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THE CLIFFORD-KLEIN SPACE FORMS OF INDEFINITE METRIC

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1. Introduction

The spherical space form problem of Clifford-Klein is the classification problem for complete connected riemannian manifolds of constant positive curvature. E. Calabi and L. Markus have recently considered the classification problem for complete connected Lorenz n-manifolds M_1^n of constant positive curvature, and have [1; Theorems 2 and 3] reduced it to the spherical space form problem for riemannian $(n-1)$-manifolds, when $n \geqq 3$. We will extend their ideas to more general signatures of metric, and reduce the classification problem for complete connected pseudo-riemannian n-manifolds M_s^n of constant positive curvature, and with $s \neq n-1$ and $2s \leqq n$, to the spherical space form problem for riemannian $(n-s)$-manifolds. As the spherical space form problem is solved in dimension 3 [3], and is trivial in even dimensions, this gives a classification (up to global isometry) of the complete connected pseudo-riemannian manifolds M_s^n of constant positive curvature with $s \neq n-1$, $2s \leqq n$, and either $n-s=3$ or $n-s$ even.

I am indebted to L. Markus and E. Calabi for showing me the manuscript of their paper [1], which is the basis for this note.

2. Pseudo-riemannian manifolds

In order to establish terminology and notation, we recall some basic facts about pseudo-riemannian manifolds. A *pseudo-riemannian metric* Q on a differentiable manifold M is a differentiable field of non-degenerate bilinear forms Q_p on the tangent-spaces M_p of M; a *pseudo-riemannian manifold* is a differentiable manifold with a pseudo-riemannian metric. We will consider only those pseudo-riemannian manifolds (M, Q) where each Q_p has the same signature[1], say $-dx_1^2 - \cdots - dx_s^2 + dx_{s+1}^2 + \cdots + dx_n^2$; (M, Q) is then denoted M_s^n. In the riemannian case $(s=0)$, we will simply write M^n. A pseudo-riemannian manifold M_s^n carries a unique linear connection—the Levi-Civita connection on the tangent-bundle of M_s^n—with torsion zero, and such that parallel translation is an isometry of tangent-spaces. In local coordinates $x \to (x_1, \cdots, x_n)$, we set $q_{ij}(x) = Q_x\big((\partial/\partial x_i), (\partial/\partial x_j)\big)$

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[1] This is automatic if M is connected.

and lower indices of the curvature tensor $R_i{}^m{}_{kl}(x)$ of the Levi-Civita connection, using the classical formula $R_{ijkl}(x) = \sum_m q_{jm}(x) R_i{}^m{}_{kl}(x)$; M_s^n has *constant curvature* K if

$$R_{ijkl}(x) = K[q_{jk}(x)q_{il}(x) - q_{ik}(x)q_{jl}(x)]$$

identically in x, for every (i, j, k, l) and every local coordinate system. M_s^n is *complete* if its Levi-Civita connection is complete, i.e., if every geodesic can be extended to arbitrary values of the affine parameter. An *isometry* is a diffeomorphism d: $N_s^n \to M_s^n$ where every tangent map is a linear isometry; a *pseudo-riemannian covering* is a covering f: $N_s^n \to M_s^n$ of connected pseudo-riemannian manifolds where every tangent map is a linear isometry.

The indefinite orthogonal group $O^s(n+1)$ is the group of all linear transformations of the real vector space R^{n+1} which preserve the bilinear form $b_s^n(x, y) = -x_1y_1 - \cdots - x_sy_s + x_{s+1}y_{s+1} + \cdots + x_{n+1}y_{n+1}$. The quadric $b_s^n(x, x) = 1$ carries a pseudo-riemannian metric such that it is a pseudo-riemannian manifold S_s^n of constant curvature $+1$, and $O^s(n+1)$ is its full group of isometries [4,§ 4]. We define $\tilde{S}_s^n$ to be S_s^n if $s < n-1$, $\tilde{S}_{n-1}^n$ to be the universal pseudo-riemannian covering manifold of S_{n-1}^n, and $\tilde{S}_n^n$ to be the component of $(0, \cdots, 0, 1)$ in S_n^n. We have proved [4, Theorem 5] that every complete connected pseudo-riemannian manifold M_s^n of constant curvature $+1$ admits a universal pseudo-riemannian covering f: $\tilde{S}_s^n \to M_s^n$, and is thus isometric to the quotient manifold $\tilde{S}_s^n/D$ where the subgroup D of the group of isometries of $\tilde{S}_s^n$ is just the group of deck transformations of f. Thus we define a *pseudo-spherical space form* of signature $(s, n-s)$ to be a complete connected pseudo-riemannian manifold M_s^n of constant curvature $+1$. A spherical space form is just a pseudo-spherical space form $M^n = M_0^n$, and, in the terminology of [1], a relativistic spherical space form is a pseudo-spherical space form M_1^n.

3. The reduction

This reduction was inspired by that of Calabi and Markus [1] where they do the case $s = 1$; the proof of our Theorem 1 being virtually identical to their proof for $s = 1$, while the methods of our Theorem 2 are different from their quite geometric techniques.

