Research Statement

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1 General Research Interests

I am generally interested in Gromov-Witten theory and adjacent fields, broadly within the areas of algebraic geometry and mathematical physics. My work thus far has consisted of developing new classes of Gromov-Witten invariants, and relating them to more well-understood classes of invariants, and examining their applications to enumerative problems and to understanding the geometry of the Gromov-Witten moduli spaces. Below I will provide general high-level accounts of the projects I've worked on. Everything discussed here works at the level of S_n-equivariant Gromov-Witten invariants, but is discussed in a lower level of generality to maintain ease of reading.

2 Twisted K-Theoretic Invariants

The following discusses work contained in [7]. Given a smooth projective variety X, denote the moduli space of stable maps by $\overline{\mathcal{M}}_{g,n,d}(X)$.

K-theoretic Gromov-Witten invariants (with gravitational descendants) are defined as holomorphic Euler characteristics on $\overline{\mathcal{M}}_{g,n,d}(X)$, twisted by the *virtual structure sheaf* $\mathcal{O}^{vir}$. Namely, given Laurent polynomials $\alpha_i \in K^*(X)[q^{\pm 1}]$, the K-theoretic invariants are defined, using correlator notation, as:

$$\langle \alpha_1, \dots, \alpha_n \rangle_{g,n,d} := \chi(\overline{\mathcal{M}}_{g,n,d}(X); \mathcal{O}^{vir} \otimes \prod ev_i^* \alpha_i(L_i))$$

These invariants are packaged into genus-g generating functions $\mathcal{F}_{g,X} := \sum_d Q^d \sum_n \frac{1}{n!} \langle t, \dots, t, \dots, \rangle_{g,n,d}$, representing all counts of connected curves in X. The total descendant potential represents the contribution from disconnected curves, and is defined as $\mathcal{D}_X := exp(\sum_g \hbar^{g-1} \mathcal{F}_{g,X})$.

Twisted K-theoretic invariants are defined by tensoring $\mathcal{O}^{vir}$ with some additional classes from $K^*(\overline{\mathcal{M}}_{g,n,d}(X))$. Given an invertible K-theoretic characteristic class C, denote the generating function twisted by $C(T^{vir})$ as $\mathcal{D}_X^C$. I proved a twisting theorem providing a relationship between $\mathcal{D}_X^C$ and $\mathcal{D}_X$. When written directly, the formula is quite complicated, but it is easier to express in terms of Givental's symplectic loop space formalism [5], where $\mathcal{D}_X$ and $\mathcal{D}_X^C$ can be realized as quantum states (denoted $\langle \mathcal{D}_X^C \rangle, \langle \mathcal{D}_X \rangle$ respectively) of a certain infinite-dimensional symplectic space $\mathcal{K}$, with a choice of Lagrangian polarization $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$.

Theorem.

$$\langle \mathcal{D}_X^C \rangle = \langle \nabla \mathcal{D}_X \rangle$$

Where ∇ is the composition of a quantized multiplication operator, a change of the origin, and a quantized symplectic transformation changing $\mathcal{K}^-$.

2.1 Emulating Cobordism Theory in K-Theory

As a consequence of this, we can use K-theory to emulate other complex-oriented cohomology theories. Let $MU^*(X)$ denote complex cobordism theory, the cohomology theory determined by the Thom spectrum. Over $\mathbb{C}$, MU^* is universal among all complex-oriented cohomology theories.

By the universal topological Riemann-Roch theorem [4], there is a "K-theoretic Chern-Dold character" $Ch_K : MU^*(X) \to K^*(X) \otimes MU^*(pt)$, satisfying $\pi_* \alpha = chi(X; Ch_m(\alpha) Td_K(TX))$, where $Td_K()$ is the universal multiplicative characteristic class in K-theory. We can thus interpret the inputs to twisted invariants as coming from some specialization of $MU^*(X)$ via Ch_K, and twisted K-theoretic correlators correspond to the left hand side of the universal Riemann-Roch theorem.

