VINCENT's Conjecture on CLIFFORD Translations of the Sphere.

VINCENT's Conjecture on CLIFFORD Translations of the Sphere

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I. Introduction and statements of theorems

G. VINCENT has suggested the possibility that every finite group of CLIFFORD translations of a sphere is either cyclic or binary polyhedral [2, § 10.5]. In a recent *Comptes rendus* note [3] I stated that this is the case; the purpose of this note is to supply a proof.

S^n is the unit sphere in EUCLIDean space R^{n+1}, and carries the induced RIEMANNian structure; hence the group of isometries of S^n is the orthogonal group $O(n+1)$. Recall that an isometry f of S^n is a *CLIFFORD translation* if the distance between a point $x \in S^n$ and its image $f(x)$ is independent of x. This just means that either $f = \pm I$ ($I =$ identity) or $n + 1 = 2m$ and there is a unimodular complex number λ such that f has m eigenvalues equal to λ and m eigenvalues equal to the complex conjugate $\bar{\lambda}$ of λ.

We recall the binary polyhedral groups. The *polyhedral groups* are the dihedral groups $\mathcal{D}_m$, the tetrahedral group $\mathcal{T}$, the octahedral group $\mathcal{O}$ and the icosahedral group $\mathcal{I}$—the respective groups of symmetries of the regular m-gon, the regular tetrahedron, the regular octahedron and the regular icosahedron. Each polyhedral group can, in a natural fashion, be considered as a subgroup of the special orthogonal group $SO(3)$. Let $\pi : Spin(3) \to SO(3)$ be the universal covering. The *binary polyhedral groups*[2]) are the *binary dihedral groups* $\mathcal{D}_m^* = \pi^{-1}(\mathcal{D}_m)$, the *binary tetrahedral group* $\mathcal{T}^* = \pi^{-1}(\mathcal{T})$, the *binary octahedral group* $\mathcal{O}^* = \pi^{-1}(\mathcal{O})$, and the *binary icosahedral group* $\mathcal{I}^* = \pi^{-1}(\mathcal{I})$.

We can now state

Theorem 1 *(conjectured by VINCENT). If Γ is a finite group of CLIFFORD translations of a sphere, then Γ is either a cyclic group or a binary polyhedral group.*

In fact, one can add

Theorem 2. *Let Γ be a finite group of CLIFFORD translations of a sphere $S^n \subset R^{n+1}$. If Γ is cyclic of order 1 or 2, then $\Gamma = \{I\}$ or $\{\pm I\}$. If Γ is cyclic of order $q > 2$, then $n + 1$ is even (say $n + 1 = 2s$) and Γ is the*

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[1]) This work was done while the author held a National Science Foundation fellowship.

[2]) This definition was brought to my attention by J. TITS.

image of a representation ϱ *of the abstract cyclic group* Z_q *where* A *is a generator of* Z_q *and* ϱ *is* $SO(2s)$*-equivalent to the representation*

$$A^t \to \begin{pmatrix} R(t/s) & & \\ & \ddots & \\ & & R(t/s) \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}.$$

If Γ *is binary polyhedral and noncyclic, then* 4 *divides* $n+1$ *(say* $n+1=4s$*) and* Γ *is the image of a representation* ϱ *of an abstract binary polyhedral group* $\mathcal{P}^* \cong \Gamma$ *where* ϱ *is* $SO(4s)$*-equivalent to a sum of* s *copies of the* $SO(4)$*-representation*

$$\mathcal{P}^* \subset Spin(3) = SU(2) \subset SO(4)\ .$$

Finally, the images of these representations are finite groups of CLIFFORD *translations of* S^n.

Using Theorem 2 we will prove

Theorem 3. *Let* Γ *be a finite group of* CLIFFORD *translations of a sphere* $S^n \subset R^{n+1}$. *Then the centralizer of* Γ *in* $O(n+1)$ *is transitive on* S^n.

Theorem 4. *Let* Γ *be a finite subgroup of* $O(n+1)$. *Then these are equivalent:*

(1) Γ *is a group of* CLIFFORD *translations of* S^n.

(2) Γ *is the image, by one of the representations described in Theorem* 2, *of a cyclic or binary polyhedral group.*

(3) *The centralizer of* Γ *in* $O(n+1)$ *is transitive on* S^n.

