THE MANIFOLDS COVERED BY A RIEMANNIAN HOMOGENEOUS MANIFOLD.*

By Joseph A. Wolf.[1]

Introduction. The sphere is known to be the universal covering for complete connected Riemannian manifolds of constant positive curvature. More precisely, if M is an n-dimensional complete connected Riemannian manifold of constant sectional curvature $k^2 > 0$ with $k > 0$, and if S^n is the sphere of radius k^{-1} in Euclidean space R^{n+1}, with the induced metric, then there is a covering of M by S^n such that the covering projection is a local isometry. Because of this phenonmenon, the complete connected Riemannian manifolds of constant positive curvature are called the "spherical space-forms." In his thesis, G. Vincent [14] attempted to classify them. Following this line of investigation, we take a compact connected Riemannian homogeneous manifold M and ask which Riemannian manifolds admit M as a Riemannian covering manifold. In Chapter I, this problem is reduced to a problem on discrete subgroups of compact Lie groups:

Given a compact Lie group G and a closed subgroup K, find all finite subgroups Γ of G such that Γ meets the union of the conjugates of K only at the identity element of G.

For the most part we restrict our attention to the case where Γ lies in the identity component of G, or, equivalently, where G is connected. In Chapter II we obtain some bounds on the ranks of abelian subgroups of Γ, and see that the problem of classifying these groups Γ is inaccessible unless $\text{rank.}\, G - \text{rank.}\, K \leqq 1$.

Chapter II ends with a sharper bound on the ranks of abelian subgroups of Γ, in case $\text{rank.}\, G - \text{rank.}\, K = 1$, which implies that every abelian subgroup of Γ is cyclic if the semisimple part of G is simply connected and Γ lies in the identity component of G. We remark that H. Zassenhaus [16] and M. Suzuki [13] have given a complete classification of the finite groups

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with all abelian subgroups cyclic. Under certain conditions on G, K and the order of Γ (Corollary 10.1) it follows that Γ itself is cyclic.

In Chapter III we obtain an arithmetic criterion (Theorem 6), assuming G and K connected, for an arbitrary given finite cyclic subgroup of G to act freely on G/K. This criterion involves the Weyl group of G and the position of K in G. It is applied to an arbitrary finite subgroup Σ of G by finding cyclic subgroups of Σ such that every element of Σ is $\mathrm{ad}(G)$-conjugate to an element of one of these cyclic subgroups. Applying to the case where G is a classical group we obtain some information on elements of order 2 in Γ, assuming $\mathrm{rank}.\, G - \mathrm{rank}.\, K = 1$. Finally, we sharpen the bound on the ranks of abelian subgroups of Γ in case G is a special orthogonal group. In Chapter V we apply the arithmetic criterion again to the case where G is a classical group. Chapter VI is a consideration of the case where $\mathrm{rank}.\, G = \mathrm{rank}.\, K$, i.e., the Euler characteristic $\chi(M) \neq 0$, and shows that M covers only a finite number, up to isometry, of Riemannian manifolds.

Our problem can be considered as a generalization of the classical Clifford-Klein problem of finding all spherical space-forms, in that we have replaced the sphere by an arbitrary (for Theorems 1 to 4 and Theorem 6), or at least more general, compact connected Riemannian homogeneous manifold. Another direction of generalization is that of considering finite groups which admit a free topological action on a space similar in some way to a sphere. In this regard, we mention some of the work of P. A. Smith [11], P. E. Conner [5], J. Milnor [8] and A. Heller [6].

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Chapter I. Reduction to a problem on Lie groups.

I. 1. Covering spaces. In order to establish notation and terminology we will recall some well-known facts and definitions concerning covering spaces. All spaces will be Hausdorff and all maps will be continuous.

A *covering* is a map $p: X \to X'$ of arcwise connected, locally simply connected spaces where every element of X' has a neighborhood U such that p maps each component of $p^{-1}(U)$ homeomorphically onto U. $p^{-1}(x')$ is the *fibre* over $x' \in X'$. All fibres have the same cardinality, the *multiplicity* of the covering. A *finite covering* is a covering of finite multiplicity. p induces a monomorphism of fundamental groups, and the covering is *normal* if $p \cdot \pi_1(X)$ is a normal subgroup of $\pi_1(X')$. This is independent of choice of basepoints. If H is a subgroup of $\pi_1(X')$, there is a covering $q\colon Y \to X'$ and a choice of basepoint in Y such that $q \cdot \pi_1(Y) = H$.

THEOREM 1. *If $p\colon X \to X'$ is a finite covering, there is a finite normal covering $q\colon X'' \to X$, where $pq\colon X'' \to X'$ is a finite normal covering.*

Proof. The multiplicity of p being equal to the index of $H = p \cdot \pi_1(X)$ in $\pi_1(X')$, the normalizer N of H in $\pi_1(X')$ has finite index in $\pi_1(X')$. Consequently ([7], p. 82) H has only a finite number of conjugates in $\pi_1(X')$, so ([7], p. 62) the intersection J of the conjugates of H has finite index in $\pi_1(X')$. Let $q\colon X'' \to X$ be a covering, where $q \cdot \pi_1(X'') = p^{-1}(J)$, and the normality conditions follows from the construction of J. *QED.*

An action of a group Γ on a space X is *effective* if the identity element of Γ is the only element inducing the identity transformation of X, is *free* if the identity element of Γ is the only element which leaves fixed a point of X, and is *properly discontinuous* if every point of X has a neighborhood which does not meet any of its transforms under Γ. If X is compact, the action of Γ is properly discontinuous if and only if Γ is finite and acts freely. The set $\Gamma(x)$, the *orbit* of a point x of X, is the set of images of x under Γ. The space X/Γ of orbits is given the quotient topology for the natural projection $x \to \Gamma(x)$; the natural projection $X \to X/\Gamma$ is a covering if and only if Γ acts properly discontinuously on X.

A *deck transformation* of a covering $p\colon X \to X'$ is a homeomorphism $\gamma\colon X \to X$, where $p \cdot \gamma = p$. The group Γ of all deck transformations acts properly discontinuously on X, and ([12], § 14) p is a normal covering if and only if Γ is simply transitive on each fibre, i. e., if and only if $p\colon X \to X'$ is a principal bundle with group Γ, i. e., if and only if $X' = X/\Gamma$.

A *Riemannian covering* is a covering $p\colon M \to M'$, where M and M' are Riemannian manifolds and p is a local isometry. If just one of M and M' is a Riemannian manifold, the requirement that p be a local isometry gives a Riemannian structure to the other and makes p a Riemannian covering. We easily see that a deck transformation is an isometry of M, because p is a local isometry.

A *Riemannian homogeneous manifold* is a Riemannian manifold whose group of isometries is transitive.

THEOREM 2. *If* $q: M'' \to M$ *is a Riemannian covering and* M *is Riemannian homogeneous, then* M'' *is Riemannian homogeneous.*

Proof. A one-parameter group of isometries of M is a homotopy and can be lifted to M'' by the covering homotopy theorem. The lifted homotopy consists of isometries of M'' because q is a local isometry. It follows that the group of isometries of M'' is locally transitive and therefore transitive. *QED.*

I. 2. Reduction to a problem on discrete subgroups of compact Lie groups.

THEOREM 3. *Let* M *be a compact connected Riemannian homogeneous manifold,* G *the group of isometries of* M, K *an isotropy subgroup of* G, *and* Γ *a subgroup of* G. *Then* Γ *is a properly discontinuous group of isometries of* M *if and only if* Γ *is finite and* $\Gamma \cap \mathrm{ad}(G)K = 1$, *where* 1 *is the identity element of* G *and* $\mathrm{ad}(G)K$ *is the set of all* $\mathrm{ad}(g)k = gkg^{-1}$ *with* $g \in G$ *and* $k \in K$.

Proof. M is compact and Γ is a group of isometries of M, so Γ is a properly discontinuous group of isometries of M if and only if Γ is finite and acts freely on M. G is transitive on M, so the isotropy subgroups of G are the subgroups $\mathrm{ad}(g)K$ with $g \in G$. Γ acts freely on M if and only if it meets each isotropy subgroup only at 1. Hence Γ acts freely if and only if $\Gamma \cap \mathrm{ad}(G)K = 1$. *QED.*

Using Theorems 1, 2 and 3, we see that the original problem

Given a compact connected Riemannian homogeneous manifold M, *find all Riemannian manifolds which admit a Riemannian covering by* M.

is reduced to the problem

Given a compact Lie group G *and a closed subgroup* K, *find all finite subgroups* Γ *of* G *such that* $\Gamma \cap \mathrm{ad}(G)K = 1$.

by taking G to be the group of isometries of a finite Riemannian covering manifold of M and K to be an isotropy subgroup of G. We then note that G and K are both compact, each has only a finite number of components, and K meets every component of G.

Chapter II. Necessary conditions for a finite group to act as a properly discontinuous group of isometries of a compact connected Riemannian homogeneous manifold.

Given a compact Lie group G and a closed subgroup K, we will find some necessary conditions on finite subgroups Γ of G for $\Gamma \cap \mathrm{ad}(G)K = 1$. These will involve the ranks of G, K, and some subgroups of Γ.

The *rank* of a finite abelian group B is the minimal number of factors in a direct product decomposition of B into cyclic groups, and is denoted rank. B. For example, if p is a prime, the elementary p-group with p^h elements, $(Z_p)^h = Z_p \times \cdots \times Z_p$, the direct product of h copies of the cyclic group of order p, has rank h. The rank of B is the maximum of the ranks of its elementary p-subgroups, and B is cyclic if and only if rank. $B \leqq 1$. If p is a prime, the *p-rank* of B, denoted p-rank. B, is the rank of a p-Sylow subgroup of B. It is the maximal integer h such that B has a subgroup isomorphic to $(Z_p)^h$.

The *rank* of a compact Lie group H, denoted rank. H, is, as usual, the common dimension of the maximal toral subgroups of H.

II. 1. A bound on the ranks of certain abelian subgroups.

THEOREM 4. *Let K be a closed subgroup of a compact Lie group G.*

1. *Given a finite subgroup Γ of G such that $\Gamma \cap \mathrm{ad}(G) = 1$ and an abelian subgroup B of Γ which lies in a torus of G, we have* rank. $B \leqq$ rank. G — rank. K.

2. *The above bound is the best possible in the sense that there is a positive integer $m(G,K)$ such that, given a finite abelian group A with* rank. $A \leqq$ rank. G — rank. K *and $m(G,K)$ prime to the order of A, a torus of G has a subgroup A' which is isomorphic to A and such that $A' \cap \mathrm{ad}(G)K = 1$.*

Proof. Let T' be a maximal torus of K, T a maximal torus of G which contains T', $n =$ rank. G and $k =$ rank. K. We replace B by a conjugate which lies in T and still have $B \cap \mathrm{ad}(G)K = 1$, hence $B \cap T' = 1$. It follows that the canonical map of T onto the $(n-k)$-torus T/T' maps B monomorphically. Since a finite subgroup of an $(n-k)$-torus has rank at most $n-k$, we conclude rank. $B \leqq n-k$. This proves the first statement.

