Math 54 – Worksheet 6

GSI: Oltman, (10/17/19)

(5.1,5.2,5.3,5.4) Eigenvalues, Eigenvectors, Characteristic equation, Diagonalization, Geometry of Diagonalization

Problem 1. *Compute the eigenvalues and eigenvectors for* $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Solution 1. The eigenvalues are the diagonal entries, so we have $\lambda_1 = 1$ and $\lambda_2 = 2$. For λ_1 we get the eigenvector by computing the basis for the null space for:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

which has a basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. A similar computation gives us an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for λ_2

Problem 2. *Determine the eigenvalues for:*

$$A = \begin{pmatrix} 186 & 324 \\ 117 & 303 \end{pmatrix}$$

this problem is courtesy of my officemate Adam.

Solution 2. This is a long computation. The eigenvalues are not important.

Problem 3. *Compute* D^6 *for* $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$

Solution 3. This is just:

$$\begin{pmatrix} 5^6 & 0 \\ 0 & 3^6 \end{pmatrix}$$

Problem 4. *Compute* A^6 *for* $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$, *given the information that*

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution 4. If you trust the matrix multiplication then you need to verify that $\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$. This is true. Therefore this is a diagonlization. So we have that:

$$A^6 = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5^6 & 0 \\ 0 & 3^6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

Then you can compute the final answer by matrix multiplication

Problem 5. *What are the eigenvalues and eigenvectors for the above matrix?*

Solution 5. So we are given that $A = PDP^{-1}$, therefore $AP = PD$, so we know that the columns of P are the eigenvectors and the entries in D are the eigenvalues. Therefore we have:

$$\lambda = 5 \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda = 3 \Rightarrow v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Problem 6. *Compute the eigenvalues and eigenvectors for:*

$$A = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution 6. If you compute the characteristic polynomial, then you get (already factored) $(5-\lambda)^2(4-\lambda) = 0$. For $\lambda = 5$ we get eigenvectors:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda = 4$ we get:

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Problem 7. *Define* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(x) = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$. *Find a basis* $\mathcal{B}$ *such that* $[T]_{\mathcal{B}}$ *is diagonal.*

Solution 7. This is all down to finding the eigenvectors and eigenvalues which end up being $\lambda = 1, 3$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Then under the basis of these eigenvectors we have that:

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Problem 8. *Determine if the following are true or false*

1. *A matrix can have* 0 *as an eigenvalue*
2. *A matrix can have* $\vec{0}$ *as an eigenvector*
3. *If* A, B, C *are square matrices such that* $A = BCB^{-1}$ *then* A *and* C *have the same eigenvalues*
4. *If* A, B, C *are square matrices such that* $A = BCB^{-1}$ *then* A *and* C *have the same eigenvectors*

Solution 8.

1. True, this means the matrix is not invertible
2. False, by definition
3. True, this is a theorem probably

4. False, take the matrix in problem 4. Then A has an eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ but D has eigenvectors as the standard basis vectors.

Problem 9. *Determine all possible eigenvalues of A if $A^2 = 1$.*

Solution 9. The only possibilities are 1 and -1.

To see this, suppose that $Av = \lambda v$. Then:

$$v = \mathbb{I}v = A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda^2 v$$

This is true if and only if $\lambda = 1$ or $\lambda = -1$. To see that such matrices exist consider:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$