

4.1, 4.2, 4.3, 4.4, 4.5, 4.6 (Abstract Vector spaces and subspaces, bases and coordinates, dimension and rank).

Problem 1. Let \mathbb{P}_2 be the vector space of all polynomials of degree less than or equal to 2. Prove that this is a vector space.

Problem 2. Consider all polynomials in \mathbb{P}_2 prove whether or not the following are subspaces of \mathbb{P}_2

1. All polynomials such that $p(0) = 0$
2. All polynomials such that $p(0) = 1$.
3. All polynomials with positive coefficients.

Problem 3. Consider the linear map $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ defined as:

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$$

determine a basis for the null space of T . Determine a basis for the column space of T . What is the rank of T ? Verify the Rank Theorem: $\text{Rank}(T) + \dim \text{Null}(T) = \dim(\mathbb{P}_2)$

Problem 4. Determine the roots of all the polynomials in the null space of the linear map $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ with T defined as:

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) + p(2) \end{pmatrix}$$

Problem 5. Let's consider the vector space of fruit: \mathcal{F} . Obviously the vector space of fruit has a basis of three elements: $\{\text{apple}, \text{orange}, \text{banana}\}$. We will consider the monkey operator M , which is obviously a linear map from \mathcal{F} to \mathcal{F}

The monkey operator will eat the banana, so if M acts on the banana, we get zero. The monkey operator cannot distinguish apples from oranges, so $M(\text{apple}) = \text{orange}$ and (I'm not super sure how) $M(\text{orange}) = 2 \cdot \text{apple}$.

1. Determine what the monkey does to the sum of an apple, an orange, and a banana
2. Determine the dimension of the null space of the monkey and the rank of the monkey
3. Suppose we have 100 identical monkeys in a line. The first monkey operates on 3 bananas, 2 oranges, and one apple. The second monkey operates on the outcome of what the first monkey produced. The third acts on the that result, and so on. What do we have after the last monkey is done?
4. Same question as before but with 101 monkeys.
5. Suppose you know that after a monkey has finished its operation, you are left with 3 apples and 2 oranges, do you know exactly what the monkey acted on?
6. Consider the estranged monkey \tilde{M} which does the same thing as the monkey operator, but eats only half the banana, that is, it sends a banana to half a banana. Repeat problem 5.5 with the estranged monkey.
7. Suppose you know that after the estranged monkey is done, it leaves x bananas, y oranges, and z apples. What did it start with?