Permutation-Equivariant Euler Theory

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1 Introduction

There are many different generalizations of the notion of Euler characteristics to orbifolds (a family of which are documented in [2]). They are given different names throughout the literature, to eliminate ambiguity, We will specify the two we use here below. The *ordinary Euler characteristic* of an orbifold $\mathcal{X}$ with coarse space X, is just $\chi(X)$. It is integral and behaves well with respect to covering maps, provided ones uses the degree of the map on coarse spaces, rather than the orbifold degree. We denote it by $\chi(X)$.

The other Euler characteristic we consider is the *Euler-Satake characterstic.* Given an effective orbifold $\mathcal{X}$ with underlying topological space $\mathcal{X}$, we can choose a presentation $\mathcal{X}$ as a quotient of a manifold M by an almost-free action of a Lie Group G.

An *orbifold triangulation* S on $\mathcal{X}$ is a triangulation on X such that each point in the interior of a simplex Δ has the same isotropy group G_Δ under the G-action, and that each simplex is small enough to be contained in a single orbifold chart of $\mathcal{X}$. Such a triangulation always exists, by a result of Illman [7].

Definition 1.1 ([10])**.** The *Euler-Satake Characteristic* $\chi^{ES}(\mathcal{X})$ is defined to be (for a choice of orbifold triangulation S):

$$\sum_{\Delta\in S}\frac{(-1)^{dim(\Delta)}}{|G_\Delta|}$$

By the same arguments as the ordinary cohomological Euler characterstic for manifolds, χ^{ES} is independent of the choice of triangulation.

The Euler-Satake characteristic is related to its ordinary Euler characteristic by the following theorem:

Theorem 1.2. *For an arbitrary orbifold $\mathcal{X}$:*
$\chi(\mathcal{X}) = \chi^{ES}(\mathcal{I}\mathcal{X})$

Proof. For a simplex δ with isotropy group G_δ, it appears in C_{G_δ} components of the inertia orbifold, where C_{G_δ} is the class number, and in the component corresponding to class c is counted with multiplicity $\frac{1}{|c|}$. Thus the total contribution of preimages of δ in $I\mathcal{X}$ is $\sum_{c\in Cl(G_\delta)}\frac{1}{|c|} = 1$.

□

If $\mathcal{X}$ is a compact almost complex orbifold, the Euler-Satake characteristic satisfies the Poincare-Hopf and Chern-Gauss-Bonnet theorems.

Theorem 1.3 ([10])**.** *For V a vector field with isolated singularities on $\mathcal{X}$:*

$$\chi^{ES}(\mathcal{X}) = \sum_{\partial\in Sing(V)} Ind_p^{orb}(V)$$

(The orbifold index of a vector field at a point is its index in an orbifold chart containing that point, divided by the order of the isotropy group at that point).

Theorem 1.4 ([10])**.**

$$\chi^{ES}(\mathcal{X}) = \int_{\mathcal{X}} c_{top}(T\mathcal{X})$$

As a corollary, $\chi(\mathcal{X}) = \int_{IX} c_{top}(IX)$, and both Euler characteristics are invariant under deformations.

Harer-Zagier [5] famously computed the Euler-Satake charactersitics of the moduli spaces of $n-$pointed smooth algebraic curves curves $M_{g,n}$. This result was extended by Bini-Harer [1] to the ordinary Euler characteristics of the moduli spaces and their stable compactifications $\overline{M}_{g,n}$, by expressing them in terms of the corresponding Euler-Satake characteristics. racteristics computed by Harer-Zagier.

Remark. Those authors refer to the Euler-Satake characteristic is referred to as the "orbifold Euler characteristic". Since that term refers to other invariants in different parts of the literature, we do not use it.

In this work, we develop versions of Gromov-Witten theory based on these invariant, meaning that we define correlators to represent the Euler and Euler-Satake characteristics of $M_{g,n,d}(X)$ and its compactification by stable maps $\overline{M}_{g,n,d}(X)$ (henceforth abbreviated to $X_{g,n,d}$), as well as subloci of those determined by constraints on the map. The versions of this theory developed using the Euler characteristic result in integer invariants, regardless of the choice of target space, and thus provide an integral alternative to the usual Gromov-Witten invariants. When $X_{g,n,d}$ is a smooth orbifold (i.e. $g = 0$, X convex), we directly interpret these correlators in terms of Euler characteristics of subspaces of $X_{g,n,d}$.

Having done this, we give formulas relating the generating functions of the ordinary Euler characteristics to those of the Euler-Satake characteristics, generalizing the formula of Bini-Harer.

2 Gromov-Witten Invariants

2.1 Motivation: Computing Euler Characteristics of Fiber Products

For the rest of this text, we use the following notation:

Given a map f between orbifolds, If will denote the induced map between inertia orbifolds.

Given a bundle V. $c(V)$ denotes the total Chern class of that bundle.

Given a morphism $f : X \to Y$, the relative tangent bunlde T_f is $T_X - f^*T_Y$.

The motivation for the definition of the Gromov-Witten invariants we consider comes from the following theorem:

Theorem 2.1. *Given a fiber diagram:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{h} & Z \end{array}$$

Where $\mathcal{X}$ is a compact almost-complex orbifold, Y and Z are compact almost-complex manifolds.

If $\mathcal{M}$ (which is a priori an orbispace), has the structure of a smooth orbifold, we have:

$$\chi(\mathcal{M}) = \int_{I\mathcal{X}} c(TI\mathcal{X}) Ih^* g_* c(T_g)$$

Proof. Taking inertia orbifolds on all sides gives that:

$$\begin{array}{ccc} I\mathcal{M} & \xrightarrow{If} & Y \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} \\ I\mathcal{X} & \xrightarrow{Ih} & Z \end{array}$$

is also a fiber diagram.

Thus

$$Ih^*g_*c(T_g) = j_*If^*c(T_g) = Ij_*c(If^*T_g).$$

Another consequence of the above diagram being a fiber square is that $If^*T_g = T_{Ij}$, so we can rewrite

$$\int_{I\mathcal{X}} c(TI\mathcal{X})Ih^*g_*c(T_g) = \int_{I\mathcal{X}} c(TI\mathcal{X})Ij_*(T_{Ij}).$$

By the projection formula above is equal to:

$$\int_{I\mathcal{M}} c(Ij^*TI\mathcal{X})c(T_{Ij}).$$

Since $T_{Ij} = TI\mathcal{M} - Ij^*TI\mathcal{X}-$, this integral is equal to

$$\int_{I\mathcal{M}} c(TI\mathcal{M}) = \chi(\mathcal{M}).$$

□

Remark. If we used the top Chern class instead of the total one, this computation would not necessarily possible due to the potential of division by 0.

