Math 54 – Worksheet 2
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Problem 1. *Determine if the following vectors are linearly independent:*

$$\begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution 1. No, any collection of vectors that include the zero vector is linearly dependent. You should be able to prove this!

Problem 2. *Consider the two vectors u, v:*

$$u = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

1. *come up with a vector w so that u, v, w are linearly dependent*
2. *come up with a vector t so that t, v, w are linearly independent*
3. *come up with a vector s so that s, t, v, w are linearly independent*

Solution 2.

1. Let $w = u$
2. Add the standard basis vectors into our collection, and preform our algorithm to select a linearly independent spanning subset:
$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$
Row reduce this to find the pivot columns which correspond to linearly independent vectors in our original set.
$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1 & 1 & 2/3 & 5/3 \\ 0 & 0 & 1 & -4/3 & -7/3 \end{pmatrix}$$
Therefore the first three vectors are a linearly independent spanning set. So we get one example of t is $(1, 0, 0)$.
3. This is not possible. You can have at most three linearly independent vectors in $\mathbb{R}^3$

Problem 3. *Consider the matrix:*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix}$$

1. *row reduce A to row echelon form*

2. *Find the basis for the column space of A.*
3. *Find a basis for the null space of A.*
4. *Find a $\vec{b}$ such that $A\vec{x} = \vec{b}$ as no solution*
5. *Find a $\vec{b}$ such that $A\vec{x} = \vec{b}$ has one unique solution.*
6. *Find a $\vec{b}$ such that $A\vec{x} = \vec{b}$ has infinitely many solutions*

Solution 3.

1. row reduce (I'm going all the way to RREF, but this isn't necessary)

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2. The pivot columns tell you that the first two columns of A form a basis for $col(A)$
3. This is asking to solve $Ax = 0$. Using what we just did we have the augmented matrix:

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we can write the solution as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

Therefore the basis for the null space is just the single vector $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

4. Here we want to find a b not in the column space of b. That is a vector not a linear combination of the first two columns of A. Why not just come up with a linearly independent vector to the basis of $col(A)$? To do this row reduce:

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 7 & 8 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & -4/9 & 2/9 \end{pmatrix}$$

the first three columns are pivots, therefore $(1, 0, 0)$ is linearly independent from the two basis vectors, and therefore not in the span. Therefore $(1, 0, 0)$ is not in the column space of A.

5. This is not possible. Suppose $b \in Col(A)$, then we can write b is a linear combination of the first two columns of A. So if the columns of A are v_1, v_2, v_3, then suppose we can write $b = av_1 + bv_2$. But since v_3 is a linear combination of v_1 and v_2 (say $v_3 = cv_1 + dv_2$), then $b = v_3 + (a - c)v_1 + (b - d)v_2$. So any solution is not unique.
6. Pick anything in the column space, then by the above reasoning, we have infinitely many solutions.

Problem 4 (HARD)**.** *Let $\vec{x}, \vec{y}, \vec{z}$ be three different real numbers. Prove that:*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} \right\}$$

are linearly independent.

Prove the general statement: given $x_1, \ldots, x_n$, n distinct real numbers, then:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ldots, \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix} \right\}$$

are linearly independent vectors

Solution 4. Label the vectors $v_1, \ldots, v_n$. If we had constants $a_1, \ldots, a_n$ such that $\sum_1^n a_i v_i = 0$, then we have for all i: $\sum_{j=1}^n a_i x^{j-1} = 0$. Define the polynomial $p(t) = \sum_{i=1}^n a_i t^{i-1}$, then by the above property, we have that $p(x_i) = 0$ for all i, therefore p has n distinct root, but it is a degree $n-1$ polynomial, therefore it must be zero, therefore $a_i = 0$ for all i, therefore the vectors are linearly independent.