Let K_0 be the identity component of K, G_0 the identity component of G, $W = \{w_1, \cdots, w_q\}$ an enumeration of the Weyl group of G_0 with respect to T, and $\{a_1, \cdots, a_t\}$ a set of automorphisms of G_0 which preserve T such that the automorphism group $\mathrm{ad}(G)$ of G_0 can be written as the union of the $a_i \cdot \mathrm{ad}(G_0)$. Two elements of T are $\mathrm{ad}(G_0)$-conjugate if and only if they are W-conjugate, and it follows that an element of T lies in $\mathrm{ad}(G)K_0$ if and only if it lies in one of the $T_{ij} = a_i(w_j(T'))$. As there are only a finite number of the k-tori T_{ij}, there exists an $(n-k)$-torus V in T which intersects each T_{ij} in a finite group. Let $m(G,K)$ be the product of the primes occurring in the orders of these finite groups and in the order b of K/K_0. Let $\beta \in V$ have order prime to $m(G,K)$ and lie in $\mathrm{ad}(G)K$. Then $\beta^b \in \mathrm{Ad}(G)K_0$, so $\beta^b \in T_{ij}$ for some (i,j). Since the order of β^b is also prime to $m(G,K)$, this implies, by the definition of $m(G,K)$, that $\beta^b = 1$. Since the order of β is prime to b, this implies $\beta = 1$.

We can find a subgroup A' of V which is isomorphic to A because V is an $(n-k)$-torus and A is a finite abelian group of rank at most $n-k$. The considerations above show that $A' \cap \mathrm{ad}(G)K = 1$ if the order of A, hence of A', is prime to $m(G,K)$. *QED.*

In Chapter III we will see examples where K, and even G, is connected and $m(G,K)$ must be even, hence $m(G,K) > 1$.

II. 2. The work of A. Borel on torsion and subgroups which lie in a torus. A. Borel has proved ([1], Chapter XII) that if G is a compact connected Lie group with classifying space B_G, p is a prime, and the integral cohomology ring $\boldsymbol{H}^*(B_G, \boldsymbol{Z})$ has no p-torsion, then every elementary p-subgroup (subgroup isomorphic to some $(Z_p)^h$) of G lies in a torus of G. A case by case check proves the converse. Borel has also shown that $\boldsymbol{H}^*(G,\boldsymbol{Z})$ has p-torsion if and only if $\boldsymbol{H}^*(B_G,\boldsymbol{Z})$ has p-torsion, using known results and checking the case $p=5$ for $\boldsymbol{E}_6$, $p=5$ and $p=7$ for $\boldsymbol{E}_7$ and $p=7$ for $\boldsymbol{E}_8$. The summary of the situation is that the following are equivalent:

1. $\boldsymbol{H}^*(G,\boldsymbol{Z})$ has no p-torsion.

2. $\boldsymbol{H}^*(B_G,\boldsymbol{Z})$ has no p-torsion.

3. G_{ss} being the semisimple part of G, $\boldsymbol{H}^*(G_{ss},\boldsymbol{Z})$ has no p-torsion.

4. $\pi_1(G_{ss})$ has order prime to p, and, if G' is a simple factor of the universal covering group of G_{ss}, then $\boldsymbol{H}^*(G',\boldsymbol{Z})$ has no p-torsion.

Finally, if H is a compact, connected, simple, simply connected Lie group, then $\boldsymbol{H}^*(H,\boldsymbol{Z})$ has p-torsion in precisely these cases:

1. $p=2$ and $H = \boldsymbol{E}_8, \boldsymbol{E}_7, \boldsymbol{E}_6, \boldsymbol{F}_4, \boldsymbol{G}_2$, or $\boldsymbol{Spin}(n)$ with $n \geqq 7$.

2. $p = 3$ and $H = \boldsymbol{E}_8$, $\boldsymbol{E}_7$, $\boldsymbol{E}_6$, or $\boldsymbol{F}_4$.

3. $p = 5$ and $H = \boldsymbol{E}_8$.

An immediate consequence of Theorem 4 and this work of A. Borel is:

THEOREM 4'. *Let G be a compact Lie group of rank n, K a closed subgroup of* rank k, *and Γ a finite subgroup of the identity component G_0 of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$. Then if p is a prime for which $\boldsymbol{H}^*(G_0, \boldsymbol{Z})$ has no p-torsion, every abelian subgroup of Γ has* p-rank $\leqq n - k$. *If $\boldsymbol{H}^*(G_0, \boldsymbol{Z})$ is torsion-free, every abelian subgroup of Γ has* rank $\leqq n - k$.

It is now clear that the problems in applying Theorem 4 are closely related to the existence of p-torsion in G. This is of two sorts—p-torsion from the fundamental group of G and p-torsion from the simply connected versions of the simple factors of G. Finally, we can only hope to classify our groups Γ in case rank. G — rank. $K \leqq 1$, due to the present rate of the theory of finite groups. We will see, however, that p-torsion in G is of little importance in case rank. $G =$ rank. K, and that only the p-torsion from $\pi_1(G)$ is of importance in case rank. G — rank. $K = 1$.

In addition to the results mentioned above, A. Borel has shown [2]

Let G be a compact connected Lie group, $\pi_1(G)$ torsion-free and $x \in G$. The centralizer of x in G is connected.

As the identity component of the centralizer of x in G is the union of the maximal tori of G which contain x, it easily follows, if $\pi_1(G)$ is torsion-free, that every abelian subgroup of G with 2 generators lies in a torus of G. We will depend heavily on this result of A. Borel in the next section.

II. 3. A further bound on the ranks of abelian subgroups. The main purpose of this section is to prove:

THEOREM 5. *Let G be a compact connected Lie group, K a closed subgroup with* rank. G — rank. $K = 1$, *Γ a finite subgroup of G with $\Gamma \cap \mathrm{ad}(G)K = 1$, p a prime, and $h(p)$ the* p-rank *of $\pi_1(G)$. Then every abelian subgroup of Γ has* p-rank $\leqq h(p) + 1$. *If $h(p) = 2$, then every abelian subgroup of Γ has* p-rank $\leqq 2$.

We will first need a few lemmas. The first two of these lemmas are known, but not well-known, so it seems best to write them out.

LEMMA 5.1. *Let G be a compact connected Lie group, G_{ss} the semi-*

simple part of G, *and* $Z(G)_0$ *the identity component of the center of* G. *There is a covering* $\phi\colon G_{ss}\times Z(G)_0\to G$ *given by* $\phi(g,t)=gt$. ϕ *is an epimorphism of compact connected Lie groups and the kernel,* ker. ϕ, *of* ϕ *is the set of all* (g,g^{-1}) *with* $g\in G_{ss}\cap Z(G)_0$.

Proof. G_{ss} has finite center and $G=G_{ss}\cdot Z(G)_0$. Note that ker. ϕ is finite and lies in the center of $G_{ss}\times Z(G)_0$.

LEMMA 5.2. *Let* G *be a compact connected Lie group. As a topological space,* G *is homeomorphic with* $G_{ss}\times Z(G)_0$. *As* $Z(G)_0$ *is a torus, it follows that the torsion subgroup of* $\pi_1(G)$ *is isomorphic to* $\pi_1(G_{ss})$, *and in particular* p-rank. $\pi_1(G)=p$-rank. $\pi_1(G_{ss})$ *for every prime* p.

Proof. We proceed by induction on the dimension s of the torus $Z(G)_0$, and the lemma is trivial if $s=0$. If $s=1$, we consider the principal bundle $G\to G/G_{ss}=Z(G)_0/(G_{ss}\cap Z(G)_0)$ with connected group G_{ss} and base which is a 1-sphere. Since this is a trivial bundle ([12], p. 99), G is homeomorphic to $G_{ss}\times(\text{1-sphere})$, which is homeomorphic to $G_{ss}\times Z(G)_0$. Now assume $s>1$. Take a subgroup H of G which is generated by G_{ss} and an $(s-1)$-torus T in $Z(G)_0$. H is homeomorphic to $G_{ss}\times T$ by induction. As before, the principal fibre bundle $G\to G/H$ tells us that G is homeomorphic to $H\times(\text{1-sphere})$, so G is homeomorphic to $G_{ss}\times Z(G)_0$.

Now note that $\pi_1(G)=\pi_1(G_{ss})\times\pi_1(\text{torus})$ and $\pi_1(\text{torus})$ is a free abelian group. *QED.*

LEMMA 5.3. *Let* G *be a compact connected Lie group,* p *a prime, and* $h(p)$ *the* p-rank *of* $\pi_1(G)$. *Let* $\phi\colon G_{ss}\times Z(G)_0\to G$ *be the covering given by* $\phi(g,t)=gt$, $\mu\colon G'\to G_{ss}$ *the universal covering of* G_{ss}, *and* $\theta\colon G'\times Z(G)_0\to G$ *the composition* $\phi\cdot(\mu\times 1)$. *Then every* $(Z_p)^{h(p)+2}$ *in* G *contains a* $(Z_p)^2$ *which is the* θ-*image of an abelian subgroup of* $G'\times Z(G)_0$. *If* $h(p)=2$, *every* $(Z_p)^3$ *in* G *contains a* $(Z_p)^2$ *which is the* θ-*image of an abelian subgroup of* $G'\times Z(G)_0$.

Proof. Let $\phi\beta_1,\cdots,\phi\beta_{h(p)+2}$ generate a $(Z_p)^{h(p)+2}$ in G, $N=\text{ker.}\,\phi$, and $\beta_j=(b_j,t_j)\in G_{ss}\times Z(G)_0$. As $[\beta_i,\beta_j]\in N$, where $[\ ,\ ]$ is the ordinary commutator, we know from the form of the elements of N that $[b_i,b_j]=1\in G_{ss}$. As $\beta_j^p\in N$ we also know that b_j^p is central in G_{ss}. Now take elements $c_j\in G'$ with $\mu(c_j)=b_j$. As b_j^p is central in G_{ss}, c_j^p is central in G'. Since the b_j commute with each other, the commutators $[c_i,c_j]$ lies in ker. μ and are thus central in G'.