(4) *The quotient* S^n/Γ *is a* RIEMANN*ian homogeneous manifold.*

II. Proof of VINCENT's conjecture

We must give an abstract characterization of finite groups of CLIFFORD translations of a sphere.

Definition. *Let* $\varphi : \Gamma \to U(q)$ *be a faithful unitary representation of an abstract finite group* Γ *such that, for every* $\gamma \in \Gamma$, *either* $\varphi(\gamma) = \pm I$ *or* q *is even (say* $q = 2s$*) and there is a unimodular complex number* λ *such that* $\varphi(\gamma)$ *is* $U(q)$*-conjugate to*

$$\begin{pmatrix} \lambda & & & & \\ & \bar{\lambda} & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \bar{\lambda} \end{pmatrix}.$$

Then φ is a CLIFFORD *representation of* Γ. *Let* Δ *be an abstract finite group which has a* CLIFFORD *representation. Then* Δ *is a* CLIFFORD *group.*

Note that a CLIFFORD representation φ of Γ gives a representation $\Gamma \xrightarrow{\varphi} U(q) \subset SO(2q)$ of Γ as CLIFFORD translations of S^{2q-1}, and a finite group Δ of CLIFFORD translations of S^n admits a CLIFFORD representation $\Delta \subset O(n+1) \subset U(n+1)$.

Lemma 1. *Let* Γ *be a noncyclic* CLIFFORD *group. Then*

(1) *Every abelian subgroup of* Γ *is cyclic.*

(2) *Given primes* p *and* q, *every subgroup of* Γ *of order* pq *is cyclic.*

(3) Γ *has a unique element of order* 2. *It generates the center of* Γ.

(4) *If* α *and* α^t *are conjugate elements of* Γ, *then* $\alpha = \alpha^t$ *or* $\alpha^{-1} = \alpha^t$.

Proof. Statements (1), (2) and the uniqueness of elements of order 2 in Γ are well known to follow from the fact that Γ has a free action on a sphere; see [2], [4] or [5], for example. As Γ has even order [2, § 10.5], (3) follows when we show that a central element $\neq 1$ of Γ has order 2.

Let φ be a CLIFFORD representation of Γ. Looking at characters, we see that the irreducible components of φ are equal and are CLIFFORD representations, so we may assume φ irreducible. If $\gamma \neq 1$ is central in Γ, SCHUR's lemma shows that $\varphi(\gamma)$ is scalar,

$$\varphi(\gamma) = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}.$$

Hence $\lambda = \bar{\lambda}$ so $|\lambda| = 1$ implies $(\gamma \neq 1)$ $\varphi(\gamma) = -I$, so that $\gamma^2 = 1$ and (3) is proved. In (4), we may assume α not central in Γ, so

$$\varphi(\alpha) = \begin{pmatrix} \lambda & & & \\ & \bar{\lambda} & & \\ & & \ddots & \\ & & & \lambda & \\ & & & & \bar{\lambda} \end{pmatrix} \quad \text{and} \quad \varphi(\alpha^t) = \varphi(\alpha)^t = \begin{pmatrix} \lambda^t & & & \\ & \bar{\lambda}^t & & \\ & & \ddots & \\ & & & \lambda^t & \\ & & & & \bar{\lambda}^t \end{pmatrix}$$

have the same eigenvalues. Thus either $\lambda = \lambda^t$ and $\alpha = \alpha^t$, or $\bar{\lambda} = \lambda^t$ and $\alpha^{-1} = \alpha^t$. Q.E.D.

Lemma 2. *Let* Γ_1 *be a normal subgroup of a* CLIFFORD *group* Γ, *assume* Γ_1 *cyclic or binary dihedral* $\mathcal{D}_m^*(m \neq 2)$, *and suppose* Γ *generated by* Γ_1 *and some element* $\tau \in \Gamma$. *Then* Γ *is cyclic or binary dihedral.*

Proof. First suppose Γ_1 cyclic of order $m : \alpha^m = 1$. $\tau\alpha\tau^{-1} = \alpha$ or α^{-1} by Lemma 1. If $\tau\alpha\tau^{-1} = \alpha$, Γ is abelian and thus cyclic by Lemma 1. Now assume $\tau\alpha\tau^{-1} = \alpha^{-1} \neq \alpha$. τ is not central in Γ so $\tau^2 \neq 1$, but τ^2 is central in Γ and Γ is not cyclic, so τ has order 4. Thus Γ is binary dihedral $\mathcal{D}_m^*$ if m is odd, $\mathcal{D}_s^*$ if $m = 2s$.