If X is a homogenous Kahler target space to Gromov-Witten theory, then the Gromov-Witten moduli spaces $X_{g,n,d}$ are genuine orbifolds. Given (almost)-holomorphic maps $f_i : Y_i \to X$, we can define the orbispace M_f to be the top left corner of the fiber diagram:

Remark. If $f_i : Y_i \to X$ embeds Y as the zero section of a bundle V on X, then $f_{i*}c(T_{f_i})$ is equal to $\frac{c_{top}(V)}{c(V)}$, the "Chern-Euler theoretic Euler class" of V.

2.2 Application to Gromov-Witten Moduli Spaces

Let $X_{g,n,d}$ be the moduli space of degree d maps from n-pointed stable curves to X. Let $f_i : Y_i \to X$ be (almost)-holomorphic maps from (almost)-complex manifolds Y_i. Consider the diagram:

$$\begin{array}{ccc} M_f & \xrightarrow{\alpha} & \prod_i Y_i \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \prod_i f_i} \\ X_{g,n,d} & \xrightarrow{\prod_i ev_i} & X^n \end{array}$$

If $X_{g,n,d}$ and M_f are orbifolds, by the the theorem in the previous section we can conclude:

$$\chi(M) = \int_{IX_{g,n,d}} c(TIX_{g,n,d}) \prod_i Iev_i^* f_{i*}c(T_{f_i}).$$

Both quantities only depend on the complex bordism class of f_i, denoted $[f_i]$. (More precisely, they depend on the image of $[f_i]$ under the map $\phi : MU^*(X) \to H^*(X)$ induced by the map $MU^*(pt) \to \mathbb{Z}$ sending a class of a manifold to its topological Euler characteristic).

3 Permutation-Equivariant Invariants

Based on this idea, we will define a theory of Gromov-Witten invariants encompassing the above Euler characteristics, and the Euler characteristics of their S_n-fixed loci.

Given an element $h \in S_n$, with cycle structure given by an integer vector ℓ, with ℓ_r cycles of length r. Take inputs $\alpha_1, \dots, \alpha_{|\ell|} \in MU^*(X)$, such that for all α in the pth cycle of length q, they are all the same, equal to $\alpha_{p,q}$.

Define the S_n-equivariant Euler-theoretic correlator $\overline{\langle \alpha_{1,1}, \dots, \alpha_{\ell_1,1}, \dots, \alpha_{k,\ell_k} \rangle}^E_{g,\ell,d}$to be

$$\prod_r r^{-\ell_r} \int_{IX^h_{g,|\ell|,d}} c_{total}(T^{vir}) \prod_{j=1}^{k} \prod_{i=1}^{\ell_j} I\hat{ev}^*_{i,j}(\phi(\alpha_{i,j}))$$

Here $\hat{ev}_{i,j}$ denotes the restriction of *any* of the evaluation maps from marked points permuted by the ith length-j cycle of h. All such resctrictions are equal since we restrict to the h-fixed locus. By construction these correlators are polyadditive.

Package these correlaotrs into a genus-g generating function $\mathcal{F}^E_{g,X} := \sum_d Q^d \sum_\ell \frac{1}{\prod_r \ell_r!} \langle \boldsymbol{t_1}, \dots, \boldsymbol{t_r}, \dots \rangle^E_{g,\ell,d}$. Here the input ${}_{t_r} = \sum \phi_a \boldsymbol{t_{a,r}}$ for ϕ_a running a basis of $MU^*(X)$.

Assume we have h acting on some disconnected curve with components $C_1, \dots, C_k$. The corresponding moduli space only has non-trivial h-fixed locus if h cyclically permutes the k components, and h^b acts on each as a permutation of the marked points on that curve. The union of the curves also has Euler characteristic $k(2-2g)$.

Motivated by this fact, we define the total descendant potential $\mathcal{D}^E_X$ to be

$$exp(\sum_g \sum_k \hbar^{k(g-1)} R_k(\mathcal{F}^E_{g,X})/k)$$

. The R^k changes the input $(\boldsymbol{t_1}, \dots, \boldsymbol{t_r}, \dots)$ into $(\boldsymbol{t_k}, \dots, \boldsymbol{t_{rk}}, \dots)$, which are there to correct for the fact that the cycles of h are all k times longer than the corresponding cycle of h^k.

3.1 Incomplete Invariants

An advantage of the Euler characteristic is that it is additive, so to define an integral representing contributions from the interior of the moduli space, we can simply subtract the corresponding Euler characteristics from the boundary. We call these "incomplete" Euler invariants. They are denoted without bars, i.e. the incomplete Euler correlator is denoted $\langle [f_1], \dots, [f_n] \rangle^E_{g,\ell,d}$.

Denote $\frac{\partial}{\partial t_{a,r}}$ by $\partial_{a,r}$, and let $g^{a,b}$ be the matrix of the MU^*-theoretic Poincare pairing. To define them explicitly, introduce the following notation: Given a permutation h with cycle structure ℓ, boundary components of the associated moduli space $X^h_{g,n,d}$ arise from curves splitting into components along cycles of nodes, rather than individual nodes themselves.

A given codimension-d boundary stratum is a fiber product of lower-dimensional moduli spaces : $X^{h_1}_{g_1,|\ell_1|,d_1} \times_{(X_{0,3,0})^{r_1}} X^{h_2}_{g_2,|\ell_2|,d_2} \times \dots X^{h_k}_{g_k,|\ell_k|,d_k}$, together with an assignment I of the locations of the original marked points, and cycles corresponding to genus reductions. Call the set of such I $B(X^h_{g,|\ell|,d}))$, each $I \in B(X^h_{g,|\ell|,d})$ determines a unique closed boundary stratum X_I, as well as an "open" boundary stratum M_I, obtained by replacing the moduli spaces in the fiber product with their uncompactified counterparts.