Let u and v be elements of a group H such that $w=[u,v]$ commutes with u. $uv = wvu$ and we assume $u^{n-1}v = w^{n-1}vu^{n-1}$ by induction on n. Hence $u^n v = uu^{n-1}v = uw^{n-1}vu^{n-1} = w^{n-1}uvu^{n-1} = w^{n-1}wvuu^{n-1} = wv^n u^n$ in general. In other words, $[u^n, v] = [u, v]^n$ if u commutes with $[u, v]$. Since $[c_i, c_j]$ is central in G', it commutes with c_i, and consequently $[c_i, c_j]^p = [c_i^p, c_j]$ which equals 1 because c_i^p is central in G'.

$M = \ker. \mu$ is isomorphic to $\pi_1(G_{ss})$ so, by Lemma 5.2 and the definition of $h(p)$, M does not contain a $(Z_p)^{h(p)+1}$. Now set $y_j = [c_{h(p)+2}, c_j]$ for $1 \leqq j \leqq h(p) + 1$. We have just seen that $y_j^p = 1$. As $\mu[c_i, c_j] = [\mu c_i, \mu c_j] = [b_i, b_j] = 1$, $y_j \in M$. It follows that the y_j generate an elementary p-subgroup Y in M of rank at most $h(p)$. Since there are $h(p) + 1$ of the y_j, we have a relation $y_1^{v_1} y_2^{v_2} \cdots y_{h(p)+1}^{v_{h(p)+1}} = 1$, v_j integers not all divisible by p. Set $c = c_1^{v_1} c_2^{v_2} \cdots c_{h(p)+1}^{v_{h(p)+1}}$ and $t = t_1^{v_1} t_2^{v_2} \cdots t_{h(p)+1}^{v_{h(p)+1}}$ and notice that the fact that $[c_i, c_j c_k] = [c_i, c_j] \cdot [c_i, c_k]$, a consequence of the fact that each $[c_i, c_q]$ is central in G', gives us $[c_{h(p)+2}, c] = 1$. We now have elements $\sigma = (c, t)$ and $\tau = (c_{h(p)+2}, t_{h(p)+2})$ in $G' \times Z(G)_0$ such that σ and τ generate an abelian group in $G' \times Z(G)_0$ whose θ-image is a $(Z_p)^2$ inside our original $(Z_p)^{h(p)+2}$ in G.

Now suppose $h(p) = 2$. As before, we have a $(Z_p)^3$ in G generated by $\phi\beta_1, \phi\beta_2, \phi\beta_3$; we have $\beta_j = (b_j, t_j)$; and we have $\mu(c_j) = b_j$. We set $y_1 = [c_1, c_2]$, $y_2 = [c_2, c_3]$, $y_3 = [c_3, c_1]$ and the y_j generate an elementary p-subgroup of M which, by definition of $h(p)$, has rank $\leqq 2$. This gives us a relation of the form $y_1^{v_1} y_2^{v_2} y_3^{v_3} = 1$, where the v_j are integers not all divisible by p. We can assume that v_1 is not divisible by p, so there are integers r and s such that $y_1 = y_2^r y_3^s$. If p divides s, $[c_1 c_3^r, c_2] = 1$. If p doesn't divide s, there is an integer u with $us \equiv -r \pmod p$, and $[c_1 c_2^u, c_2 c_3^s] = 1$. In either case we get an abelian group from the c_j whose θ-image is a $(Z_p)^2$ inside our original $(Z_p)^3$ in G. *QED.*

Proof of Theorem 5. Let B be an abelian subgroup of Γ with p-rank. $B > h(p) + 1$. Then B contains a $(Z_p)^{h(p)+2}$. By Lemma 5.3 we have a $(Z_p)^2$ in B which is the θ-image of an abelian subgroup S of $G' \times Z(G)_0$. S is generated by two elements. By a theorem of A. Borel, mentioned in § II.2, S lies in a torus of $G' \times Z(G)_0$, so $\theta(S)$ lies in a torus of G. Hence Γ contains a $(Z_p)^2$ which lies in a torus of G. As rank. G — rank. $K = 1$ and $\Gamma \cap \mathrm{ad}(G)K = 1$, this contradicts Theorem 4. The proof that $h(p) = 2$ implies p-rank. $B \leqq 2$ is identical. *QED.*

COROLLARY 5.1. *Let G be a compact connected Lie group which has torsion-free fundamental group, i.e., such that G_{ss} is simply connected. Let K be a closed subgroup of G such that* rank. G — rank. $K = 1$ *and let Γ be a* *finite subgroup of G such that* $\Gamma \cap \mathrm{ad}(G)K = 1$. *Then every abelian subgroup of* Γ *is cyclic. The odd Sylow subgroups of* Γ *are cyclic and the 2-Sylow subgroups are cyclic or generalized quaternionic, i.e., given by two generators A and B with the relations*

$$A^{2^{a-1}} = 1, \quad A^{2^{a-2}} = B^2, \quad BAB^{-1} = A^{-1}, \quad a \text{ integer}, \quad a > 2.$$

Proof. Let V be an abelian subgroup of Γ and write V as a product of p-subgroups. By Theorem 5, each of these p-subgroups has rank $\leqq 1$, hence is cyclic. Since V is a product of cyclic subgroups of pairwise relatively prime orders, it follows that V is cyclic. The rest is known ([14], Chapter I) to follow. *QED.*

Chapter III. An arithmetic criterion and first application to the classical groups.

III. 1. Angular parameters and the arithmetic criterion. Let G be a compact connected Lie group of rank n, K a closed connected subgroup of rank k, T a maximal torus of G which contains a maximal torus T' of K, $W = \{w_1, \cdots, w_q\}$ an enumeration of the Weyl group of G relative to T, and $T_i = w_i(T')$. We choose an *integral basis* of the Lie algebra T of T, i.e., an ordered basis $\mathsf{X} = \{X_1, \cdots, X_n\}$ of T such that $\exp(\sum_s a_s X_s) = 1$ if and only if each a_s is an integer. The Lie algebra T_i of T_i can be described as the set of all elements $\sum_s a_s X_s$ of T such that $\sum_s v_{ijs} a_s = 0$ for $1 \leqq j \leqq n - k$, where each $\{v_{ij1}, \cdots, v_{ijn}\} = V_{ij}$ is an ordered set of relatively prime integers. The v_{ijs} can be chosen to be rational because each T_i is closed in T, and the obvious normalization transforms each V_{ij} into a set of relatively prime integers.

DEFINITION. *The* $q(n-k)$ *ordered sets* V_{ij} *of relatively prime integers are the angular parameters of K in G relative to* X.

We remark that, for a given K and G, the choice of X does not specify the angular parameters of K in G uniquely.

Let Γ be a finite subgroup of G. We choose cyclic subgroups $\{\gamma_t\} = \Gamma_t$ of Γ such that every element of Γ is $\mathrm{ad}(G)$-conjugate to an element of one of the Γ_t. Then $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\Gamma_t \cap \mathrm{ad}(G)K = 1$ for each t. The angular parameters of K in G give us an arithmetic criterion for $\Gamma_t \cap \mathrm{ad}(G)K = 1$:

THEOREM 6. *Let G be a compact connected Lie group of* rank n, *K a* *closed connected subgroup of* rank k, $V_{ij} = \{v_{ij1}, v_{ij2}, \cdots, v_{ijn}\}$ *the angular parameters of* K *in* G *relative to an integral basis* $\mathsf{X} = \{X_1, \cdots, X_n\}$ *of the Lie algebra* T *of a maximal torus of* G, $B = \{\beta\}$ *a cyclic subgroup of order* m *in* G, *and* $X = \sum_s a_s X_s \in \mathsf{T}$ *such that* $\exp(X)$ *is* $\mathrm{ad}(G)$*-conjugate to* β. *Then each* $b_s = m a_s$ *is an integer, and* $B \cap \mathrm{ad}(G)K = 1$ *if and only if each*

$$V_i = \{m, \textstyle\sum_s v_{i1s} b_s, \sum_s v_{i2s} b_s, \cdots, \sum_s v_{i\,n-k\,s} b_s\}$$

is a set of relatively prime integers.

Proof. Each $b_s = m a_s$ is an integer because X is an integral basis of T and $\exp(mX) = \exp(\sum_s b_s X_s)$ is conjugate to $\beta^m = 1$.

We will use the notation leading to the definition of the angular parameters of K in G relative to X. An element of T lies in $\mathrm{ad}(G)K$ if and only if it lies in one of the T_i, so $B \cap \mathrm{ad}(G)K = 1$ if and only if $\exp(rX) \notin T_i$ for any i whenever $r \not\equiv 0 \pmod m$. X being an integral basis of T, $\exp(rX) \in T_i$ if and only if there is a choice a_{is} of integers such that $rX + \sum_s a_{is} X_s = \sum_s (r a_s + a_{is}) X_s$ lies in T_i, i.e., such that $\sum_s v_{ijs}(r b_s + m a_{is}) = 0$ for $1 \leqq j \leqq n-k$. Reducing modulo m this says that $r \sum_s v_{ijs} b_s \equiv 0$ for $1 \leqq j \leqq n-k$. If $r \not\equiv 0 \pmod m$, this implies that V_i is not a set of relatively prime integers.

Now suppose that V_i is not a set of relatively prime integers. Then there is an integer $r \not\equiv 0 \pmod m$ such that $r \sum_s v_{ijs} b_s \equiv 0 \pmod m$ for $1 \leqq j \leqq n-k$. We will show that $\exp(rX) \in T_i$, so $\beta^r \in \mathrm{ad}(G)K$. Let U_{ij} be the $(n-1)$-torus whose Lie algebra U_{ij} is the hyperplane $\sum_s v_{ijs} x_s = 0$ in T, where the x_s are coordinates relative to the basis X. $T_i = \bigcap_j U_{ij}$. V_{ij} being a set of relatively prime integers, we have integers c_{ijs} with $\sum_s c_{ijs} v_{ijs} = 1$. The congruences $r \sum_s v_{ijs} b_s \equiv 0 \pmod m$ gives us integers t_{ij} with $m t_{ij} + r \sum_s v_{ijs} b_s = 0$, so we have integers $a_{ijs} = c_{ijs} t_{ij}$ such that $\sum_s v_{ijs}(r b_s + m a_{ijs}) = 0$, for $1 \leqq j \leqq n-k$. This just says that $\exp(rX) \in U_{ij}$ for $1 \leqq j \leqq n-k$, so $\exp(rX) \in T_i$. *QED.*

Theorem 6 can be used to check $\Gamma \cap \mathrm{ad}(G)K = 1$ provided that the intersection of K with the identity component G_0 of G is connected and $\Gamma \subset G_0$. Let $\{f_t\}$ be automorphisms of G such that the automorphism group $\mathrm{ad}(G)$ of G is the union of the $\mathrm{ad}(G_0) \cdot f_t$. Let $K_t = f_t(K) \cap G_0$. Given $\Gamma \subset G_0$, $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\Gamma \cap \mathrm{ad}(G_0)K_t = 1$ for every t. We can check the $\Gamma \cap \mathrm{ad}(G_0)K_t = 1$ with Theorem 6 and thus check $\Gamma \cap \mathrm{ad}(G)K = 1$.