Now suppose Γ_1 binary dihedral $\mathcal{D}_m^*$ with $m \neq 2 : \alpha^m = 1 = \beta^4$, $\beta\alpha\beta^{-1} = = \alpha^{-1}$ for m odd; $\alpha^{2m} = 1$, $\beta^2 = \alpha^m$, $\beta\alpha\beta^{-1} = \alpha^{-1}$ for m even. As $m \neq 2$, the cyclic group $\{\alpha\}$ is a characteristic subgroup of Γ_1, hence a normal subgroup of Γ. Thus $\tau\alpha\tau^{-1}$ is either α or α^{-1}. β^2 is central in Γ because it has order 2, so the subgroup Γ' generated by β^2, α, and either τ or $\tau\beta$ is abelian and thus cyclic. Γ is generated by Γ' and β. $\tau\beta\tau^{-1}$ has order 4, hence is of the form $\beta\alpha^u$ or $\beta^3\alpha^u$; thus $\beta^{-1}\tau\beta$ is of the form $\alpha^u\tau$ or $\alpha^u\tau\beta^2$ and $\beta^{-1}(\tau\beta)\beta$ is of the form $\alpha^u(\tau\beta)$ or $\alpha^u(\tau\beta)\beta^2$. Thus Γ_1 is normal in Γ and we are done by the first paragraph of the proof. Q.E.D.

The next lemma depends on a procedure of H. ZASSENHAUS [5, proof of Satz 7] which depends on his result [5, Satz 6]: *Let G be a finite solvable group of order not divisible by 2^{s+1}, and which contains an element of order $2^{s-1} (s > 1)$. Then G has a normal subgroup G_1, with cyclic 2-SYLOW subgroup, such that G/G_1 is the cyclic group Z_2 of order 2, the alternating group $\mathcal{A}_4$ on 4 letters, or the symmetric group $\mathcal{S}_4$ on 4 letters.* The lemma also uses a result of G. VINCENT [2, Théorème X] which implies that a CLIFFORD group with all SYLOW subgroups cyclic is either cyclic or binary dihedral $\mathcal{D}_m^*$ (m odd).

Lemma 3. *A solvable CLIFFORD group is cyclic, binary dihedral, binary tetrahedral or binary octahedral.*

Proof. Let Γ be a solvable CLIFFORD group. We recall [2,5] that the odd SYLOW subgroups of Γ are cyclic and the 2-SYLOW subgroups are either cyclic or generalized quaternionic (binary dihedral $\mathcal{D}_m^*$ where $m > 1$ is a power of 2) because every abelian subgroup of Γ is cyclic. If the 2-SYLOW subgroups of Γ are cyclic, we are done by the above-mentioned result of VINCENT. Otherwise, Γ has order $2^s n$ with n odd and $s > 2$, and an element of order 2^{s-1}. Using the above-mentioned result of ZASSENHAUS, we take a normal subgroup Γ_1 of Γ with all SYLOW subgroups cyclic and $\Gamma/\Gamma_1 = Z_2$, $\mathcal{A}_4$ or $\mathcal{S}_4$. Note that Γ_1 is either cyclic or $\mathcal{D}_m^*$ (m odd) by the result of VINCENT.

Case 1: $\Gamma/\Gamma_1 = Z_2$. By Lemma 2, Γ is cyclic or binary dihedral.

Case 2: $\Gamma/\Gamma_1 = \mathcal{A}_4$. As the 2-SYLOW subgroups of Γ are generalized quaternionic and those of Γ/Γ_1 are $Z_2 \times Z_2$, Γ_1 must have some even order $2t$. Γ/Γ_1 is given in generators and relations by $\hat{\mu}^2 = \hat{\nu}^2 = \hat{\omega}^3 = 1$, $\hat{\mu}\hat{\nu} = \hat{\nu}\hat{\mu}$, $\hat{\omega}\hat{\mu}\hat{\omega}^{-1} = \hat{\nu}$ and $\hat{\omega}\hat{\nu}\hat{\omega}^{-1} = \hat{\nu}\hat{\mu}$. We choose representatives μ, ν, ω in Γ for $\hat{\mu}$, $\hat{\nu}$, $\hat{\omega}$ in Γ/Γ_1.