We denote the contribution to the correlator (with original inputs $\alpha = (\alpha_{\boldsymbol{1},\boldsymbol{1}}, \dots))$ by $C^E_\alpha(X_i)$, which is defined by:

$C_\alpha(X_I) := \prod_{i=1}^k \overline{\langle \alpha_I \rangle}^E_{g_i,n_i,d_i}$.

Here the insertion α_I into a correlator means that inputs are determined according to the assignment I, with inputs corresponding to a cycle of nodes of length r receiving $\sum r g^{a,b} \phi_a \otimes \phi_b$ (either as an individual inpout in the case of nodes corresponding to a genus reduction, or split between the two correlators for reducible nodes. We will omit the subscript α when the inputs are obvious. Using this notation, we can formally define the incomplete correlators:

$$\langle \alpha_{1,1}, \dots \rangle^E_{g,n,d} := \sum_I (-1)^d C^E_\alpha(X_I).$$

The factor r in the input for nodes accounts for the normalization coefficient $\prod_r r^- \ell_r$, as each node contributes an additional cycle of length r which is not present in ℓ.

Remark. We can also rewrite these equations in terms of the "open" boundary strata, which will be useful later. Given the same labelling data, let M_I be the "open" part (i.e. ignoring the singular locus in each term in the fiber product) of X_I. Define $C(M_I)$ in the same way as $C(X_I)$, but with all complete correlators replaced with incomplete ones. Since the open boundaries do not overlap we can rewrite the previous equation as:

$$\langle \alpha_1, \dots, \alpha_n \rangle^E_{g,n,d} = \overline{\langle \alpha_1, \dots, \alpha_n \rangle}^E_{g,n,d} - \sum_I C^E(M_I).$$

Theorem 3.1. *The operator* $\Delta := exp(-\bigoplus_r \frac{1}{2} r \hbar^r \sum_{a,b} g^{a,b} \partial_{a,1} \partial_{b,1})$ *applied to* $\mathcal{D}^E_X$ *correctly subtracts the contributions of the integrals from the boundary components.*

We will first consider the case where $\boldsymbol{t_r} = 0$ for all $r \geq 1$, i.e. we consider the ordinary, rather than S_n-equivariant theory.

First note

$$\sum_{a,b} g^{ab} \hbar \partial_a \partial_b exp(\sum \hbar^{g-1} \mathcal{F}^E_{g,X}) = \sum_{a,b} g^{ab} \mathcal{D}^E_X(\boldsymbol{t})(\hbar \sum \hbar^{g-1} \mathcal{F}^E_{g,X}(\boldsymbol{t}, \phi_a) \mathcal{F}^E_{g,X}(\boldsymbol{t}, \phi_b) + \mathcal{D}^E_X(\boldsymbol{t})(\sum \hbar^g) \mathcal{F}_{g,E}(\mathbf{t}, \phi_{\mathbf{a}}, \phi_{\mathbf{b}}),$$

This is precisely the generating function for invariants of disconnected curves, where the contribution from one component is replaced with the contribution from its codimension-1 boundary.

The placement of the extra $\hbar$ ensures that a product of correlators from curves with genus g_1 and g_2 each with one fixed insert has coefficient $\hbar^{g_1+g_2-1}$, so it is treated as coming from a degeneration of a curve with genus $g_1 + g_2$, and similarly, a correlator with 2 fixed inserts has coefficient $\hbar g$, so it is treated as coming from a genus $g + 1$ curve. The factor $1/2$ accounts for the symmetry between the branches.

Similarly $\frac{1}{n!}(\frac{\hbar}{2} \partial_a \partial_b)^n$ accounts for contributions from the virtual codimension n boundary components, the $\frac{1}{n!}$ coefficient corrects for the order in which the insertions are taken.

The argument is essentially the same for the general case. In the case of invariants coming from a permutation h with cycle structue ℓ, the codimension-1 boundary strata of $X^h_{g,|\ell|,d}$ correspond to a cycle of marked points coinciding with another cycle. h restricts on each component to a symmetry h_1, h_2, with cycle structures $(\ell_1, C), (\ell_2, C)$, where C represents a single cycle of length r.

Splitting along such a cycle is accounted for entirely by the operator $\Delta := exp(-\frac{1}{2} \hbar^r r \sum_{a,b} g^{a,b} \partial_{a,r} \partial_{b,r})$, the $\hbar^r$ to account for the change in Euler characteristic by r points, and the additional factor of r to correct the normalization term $\prod_r r^{-\ell_r}$, since the terms associated to (ℓ_1, C) and (ℓ_2, C) differ from the term associated to ℓ by a factor of r.

4 Geometric Interpretation and Stratified Transversality

For now we restrict our context to the case where $X_{g,n,d}$ is an orbifold, i.e. when $g = 0$ and X is convex. Consider a generic diagram of the form:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{ev} & \prod_i Y_i \\ \downarrow j & & \downarrow F \\ X_{0,n,d} & \xrightarrow{ev} & X \end{array}$$

Even in the case we have described, M is not necessarily an orbifold, so the arguments from the previous section cannot be literally employed to show that the correlator $\overline{\langle[f_1],\ldots,[f_n]\rangle}^{E}_{0,n,d}$ computess $\chi(M)$. Nonetheless, there is a literal geometric interpretation for this value. By a theorem of Trottman-Murolo-du Plessis [9], we can arrange such that $h \times j$ is stratified transverse to the diagonal in $Z \times Z$ by deforming the diagonal by the flow of some vector field. This deformation, called Δ', is the graph of some map $h : \mathbb{Z} \to Z$, so we can equivalently replace $\prod_i f_i$ with its composition with h.

Let X_i be a stratum of $X_{g,n,d}$, it is a smooth manifold. Further, the orbifold structure on X gives it the structure of a smooth orbifold, by lifting the manifold charts of X_i to their preimages in the orbifold charts of X. As a consequence of stratified transversality, the same is true for the corresponding stratum M_i on the fiber product. All strata are stably-almost complex.

Choose a normal neighborhood U_i of X_i in $X_{g,n,d}$, by the orbifold tubular neighborhood theorem, we can take U_i to be an orbifold as well. For U_i sufficiently small the fiber product $V_i := U_i \times_X Y$ is an orbifold neighborhood of M_i.