The application of Theorem 6 is simplified when the Weyl group W of G acts on the integral basis X by signed permutations: the angular parameters can then be chosen so that each V_{ij} is obtained from V_{1j} by the same signed permutations. We will use this trick when G is a classical group.

III. 2. Even and odd subgroups of the classical groups. By the *classical groups* we mean the unitary groups $\boldsymbol{U}(n)$, the special unitary groups $\boldsymbol{SU}(n)$, the symplectic groups (often called the unitary symplectic groups) $\boldsymbol{Sp}(n)$, the special orthogonal groups $\boldsymbol{SO}(n)$, and the spin groups (universal covering groups of the special orthogonal groups) $\boldsymbol{Spin}(n)$. They are all compact connected Lie groups. $\boldsymbol{U}(n)$ has rank n, semisimple part $\boldsymbol{SU}(n)$ and fundamental group infinite cyclic. $\boldsymbol{SU}(n+1)$ has rank n, is simple for $n \geqq 1$, and is simply connected. $\boldsymbol{Sp}(n)$ has rank n, is simple for $n \geqq 1$, and is simply connected. $\boldsymbol{Sp}(n)$ can be viewed as all elements of $\boldsymbol{U}(2n)$ which preserve an antisymmetric nondegenerate 2-form on complex Euclidean space C^{2n}. Given an orthonormal basis $\{e_1, \cdots, e_{2n}\}$ of C^{2n}, we'll use the form

$$A\left(\sum_1^{2n} x_j e_j, \sum_1^{2n} y_j e_j\right) = \sum_1^{n} (x_j y_{j+n} - y_j x_{j+n}).$$

$\boldsymbol{SO}(2n$ or $2n+1)$ has rank n. $\boldsymbol{SO}(k)$ is semisimple for $k \geqq 3$, simple for $4 \neq k \geqq 3$, has fundamental group Z_2 if $k \geqq 3$, and has universal covering group $\boldsymbol{Spin}(k)$.

Given a classical group G, we have a canonical choice of a maximal torus T of G:

1. $G = \boldsymbol{U}(n)$. T is the set of all matrices $\operatorname{diag}\{a_1, \cdots, a_n\}$, where each a_j is a unimodular complex number.

2. $G = \boldsymbol{SU}(n)$. T is the set of all $\operatorname{diag}\{a_1, \cdots, a_n\}$ of determinant 1, where each a_j is a unimodular complex number.

3. $G = \boldsymbol{Sp}(n)$. T is the set of all matrices $\begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}$, where $\bar{D}$ is the complex conjugate of D and D is in the canonical maximal torus of $\boldsymbol{U}(n)$.

4. $G = \boldsymbol{SO}(2n$ or $2n+1)$. T is the set of all matrices $\operatorname{diag}\{R(t_1), \cdots, R(t_n), (1)\}$, where $R(t) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$ and the (1) appears only if $G = \boldsymbol{SO}(2n+1)$.

5. $G = \boldsymbol{Spin}(2n$ or $2n+1)$. T is the complete inverse image of our chosen maximal torus in $\boldsymbol{SO}(2n$ or $2n+1)$.

If G is not a $\boldsymbol{Spin}$ or special unitary group, we have a canonical choice of integral basis $\mathsf{X}_G = \{X_1, \cdots, X_n\}$ of the Lie algebra T of T:

1. $G = \boldsymbol{U}(n)$. $\exp(tX_j) = \operatorname{diag}\{1, \cdots, 1, \exp(2\pi it), 1, \cdots, 1\}$, where the $\exp(2\pi it)$ is in the j-th place and $i^2 = -1$.

2. $G=\boldsymbol{Sp}(n)$. $\exp(tX_j)=\begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}$, where $\bar{D}$ is the complex conjugate of D and $D=\operatorname{diag}\{1,\cdots,1,\exp(2\pi it),1,\cdots,1\}$ has the $\exp(2\pi it)$ in the j-th place.
3. $G=\boldsymbol{SO}(2n$ or $2n+1)$. Let I_2 be the 2×2 identity matrix; $\exp(tX_j)=\operatorname{diag}\{I_2,\cdots,I_2,R(t),I_2,\cdots,I_2,(1)\}$, where $R(t)$ is the j-th block.

The Weyl group W of G acts on X_G by signed permutations:

1. $G=\boldsymbol{U}(n)$. W acts on X_G by all permutations.
2. $G=\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n+1)$. W acts on X_G by all signed permutations.
3. $G=\boldsymbol{SO}(2n)$. W acts on X_G by all signed permutations where the the number of sign changes is even.

Let K be a closed connected subgroup of rank k in a classical group $G=\boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n$ or $2n+1)$. Replacing K by a conjugate, we have a maximal torus T' of K which lies in our canonical maximal torus T of G. The Lie algebra T' of T' is the intersection of $n-k$ hyperplanes $\sum_s v_{js}x_s=0$, where the x_s are coordinates in T relative to the canonical integral basis X_G; we can assume that each $V_j=\{v_{j1},v_{j2},\cdots,v_{jn}\}$ is a set of relatively prime integers. If $W=\{w_1,\cdots,w_q\}$ is an enumeration of the Weyl group of G relative to T, W envisaged as a group of signed permutations on n-tuples from its action on X_G, the angular parameters of K in G relative to X_G are the $V_{ij}=w_i(V_j)$.

DEFINITION. *The $n-k$ ordered sets V_j of relatively prime integers are the canonical parameters of K in $G=\boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n$ or $2n+1)$.*

Let K be a closed connected subgroup of rank $k-1$ in $\boldsymbol{SU}(n)$. Viewing $\boldsymbol{SU}(n)$ as a subgroup of $\boldsymbol{U}(n)$, K has canonical parameters $V_1,\cdots,V_{n-k+1}$ in $\boldsymbol{U}(n)$. We may assume that $V_{n-k+1}=\{1,1,\cdots,1\}$.

DEFINITION. *The $n-k$ ordered sets $V_1,\cdots,V_{n-k}$ of relatively prime integers are the canonical parameters of K in $\boldsymbol{SU}(n)$.*

Let K be a closed connected subgroup of rank k in $\boldsymbol{Spin}(2n$ or $2n+1)$ and let $f\colon \boldsymbol{Spin}\to\boldsymbol{SO}$ be the natural projection. We will use the canonical parameters $V_1,\cdots,V_{n-k}$ of $f(K)$ in $\boldsymbol{SO}(2n$ or $2n+1)$ for the canonical parameters of K in $\boldsymbol{Spin}(2n$ or $2n+1)$:

DEFINITION. *The $n-k$ ordered sets V_j of relatively prime integers are the canonical parameters of K in $\boldsymbol{Spin}(2n$ or $2n+1)$.*

Given a closed connected subgroup K of a classical group G and an integral basis X of the Lie algebra of a maximal torus of G, we can always construct the angular parameters of K in G relative to X from the canonical parameters of K in G.

The fact that the Weyl group acts on the canonical parameters by signed permutations allows us to define:

DEFINITION. *Let K be a closed connected subgroup of a classical group $G = \boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{SO}(2n$ or $2n+1)$ or $\boldsymbol{Spin}(2n$ or $2n+1)$ such that* rank. G — rank. $K = 1$. *Let $V = \{v_1, \cdots, v_n\}$ be the canonical parameter of K in G and set $v = v_1 v_2 \cdots v_n$. Then K is an even subgroup of G if v is an even integer; K is an odd subgroup of G if v is an odd integer.*

The most familiar examples of even subgroups are

$$\boldsymbol{U}(n-1) \subset \boldsymbol{U}(n),\ \boldsymbol{SU}(n-1) \subset \boldsymbol{SU}(n),\ \boldsymbol{Sp}(n-1) \subset \boldsymbol{Sp}(n),$$
$$\boldsymbol{SO}(2n-1) \subset \boldsymbol{SO}(2n) \text{ and } \boldsymbol{Spin}(2n-1) \subset \boldsymbol{Spin}(2n).$$

In these examples the canonical parameter can be taken to be $\{1, 0, \cdots, 0\}$.

III. 3. The orthogonal groups. If G is a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n+1)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{Spin}(2n$ or $2n+1)$ of rank n, K is a closed connected subgroup of rank $n-1$ and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then Corollary 5.1 tells us that every abelian subgroup of Γ is cyclic. If, however, $G = \boldsymbol{SO}(2n$ or $2n+1)$, then we only know that every abelian subgroup of Γ is of the form $Z_u \times Z_v$, where u is a power of 2. As it is known [11] that a $(Z_2)^2$ cannot act freely on the sphere $S^{2n-1} = \boldsymbol{SO}(2n)/\boldsymbol{SO}(2n-1)$, there is, at least for some choices of K, room for improvement:

THEOREM 7. *Let G be a special orthogonal group $\boldsymbol{SO}(q) = \boldsymbol{SO}(2n$ or $2n+1)$ of rank n and let K be a closed connected subgroup of rank $n-1>0$. If K is odd, G has a subgroup B isomorphic to $(Z_2)^2$ with $B \cap \mathrm{ad}(G)K = 1$. If K is even and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then every abelian subgroup of Γ is cyclic.*

Proof. Let $V = \{v_1, \cdots, v_n\}$ be the canonical parameter of K in G and let $b \in G$ have order 2. The eigenvalues of b are all 1 or -1. As det. $b = 1$, the multiplicity of the eigenvalue -1 is some even number $2s$. It is clear that b is $\mathrm{ad}(G)$-conjugate to $\exp(\frac{1}{2}X_1 + \frac{1}{2}X_2 + \cdots + \frac{1}{2}X_s)$, where $\mathsf{X}_G = \{X_1, \cdots, X_n\}$ is our canonical integral basis, so the arithmetic criterion (Theorem 6) says that $b \in \mathrm{ad}(G)K$ if and only if some sum of s of the v_j, without repetitions, is even. When the v_j are all odd this means that

$b \in \mathrm{ad}(G)K$ if and only if s is even; when one of the v_j is even and $s < n$, this implies that $b \in \mathrm{ad}(G)K$.