First suppose that Γ_1 is cyclic: $\alpha^{2t} = 1$. Lemma 1 shows that one of $\nu\mu$, ν and μ commutes with α, so we can assume $\mu\alpha = \alpha\mu$. Then μ and α generate a cyclic group of order $4t$, which is normal in the group Γ' generated by μ, α and ν. Lemma 2 shows that Γ' is either cyclic order $8t$ or binary dihedral $\mathcal{D}^*_{2t}$ of order $8t$. Note that Γ' is normal in Γ. If $t \neq 1$, Lemma 2 shows that Γ is binary dihedral. If $t = 1$, $\Gamma' = \mathcal{D}^*_2$ has automorphism group $\mathcal{S}_4$, so an automorphism of Γ' of order $3k$ has order 3, and thus ω^3 is central in Γ. Replacing ω by $\alpha\omega$ if necessary, we see that Γ is the binary tetrahedral group $\mathcal{T}^* : \mu^4 = 1$, $\mu^2 = \nu^2 = \alpha$, $\omega^3 = 1$, $\mu\nu\mu^{-1} = \nu^{-1}$, $\omega\mu\omega^{-1} = \nu$ and $\omega\nu\omega^{-1} = \nu\mu$.

Now suppose that $\Gamma_1 = \mathcal{D}^*_m$ (m odd): $\alpha^m = \beta^4 = 1$, $\beta\alpha\beta^{-1} = \alpha^{-1}$. The cyclic group generated by α is characteristic in Γ_1, hence normal in Γ. As before we can assume $\mu\alpha = \alpha\mu$, so μ and α generate a cyclic group, evidently normal in the group Γ' generated by μ, ν and α. By Lemma 2, Γ' is either cyclic or binary dihedral. As the order of Γ' is not 8 and Γ' is normal in the group Γ'' generated by Γ' and β, Γ'' is binary dihedral by Lemma 2. Γ'' is normal in Γ because it is generated by Γ_1, μ and ν; a final application of Lemma 2 shows that Γ is binary dihedral.

Case 3: $\Gamma/\Gamma_1 = \mathcal{S}_4$. We have a natural homomorphism $\psi : \Gamma \to \mathcal{S}_4$ of Γ onto $\mathcal{S}_4$ with kernel Γ_1, and we set $\Gamma' = \psi^{-1}(\mathcal{A}_4)$. Γ' is a normal subgroup of index 2 in Γ. By Case 2, Γ' is either binary dihedral $\mathcal{D}^*_q$ ($q \neq 2$) or binary tetrahedral $\mathcal{T}^*$. If $\Gamma' = \mathcal{D}^*_q (q \neq 2)$, Lemma 2 shows $\Gamma = \mathcal{D}^*_{2q}$. If $\Gamma' = \mathcal{T}^*$, then Γ_1 is cyclic order 2, is the center of Γ' and is the center of Γ. It is now easy to see that Γ is the binary octahedral group $\mathcal{O}^*$. Q.E.D.

It now remains only to show that a non-solvable CLIFFORD group is the binary icosahedral group $\mathcal{J}^*$. Our proof depends on the isomorphism of $\mathcal{J}^*$ with the group $SL(2,5)$ of unimodular 2×2 matrices over the field Z_5 of 5 elements, as well as a result of M. SUZUKI which implies [1, Theorem E] that a non-solvable group with every abelian subgroup cyclic has a normal subgroup isomorphic to some $SL(2,p)$ with $p > 3$ prime.

