $\dim. M_0>2$, *or if* $\dim. M_0=2$ *and* M' *is noncompact, then there are continuum many affinely inequivalent diffeomorphic Riemannian manifolds covered by* M *which have noncompact homogeneous holonomy groups. If* $\dim. M_0<2$, *or if* $\dim. M_0=2$ *and* M' *is compact, then every Riemannian manifold covered by* M *has compact homogeneous holonomy group. If* $\dim. M_0=q-1<2$ *and* r *is the order of* $I(M')/I_0(M')$, *then the number of components of the homogeneous holonomy group of a Riemannian manifold covered by* M *is a divisor of* qr.

Proof.[9] The last statement follows from Theorem 7.1 and the fact the group of translations of M_0 has finite index q in $I(M_0)$. This also proves the second statement except when $\dim. M_0 = 2$. Let $\dim. M_0 = 2$, let M' be compact, and let Γ be the group of deck transformations of a Riemannian covering $M \to N$. $I(M')$ is compact, whence the projection Γ_0 of Γ on $I(M_0)$ is discrete. We wish to show that the group H of rotation parts of elements of Γ_0 is finite. Let U be the linear subspace of the vectorspace M_0 which is spanned by the translation parts of elements of Γ_0. If $\dim. U = 2$, then finiteness of H follows from the Bieberbach theorem [3] (or from Theorem 5.1). If $\dim. U = 1$, then M_0 has an orthonormal basis $\{u, v\}$ where u spans U. As H normalizes U, every element of H has matrix $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in this basis; thus H is finite. If $\dim. U = 0$, then $\Gamma_0 = H$ lies in a compact group, and thus finite because Γ_0 is discrete. Now H is finite in any case, and the last part of the second statement follows from Theorem 7.1.

7.4. For each real number t, we define $g_t = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$, the rotation with eigenvalues $\exp(\pm 2\pi\sqrt{-1}t)$. Now suppose $\dim. M_0 \geqq 2$, view M_0 as a vectorspace, and let $\{v_1, \cdots, v_r\}$ be an orthonormal basis of M_0; let A_t be the linear transformation $\begin{pmatrix} g_t & 0 \\ 0 & I_{r-2} \end{pmatrix}$ of M_0. If $\dim. M_0 > 2$, then let γ_t be the isometry $(m_0, m') \to (A_t m_0 + v_3, m')$ of $M = M_0 \times M'$; if $\dim. M_0 = 2$ and M' is noncompact, then we have a tranvection τ in a noncompact irreducible factor of M', and we define γ_t to be the isometry $(m_0, m') \to (A_t m_0, \tau m')$ of M. In either case, γ_t generates an infinite cyclic subgroup Γ_t of $I(M)$ which acts freely and properly discontinuously on M. Thus $N_t = M/\Gamma_t$ is a Riemannian manifold covered by M. In both cases $\gamma_t = \beta_t \alpha_t$ where β_t is a transvection of M along some geodesic σ through our basepoint x, and where α_t is an isometry of M with $\alpha_t(x) = x$. The element of $H(N_t)/H_0(N_t)$ determined by γ_t being represented on the tangentspace M_x by the differential of γ_t follows by parallel translation along σ from $\beta_t(x) = \gamma_t(x)$ to x, this element is represented on M_x by the differential of α_t. By construction, this differential is given by A_t on $(M_0)_x$ and is the identity on $(M')_x$; thus we may view the linear transformation A_t as a generator of $H(N_t)/H_0(N_t)$. In particular, N_t has compact homogeneous holonomy group if and only if A_t has finite order, i.e., if and only if t is

[9] As the proof will show, the essential case in when M is irreducible. This was originally handled by a lemma developed in discussions with H. C. Wang; in the present context, however, it is easier to appeal to Theorem 7.1.

rational. Affine equivalence induces isomorphism of holonomy groups as groups of linear transformations; thus the first statement follows from the fact that the N_t are mutually real-analytically homeomorphic and we can choose continuum many algebraically independent irrational numbers t.

q. e. d.

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