Math 54 – Worksheet 5
GSI: Oltman, (9/24/19)

4.1, 4.2, 4.3, 4.4, 4.5, 4.6 (Abstract Vector spaces and subspaces, bases and coordinates, dimension and rank).

Problem 1. *Let $\mathbb{P}_2$ be the vector space of all polynomials of degree less than or equal to 2. Prove that this is a vector space.*

Solution 1. $\mathbb{P}_2$ contains the zero vector. If you add or scale two degree 2 or less polynomials, you get another polynomial in $\mathbb{P}_2$

Problem 2. *Consider all polynomials in $\mathbb{P}_2$ prove whether or not the following are subspaces of $\mathbb{P}_2$*

1. *All polynomials such that $p(0) = 0$*
2. *All polynomials such that $p(0) = 1$.*
3. *All polynomials with positive coefficients.*

   Solution 2.

1. Yes, this is the null space of a linear map
2. No, this isn't, it doesn't contain the zero vector.
3. If you scale them by -1, you will no long be in this set.

Problem 3. *Consider the linear map $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined as:*

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$$

determine a basis for the null space of T. Determine a basis for the column space of T. What is the rank of T? Verify the Rank Theorem: $Rank(T) + \dim Null(T) = \dim(\mathbb{P}_2)$

Solution 3. Consider the basis $\{1, x, x^2\}$, then we have that $T(1) = (1, 1), T(x) = (0, 1), T(x) = (0, 1)$. Therefore we can write the standard matrix as:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The null space of this matrix has a basis element $(0, 1, -1)$. Therefore the basis for the null space T is $x - x^2$. The basis for the column space is the same as the basis of the column space of A which is just $\mathbb{R}^2$, so a basis could be $\{(1, 0), (0, 1)\}$. The rank of T is just the dimension of the image of T which is 2.

Since $\dim(\mathbb{P}_2) = 3$, we indeed have that $2 + 1 = 3$.

Problem 4. *Determine the roots of all the polynomials in the null space of the linear map $T : \mathbb{P}_2 \to \mathbb{R}^2$ with T defined as:*

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) + p(2) \end{pmatrix}$$

Solution 4. The standard matrix for this map will just be:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \end{pmatrix}$$

The null space for this is just $a = 0, b = -t(5/3), c = t$. Therefore the null space of T is just $t((-5/3)x + x^2)$. The roots of this are just 0 and 5/3

Problem 5. *Let's consider the vector space of fruit: $\mathcal{F}$. Obviously the vector space of fruit has a basis of three elements: $\{apple, orange, banana\}$. We will consider the monkey operator M, which is obviously a linear map from $\mathcal{F}$ to $\mathcal{F}$*

The monkey operator will eat the banana, so if M acts on the banana, we get zero. The monkey operator cannot distiguish apples from oranges, so $M(apple) = orange$ and (I'm not super sure how) $M(orange) = 2 \cdot apple$.

1. *Determine what the monkey does to the sum of an apple, an orange, and a banana*
2. *Determine the dimension of the null space of the monkey and the rank of the monkey*
3. *Suppose we have 100 identical monkeys in a line. The first monkey operates on 3 bananas, 2 oranges, and one apple. The second monkey operates on the outcome of what the first monkey produced. The third acts on the that result, and so on. What do we have after the last monkey is done?*
4. *Same question as before but with 101 monkeys.*
5. *Suppose you know that after a monkey has finished its operation, you are left with 3 apples and 2 oranges, do you know exactly what the monkey acted on?*
6. *Consider the estranged monkey $\tilde{M}$ which does the same thing as the monkey operator, but eats only half the banana, that is, it sends a banana to half a banana. Repeat problem 5.5 with the estranged monkey.*
7. *Suppose you know that after the estranged monkey is done, it leaves x bananas, y oranges, and z apples. What did it start with?*

Solution 5.

1. you get an orange and two apples.
2. Writing M is a matrix with respect to this basis, we get:

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   There are two pivot columns therefore the rank is 2, by the rank theorem, the dimension of the null space of M is just 1.

3. The input is just $\vec{v} = (3, 2, 1)$, we need to compute $A^{100}\vec{v}$, then convert this back to apples, oranges and bananas. Note that:

$$A^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore if you do this enough times you realize that:

$$(A^2)^m = \begin{pmatrix} 2^m & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore:

$$A^{100} = (A^2)^{50} = \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore:

$$A^{100}\vec{v} = \begin{pmatrix} 3 \cdot 2^{50} \\ 2 \cdot 2^{50} \\ 0 \end{pmatrix}$$

So we get $3 \cdot 2^{50}$ apples, $2 \cdot 2^{50}$ oranges, and no bananas.

4. We just need to compute AA^{100}:

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2^{51} & 0 \\ 2^{50} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then do the same as above.

5. No because A isn't invertible.

6. $\tilde{M}$ is invertible, the matrix is just:

$$\tilde{A} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Then you can invert $\tilde{A}$ (or just solve a linear system which is probably easier) to get:

$$\tilde{A}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Then compute $\tilde{A}^{-1}\vec{v}$ with $\vec{v} = (3, 2, 0)$, to get: 2 apples, $3/2$ oranges, and no bananas

7. Just multiply the inverse matrix with the vector (x, y, z) to get y apples, $x/2$ oranges and $2z$ bananas.