Lemma 4. *If p is a prime and $SL(2,p)$ is a CLIFFORD group, then $p = 3$ or $p = 5$.*

Proof. Let ω be a generator of the multiplicative group of non-zero elements of the field Z_p of p elements, and set

$$\nu = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ in } SL(2,p).$$

$\nu\alpha\nu^{-1} = \alpha^{(\omega^2)}$ so $\omega^2 \equiv \pm 1 \pmod{p}$ by Lemma 1. Hence $\omega^4 \equiv 1 \pmod{p}$

so, as ω has order $p-1$ in the multiplicative group, $p-1$ divides 4. Thus p is 2, 3, or 5. $p \neq 2$ because $SL(2,2)$ has several elements of order 2. Q.E.D.

Lemma 5. *Let* Γ *be a* CLIFFORD *group, and suppose that* Γ *has a normal subgroup* Γ_1 *isomorphic to* $SL(2,5)$. *Then* $\Gamma = \Gamma_1$.

Proof. Given $\gamma \in \Gamma$, let $ad(\gamma)$ denote the automorphism $\alpha \to \gamma\alpha\gamma^{-1}$ of Γ_1. Let $\gamma \in \Gamma$ and assume that $ad(\gamma)$ is an inner automorphism of Γ_1. There is a $\gamma' \in \Gamma_1$ with $ad(\gamma\gamma') = 1$, so $\gamma\gamma'$ is central in the noncyclic CLIFFORD group generated by $\gamma\gamma'$ and Γ_1. Thus $\gamma\gamma' \in \Gamma_1$, for either $\gamma\gamma' = 1$, or $\gamma\gamma'$ is the unique element of Γ of order 2, and that is contained in Γ_1. Thus $\gamma' \in \Gamma_1$ implies $\gamma \in \Gamma_1$. It follows that Γ/Γ_1 is isomorphic to a group of outer automorphisms of $SL(2,5)$. The group of outer automorphisms of $SL(2,5)$ has order 2, so Γ_1 has index 1 or 2 in Γ.

Now assume $\Gamma \neq \Gamma_1$, and let $\sigma \in \Gamma$ such that $ad(\sigma)$ is the outer automorphism of $SL(2,5) = \Gamma_1$ which is conjugation by $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. σ cannot have order 2 but $\sigma^2 = -I \in SL(2,5)$, being central in Γ. In $SL(2,5)$ we have

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As $ad(\sigma)\alpha = \beta^3$ and $\gamma\alpha\gamma^{-1} = \beta^{-1}$, β is conjugate in Γ to $\beta^{-3} = \beta^2$. As Γ is CLIFFORD, it follows that $\beta = I$ or β has order 3. This is a contradiction. Q.E.D.

Lemma 6. *Let* Γ *be a non-solvable* CLIFFORD *group. Then* Γ *is a binary* icosahedral group $\mathcal{J}^*$.

Proof. Lemmas 4 and 5 and the result mentioned of SUZUKI [1, Theorem E] show $\Gamma \cong SL(2,5)$. But $SL(2,5) \cong \mathcal{J}^*$. Q.E.D.

Theorem 1 is an immediate consequence of Lemmas 3 and 6.

III. Representations of CLIFFORD groups

Given an abstract CLIFFORD group Γ, we will find the faithful orthogonal representations $\varphi : \Gamma \to O(n+1)$ such that $\varphi(\Gamma)$ is a group of CLIFFORD translations of S^n. This will provide proofs of Theorems 2 and 3.

Lemma 7. Let γ *generate a cyclic group* Γ *of finite order* q *and let* $\psi : \Gamma \to O(n+1)$ *be a faithful orthogonal representation such that* $\psi(\Gamma)$ *is a group of* CLIFFORD *translations of* S^n. *If* $q \leq 2$, $\psi(\Gamma) = \{I\}$ *or* $\{\pm I\}$. *If*

$q > 2$, then $n + 1 = 2s$ and ψ is $O(n+1)$-equivalent to a sum of s copies of one of the representations given by

$$\sigma_t(\gamma) = R(t/q) = \begin{pmatrix} \cos(2\pi t/q) & \sin(2\pi t/q) \\ -\sin(2\pi t/q) & \cos(2\pi t/q) \end{pmatrix}, \quad t \text{ prime to } q.$$

Conversely, $\{I\}$, $\{\pm I\}$ and $O(2s)$-conjugates of images of sums of s copies of a σ_t are groups of CLIFFORD translations.