Suppose K is odd. Then each of the v_j is odd, so we must exhibit a $(Z_2)^2$ in G in which every element $\neq 1$ has the eigenvalue -1 of multiplicity congruent to 2 modulo 4. Let $\boldsymbol{I}_t$ be the $t \times t$ identity matrix; then such a $(Z_2)^2$ is given by generators

$$b_1 = \mathrm{diag}\{-1, -1, 1, \boldsymbol{I}_{q-3}\}, \qquad b_2 = \mathrm{diag}\{1, -1, -1, \boldsymbol{I}_{q-3}\}$$

Suppose that K is even, so one of the v_j is even. By Theorem 5 we need only show that Γ contains no $(Z_2)^2$, so we must show that a $(Z_2)^2$ in G has an element $\neq 1$ with eigenvalue -1 of multiplicity not equal to $2n$. A $(Z_2)^m$ in $\boldsymbol{SO}(q)$ is conjugate to a group of diagonal matrices. It follows that G contains a $(Z_2)^2$ where every element $\neq 1$ has eigenvalue -1 with multiplicity $2n$ only if $q = 3$. That case was ruled out by the assumption rank. $K > 0$. *QED.*

III. 4. Elements of order 2 which act freely.

THEOREM 8. *Let G be a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{SO}(2n)$ or $\boldsymbol{Spin}(2n)$ and let K be an even subgroup (hence closed and connected, and* rank. G — rank. $K = 1$). *Let Γ be a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$. Then Γ has at most one element of order 2, and an element of order 2 in Γ is central in G. Let H be a closed connected subgroup of $\boldsymbol{SU}(n)$ such that* rank. $\boldsymbol{SU}(n)$ — rank. $H = 1$ *and let Σ be a finite subgroup of $\boldsymbol{SU}(n)$ such that $\Sigma \cap \mathrm{ad}(\boldsymbol{SU}(n))H = 1$. Then both n and H are even if Σ has an element of order 2.*

Proof. Suppose $G \neq \boldsymbol{Spin}(2n)$ and let $\gamma \in \Gamma$ have order 2. As in the proof of Theorem 7, the arithmetic criterion shows that γ has the eigenvalue -1 with multiplicity $2n$ if $G = \boldsymbol{SO}(2n)$ or $\boldsymbol{Sp}(n)$, and with multiplicity n if $G = \boldsymbol{U}(n)$ or $\boldsymbol{SU}(n)$. Hence γ is conjugate to $-\boldsymbol{I}$, the negative of the identity matrix in G. As $-\boldsymbol{I}$ is central in G, $\gamma = -\boldsymbol{I}$ and is central in G.

Now suppose that $G = \boldsymbol{Spin}(2n)$ and $f\colon G \to \boldsymbol{SO}(2n)$ is the natural map. Let -1 denote the element of order 2 in ker. f. If -1 is in Γ or K, then Γ or K consists of whole f-fibres and we have $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$. If -1 is in neither Γ nor K, then either $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$ or Γ has an element $\gamma \neq 1$ such that $\boldsymbol{I}_{2n} \neq f(\gamma) \in f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K)$. We will show that this last alternative does not occur. For if it does, K has a conjugate K' such that $-\gamma \in K'$. γ has order 2, for $\gamma \notin$ ker. f but $\gamma^2 = (-\gamma)^2 \in \Gamma \cap K'$. We can pass to a conjugate of γ and assume

$\gamma = e_1 \cdot e_2 \cdot \cdots \cdot e_{2s}$, where the e_j are an orthonormal basis of Euclidean space $\boldsymbol{R}^{2n}$, taken as generators of the Clifford algebra $\boldsymbol{C}(\boldsymbol{R}^{2n})$, and dots denote Clifford multiplication. If $s = n$, γ is central and thus, by Theorem 5, the only element of Γ of order 2. If $s < n$, let $\beta = e_{2s} \cdot e_{2s+1} \in \boldsymbol{Spin}(2n)$ and $\mathrm{ad}(\beta)\gamma = -\gamma$. This implies that both γ and $-\gamma$ are in $\mathrm{ad}(G)K$, which contradicts $\Gamma \cap \mathrm{ad}(G)K = 1$. Now we can assume that $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$. $f(K)$ is an even subgroup of $\boldsymbol{SO}(2n)$ because K is even in $\boldsymbol{Spin}(2n)$, so an element of $f(\Gamma)$ of order 2 is $-\boldsymbol{I}_{2n}$. It follows that an element of Γ of order 2 lies in $f^{-1}(\{\pm \boldsymbol{I}_{2n}\})$, hence is central in $\boldsymbol{Spin}(2n)$. Uniqueness follows from Theorem 5.

Let $G = \boldsymbol{SU}(n)$ and let $\sigma \in \Sigma$ have order 2. If H is odd, the arithmetic criterion implies that the eigenvalue -1 of σ has odd multiplicity, contradicting $\det.\sigma = 1$. Thus H is even. If n is odd, we again contradict $\det.\sigma = 1$, for, H being even, the arithmetic criterion says that $\sigma = -\boldsymbol{I}_n$. *QED.*

Chapter IV. Finite subgroups of classical groups which have all abelian subgroups cyclic.

Theorems 5 and 7 tell us that if G is a classical group and K is a closed connected subgroup, assumed to be an even subgroup if G is special orthogonal, such that $\mathrm{rank}.G - \mathrm{rank}.K = 1$, and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then every abelian subgroup of Γ is cyclic. For this reason, we'll examine the finite groups with all abelian subgroups cyclic.

IV. 1. Classification of finite groups with all abelian subgroups cyclic. The finite groups with all abelian subgroups cyclic fall into two classes ([14], Chapter I)—those with all Sylow subgroups cyclic, and those with odd Sylow subgroups cyclic and 2-Sylow subgroups generalized quaternionic. H. Zassenhaus ([16], p. 198, p. 202) and M. Suzuki ([13], p. 689) have given a complete classification of these groups in terms of generators and relations. We will not use this classification, but rather will rely on a simpler description given in H. Zassenhaus' book ([17], p. 175) for the finite groups with all Sylow subgroups cyclic, and on the fact that a finite group with all abelian subgroups cyclic has all Sylow subgroups cyclic if its 2-Sylow subgroups are not generalized quaternionic. For reference, the generalized quaternionic groups are the groups $\boldsymbol{Q}2^a$ of order 2^a, $a \geqq 3$, given by

$$A^{2^{a-1}} = 1, \quad B^2 = A^{2^{a-2}}, \quad BAB^{-1} = A^{-1}, \quad a \text{ integer}, \quad a \geqq 3.$$

A finite group of order N with all Sylow subgroups cyclic is given by $A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$, $r^n \equiv 1 \pmod{m}$.

Our plan of attack is to calculate representations of these groups in the classical groups, and find conditions under which the image of a representation acts freely on an appropriate coset space.

IV. 2. Classical representations of the generalized quaternionic groups. Following G. Vincent ([14], Chapter III), elementary techniques of representation theory tell us that the irreducible unitary representations of the generalized quaternionic group $\boldsymbol{Q}2^a$ are:

1. The 4 $\boldsymbol{U}(1)$-representations given by $A \to \pm 1$ and $B \to \pm 1$.
2. The $2^{a-2} - 1$ $\boldsymbol{U}(2)$-representations S_r, $1 \leqq r < 2^{a-2}$, given by

$$S_r\colon A \to \begin{pmatrix} u^r & 0 \\ 0 & u^{-r} \end{pmatrix} \text{ and } B \to \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}, \text{ where } u = \exp(2\pi i/2^{a-1}).$$

Note that S_r is faithful if and only if r is odd. Let S denote the $\boldsymbol{U}(1)$- representation $A \to 1$ and $B \to -1$.

It follows that a special unitary representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{U}(1)$-representations plus a sum of some of

1. The 2^{a-3} $\boldsymbol{SU}(2)$-representations S_r, where r is odd.
2. The 2^{a-3} $\boldsymbol{SU}(3)$-representations $S_r + S$, where r is even.
3. The $\boldsymbol{SU}(4)$-representations $S_{r_1} + S_{r_2}$, where r_1 and r_2 are even.

Similarly, a symplectic representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{U}(1)$-representations plus a sum of some of

1. The 2^{a-3} $\boldsymbol{Sp}(1)$-representations S_r, where r is odd.
2. The 2^{a-3} $\boldsymbol{Sp}(1)$-representations $S_r + S_r^*$, where r is even and S_r^* is the complex conjugate representation of S_r.

A unitary, special unitary or symplectic representations of $Q2^a$ is faithful if and only if it has a summand S_r with r odd.

S_r is unitarily equivalent to its conjugate representation S_r^*, and is equivalent to a real representation if and only if r is even. As before, we set $R(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \in \boldsymbol{SO}(2)$; the irreducible orthogonal representations of $\boldsymbol{Q}2^a$ are:

1. The 4 $\boldsymbol{O}(1)$-representations given by $A \to \pm 1$ and $B \to \pm 1$.

2. The $2^{a-3}-1$ $\boldsymbol{O}(2)$-representations S_r, r even, unitarily equivalent to the corresponding $\boldsymbol{U}(2)$-representations, given by $S_r\colon A\to R(r/2^{a-1})$ and $B\to\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
3. The 2^{a-3} $\boldsymbol{O}(4)$-representations T_r, r odd, equivalent to $S_r+S_r{}^*$, given by $T_r\colon A\to\begin{pmatrix}R(r/2^{a-1}) & 0\\ 0 & R(-r/2^{a-1})\end{pmatrix}$ and $B\to\begin{pmatrix}0 & I_2\\ -I_2 & 0\end{pmatrix}$.

A special orthogonal representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{O}(1)$-representation plus a sum of some of

1. The $\boldsymbol{SO}(3)$-representations S_r+S (r even, of course).
2. The $\boldsymbol{SO}(4)$-representations $S_{r_1}+S_{r_2}$ with r_1 and r_2 even.
3. The $\boldsymbol{SO}(4)$-representations T_r (r odd, of course).

An orthogonal or special orthogonal representation of $\boldsymbol{Q}2^a$ is faithful if and only if it has a summand T_r.

Each of the T_r can be lifted to a faithful $\boldsymbol{Spin}(4)$-representation of $\boldsymbol{Q}2^a$. Let T be one of the T_r and let $\{e_j\}$ be the orthonormal basis of $\boldsymbol{R}^4$ with respect to which our matrices are written; the $\{e_j\}$ generate the Clifford algebra $\boldsymbol{C}(\boldsymbol{R}^4)$. We choose $T'(A)\in\boldsymbol{Spin}(4)$ over $T(A)$ and $T'(B)\in\boldsymbol{Spin}(4)$ over $T(B)$. We then have

$$T'(A)=\pm(\cos x+e_2\cdot e_1\sin x)\cdot(\cos x-e_4\cdot e_3\sin x),$$

$$T'(A)^{-1}=\pm(\cos x-e_2\cdot e_1\sin x)\cdot(\cos x+e_4\cdot e_3\sin x)$$

and

$$T'(B)=\pm\tfrac{1}{2}(1+e_3\cdot e_1)\cdot(1+e_4\cdot e_2),$$

where dots denote Clifford multiplication, and $x=\pi r/2^{a-1}$. A short calculation shows that $T'(A)^{2^{a-2}}=e_1\cdot e_2\cdot e_4\cdot e_3=T'(B)^2$. Another calculation shows that $T'(B)\cdot T'(A)=T'(A)^{-1}\cdot T'(B)$. It follows that $T'(A)$ and $T'(B)$ generate a $\boldsymbol{Q}2^a$ in $\boldsymbol{Spin}(4)$, so T' extends to a $\boldsymbol{Spin}(4)$-representation of $\boldsymbol{Q}2^a$. T' is faithful because it covers a faithful representation.