In this manner, we can treat twisted invariants as if they were pushforwards from $MU^*(\overline{\mathcal{M}}_{g,n,d}(X))$ (although the latter is not well-defined, since $\overline{\mathcal{M}}_{g,n,d}(X)$ is not a manifold). This procedure was done by Coates-Givental in [2], using cohomological invariants rather than K-theoretic ones, and given the name "fake quantum cobordism theory". Since K-theory interacts differently with the orbifold structure of $\overline{\mathcal{M}}_{g,n,d}(X)$, the emulating bordism theory in cohomology and K-theory yields different results. As such, we give the latter the name "multiplicative quantum cobordism theory", since according to the philosophy of chromatic homotopy theory, Ch_K is determined by the formal group law associated to K-theory, which is the multiplicative one.

2.2 Example: Hirzebruch K-Theory

Of particular interest is the case where C is the S^1-equivariant K-theoretic Euler class, i.e.

$$C(V) = \Lambda_{-y}(V) := \sum (-y)^p \bigwedge^p V^*$$

. Equivalently, it is the class determined on line bundles by $C(L) = 1 - yL^{-1}$, and extended multiplicatively. The resulting modification of quantum K-theory appears in calculating the K-theoretic invariants of the Grassmanian, as discussed in [6]. The resulting cohomology theory is determined from $MU^*(X)$ by replacing the bordism class of a manifold by its Hirzebruch χ_y-genus, as such the resulting theory of invariants is called "Hirzebruch K-theory".

When $y = 0$, we recover ordinary K-theory. Under the limit $y \mapsto 1$, the symplectic formalism degenerates, and the characteristic class C is no longer invertible (for examples it vanishes for the trivial bundle). The associated genus is the complex bordism invariant that is equal to the topological Euler characteristic in the case where the manifold is almost complex. My current project involves investigating what happens in this particular case, and will be discussed in the next section.

3 Euler-Theoretic Invariants

The following discusses currently unpublished work. A very preliminary draft discussing this theory is available on my website. If $\mathcal{X}$ is an orbifold, its ordinary Euler characteristic, denoted $\chi(\mathcal{X})$ is just the Euler characteristic of its coarse moduli space. Its orbifold Euler characteristic, denoted $\chi^{orb}(X)$, is a weighted count of simplices by isotropy groups, and is not in general in integer. If X is compact and complex, both of these quantities satisfy Chern-Gauss Bonnet-Theorems. $\chi^{orb}(\mathcal{X}) = \int_{\mathcal{X}} c(T\mathcal{X})$. Here c denotes the total Chern class.

These formulas can be written down equally well in the case where $\mathcal{X}$ is a smooth complete Deligne-Mumford stack, and can be easily adapted to the case where $\mathcal{X}$ is virtually smooth. Given such an $\mathcal{X}$, define its virtual Euler characteristic $\chi^{vir}(\mathcal{X})$ as $\int_{[I\mathcal{X}]^{vir}} c(T^{vir} I\mathcal{X})$.

Euler-theoretic Gromov-Witten invariants are a class of invariants representing the virtual Euler characteristics of certain moduli spaces of stable maps. Given holomorphic maps $f_i : Y_i \to X$, consider the following diagram:

$$\begin{array}{ccc} M_f & \xrightarrow{\alpha} & \prod_i Y_i \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \prod_i f_i} \\ \overline{\mathcal{M}}_{g,n,d}(X) & \xrightarrow{\prod_i ev_i} & X^n \end{array}$$

If $[f_i]$ are the complex bordism classes of f_i (relative to X), we define the Euler-theoretic invariant $\overline{\langle [f_1], \ldots, [f_n] \rangle}^E_{g,n,d}$ to be:

$$\overline{\langle [f_1], \ldots, [f_n] \rangle}^E_{g,n,d} := \int_{[I\overline{\mathcal{M}}_{g,n,d}(X)]^{vir}} c(T^{vir} I\overline{\mathcal{M}}_{g,n,d}(X)) \prod_{j=1}^{k} \prod_{i=1}^{n} Iev^*_{i,j}(f_{i*} c(T_{f_i})$$

Denote the all-genus generating function for these invariants $\mathcal{D}^E_X$.

