Math 54 – Worksheet 3
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Problem 1. *Let* $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, *and* $v_2 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$. *And let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation that maps* x *into* $x_1v_1 + x_2v_2$. *Find a matrix* A *such that* $T(x) = Ax$ *for each* x.

Solution 1.

$$\begin{pmatrix} -2 & 7 \\ 5 & -3 \end{pmatrix}$$

Problem 2. *Show that the transformation* T *defined by* $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ *is not linear.*

Solution 2. $T(0) = (0, 4, 0) \neq 0$

Problem 3. *In this problem we will consider the space of polynomials (up to degree 3) as a vector space. We will identify the arbitrary polynomial* $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ *with the vector* $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

1. *Write the constant polynomial* $p(x) = 1$ *as a vector in this way.*
2. *Compute the derivative of* $p(x)$ *and write this new polynomial as a vector*
3. *Repeat this process for the polynomials* $p(x) = x, x^2, x^3$.
4. *Put all the vectors you found in step* 2 *into a* 4×4 *matrix,* D. *Verify the following claim*

   Claim 1. *Given an arbitrary polynomial of degree at most* 4. *If you first convert this into a vector, apply* D *to it, then convert it back into a polynomial, you get the derivative of your original polynomial.*
5. *think about this*

Solution 3.

1. $(1, 0, 0, 0)$
2. $p' = 0$, so the new vector is $\vec{0}$.
3. We get:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

4.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Given an arbitrary polynomial as $p = \sum_0^3 a_n x^n$, we can convert it into a vector $(a_0, \ldots, a_3)$, if we multiply it by D, we get:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix}$$

Which can be converted to the polynomial $a_1 + 2a_2x + 3a_3x^2 = p'(x)$

Problem 4. *Let $\vec{v}_1 = (4, 5)$ and $\vec{v}_2 = (1, 3)$ and suppose that T is a linear map that sends $\vec{v}_1$ to $\vec{v}_2$ and $\vec{v}_2$ to $\vec{v}_1$. Find $T((1, 1))$.*

Solution 4. We need to write $(1, 1)$ as a linear combination of v_1 and v_2. This can be done by row reducing the augmented matrix:

$$\begin{pmatrix} 4 & 1 & 1 \\ 5 & 3 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2/7 \\ 0 & 1 & -1/7 \end{pmatrix}$$

Therefore $(1, 1) = (2/7)v_1 + (-1/7)v_2$. Therefore:

$$T(1, 1) = (2/7)T(v_1) - (1/7)T(v_2) = (2/7)v_2 - (1/7)v_1$$

And you get some vector.