Let $V=S_{r_1}+S_{r_2}$, $r_i=2u_i$, a non-faithful $\boldsymbol{SO}(4)$-representation of $\boldsymbol{Q}2^a$. If $V'(A)\in\boldsymbol{Spin}(4)$ lies over $V(A)$ and $V'(B)\in\boldsymbol{Spin}(4)$ lies over $V(B)$, a short calculation shows that $V'(B)^2=-1$, $V'(B)\cdot V'(A)\cdot V'(B)^{-1}=V'(A)^{-1}$, and $V'(A)^{2^{a-2}}=-1$ if and only if u_1+u_2 is odd. In other words, V' extends to a $\boldsymbol{Spin}(4)$-representation $V'$of $\boldsymbol{Q}2^a$ if and only if one of the u_i is odd and the other is even. In that case, V' is faithful and -1 is the element of order 2 in $V'(\boldsymbol{Q}2^a)$.

Let $U=S_r+S$, $r=2u$, a non-faithful $\boldsymbol{SO}(3)$-representation of $\boldsymbol{Q}2^a$.

Choosing $U'(A)$ and $U'(B)$ in $\boldsymbol{Spin}(3)$ over $U(A)$ and $U(B)$, we see that $U'(B)^2 = -1$, $U'(B)\cdot U'(A)\cdot U'(B)^{-1} = U'(A)^{-1}$, and $U'(A)^{2^{a-2}} = -1$ if and only if u is odd. Thus U' extends to a $\boldsymbol{Spin}(3)$-representation U' of $\boldsymbol{Q}2^a$ if and only if u is odd; in that case, U' is faithful and -1 is the element of order 2 in $U'(\boldsymbol{Q}2^a)$.

IV. 3. Unitary representations of finite groups which have all Sylow subgroups cyclic. Let Γ be a finite group of order N with every Sylow subgroup cyclic. Γ is given by two generators A and B with relations $A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$, $r^n \equiv 1 \pmod m$. Note that m is odd; if m were even, r would be odd because A and A^r have the same order, so $2 \mid ((r-1)n, m)$, where we denote *a divides b* by $a \mid b$. Note also that not r but only the mod m residue class of r is important. Let ϕ be the Euler ϕ-function and let G_m be the multiplicative group of integers prime to m, taken modulo m. As m is odd, there can be no confusion with the exceptional Lie group $\boldsymbol{G}_2$. Given $C \in \Gamma$, let $\{C\}$ denote the cyclic subgroup of Γ generated by C. Let d be the order of r in G_m. As $r^n \equiv 1 \pmod m$, $d \mid n$ and we can write $n = n'd$. If $m_i \mid m$, set $d_i =$ order of r in G_{m_i}, $n = d_i n_i'$. G. Vincent has proved ([14], p. 156):

Γ has exactly $\phi(m_j)n'd/d_j^2$ irreducible unitary representations of degree d_j. On restricting one of these representations to $\{A\}$, it has kernel $\{A^{m_j}\}$. As m_j runs over all divisors of m, including 1 and m, these representations of degree d_j give all irreducible unitary representations of Γ.

This, together with other results of G. Vincent ([14], Chapter III) make it fairly easy to verify that the irreducible unitary representations of degree d_j of Γ are given by:

$$f_{jk}(A) = \operatorname{diag}\{\exp(2\pi i/m_j), \exp(2\pi i r/m_j), \cdots, \exp(2\pi i r^{d_j-1}/m_j)\}$$

$$f_{jk}(B) = \exp(2\pi i k/n)\cdot \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ for } 1 \leqq k \leqq \phi(m_j)n'd/d_j^2.$$

Chapter V. A second application of the arithmetic criterion to the classical groups.

V. 1. Generalized quaternionic subgroups of classical groups which act freely. Suppose G is a classical group of rank n and K is a closed connected subgroup of rank $n-1$, assumed to be an even subgroup if G is a special orthogonal group. A generalized quaternionic subgroup Γ of G can be considered to be the image of a faithful G-representation of the appropriate $\boldsymbol{Q}2^a$. Since we know these representations explicitly, we can apply the arithmetic criterion to check whether $\Gamma \cap \mathrm{ad}(G)K = 1$. The importance of this procedure is that if no generalized quaternionic subgroup of G can act freely on G/K, then every finite subgroup of G that acts freely on G/K has all Sylow subgroups cyclic. Such groups have particularly simple structure among the finite groups with all abelian subgroups cyclic.

THEOREM 9.1. *Let G be a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, or $\boldsymbol{SO}(2n$ or $2n+1)$, K a closed connected subgroup with* rank. G — rank. $K = 1$, *$V = \{v_1, \cdots, v_n\}$ the canonical parameter of K in G, and q the number of v_j which are odd. Then G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if*

n is even and q is odd, if $G \neq \boldsymbol{Sp}(n)$; q is odd or $q = n$, if $G = \boldsymbol{Sp}(n)$. If G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ and if B is any generalized quaternionic group, then G has a subgroup B' isomorphic to B such that $B' \cap \mathrm{ad}(G)K = 1$. If K is an even subgroup in case $G = \boldsymbol{SO}(2n$ or $2n+1)$, if

n is odd or q is even, if $G \neq \boldsymbol{Sp}(n)$; q is even and $q < n$, if $G = \boldsymbol{Sp}(n)$, and if Σ is a finite subgroup of G such that $\Sigma \cap \mathrm{ad}(G)K = 1$, then every Sylow subgroup of Σ is cyclic.

Proof. Let Γ be a generalized quaternionic subgroup of $G = \boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n$ or $2n+1)$, considered as the image of a faithful G-representation F of $\boldsymbol{Q}2^a$. Checking the various possible summands of F, we see that every element of Γ is $\mathrm{ad}(G)$-conjugate to a power of $F(A)$ or $F(B)$; it follows that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\{F(A)\} \cap \mathrm{ad}(G)K = 1 = \{F(B)\} \cap \mathrm{ad}(G)K$. We will apply the arithmetic criterion (Theorem 6) to these two cyclic subgroups of Γ. When we do this, the integer m in the formulation of the arithmetic criterion will be a power of 2, as Γ is a 2-group, so we may ignore the integers b_j which are even. As the kernel of a non-faithful G-representation of $\boldsymbol{Q}2^a$ contains the element $B^2 = A^{2^{a-2}}$ of order 2, this means we need only consider the faithful summands of F. Replacing Γ by an $\mathrm{ad}(G)$-conjugate if necessary, we can assume $F = S_{r_1} + \cdots + S_{r_s}$ if $G = \boldsymbol{U}(n)$ or $\boldsymbol{Sp}(n)$ and $F = T_{r_1} + \cdots + T_{r_s}$ if $G = \boldsymbol{SO}(2n$ or $2n+1)$, where the r_j are odd. An application of the arithmetic criterion now tells us that the condtion for $\Gamma \cap \mathrm{ad}(G)\mathrm{K} = 1$ is that, for every element g of the group $\boldsymbol{S}(n)$ of all permutations on $\{1,2,\cdots,n\}$, $\sum_j r_j(v_{g(2j-1)}-v_{g(2j)})$ and $\sum_j(v_{g(2j-1)}-v_{g(2j)})$ are odd if $G=\boldsymbol{U}(n)$, $\sum_j r_j(\pm v_{g(j)})$ and $\sum_j(\pm v_{g(j)})$ are odd for any arrangement of $\pm$ signs if $G=\boldsymbol{Sp}(n)$, and

$$\textstyle\sum_j r_j(\pm v_{g(2j-1)}-(\pm v_{g(2j)})) \text{ and } \sum_j(\pm v_{g(2j-1)}-(\pm v_{g(2j)}))$$

are odd for any arrangement of $\pm$ signs, requiring only that the number of minus signs be even if $G=\boldsymbol{SO}(2n)$, if $G=\boldsymbol{SO}(2n \text{ or } 2n+1)$. As the r_j are odd, the first number has the same residue mod 2 as the second, so we may ignore the first one in each case. Similarly, we may ignore signs. We now see that the condition for $\Gamma\cap\mathrm{ad}(G)K=1$ is that, for every $g\in\boldsymbol{S}(n)$, $\sum_{j=1}^{2s}v_{g(j)}$ is odd if $G\neq\boldsymbol{Sp}(n)$; $\sum_{j=1}^{s}v_{g(j)}$ is odd if $G=\boldsymbol{Sp}(n)$. This is independent of a and of the r_j, and depends only on n, s and q. It happens if and only if $n=2s$ and q is odd if $G\neq\boldsymbol{Sp}(n)$; q is odd or $q=n$ if $G=\boldsymbol{Sp}(n)$.

The theorem is now proved except for $G=\boldsymbol{SU}(n)$. Suppose $G=\boldsymbol{SU}(n)\subset\boldsymbol{U}(n)$ and let K' be a closed connected subgroup of rank $n-1$ in $\boldsymbol{U}(n)$ such that $K=K'\cap\boldsymbol{SU}(n)$. If Γ is a subgroup of $\boldsymbol{SU}(n)$, $\Gamma\cap\mathrm{ad}(\boldsymbol{SU}(n))K=\Gamma\cap\mathrm{ad}(\boldsymbol{U}(n))K'$, so we are done because a generalized quaternionic subgroup of $\boldsymbol{U}(n)$ can be assumed, for purposes of checking $\Gamma\cap\mathrm{ad}(\boldsymbol{U}(n))K'=1$, to be the image of a sum of S_{r_j} with r_j odd, and hence can be assumed to lie in $\boldsymbol{SU}(n)$. *QED.*

The situation with the ***Spin*** groups is more complicated because of the relative abundance of faithful ***Spin***-representations of the generalized quaternionic groups.