Proof. The statement for $q \leq 2$ is clear; assume $q > 2$. As $\psi(\gamma)$ is a CLIFFORD translation of order q, it has $(n + 1 = 2s)$ s eigenvalues $\exp(2\pi i t/q)$ and s eigenvalues $\exp(-2\pi i t/q)$, where t is prime to q. Thus $\psi(\gamma)$ is $O(n+1)$-conjugate to $\begin{pmatrix} R(t/q) & & \\ & \ddots & \\ & & R(t/q) \end{pmatrix}$, so ψ is $O(n+1)$-equivalent to $\sigma_t \oplus \cdots \oplus \sigma_t$. The rest is clear. Q.E.D.

Lemma 8. *An irreducible CLIFFORD representation φ of a non-cyclic group Γ has degree 2.*

Proof. Γ is binary polyhedral. Suppose first that $\Gamma = \mathscr{D}_m^*$. $m > 1$ as $\mathscr{D}_1^*$ is cyclic. $\mathscr{D}_m^*$ has $m + 3$ conjugacy classes of elements, hence $m + 3$ inequivalent irreducible unitary representations, say of degrees d_j. The commutator subgroup has index 4 so we may assume $d_1 = d_2 = d_3 = d_4 = 1$, and the other $d_j > 1$. $\Sigma d_j^2 = 4m$ as $\mathscr{D}_m^*$ has order $4m$, so each d_j is 1 or 2. φ has even degree as Γ is non-cyclic, so the degree of φ is 2.

Now suppose $\Gamma = \mathscr{T}^*$ binary tetrahedral group. As above, we see that the degrees of the irreducible representations are 1, 2 and 3. As φ has even degree, it has degree 2.

Suppose that $\Gamma = \mathscr{O}^*$. $\mathscr{O}^*$ has a subgroup $\mathscr{T}^*$ of index 2 such that φ is irreducible if and only if its restriction to $\mathscr{T}^*$ is irreducible. Hence φ has degree 2.

Finally, suppose that $\Gamma = \mathscr{J}^*$. $\mathscr{J}^*$ has 9 conjugacy classes, order 120, and presentation: $\alpha^{10} = 1$, $\alpha^5 = \gamma^3$, $\gamma\alpha\gamma^{-1} = \alpha^{-1}\gamma$. As φ has even degree $q = 2r$, $\varphi(\alpha)$ has r eigenvalues $\exp(2\pi i\, v/10)$ and r eigenvalues $\exp(-2\pi i\, v/10)$, for some integer v prime to 10. Thus the character χ_φ of φ is determined on 6 conjugacy classes by r and v: $\chi_\varphi(1) = 2r$, $\chi_\varphi(\alpha) = 2r\cos(\pi v/5)$, $\chi_\varphi(\alpha^2) = 2r\cos(2\pi v/5)$, $\chi_\varphi(\alpha^3) = 2r\cos(3\pi v/5)$, $\chi_\varphi(\alpha^4) = 2r\cos(4\pi v/5)$ and $\chi_\varphi(\alpha^5) = -2r$.

Let b be an eigenvalue of $\varphi(\gamma)$. As $\varphi(\gamma)^3 = \varphi(\alpha)^5 = -I$, b is a cube root of -1. $\varphi(\gamma) \neq I$ so $b = \exp(2\pi i/6)$ or $b = \exp(-2\pi i/6)$. Thus $\chi_\varphi(\gamma) = r(b + \bar{b}) = 2r \cdot \cos(\pi/3) = r$ and $\chi_\varphi(\gamma^2) = r(b^2 + \bar{b}^2) = 2r \cdot \cos(2\pi/3)$

$= -r$. Finally χ_φ is zero on the conjugacy class consisting of elements of order 4, so χ_φ is determined on all 9 conjugacy classes—hence is completely determined—by r and v. We notice that χ_φ is precisely r times the character of one of the representations $\mathcal{T}^* \subset Spin(3) = SU(2) \subset U(2)$, so the irreducibility of φ implies $r = 1$. Q.E.D.