If the above diagram is (virtually) transverse, this integral is in fact equal to the virtual Euler characteristic of $\mathcal{M}_{\{}$.

3.1 Enumerative Applications

If f_i are embeddings and the dimension of M_f is 0, then the Euler characteristic of M_f is a count of the stable maps with the ith marked point passing through Y_i. As such, in this special case Euler-theoretic invariants have the same enumerative interpretations as the usual Gromov-Witten invariants, so they can be used as a framework for enumerative problems.

Theorem. *Considered in this manner, Euler theoretic invariants have the following enumerative properties:*

- *The invariants provide "counts" even for enumerative problems with nonzero expected dimension.*
- *Boundary contributions can be easily removed by replacing the tangent bundle with its logarithmic counterpart, and doing so is equivalent to subtracting the virtual Euler characteristics of the boundary contributions.*
- *(Conjectural) The invariants take integer values for all possible targets X.*

I have shown the invariants are integers in the cases where $\overline{\mathcal{M}}_{g,n,d}(X)$ is genuinely (rather than virtually) smooth, which includes $X = pt$ in all genera, and $g = 0$, X convex. In this case, the invariants compute the genuine Euler characteristic $\chi(M_f)$, where $\prod f_i$ has been deformed so M_f is a stratified orbifold. However even absent such an interpretation, I expect integrality to remain.

As currently defined, Euler-theoretic invariants are difficult to compute. However, as alluded to earlier, they are computable from Hirzebruch K-theoretic invariants. Recall that given inputs $V_i \in K^0(X)$, Hirzebruch-theoretic correlators are defined as:

$$\langle V_1, \dots, V_n \rangle^y_{g,n,d} := \chi(\overline{\mathcal{M}}_{g,n,d}(X); \mathcal{O}^{vir} \otimes \Lambda_{-y}(T^{vir}) \prod_i ev_i^* V_i).$$

Theorem.

$$\lim_{y \mapsto 1} \langle f_{1*}\Lambda_{-y}(T_{f_i}), \dots \rangle^y_{g,n,d} = \overline{\langle [f_1], \dots, [f_n] \rangle}^E_{g,n,d}$$

In addition, they are computable via integration on $\overline{\mathcal{M}}_{g,n,d}(X)$, rather than its inertia orbifold. Define *fake Euler-theoretic invariants* in the same manner as genuine ones, but integrate over $\overline{\mathcal{M}}_{g,n,d}(X)$ instead of $I\overline{\mathcal{M}}_{g,n,d}(X)$. By a theorem of Coates, these fake invariants can be recovered from ordinary cohomological ones.

Naming the corresponding potentials $\mathcal{D}^{fake,E}$, the relationship between genuine and fake potentials has the form of a Wick-type summation over a certain class of labelled graphs. ∇_e is an operator for each edge, and vertices are labelled by positive integers M,

Theorem.

$$\mathcal{D}^E_X = exp(\sum_{edges\ e} \nabla_e) \bigotimes_{labelled\ vertices\ with\ value\ M} \mathcal{D}^{fake,E}_{X \times B\mathbb{Z}_M}.$$

The invariants of $X \times B\mathbb{Z}_M$ can be converted into ones for X using a result of Jarvis-Kimura in [8]. In the case where $\overline{\mathcal{M}}_{g,n,d}(X)$ is genuinely smooth, and all inputs are set to 1, the above formula expresses the ordinary Euler characteristic of $\overline{\mathcal{M}}_{g,n,d}(X)$ in terms of its orbifold Euler characteristics. For the case $X = pt$, this is equivalent Bini-Harer's formula (50) in [1].