LEMMA 9.1. *Let K be a closed connected subgroup of rank $n-1>0$ in $G=\boldsymbol{Spin}(2n$ or $2n+1)$ and let -1 be the element of order 2 in the kernel of the natural projection $f\colon\boldsymbol{Spin}\to\boldsymbol{SO}$. Then K is an even subgroup of G if and only if it contains -1.*

Proof. K contains -1 if and only if -1 lies on a 1-parameter subgroup of K, as K is connected. Let Y be a 1-parameter subgroup of K. As -1 is central in G we may assume that $(f\,Y)(t)=\exp(\sum_s ta_sX_s)$, where the exponential is taken in $\boldsymbol{SO}(2n$ or $2n+1)$ and $\{X_1,\cdots,X_n\}$ is our canonical integral basis for $\boldsymbol{SO}(2n$ or $2n+1)$. We may assume Y normalized so that $(f\cdot Y)(t)=\boldsymbol{I}$ if and only if t is an integer; the a_s are then integers and -1 lies on Y if and only if $-1=Y(1)$. $Y(1)=-1$ if and only if the number of odd a_j is odd.

Let $V=\{v_1,\cdots,v_n\}$ be the canonical parameter of K in G, i.e., the canonical parameter of $f(K)$ in $\boldsymbol{SO}$. If K is odd, $\sum_s v_sa_s=0$ implies that an even number of a_s are odd and $-1 \neq Y(1)$. If K is even, we can assume v_1 even and v_2 odd; we construct a conjugate X of a 1-parameter subgroup of K which contains -1 by $(f \cdot X)(t) = \exp(t(v_2 X_1 - v_1 X_2))$. *QED.*

We mention an interesting consequence of Lemma 9.1:

COROLLARY 9.1. *Let K be a closed connected subgroup of rank $n-1>0$ in $G = \boldsymbol{SO}(2n$ or $2n+1)$. Then G/K is simply connected if and only if K is an even subgroup of G; $\pi_1(G/K) = Z_2$ if K is odd.*

Proof. The universal covering of G/K is $\boldsymbol{Spin}(2n$ or $2n+1)/K'$, where K' is the identity component of $f^{-1}(K)$. *QED.*

THEOREM 9.2. *Let K be a closed connected subgroup of rank $n-1>0$ in $G = \boldsymbol{Spin}(2n$ or $2n+1)$, $V = \{v_1, \cdots, v_n\}$ the canonical parameter of K in G, and q the number of v_j which are odd.*

Suppose K is even. Then G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if n is even (say $n = 2s$) and q is odd, and any such Γ is $\mathrm{ad}(G)$-conjugate to the image of a faithful G-representation F' of a $\boldsymbol{Q}2^a$, where $F' = T'_{r_1} + \cdots + T'_{r_s}$ for some choice of odd integers r_j. If $n = 2s$, q is odd, $a \geqq 3$, $\{r_1, \cdots, r_s\}$ are odd integers and

$$\Gamma = (T'_{r_1} + \cdots + T'_{r_s})(\boldsymbol{Q}2^{a'}),$$

then $\Gamma \cap \mathrm{ad}(G)K = 1$. If n is odd or q is even, and Σ is a finite subgroup of G such that $\Sigma \cap \mathrm{ad}(G)K = 1$, then every Sylow subgroup of Σ is cyclic.

Suppose K is odd. Given $a \geqq 3$, G has a subgroup Γ isomorphic to $\boldsymbol{Q}2^a$ such that $\Gamma \cap \mathrm{ad}(G)K = 1$, -1 is the element of order 2 in Γ, $f(\Gamma)$ is a dihedral 2-subgroup of $\boldsymbol{SO}(2n$ or $2n+1) = G'$ such that $f(\Gamma) \cap \mathrm{ad}(G')f(K) = 1$ and $f(\Gamma)$ is $\mathrm{ad}(G')$-conjugate to the image of a non-faithful G'-representation F of $\boldsymbol{Q}2^a$ which is a sum of representations of the forms $S_{2r} + S_{4s}$, r odd, and $S_{2t} + S$, t odd.

Proof. $f \colon G = \boldsymbol{Spin} \to \boldsymbol{SO} = G'$ being the natural projection, -1 is the element of order 2 in $\ker . f$ and $K' = f(K)$. Suppose K is even, so $-1 \in K$; given a subgroup Γ of G, $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $f(\Gamma) \cap \mathrm{ad}(G')K' = 1$ and $-1 \notin \Gamma$. A generalized quaternionic subgroup Γ of G not containing -1 is $\mathrm{ad}(G)$-conjugate to the image of a faithful G-representation $F' = T'_{r_1} + \cdots + T'_{r_s} + t'$, t' not faithful, of $\boldsymbol{Q}2^a$. $f \cdot F'$ is a faithful G'-representation $F = T_{r_1} + \cdots + T_{r_s} + t$, t not faithful, of $\boldsymbol{Q}2^a$. In the proof of Theorem 9.1 we saw that $f(\Gamma) \cap \mathrm{ad}(G')K' = 1$ if and only if $n = 2s$ and q is odd, and this is independent of the choices of a and of the odd integers r_j.

If $n=2s$, then t, and hence t', is trivial because F represents by matrices of determinant 1.

Now suppose K is odd. Given $a \geqq 3$ we will construct a subgroup Γ' of G', isomorphic to the dihedral group $\boldsymbol{D}2^{a-1}$ of 2^{a-1} elements, such that $\Gamma' \cap \mathrm{ad}(G')K' = 1$ and $\Gamma = f^{-1}(\Gamma')$ is isomorphic to $\boldsymbol{Q}2^a$. Then -1 will be the element of order 2 in Γ. Given $\gamma \in \Gamma \cap \mathrm{ad}(G)K$, $f(\gamma) = 1$; as $-1 \notin K$ it follows that $\gamma = 1$, so $\Gamma \cap \mathrm{ad}(G)K = 1$.

Let Γ' be a dihedral 2-subgroup of G'. As $\boldsymbol{D}2^{a-1}$ is the quotient of $\boldsymbol{Q}2^a$ by the subgroup generated by $B^2 = A^{2^{a-2}}$, we may view Γ' as the image of a representation $F = S_{2r_1} + \cdots + S_{2r_u} + S_{4t_1} + \cdots + S_{4t_v} + s$, r_j odd and s a sum of $\boldsymbol{O}(1)$-representations, of $\boldsymbol{Q}2^a$. Let $2w'$ and w be the multiplicities of the eigenvalue -1 of $s(A)$ and $s(B)$; the eigenvalue -1 of $F(B)$ has multiplicity $u+v+w=2x$. K' is odd because K is odd; as in the proof of Theorem 9.1 the arithmetic criterion shows that $\{F(A)\} \cap \mathrm{ad}(G')K' = 1$ if and only if u is odd when $a > 3$, if and only if $u + w'$ is odd when $a = 3$. It also shows that $\{F(B)\} \cap \mathrm{ad}(G')K' = 1$ if and only if x is odd.

The representations F which lift to ***Spin*** are of the form

$$(*) \qquad \begin{aligned} F = (S_{2r_1} + S_{4s_1}) + \cdots + (S_{2r_p} + S_{4s_p}) \\ + (S_{2t_1} + S) + \cdots + (S_{2t_q} + S) \end{aligned}$$

with r_j, t_j odd and where $S\colon A \to 1$, $B \to -1 \in \boldsymbol{O}(1)$. In this case, every element of Γ' is conjugate to a power of $F(A)$ or $F(B)$ and we have $u = p + q = x$, $w' = 0$; it follows that $\Gamma' \cap \mathrm{ad}(G')K' = 1$ if and only if $p+q$ is odd. rank. $K > 0$ implies $n > 1$, so we can find non-negative integers p and q with $p+q$ odd and $4p + 3q \leqq 2n$, hence a G'-representation F of the form $(*)$ with $p+q$ odd. *QED.*

V. 2. Subgroups of the unitary group which have all Sylow subgroups cyclic and act freely. Suppose that K is a closed connected subgroup of $\boldsymbol{U}(n)$ of rank $n-1$ and Γ is a finite subgroup of $\boldsymbol{U}(n)$ with all Sylow subgroups cyclic. Γ is conjugate to the image of a faithful representation of an abstract finite group Σ with all Sylow subgroups cyclic, and we can replace Γ by that conjugate. We will apply our arithmetic criterion (Theorem 6) to see whether $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$. This is of considerable interest if n is odd or the number of odd elements of the canonical parameter of K in $\boldsymbol{U}(n)$ is even. for then every finite subgroup B of $\boldsymbol{U}(n)$ such that $B \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$ has every Sylow subgroup cyclic.

Let N be the order of Σ. We represent Σ by generators and relations:

$A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$ and $r^n \equiv 1 \pmod{m}$. If Σ is cyclic, $m = 1$ and this becomes $B^N = 1$.

If Γ is cyclic of order t and has a generator γ with eigenvalues

$$\exp(\pi i r_1/t), \cdots, \exp(2\pi i r_n/t),$$

and $V = \{v_1, \cdots, v_n\}$ is the canonical parameter of K in $\boldsymbol{U}(n)$, a direct application of the arithmetic criterion shows that $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$ if and only if $\sum_s r_s v_{g(s)}$ is prime to t for every element g of the permutation group $\boldsymbol{S}(n)$. We will, then, ignore this case and henceforth assume that Σ is not cyclic.

In the notation of § IV. 3, we can assume that Γ is the image of the faithful representation $F = \sum_{j=1}^{a} \sum_{p=1}^{b_j} f_{jk_{jp}}$ of Σ in $\boldsymbol{U}(q)$.

Given an integer u and a divisor m_j of m, we define $u^{(j)} = (u, d_j)$, $0 \leqq u_j < d_j$ and $u_j \equiv u \pmod{d_j}$, $d_j^{(u)} \cdot u^{(j)} = d_j$, and

$$r^{(u)} = 1 + r^{u_j} + r^{2u_j} + \cdots + r^{(d_j^{(u)}-1)u_j}.$$

Given a second integer v, we define $h(u,v)$ to be the order of $B^u A^v$ in Σ. A calculation shows that $f_{jk_{jp}}(B^u A^v)$ has eigenvalues

$$\exp(2\pi i[(h(u,v)/nm_j)(k_{jp}um_j + u^{(j)}m_j n_j' e + n_j' v u^{(j)} r^{(u)} r^t)]/h(u,v))$$

for $0 \leqq e < d_j^{(u)}$ and $0 \leqq t < u^{(j)}$. If u is prime to d_j, a calculation shows that this means that $f_{jk_{jp}}(B^u A^v)$ has eigenvalues $\exp(2\pi i[k_{jp}u - en_j']/n)$ for $0 \leqq e < d_j$, hence is $\mathrm{ad}(\boldsymbol{U}(d_j))$-conjugate to $f_{jk_{jp}}(B^u)$. If $d \mid u$, another calculation shows that the eigenvalues of $f_{jk_{jp}}(B^u A^v)$ can be written, on setting $u = wd$ so $u/n = w/n'$, as

$$\exp(2\pi i[k_{jp}wm + (mn'/m_j)vr^t]/mn') \text{ for } 0 \leqq t < d_j.$$

Now set $N_{jpe} = \sum_{c<j} d_c b_c + (p-1)d_j + e + 1$, and, given an integer u, set $N_{jpet}(u) = \sum_{c<j} d_c b_c + (p-1)d_j + eu^{(j)} + t + 1$. With this notation, an application of the arithmetic criterion now yields:

THEOREM 10. *Let K be a closed connected subgroup of rank $q-1$ in the unitary group $\boldsymbol{U}(q)$, $V = \{v_1, \cdots, v_q\}$ the canonical parameter of K in $\boldsymbol{U}(q)$, $\boldsymbol{S}(q)$ the group of all permutations on $\{1, 2, \cdots, q\}$ and Γ a finite subgroup of $\boldsymbol{U}(q)$ which is the image of a faithful representation $F = \sum_{j=1}^{a} \sum_{p=1}^{b_j} f_{jk_{jp}}$ of the abstract finite non-cyclic group Σ with all Sylow subgroups cyclic. Then $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(q))K = 1$ if and only if for every $g \in \boldsymbol{S}(q)$ we have:*

1. $\sum_{j=1}^{a} \sum_{p=1}^{b_j} \sum_{e=0}^{d_j-1} v_{g(N_{jpe})} \cdot (k_{jp} + en_j')$ *is prime to* n.