We remark that we have just seen: *If* $\varphi : \mathcal{T}^* \to U(q)$ *is an irreducible* CLIFFORD *representation, then* $q = 2$ *and* φ *is equivalent to one of the representations* $\mathcal{T}^* \subset Spin(3) = SU(2) \subset U(2)$. In fact we have

Lemma 9. *Let* $\varphi : \Gamma \to U(q)$ *be an irreducible* CLIFFORD *representation of a noncyclic group. Then* $q = 2$, Γ *is binary polyhedral, and* φ *is equivalent to one of the representations* $\Gamma \subset Spin(3) = SU(2) \subset U(2)$.

Proof. We need only check the equivalence class of φ for $\Gamma = \mathcal{D}^*_m (m > 1)$, $\mathcal{T}^*$ and $\mathcal{O}^*$. As with $\mathcal{T}^*$, we calculate the character χ_φ and see that it is the same as the character of one of the representations $\Gamma \subset Spin(3) = SU(2) \subset U(2)$. Q.E.D.

Proof of Theorem 3. Given a finite group Γ of CLIFFORD translations of $S^n \subset R^{n+1}$, we will show the centralizer G of Γ in $O(n+1)$ to be transitive on S^n. This is obvious if Γ is cyclic of order 1 or 2, so we first suppose Γ cyclic of order q $(q > 2)$. Let $2s = n + 1$, as $n + 1$ is even; let $\Gamma' \subset U(s)$ be the cyclic group generated by $\exp(2\pi i\ 1/q) I$. Γ' is central in $U(s)$ so its centralizer in $U(s)$ is transitive on the unit sphere in complex euclidean space C^s. By Lemma 7 we can assume that Γ' goes onto Γ, and its centralizer $U(s)$ into G, under the inclusion $U(s) \subset O(n+1)$ induced by an isometry of C^s onto R^{n+1} which sends the unit sphere of C^s onto S^n. Hence G is transitive on S^n.

Now suppose Γ noncyclic. Γ is isomorphic to a binary polyhedral group $\mathcal{P}^*$. Let K be the algebra of quaternions and let K' be the multiplicative group of unit quaternions. Under the inclusion and identification $\mathcal{P}^* \subset Spin(3) = K'$, we'll view $\mathcal{P}^*$ as a subgroup of K'. Let $K^s (4s = n + 1)$ be a *left* quaternionic euclidean space, so that K (hence K', hence $\mathcal{P}^*$) acts on K^s by *left* scalar multiplication and the symplectic group $Sp(s)$ acts on the right. The action of $Sp(s)$ commutes with that of $\mathcal{P}^*$, and $Sp(s)$ is transitive on the unit sphere of K^s. By Lemma 9 we can assume that $\mathcal{P}^*$ goes onto Γ, and $Sp(s)$ goes into G, under the inclusions $K' \subset O(n+1)$ and $Sp(s) \subset O(n+1)$ induced by an isometry of K^s onto R^{n+1} which sends the unit sphere of K^s onto S^n. Hence G is transitive on S^n. Q.E.D.

Proof of Theorem 2. By Lemmas 7 and 9, all that remains to be shown is that the images of the representations of Theorem 2 are actually groups of

CLIFFORD translations. Let $\Gamma \subset O(n+1)$ be the image of one of those representations. In the proof of Theorem 3, we saw that the centralizer G of Γ in $O(n+1)$ is transitive on S^n. Now let $\gamma \in \Gamma$, let $x, y \in S^n$, and let δ be the distance function on S^n determined by its RIEMANNian metric. There is an element $g \in G$ with $g(x) = y$. Hence

$$\delta(x, \gamma x) = \delta(gx, g\gamma x) = \delta(y, \gamma g x) = \delta(y, \gamma y)$$

so γ is a CLIFFORD translation of S^n. Q.E.D.

IV. Homogeneous space-forms

We will prove Theorem 4. Theorem 2 establishes the equivalence of (1) and (2), Theorem 3 shows that (1) implies (3), and the proof of Theorem 3 shows that (3) implies (1). It is obvious that (3) implies (4): the centralizer of Γ induces a transitive group of isometries of S^n/Γ. Finally, (4) implies (3) is known [3, Théorème 1]. Q.E.D.

We remark that Theorems 3 and 4 provide a proof of a result [3, Théorème 6] previously announced without proof in the *Comptes rendus*, and that Theorems 1 and 4 provide an alternative proof of the classification [3, Théorème 5] of the RIEMANNian homogeneous spherical space-forms.

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