Choosing differential forms to represent the relevant cohomology classes, and choosing an appropriate partition of unity, gives that the contribution from U_i to the correlator is $\int_{IU_i} c(TIX_{g,n,d}) \prod_i Iev_i^* c(f_* T_{f_i})$. Evaluating this integral is equivalent to evaluating $\int_{IV_i} c(f^* TIX_{g,n,d}) c(T_h)$, which computes the sum of the orbifold indics at singular points of a general vector field on IV_i. For U_i sufficiently small, we can extend a vector field F from IM_i such that it does not vanish elsewhere on V_i. Hence the contribution is the orbifold index of F, which is equal to the ordinary Euler characteristic of M_i. Adding up the contribution from each stratum thus gives $\chi(M)$. (The same procedure without passing to the inertia orbifolds computes the corresponding Euler-Satake characteristic).

As consequence, for convex X, the correlator $\overline{\langle[f_1],\ldots,[f_n]\rangle}^{E}_{0,n,d}$ computes the Euler characteristic of the fiber product $X_{0,n,d} \times_X \prod_i Y_i$, provided the map $\prod_i f_i$ is deformed generically such that $\prod_i ev_i \times \prod_i f_i$ is stratified transverse to the diagonal in $X \times X$.

Now if $n = |\ell|$ for some partition ℓ, and the $[f_i]$ are chosen such that they are compatible with ℓ, then the corresponding permutation-equivariant correlator represents the Euler characteristic of the h-fixed locus of M_f, deformed as above. Since the fixed-point loci of a finite group acting on a smooth orbifold are also smooth orbifolds, the above calculation goes through essentially unchanged.

5 Euler-Satake Invariants

We can build the exact same formalism based on the Euler-Satake characteristic, i.e. twisting the virtual fundamental class by $c_{total}(TX^{vir}_{g,n,d})$ instead of the tangent bundle to the inertia stack. We do so for now without permutation-equivariance, i.e. we define correlators:

$$\overline{\langle\alpha_1,\ldots,\alpha_n\rangle}^{ES}_{g,n,d} := \int_{X_{g,n,d}} c(TX^{vir}_{g,n,d}) \prod_i ev_i^* \alpha_i$$

We define the complete and incomplete correlators $\overline{\langle\alpha_1,\ldots,\alpha_n\rangle}^{ES}_{g,n,d}$ and $\langle\alpha_1,\ldots,\alpha_n\rangle^{ES}_{g,n,d}$ analogously to the Euler-theoretic case. Call the corresponding potentials $\mathcal{D}^{ES}_X$ and $\mathcal{ES}_X$, the operator subtracting the boundary contributions is $\Delta^{ES} := exp(-\hbar/2 \sum_{a,b} g^{a,b} \partial_a \partial_b)$.

$\mathcal{D}^{ES}_X$ is (after applying ϕ to the inputs) a generating function for ordinary cohomological Gromov-Witten invariants, twisted by a characteristic class of the virtual tangent bundle, and is determined from the ordinary cohomological potential $\mathcal{D}_X$ by a theorem of Coates. As consequence, $\mathcal{D}^{ES}_X$ is homogenous after a dilaton shift of $\frac{-z}{1-z}$.

The geometric interpretation for Euler-theoretic invariants also applies verbatim to Euler-Satake invariants.

Remark. The virtual tangent bundle to the moduli space of stable maps takes the form

$$\pi_* ev^*_{n+1} T_X - \pi_* L^{-1}_{n+1} - (\pi_* i_* \mathcal{O}_{\mathcal{Z}})^*$$

.

The first two components form the part logarithmic with respect to the singular locus, T_{log}^{vir}. By a theorem of Coates, twisting by the total Chenr class of the third component is equivalent to applying the operator Δ_{ES}^{-1}, so we can alternatively define incomplete Euler-Satake-theoretic correlators as:

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d}^{ES} = \int_{X_{g,n,d}} c(T_{log}^{vir} X_{g,n,d}) \prod_i ev_i^* \phi(\alpha_i)$$

5.1 Target $X \times B\mathbb{Z}_M$

For the case when the target space $\mathcal{X}$ is an orbifold, we avoid interpreting the notion of a bordism class of a map to $\mathcal{X}$, and instead we define the potentials $\mathcal{D}_{\mathcal{X}}^{ES}$ and $\mathcal{ES}_{\mathcal{X}}$, as Gromov-Witten potentials valued in $H^*(I\mathcal{X}) \otimes MU^*(pt)$, twisted by the classes $c(T^{vir})$ and $c(T_{log}^{vir})$ respectively. By a theorem of Tonita, the potentials are determined by $\mathcal{D}_{\mathcal{X}}$, and are homogenous after a dilaton shift of $\frac{-z}{1-z}$, but the input is only applied marked points whose input maps to the identity component of $I\mathcal{X}$.

For the case of target $X \times B\mathbb{Z}_M$, the inertia stack has M connected components each isomorphic to $X \times B\mathbb{Z}_M$, labelled by Mth roots of unity. The cohomology at each component is isomorphic to $MU^*(X)$, with unit element h_ζ. So the inputs to such a correlator are $t_\zeta = \sum \phi_a t_{a,\zeta} h_\zeta$, where ϕ_a a basis of $MU^*(X)$.

We can interpret the Euler-Satake invariants of $X \times B\mathbb{Z}_M$ as Euler-Satake invariants of X using a formula of Jarvis-Kimura [8], who proved that each connected component of $\overline{M}_{g,n,d}(X \times BG)$ is a virtual covering of $X_{g,n,d}$ with a prescribed degree, determined by the group G. Applying their result to our case yields:

Given some correlator $\overline{\langle \alpha_1 h_{\zeta_1}, \ldots, \alpha_n h_{\zeta_n} \rangle}_{g,n,d,X \times B\mathbb{Z}_M}^{ES} = \begin{cases} M^{2g-1} \overline{\langle \alpha_1, \ldots, \alpha_n \rangle}_{g,n,d,X}^{ES} & \prod_{i=1}^n \zeta_i = 1 \\ 0 & \text{otherwise} \end{cases}$

6 Limits from Hirzebruch theory

6.1 S_n-Equivariant K-theoretic Invariants

We remind the definition of S_n-equivariant K-theoretic Gromov-Witten invariants, introduced in [4].