2. $\sum_{j=1}^{a} \sum_{p=1}^{b_j} \sum_{e=0}^{d_j-1} v_{g(N_{jpe})} \cdot (mk_{jp} + (m/m_j)n'r^e)$ *is prime to* mn'.

3. Given integers u and v with $1 \leqq u < n$, $1 \leqq v < m$ *and* $1 < (u, d) < d$,

$$\sum_{j=1}^{a} \sum_{p=1}^{b_j} \sum_{e=0}^{d_j^{(u)}-1} \sum_{t=0}^{u^{(j)}-1} v_{g(N_{jpet}(u))} \cdot (h(u,v)/nm_j)$$
$$\cdot (k_{jp}um_j + u^{(j)}m_jn_j'e + n_j'vu^{(j)}r^{(u)}r^t) \not\equiv 0 \pmod{h(u,v)}.$$

To adapt these formulae to the other classical groups, we proceed as follows:

$\boldsymbol{SU}(q)$. K must have rank $q-2$ and Γ must lie in $\boldsymbol{SU}(q)$. Formulae (1, 2, 3; Theorem 10) remain unchanged.

$\boldsymbol{Sp}(q)$. $\Gamma \subset \boldsymbol{Sp}(q) \subset \boldsymbol{U}(2q)$, $\boldsymbol{S}(q)$ must be replaced by the group $\boldsymbol{S}'(q)$ of all signed permutations on V, and, for each formula of Theorem 10, the numbers following the v's fall into two sets, one of which is the negative of the other, and only one must be summed.

$\boldsymbol{SO}(2q+1)$. $\Gamma \subset \boldsymbol{SO}(2q+1) \subset \boldsymbol{U}(2q+1)$ and we proceed as for $\boldsymbol{Sp}(q)$.

$\boldsymbol{SO}(2q)$. $\Gamma \subset \boldsymbol{SO}(2q) \subset \boldsymbol{U}(2q)$, $\boldsymbol{S}(q)$ must be replaced by the group $\boldsymbol{S}''(q)$ of all signed permutations on V which involve an even number of changes of sign, and we proceed as for $\boldsymbol{Sp}(q)$.

$\boldsymbol{Spin}(2q$ or $2q+1)$. We proceed as for $\boldsymbol{SO}(2q$ or $2q+1)$.

Recall that if $G = \boldsymbol{U}(q)$, $\boldsymbol{SU}(q)$, $\boldsymbol{Sp}(q)$, $\boldsymbol{SO}(2q$ or $2q+1)$, or $\boldsymbol{Spin}(2q$ or $2q+1)$, and if $K = \boldsymbol{U}(q-1)$, $\boldsymbol{SU}(q-1)$, $\boldsymbol{Sp}(q-1)$, $\boldsymbol{SO}(2q-1$ or $2q-2)$, or $\boldsymbol{Spin}(2q-1$ or $2q-2)$, respectively, imbedded in the usual way, the canonical parameter of K in G is $\{1, 0, \cdots, 0\}$. With this in mind, we can use Theorem 10 to generalize some rather nice theorems of H. Zassenhaus [16] and G. Vincent [14]:

COROLLARY 10.1. *Let G be a classical group $\boldsymbol{U}(q)$, $\boldsymbol{SU}(q)$, $\boldsymbol{Sp}(q)$, $\boldsymbol{SO}(2q$ or $2q+1)$ or $\boldsymbol{Spin}(2q$ or $2q+1)$ and let K be a closed connected subgroup such that* rank. G — rank. $K = 1$ *and the canonical parameter of K in G is $\{1, 0, 0, \cdots, 0\}$. Let Γ be a finite subgroup of G with $\Gamma \cap \mathrm{ad}(G)K = 1$, such that the order of Γ is either the product of two primes or is prime to $2q$. Then Γ is cyclic.*

Proof. Suppose first that the order of Γ is prime to $2q$. As every abelian subgroup of Γ is cyclic and Γ has odd order, every Sylow subgroup of Γ is cyclic. Formula 2 of Theorem 10 now says that $mk_{jp}+(m/m_j)n'r^e$ is prime to mn', hence to m, so $m_j=m$ and consequently each $d_j=d$. This implies that d divides both q and the order of Γ, which are relatively prime, so $d=1$. But $d=1$ ifplies $r=1$ and thus that Γ is cyclic.

Suppose Γ has order mn with m and n prime, and that Γ is not cyclic. As Γ is not abelian, $m\neq n$. It follows that every Sylow subgroup of Γ is cyclic, so we look at Theorem 10, which, by switching m and n if necessary, is directly applicable. $d\,|\,n$ and n is prime, so $d=1$ or $d=n$. As $d=1$ implies that Γ is cyclic, $d=n$. Formula 2 of Theorem 10 then shows $m_j=m$, so $n_j'=n'=1$. Formula 1 of Theorem 10 then says that $k_{jp}+e$ is prime to n for $0\leqq e<n$, which is impossible. *QED.*

In addition to providing known information on spheres, Corollary 10.1 tells us something about the Grassmann manifolds $\boldsymbol{SO}(2q)/\boldsymbol{SO}(2q-2)$, $\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-1)$ and $\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-2)$. The formulae of Theorem 10 can yield all sorts of information by placing special conditions on the canonical parameter.

Chapter VI. Manifolds with non-zero Euler characteristic.

After stating that we would for the most part concentrate on the case $\text{rank}.\,G-\text{rank}.\,K\leqq 1$, we devoted our attention primarily to the case $\text{rank}.\,G-\text{rank}.\,K=1$. In this chapter, we will prove a theorem about the case where $\text{rank}.\,G=\text{rank}.\,K$. First recall the well-known fact ([10], p. 15) that a coset space G/K of a compact connected Lie group G by a closed subgroup K has Euler characteristic $\chi(G/K)\geqq 0$, and that $\chi(G/K)>0$ if and only if $\text{rank}.\,G=\text{rank}.\,K$. We will prove:

THEOREM 11. *Let M be a compact connected Riemannian homogeneous manifold with Euler characteristic $\chi(M)\neq 0$. Then there are only a finite number, up to isometry, of Riemannian manifolds which admit M as a Riemannian covering manifold.*

Remark. If M' admits a Riemannian covering of multiplicity n by M, we have $\chi(M)=n\cdot\chi(M')$. As $\chi(M')$ must be an integer, it is clear, intuitively speaking, that one can go down only a finite number of steps from M. The theorem says, then, that there are only a finite number of steps from M. The theorem says, then, that there are only a finite number of "directions" in which one can go down. These various "directions" will be seen to correspond roughly to the subgroups of the finite group G/G_0, where G_0 is the identity component of the group G of isometries of M.

Proof. We will first show that we may assume M simply connected, so that we will only have to consider normal coverings, i. e., coverings which are effectuated by the group of deck transformations. Let K_0 be the intersection of an isotropy subgroup K of the group G of isometries of M with the identity component G_0 of G, so $M = G_0/K_0$. K_0 contains a maximal torus of G_0 but contains no nontrivial normal subgroup of G_0; it follows that G_0 is centerless, hence semisimple, and thus has finite fundamental group. The homotopy sequence of the fibring $G_0 \rightarrow G_0/K_0 = M$ then shows that M has finite fundamental group, so the universal Riemannian covering manifold M'' is compact. We will be done if we show that only a finite number, up to isometry, of Riemannian manifolds admit a Riemannian covering by M'', so we may assume M simply connected.

We now need only show that there are only a finite number of properly discontinuous subgroups of G which give mutually non-isometric quotient manifolds of M. As conjugate subgroups of G give isometric quotients, we need only show that there are only a finite number of mutually non-conjugate properly discontinuous subgroups of G. As $\text{rank}. G = \text{rank}. K$, $\text{ad}(G)K$ contains G_0 so a properly discontinuous subgroup of G meets G_0 only at 1 and is thus isomorphic to a subgroup of the finite group G/G_0. The proof of Theorem 11 is thus reduced to:

LEMMA 11.1 (MOSTOW). *Let G be a compact Lie group and Γ a finite group. Then G contains only a finite number of conjugacy classes of isomorphs of Γ.*

Proof. Suppose the contrary and let $\{\Gamma_n\}$ be a sequence of mutually non-conjugate isomorphs of Γ which lie in G. We can assume that $\Gamma_n = \{\gamma_{1n}, \gamma_{2n}, \cdots, \gamma_{kn}\}$ ordered so that $\gamma_{jn} \rightarrow \gamma_{jm}$ is an isomorphism $\Gamma_n \rightarrow \Gamma_m$ for every m and n. As Γ is finite and G compact, we can assume that each sequence $\{\gamma_{jn}\}_n$ converges, $\{\gamma_{jn}\} \rightarrow \gamma_j$. It is clear that $\Sigma = \{\gamma_1, \cdots, \gamma_k\}$ is a subgroup of G, although we don't yet know that the γ_j are all distinct. A theorem of D. Montgomery and L. Zippin ([9], p. 216) says that Σ has a neighborhood U such that every subgroup of G in U is $\text{ad}(G)$-conjugate to a subgroup of Σ. As the Γ_n eventually lie in U, this contradicts their mutual non-conjugacy. *QED.*

The proof of Theorem 11 also furnishes a proof of:

THEOREM 11′. *Let M be a compact connected Riemannian homogeneous* *manifold. If M has finite fundamental group, there are only a finite number, up to isometry, of Riemannian manifolds with a given fundamental group which admit M as a Riemannian covering manifold. In any given case, there are only a finite number, up to isometry, of Riemannian manifolds which admit a normal Riemannian covering by M with a given group of deck transformations.*

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