THEOREM 1. *Let M_s^n be a pseudo-spherical space form with $s \neq n-1$ and $2s \leqq n$. Then M_s^n has finite fundamental group.*

PROOF. $S_s^n = \tilde{S}_s^n$ because $s < n-1$, so we have a universal pseudo-riemannian covering f: $S_s^n \to M_s^n$. The group D of deck transformations of the covering is isomorphic to the fundamental group of M_s^n, so we must prove that D is finite.

S_s^n is given by $-\sum_1^s x_j^2 + \sum_{s+1}^{n+1} x_j^2 = 1$ where the x_j are coordinates in R^{n+1} relative to a basis $\{v_1, \cdots, v_{n+1}\}$. Let V be the subspace spanned by $\{v_{s+1}, \cdots, v_{n+1}\}$. If g is any non-singular linear transformation of R^{n+1}, then $2s \leqq n < n+1$ gives us

$$\dim (V \cap gV) \geqq 2 \cdot \dim V - (n+1) > 0 .$$

If S^{n-s} denotes the "equatorial sphere" $x_1 = \cdots = x_s = 0$, $\sum_{s+1}^{n+1} x_j^2 = 1$ in S_s^n, and if $g \in D$, it follows that $g(S^{n-s})$ meets S^{n-s}.

The rest of the argument follows [1]. If D were infinite, compactness of S^{n-s} would give a point $p \in S^{n-s}$, a sequence $\{g_i\}$ of distinct elements of D, and a sequence $\{p_i\}$ in S^{n-s} such that each $g_i(p_i) \in S^{n-s}$ and $\{g_i(p_i)\} \to p$. We may pass to a subsequence and assume $\{p_i\} \to p' \in S^{n-s}$. This gives an element $g \in D$ with $g(p') = p$; so $\{g^{-1}g_i(p_i)\} \to g^{-1}(p) = p'$. As all the g_i are distinct, we may assume that none of the $g^{-1}g_i$ is $1 \in D$. Thus every neighborhood of p' in S_s^n meets one of its transforms by an element $\neq 1$ of D. This is impossible because D, being a group of deck transformations, acts freely and properly discontinuously on S_s^n. q.e.d.

Notice that the product $\mathrm{O}(s) \times \mathrm{O}(n-s+1)$ is a maximal compact subgroup of $\mathrm{O}^s(n+1)$, where $\mathrm{O}(s)$ is the ordinary orthogonal group on the first s coordinates in R^{n+1} and $\mathrm{O}(n-s+1)$ is the ordinary orthogonal group on the last $n-s+1$ coordinates in R^{n+1}.

LEMMA (É. Cartan). *Every compact subgroup of* $\mathrm{O}^s(n+1)$ *is conjugate to a subgroup of* $\mathrm{O}(s) \times \mathrm{O}(n-s+1)$.

PROOF. This follows from a technique of É. Cartan [2, § 16]. Also, A. Borel has remarked that the lemma is just simultaneous diagonalization of two quadratic forms. q.e.d.

THEOREM 2. *Let* S^{n-s}/G *be a spherical space form, view G as the image of a faithful orthogonal representation* $f\colon H \to \mathrm{O}(n-s+1)$ *of an abstract finite group H, let* $g\colon H \to \mathrm{O}(s)$ *be any orthogonal representation of H, and set* $D = (g+f)(H) \subset \mathrm{O}(s) \times \mathrm{O}(n-s+1) \subset \mathrm{O}^s(n+1)$. *Then D is free and properly discontinuous on* S_s^n, S_s^n/D *is a pseudo-spherical space form, and* S_s^n/D *has finite fundamental group if* $s \neq n-1$. *Conversely, let* M_s^n *be a pseudo-spherical space form with* $s \neq n-1$ *and with finite fundamental group (automatic if* $2s \leqq n$*). Then* M_s^n *is isometric to one of the manifolds* S_s^n/D *described above.*

PROOF. D is finite, thus properly discontinuous on S_s^n. If $h \in H$ and $(g+f)(h)$ has an eigenvalue $+1$, then either $h = 1$ or the corresponding eigenvector lies in the span of the first s coordinates on R^{n+1}. Thus D acts freely on S_s^n. This shows that S_s^n/D is a pseudo-spherical space form.

If $s \neq n-1$, then S_s^n is simply connected, and the fundamental group of S_s^n/D is finite.

For the second part of the theorem, let $f: \tilde{S}_s^n \to M_s^n$ be the universal pseudo-riemannian covering, and let D be the group of deck transformations. As $s \neq n-1$, D is a subgroup of $O^s(n+1)$. D was assumed finite, thus compact, although this is automatic if $2s \leqq n$. Using the lemma, we may replace D by a conjugate, thereby changing M_s^n by an isometry, and assume $D \subset O(s) \times O(n-s+1)$. Now write every $a \in D$ in the form (a_1, a_2) with $a_1 \in O(s)$ and $a_2 \in O(n-s+1)$. The $\{a_2\}$ generate a subgroup G of $O(n-s+1)$ which acts freely on the equatorial sphere S^{n-s} in S_s^n; thus S^{n-s}/G is a spherical space form. Set $H = D$, $f(a) = a_2$ and $g(a) = a_1$, and the theorem is proved. q.e.d.

Theorems 1 and 2 reduce the classification problem for complete connected pseudo-riemannian manifolds M_s^n ($s \neq n-1$ and $2s \leqq n$) of constant positive curvature to the classification problem for complete connected riemannian manifolds M^{n-s} of constant positive curvature.

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