**Sum of Squares of Symmetric Polynomial Inequalities**

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### Introduction

Sum of squares (SOS) relaxations are often used to certify nonnegativity of polynomials. For symmetric polynomials, there are reductions to the problem size that can be done using tools from representation theory ([4]). The primary goal here is to use these tools to prove statements about symmetric polynomial inequalities explored in [2].

### Sum of Squares

Let \( p(x) \in \mathbb{R}[x_1, \ldots, x_d] \) be a nonnegative polynomial of degree 2d. If there exist \( q_1(x), \ldots, q_d(x) \in \mathbb{R}[x_1, \ldots, x_d] \) such that

\[
p(x) = q_1(x) + q_2(x) + \cdots + q_d(x)
\]

then \( p(x) \) is a sum of squares (SOS) polynomial.

Let \( [x]_d = [1, x_1, \ldots, x_d, x_1^2, x_2^2, \ldots, x_d^2] \) be a vector with all monomials of degree at most \( d \). Consider the equation,

\[
p(x) = \sum_{i=1}^d p_i x_i^d - [x]_d Q [x]_d^T \tag{1}
\]

where \( Q \) is a symmetric \((n \times n) \times (n \times n)\) matrix of unknowns (called the Gram matrix). This defines a system of linear equations in the coefficients of \( p \) and the entries of \( Q \).

\( p \) is SOS if there exists a PSD \( Q \) that satisfies (1). That is, certifying if \( p \) is SOS is equivalent to solving the SDP feasibility problem,

\[
p_0 = \sum_{i=1}^d Q_{ii}, \quad Q \succeq 0
\]

### Symmetric Polynomial Inequalities

Let \( h_1 \) be the homogeneous symmetric polynomial with respect to a partition \( \lambda \). The corresponding term-normalized homogeneous symmetric polynomial is

\[
H_\lambda(x) = \frac{h_\lambda(x)}{h_\lambda[1, \ldots, 1]}
\]

**Conjecture:** \( H_\lambda(x) \leq H_\mu(x) \), \( x \geq 0 \) \( \Leftrightarrow \lambda \preceq \mu \).

Similar statements have been proven for the elementary, power-sum, and Schur symmetric polynomials, though not using sums of squares. This project aims to prove the conjecture via SOS programs. First, each \( x_i \) is replaced with \( y_i^2 \) to eliminate the constraint \( x \geq 0 \). Then the difference \( H_\lambda(y) - H_\mu(y) \) is the object of study.

### Symmetry-Adapted Basis

To exploit the symmetries of \( H_\lambda(y) - H_\mu(y) \), consider the usual action \( S_n \subseteq \mathbb{Q}[x_1, \ldots, x_n] \). The associated matrix representation can be block diagonalized. Amazingly, this also block diagonalizes the Gram matrix \( Q \) from (1). For example, if \( n = d = 3 \) and using the \( Q \) from the Variable Reduction section,

\[
T^T QT = \begin{pmatrix}
\mathcal{C} & \mathcal{E} \\
\mathcal{E}^T & \mathcal{F}
\end{pmatrix}
\]

where

\[
\mathcal{C} = \begin{pmatrix}
q_1 + 2q_2 & \sqrt{3} q_1 + q_3 & \sqrt{3} q_2 \\
\sqrt{3} q_1 + q_3 & q_1 + q_2 + 2q_3 + q_4 & \sqrt{6} q_2 + q_4 \\
\sqrt{3} q_2 & \sqrt{6} q_2 + q_4 & q_2 + 2q_4 + 2q_5 + q_6 + q_7
\end{pmatrix}
\]

Moreover, it changes the monomial basis in \([x]_d\) to a symmetry-adapted basis,

\[
\begin{pmatrix}
\frac{\sqrt{3}}{2} (q_1 + q_3) \\
\frac{\sqrt{3}}{2} (q_1 + q_3) \\
\sqrt{3} (q_2 + q_4)
\end{pmatrix}
\]

Now the SDP can be run on these smaller block matrices with fewer variables.

### Example

When \( n = 3 \) and \( \lambda = \mu = \{3\}, H_\lambda(y) - H_\mu(y) \) has a SOS decomposition for all three cases. For example,

\[
H_{\{3\}}(y) - H_{\{1,1,1\}}(y) = \frac{1}{8} \sqrt{3} y_1^3 + \frac{1}{8} \sqrt{3} y_2^3 + \frac{1}{18} (\sqrt{2} y_1^3 + \sqrt{2} y_2^3 + \sqrt{2} y_3^3)
\]

### Variable Reduction

A reduction in unknowns of the Gram matrix \( Q \) can be made by first restricting to the fixed-point subspace, i.e., matrices which commute with the matrix representation from the usual \( S_n \) action on \([x_1, \ldots, x_n]_d\). For example, if \( n = d = 3 \),

\[
Q = \begin{pmatrix}
\mathcal{A} & \mathcal{B} & \mathcal{C} \\
\mathcal{B} & \mathcal{D} & \mathcal{E} \\
\mathcal{C} & \mathcal{E} & \mathcal{F}
\end{pmatrix}
\]

where \( j = k \) because the matrix must be symmetric.

The number of variables is reduced from 55 to 13.

### The Trivial Block

For symmetric polynomials of degree \( 2n \) in \( n \) variables, the matrix associated to the trivial representation is

\[
Q^{(n)} = \begin{pmatrix}
\text{colsum}(\lambda_1) & \cdots & \text{colsum}(\lambda_n) \\
\text{colsum}(\lambda_1) & \cdots & \text{colsum}(\lambda_n) \\
\vdots & \ddots & \vdots \\
\text{colsum}(\lambda_1) & \cdots & \text{colsum}(\lambda_n)
\end{pmatrix}
\]

where \( \lambda_i = \sqrt{\text{Orbit}(x^{3i})} \), \( p \) is the number of partitions of \( n \), and \( \lambda_1 \) is the matrix block of \( Q \) indexed by \( x^{3i} \) and \( x^{2i} \) on the rows and columns respectively.

For example, \( 3 \text{colsum}(\lambda_1) = \frac{\sqrt{3}(c+d+e)}{6} \) for \( Q \) above.

### References