4 Current and Future Plans

4.1 Immediate Goals

I am currently working on finding some applications of the Euler theoretic invariants. To that end, I am in the process of better understanding specific examples, such as for target $\mathbb{CP}^{\ltimes}$, and comparing the J-functions to J-functions from existing theories. Ideally this will help place the Euler-theoretic invariants within Givental's symplectic loop space formalism, and from there it will be easier to find connections with other areas.

4.2 Potential Applications of Euler Invariants

There are a few potential directions that stand out in terms of applying the theory of Euler invariants. I am not sure which of these avenues will be the most fruitful, but I am currently giving them all potential consideration. One I plan to look into is better understanding the enumerative properties of Euler theoretic invariants. I'm interested in establishing integrality for all target spaces, as well as better understanding under what circumstances the invariants are enumerative.

One potential avenue for applying Euler-theoretic invariants is in relation to the work of Norbury et al. on counting lattice points in $M_{g,n}$. In [3], it is shown that the quantities $\chi^{orb}(\mathcal{M}_{g,n})$ appear as special cases of certain Eynard-Orantin invariants for a particular choice of spectral curve, and can be interpreted as solutions to particular Hurwitz problems. The resulting theory of invariants also provides answers for Hurwitz problems with higher-dimensional spaces of solutions, and the associated numbers also represent by the orbifold Euler characteristic of the solution space.

These findings suggest two directions in which to proceed. The first would be to figure out what are the analogues of Hurwitz problems that can be solved by Euler invariants. And the second is to figure out if there is an analogue of Eynard-Orantin invariants that encapsulates a generating function fro Euler-theoretic invariants. For the latter task, some modifications may be necessary to Eyanrd-Orantin's framework in order to account for the slightly more complicated dilaton equation that the Euler invariants satisfy.

I was also recently made aware of an old question posed by Harer, which is to compute the Euler characteristic of the moduli spaces of curves with level structure. If enough of the tautological classes used in defining Euler-theoretic invariants transfer over to the moduli spaces of curves with level structure, I would be able to compute this quantity (for curves with at least 3 marked points), via similar techniques to calculations I have already done. t

4.3 Additional Plans

A more long-term goal, which was the initial motivation behind working on twisted K-theoretic invariants, is to develop a theory of Gromov-Witten invariants in elliptic cohomology. From speaking to algebraic topologists in the area, it seems as if the resulting theory around equivariant transfer maps has developed to the point where it might be possible to directly define the analogue to Gromov-Witten invariants (since the G-W moduli spaces are virtually smooth orbifolds, there needs to be a notion of pushforward in elliptic cohomology that applies in this setting). The case of Hirzebruch-K-theory, would likely end up as an intermediary between quantum K-theory and the elliptic theory, in the same manner that the Hirzebruch χ_y-genus is a degenerate case of the elliptic genus.

References

[1] G. Bini and J. Harer. Euler characteristics of moduli spaces of curves. *Journal of the European Mathematical Society*, 13:487–512, 2005.

[2] Tom Coates and Alexander Givental. *Quantum Cobordisms and formal group laws*, pages 155–171. Birkhäuser Boston, Boston, MA, 2006.

[3] Norman Do and Paul Norbury. Counting lattice points in compactified moduli spaces of curves. *Geometry & Topology*, 15(4):2321 – 2350, 2011.

[4] E. Dyer, J. F. Adams, and G. C. Shepherd. *Relations between cohomology theories*, page 188–195. London Mathematical Society Lecture Note Series. Cambridge University Press, 1972.

[5] Alexander Givental. Permutation-equivariant quantum K-theory IX: Quantum Hirzebruch-Riemann-Roch in all genera. 2017.

[6] Alexander Givental and Xiaohan Yan. Quantum K-theory of grassmannians and non-Abelian localization. 2020.

[7] Irit Huq-Kuruvilla. Multiplicative quantum cobordism theory, 2021.

[8] Tyler J. Jarvis and Takashi Kimura. The orbifold quantum cohomology of the classifying space of a finite group. In *Orbifolds in Mathematics and Physics, Contemp*, 2002.