For target X, the base ring is taken to be $K^0(X) \otimes \Lambda$, for Λ some algebra equppied with an action of the Adams operations Ψ^k, extending the one on $K^0(X)$. Given $h \in S_n$ with $\ell_r(h)$ cycles of length r, with r ranging from 1 to s, h acts on $X_{g,n,d}$ by permuting the marked points.

For each r, given inputs $w_{r1}, \ldots, w_{r\ell_r}$ each of the form $\sum \phi_m q^m$, for $\phi_m \in K^0(X) \otimes \Lambda$, associate to the input w_{rk} the element $W_{rk} \in K^0(X_{g,n,d}) := \prod_{\alpha=1}^r \sum_m \mathrm{ev}_{\sigma_\alpha}^* \phi_m L_{\sigma_\alpha}^m$, where σ_α are the marked points permuted by the kth cycle of length r, and L_{σ_α} are the corresponding cotangent line bundles on $X_{g,n,d}$

Given a partition ℓ, a genus g, and a degree d, S_n-equivariant correlators are defined as follows

$$\langle w_{11}, \ldots, w_{1\ell_1}, \ldots \rangle_{g,\ell,d} := \prod_r r^{-\ell_r} str_h H^*(X_{g,n,d}, \mathcal{O}_{g,n,d}^{vir} \prod_{i=1}^s \otimes_{j=1}^{\ell_i} W_{ij}).$$

The elements of Λ act $\Psi-$linearly, i.e. scaling the rth input by $s \in \Lambda$ is equivalent to multiplication by $\Psi^r(s)$.

6.2 Hirzebruch Invariants

We can realize the Euler-theoretic correlators as limits of correlators from wquantum Hirzebruch theory, as introduced by the author in [6]. Quantum Hirzebruch theory is a variant of quantum K-theory, based on the Hirzebruch $\chi_{-y}-$genus. It is defined as follows:

Let the class $\Lambda_{-y}(V)$ be defined for bundles by $\sum_i (-y)^i \Lambda^i(V^*)$. Extend it multiplicatively to a characteristic class on virtual bundles. In this guise, it is determined by the series $1 - yq^{-1}$. On manifolds, the extension to virtual bundles is well-defined provided the coefficient ring is localized at $y-1$, since $(L-1)$ is a nilpotent class, and we can expand:

$$\frac{1}{1-yL^{-1}} = \frac{1}{1-y}\frac{1}{1-\frac{y}{1-y}L^{-1}} = \sum_{n\geq 0}\frac{y^n}{(1-y)^{n+1}}(L^{-1}-1)^n$$

However the case for orbibundles is slightly more complicated, as $(L-1)$ need not be nilpotent for a 1-dimensional orbibundle. Using the fact that Kawasaki's Chern character gives an isomorphism between $K^0(X)$ and $H^*(IX)$, we see that we the coefficients will be rational functions of y with poles at roots of unity, call the algebra of such functions A. Hirzebruch theory is S_n-equivariant K-theory, which the virtual structure sheaf $\mathcal{O}^{vir}$ tensored with $\Lambda_{-y}(T^{vir})$. Call a correlator from this theory $\langle \dots \rangle^y_{g,\ell,d}$, the inputs are tensored with rational functions of y, and we impose that $\Psi^k(y) = y^k$.

We prove that the limit as $y \mapsto 1$ of these invariants (with specifically chosen inputs), recover Euler-theoretic invariants. For ease of reading, we first prove it for ordinary invariants, then address the changes required to adapt the proof to S_n-equivariant ones.

6.3 Ordinary Limit

Given inputs $V_i \in K^0(X)\otimes A$, the ordinary Hirzebruch-theoretic correlator is:

$$\langle V_1,\dots,V_n\rangle^y_{g,n,d} := \chi^{vir}(X_{g,n,d};\Lambda_{-y}(T^{vir})\prod_i ev_i^* V_i).$$

Theorem 6.1.

$$lim_{y\mapsto 1}\langle f_{1*}\Lambda_{-y}(T_f),\dots\rangle^y_{g,n,d} = \overline{\langle [f_1],\dots,[f_n]\rangle}^E_{g,n,d}$$

Proof. We apply the Kawasaki-Riemann-Roch theorem (more precisely, the virtual version due to Tonita) to the right hand side, yielding the following:

$$\langle f_{1*}\Lambda_{-y}(T_f),\dots\rangle^y_{g,n,d} = \int_{IX_{g,n,d}} Td(TIX_{g,n,d})ch(tr(i^*\Lambda_{-y}*(TX_{g,n,d})\prod_i ev_i^* f_{i*}\Lambda_{-y}(T_{f_i})))\frac{1}{ch(tr(\Lambda N^*))}$$

(Here N is the normal bundle of the natural map $IX_{g,n,d}\to X_{g,n,d}$. The invariant part of $i^*TX_{g,n,d}$ is $TIX_{g,n,d}$, and the components with nontrivial eigenvalues form the normal bundle to i of the component. The trace operator does nothing to classes pulled back from X, since they are all necessarily g-invariant. Thus we can simplify the above expression as follows:

$$\langle f_{1*}\Lambda_{-y}(T_{f_1}),\dots\rangle^y_{g,n,d} = \int_{IX_{g,n,d}} Td(TIX_{g,n,d})ch(\Lambda_{-y}*(TIX_{g,n,d}))\prod_i Iev_i^* ch(f_{i*}\Lambda_{-y}(T_{f_i}))\frac{ch(tr(\Lambda_{-y}(N))}{ch(tr(\Lambda N^*))}$$

We can then use the (ordinary) Grothendieck-Riemann-Roch theorem on each f_i to yield:

$$\langle f_{1*}\Lambda_{-y}(T_{f_1}),\dots\rangle^y_{g,n,d} = \int_{IX_{g,n,d}} Td(TIX_{g,n,d})ch(\Lambda_{-y}*(TIX_{g,n,d}))\prod_i Iev_i^* f_{i*}ch(\Lambda_{-y}(T_{f_i})Td(T_{f_i}))\frac{ch(tr(\Lambda_{-y}(N))}{ch(tr(\Lambda N^*))}$$

Introduce the characterisitic class G defined on bundles by $G(V) = Td(V)ch(\Lambda_{-y}(V))$, then the expression becomes:

$$\int_{IX_{g,n,d}} G(TIX_{g,n,d})\prod_i Iev_i^* f_{i*}G(T_{f_i})\frac{ch(tr(\Lambda_{-y}(N))}{ch(tr(\Lambda N^*))}$$

G is a multiplicative characteristic class defined by the polynomial $\frac{x(1-ye^{-x})}{1-e^{-x}}$ (here x denotes the first Chern class of a line bundle)

We can scale G by $\frac{1}{1-y}$, while simultaneously scaling x by $(1-y)$ (this also modified the Chern character in the above expresison). Doing so does not change the top degree terms of the integrand, hence does not affect the outcome. We call the resulting characteristic class td_y.

$$\int_{IX_{g,n,d}} td_y(TIX_{g,n,d}) \prod_i Iev_i^* f_{i*} td_y(T_{f_i}) \frac{ch_y(tr(\Lambda_{-y}(N))}{ch_y(tr(\Lambda N^*))}$$

The normal bundle term becomes 1 in the limit, so the limit of the entire quantity depends on the limit of the class td_y as $y \mapsto 1$.

td_y is detemrned by the series $\frac{G((1-y)x)}{1-y} = \frac{x(1-ye^{(y-1)x})}{1-e^{(y-1)x}}$.

We can rewrite this expression as:

$$x(1 - \frac{(y-1)e^{(y-1)x}}{1-e^{(y-1)x}})$$

Using the fact that $lim_{y\to 1} \frac{z^{1-y}-1}{1-y} = ln(z)$, we see the entire expression evaluates to $1+x$, which represents the total Chern class. Thus the limit of the χ_{-y}-theoretic correlator becomes:

$$\int_{IX_{g,n,d}} c(TIX_{g,n,d}) \prod_i Iev_i^* f_{i*} c(T_{f_i})$$

Which is equal to the desired correlator.

□

6.4 General Limit

Theorem 6.2. *Given maps $f_{i,j} : Y_{i,j} \to X$ compatible with the cycle structure ℓ:*

$$lim_{y\mapsto 1} \langle f_{1,1*}\Lambda_{-y}(T_{f_{1,1}}), \dots \rangle^y_{g,\ell,d} = \overline{\langle [f_{1,1}], \dots \rangle}^E_{g,\ell,d}$$

.

For inputs $V_{i,j}$, the Hirzebruch-theoretic correlator associated to $\mathfrak{g}, \ell, d$ is equal to (if we denote the ith cycle of length j by $C(i,j)$):

$$str_h(X_{g,|\ell|,d}; \Lambda_{-y}(T^{vir}) \bigotimes_{i,j} \prod_{\alpha \in C(i,j)} ev^*_{\sigma_\alpha} V_{i,j})$$

We proceed as in the first case, but apply the Lefschetz-Kawasaki-Riemann-Roch theorem from [4], which states that:

$$str_h(\mathcal{M}, V) = \int_{IM^h} Td(IM^h) ch\big(tr_h(V)\big) \frac{1}{tr_h \Lambda N^*})$$

.

Since each input $V_{i,j}$ appears j times, corresponding to the j marked points in the ith length-j cycle, and h acts on these by cyclically permuting the copies $tr_h(\bigotimes_\alpha ev^*_{\sigma_\alpha} V_{i,j}) = \widehat{ev}^*_{i,j} \Psi^j(V_{i,j})$.

So the applying the formula in our context yields:

$$\int_{IX^h_{g,|\ell|,d}} Td(T^{vir} IX^h_{g,|\ell|,d}) ch\big(tr(\prod_{i,j} \widehat{ev}^*_{i,j} \Psi^j(f_{i,j*}\Lambda_{-y}(T_{f_{ij}})))tr_h(\frac{\Lambda_{-y}(N)}{\Lambda(N^*)})\big)$$

The only change from the previous case is what happens to the inputs, everything else will proceed essentially the same way. So we need only consider what happens to a particular input, $V = f_* \Lambda_{-y}(T_f)$, coming from some cycle of length k. The corresponding term in the expression is $ch(tr_h(\widehat{ev}^* \Psi^k(f_* \Lambda_{-y}(T_f))))$

We can apply Adams-Riemann-Roch on the target to this quantity to get: $ch(f_*(\Psi^k(\Lambda_{-y}(T_f))C_k(T_f)))$, where C_k is the Adams-Todd class determined on a line bundle q by $\frac{1-q^{-1}}{1-q^{-k}}$.

Now we can apply Grothendieck-Riemann-Roch to rewrite this expression as $f_*Td(T_f)ch(\Psi^k(\Lambda_{-y}(T_f))C_k(T_f))$, which is equal to $f_*(H(T_f))$, where $H(T_f)$ is the characteristic class determined by the expression: $\frac{(1-y^k e^{-kz})z}{1-e^{-kz}}$. Applying the same change of coordinates: i.e. scaling z by $(1-y)$ and dividing the characteristic class by $(1-y)$ gives:

$$\frac{(1-y^k e^{(y-1)kz})z}{1-e^{(y-1)kz}}$$

Expanding this expression gives:

$$\frac{1-y^k-y^k(k(y-1)z+\dots)}{-(y-1)kz+\dots}z=\frac{\frac{1-y^k}{1-y}-y^k(kz+\dots)}{-kz+\dots}z$$

The limit of this expression as $y\mapsto 1$ is $1+z$, as before, so we are in exactly the same situation as case with ordinary invariants.

6.5 Dilaton Shift for Euler theory

The Hirzebruch K-theoretic potential is dilaton shifted by the quantity $\frac{1-q}{1-yq}$ in the rth component. To determine the corresponding shift for Chern-Euler theory, we must also understand the limiting value of this quantity.

This input corresponds to a product of $\frac{1-L_i}{1-yL_i}$ of line bundles in a cycle of length r, permuted cyclically by h. The contribution of this to the Kawasaki-Riemann-Roch formula becomes $\frac{1-\zeta\widehat{L_i}^r}{1-y\zeta\widehat{L}_i^r}$, where ζ is the eigenvalue of h^r on the cotangent line to the corresponding point on the quotient curve. If we change coordinates as before and declare compute the limit as $y\to 1$, this quantity will be 1 if $\zeta\neq 1$, and $\frac{-c_1(L_i)}{1-c_1(L_i)}$ otherwise. If $\zeta=1$, then the ith marked point in $C/\tilde{h}$ is not an orbifold point, equivalently $ord_{\tilde{h}}=r$.

We will denote this shift by $\frac{-c_1(\widehat{L}_i^{inv})}{1-c_1(\widehat{L}_i^{inv})}$, and note that it takes different values depending on where it is inserted, and on what component of $IX^h_{g,n,d}$ the integral is taken. This choice of dilaton shift does not grant the potential any homogeneity properties, but will occur as a correction term that simplifies the adelic formula.

7 Adelic Formulas for Euler Invariants

The inputs to the Euler-Satake theory of $X\times B\mathbb{Z}_M$ are labelled t_ζ for $\zeta^M=1$ according to the sector in which the marked point lands. In the following argument, we consider the tensor product $\bigotimes_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}$. We relabel the t_ζ input to the potential of $X\times B\mathbb{Z}_M$ to $t_{\zeta,r}$, where $r=M/ord(\zeta)$.

We use the description of the inertia stack of $X_{g,n,d}$ introduced in [4] to prove the following "adelic formula" for the $\mathcal{D}^E_X$.

$$\mathcal{D}^E_X=\big[exp(\sum_r\frac{\hbar^r r}{2}\sum_{\zeta,\eta\neq 1}\nabla^r_{(\zeta,\eta)}\bigotimes_M\mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}\big]|_{t_{\zeta,r}=\boldsymbol{t_r}}. \tag{7.1}$$

Where

$$\nabla^r_{\zeta,\eta}=\sum_{a,b}g^{ab}\partial_{\zeta,r,a}\partial_{\eta^{-1},r,b}$$

In addition:

$$\mathcal{E}_X=\big[\bigotimes_M\mathcal{ES}_{X\times B\mathbb{Z}_M}\big]|_{t_{\zeta,r}=\boldsymbol{t_r}} \tag{7.2}$$

Remark. These formulas are not quite complete as stated, they are only true after all potentials are dilaton shifted and certain adjustments have been made to the inputs.

7.1 Adelic Description of $IX_{g,n,d}$

A connected component (henceforth referred to as a Kawasaki stratum) of this space is described by certain combinatorial data:

- A graph G dual to $\widehat{\mathcal{C}}$, the quotient of $\mathcal{C}$ by the cyclic group generated by $\tilde{h}$.
- A positive integer M_v for each vertex v, denoting the number of preimages in $\mathcal{C}$ of the irreducible component corresponding to v.
- The discrete characteristics (genus $\widehat{g}_v$, degree $\widehat{d}_v$, number of marked points and edges $\widehat{n}_v$) of the map on each irreducible component.
- A labelling of the vertices of G with eigenvalues of $\tilde{h}^r$ on the tangent lines to any preimage of the ramification points of order r. These eigenvalues will be primitive mth roots of unity for $m = \frac{M_v}{r}$.
- A labeling of the edges of G (corresponding to nodes) with pairs of eigenvalues of $\tilde{h}^r$ on any branch at a preimage to the node. We require that these eigenvalues not be inverse to each other (i.e. the node is unbalanced), so the node cannot be smoothed within the stratum.

After normalizing at the unbalanced nodes, each vertex represents a component of a Chen-Ruan moduli space of stable maps to the orbifold $X \times B\mathbb{Z}_M$, given by taking the quotient of the stable map to X by $\tilde{h}$. After doing this, the eigenvalue at a marked point also determines the sector of $I(X \times B\mathbb{Z}_M)$ in which the evaluation map at that marked point lands.

In the reverse direction, given any collection of stable maps coming from the vertices of a graph G. The above data gives gives a ramified $\mathbb{Z}_{M_v}$-principal bundle over the curve $\widehat{C}_v$, for each vertex v.

By imposing a diagonal constraint at each edge, these bundles can be glued $\mathbb{Z}_{M_v}$-equivariantly. The resulting total space is a curve $\mathcal{C}$ with a map to X of degree $\sum_v M_v \widehat{d}_v$, and a symmetry $\tilde{h}$ given by a generator of $\mathbb{Z}_{lcm(M_v)}$.

Using this description we can rewrite the CE-theoretic integral over $IX^h_{g,n,d}$, in terms of integrals over these Chen-Ruan spaces. The resulting formula takes the form of Wick's summation over graphs. As we will see, the vertex contributions end up being integrals in the Euler-Satake theory of $X \times B\mathbb{Z}_M$.

The justification for this technique is similar than other applications, albeit slightly simpler since the invariants are already defined as integrals on the inertia orbifold, so there is no need to invoke any kind of Kawasaki formula.

7.2 Proof of 7.1 and 7.2

Recall that the Chern-Euler theoretic correlator $\overline{\langle t_1, \ldots, \rangle}^E_{g,\ell,d}$ is equal to $\int_{IX^h_{g,\ell,d}} c(T^{vir} IX^h_{g,\ell,d}) \prod_i \widehat{ev}_i^* c(\boldsymbol{t})$

We will compute this integral on each component of $IX_{g,n,d}$ as in Wick's formula, first by computing it on a stratum of a single vertex, and then computing the effect of joining two vertices via an edge.

7.3 Vertices

Let $\widehat{\mathcal{M}}$ be a Kawasaki stratum of $IX_{g,n,d}$ consisting of a single vertex with 0 edges. The contribution from this stratum to the above correlator is equal to:

$$\int_{\widehat{\mathcal{M}}} c(T^{vir}\widehat{\mathcal{M}}) \prod_i \widehat{ev}_i^* \boldsymbol{t}|_{\widehat{\mathcal{M}}}$$

$\widehat{\mathcal{M}}$ is a connected component Chen-Ruan moduli space of maps to $X \times B\mathbb{Z}_M$ (meaning the sectors in which each marked point lands are pretedeterined). In this guise, it has evaluation maps to $X \times B\mathbb{Z}_M$ which we abusively also denote $\widehat{ev}$. This is justified since restricting $\widehat{ev}^*\alpha_i$ to $\widehat{\mathcal{M}}$ is equivalent to pulling by $\alpha_i h_{\zeta_i}$ via the ith evaluation map on $\mathcal{M}$.

So the above integral is equal to:

$$\int_{\widehat{\mathcal{M}}} c(T^{vir}\widehat{\mathcal{M}}) \prod_i \widehat{ev}_i^* \phi(\boldsymbol{t}) h_{\zeta(i)}$$

This is an invariant in the Euler-Satake-theory of $X \times B\mathbb{Z}_M$. (The marked points that do not appear as a result of the quotient have input 1 in their appropriate sector)

Thus the vertex contributions of Wick's formula are of the form $\bigotimes_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}$. After some adjustments to the inputs, Novikov's variables, and Planck's constant, which we delineate below:

7.4 Inputs

First, to reconcile the exponents of $\hbar$ in both sides of the formula, since $\hbar$ is weighted by half the Euler characteristic of the *covering* curve in $\mathcal{D}^E_X$, and it is based on the same invariant *quotient curve* in $\mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}$. This procedure ends up being essentially the same as what is done in [4].

Namely, on a stratum given by a graph G where the component of the quotient curve has genus $\widehat{g}_v$, $\widehat{n}_v$ marked points, and $\tilde{h}$ has order M_v, we can apply the Riemann-Hurwitz to the quotient map to determine the Euler characteristic of the covering curve $\mathcal{C}$. (Each edge e has r_e preimages on $\mathcal{C}$, and the ith marked point on the component corresponding to v has r_i preimages on $\mathcal{C}$.)

The result is

$$\frac{-\chi(\mathcal{C})}{2} = \sum_v M_v(\widehat{g}_v - 1) - \sum M_v(\frac{\widehat{n}_v}{2}) - \sum_v \sum_i \frac{r_i}{2} + \sum_e r_e.$$

This means that the necessary steps to correct the exponents of $\hbar$ are as follows:

To account for the first two terms, replace $\hbar$ with $\hbar^{M_v}$ in each vertex potential, then divide each input by $\hbar^{M_v/2}$.

To address the remaining terms, add a factor of $\hbar^r$ at each edge, and divide each input by an additional factor of $\hbar^{r/2}$. In addition, to correct the degree of the maps we replace Q with Q^{M_v}.

Under the present accounting, a marked point on the quotient curve with eigenvalue ζ receives the input $\boldsymbol{t}_{\boldsymbol{r}(\boldsymbol{\zeta})} h_\zeta$ if it represents a cycle of r marked points on the covering curve. Otherwise, it receives an input of $1h_\zeta$, note that non-orbifold marked points ($\zeta = 1, r(\zeta) = M$) can only occur as images of the original points, so they do not ever received the input of 1.

To account for both of the above possibilities for orbifold marked points, and to perform the corrections to $\hbar$ discussed previously, the necessary vertex contributions must be:

$$\bigotimes_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}\Big(\frac{1}{\sqrt{\hbar^M}}\big(\frac{\boldsymbol{t}_{\boldsymbol{M}}}{\sqrt{\hbar^M}} h_1 + \sum_{\zeta^M=1, \zeta\neq 1} \frac{\boldsymbol{t}_{\boldsymbol{r}(\boldsymbol{\zeta})} + 1}{\sqrt{\hbar^{r(\zeta)}}} h_\zeta\big), \hbar^M, Q^M\Big)$$

7.5 Edge Contributions

Recall an edge of level r of a graph G connecting vertices v_+, v_- with labels η_+, η_-, represents an $r-$tuple of nodes on the covering curve permuted cyclically by $\tilde{h}$, with $\tilde{h}^r$ acting on each branch with eigenvalues ν_+ and ν_-, which are not mutually inverse. After normalization, the remnants of e represent marked points in the Chen-Ruan spaces $\mathcal{M}_+$ and $\mathcal{M}_-$, of sector ν_+ and ν_- respectively.

As we have discussed, at each edge we multiply by $\hbar^r$ to correctly account for the Euler characteristic of the covering curve. In addition, we need to enforce that evaluation maps $\widehat{ev}_+, \widehat{ev}_-$ at each branch send the marked points connected by e to the same class in $H^*(X)$, which regarded as the $\nu_\pm$ sector in $H^*(IX \times B\mathbb{Z}_{M_\pm})$.

To account for the fact that there are r ways to equivariantly glue the two covering curves of the vertices, and the symmetry between ν_+ and ν_-, we multiply by an additional factor of $r/2$.

This means the contribution for an edge e of r with eigenvalues $\nu_\pm$ is $\frac{r\hbar^r}{2}\Delta_e^{\nu_+,\nu_-^{-1}}$, where Δ_e unglues the diagonal constraint between the sectors ν_+,ν_-^{-1}).

So the formula is, written naively:

$$exp(\sum_{edges}\frac{r\hbar^r}{2}\nabla_{e,r})\bigotimes_{vertices}\mathcal{D}^{ES}(v). \tag{7.3}$$

If e connects two vertices, $\nabla_{e,r}$ multiplies the two potentials and glues them along the ν,ν^{-1}. If e is a loop, Δ_e glues the two correlator series. Since we have the formula $e^F_{xy} = F_zF_ye^F + F_{xy}e^F$, applying a quadratic differential accounts for both kinds of insertion. For $r > 1$, the operator includes a factor of r to account for the r ways of choosing which point to label "1" when gluing the covering curves. Similarly, the operators receive a factor of $\hbar^r$ to correctly account for the Euler characteristic of the covering curve. The product of this procedure is:

$$\mathcal{D}^E_X(\boldsymbol{t},\hbar,Q) =$$

$$[e^{\sum_r \frac{\hbar^r}{2}\sum_{\nu_+\nu_-\neq 1}\sum_{a,b} g^{a,b}\partial_{\nu_+,r,a}\partial_{\nu_-^{-1},r,b}}\bigotimes_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}(\frac{1}{\sqrt{\hbar^M}}(\frac{t_{M,1}}{\sqrt{\hbar^M}}h_1+\sum_{\zeta^M=1,\zeta\neq 1}\frac{t_{r(\zeta),\zeta}+1}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta),\hbar^M,Q^M)]_{\boldsymbol{t_r}=t_{r,\zeta}} \tag{7.4}$$

After accounting for the dilaton shift, this formula can be simplified. Recall that the dilaton equation for Euler-Satake invariants states that $\mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}$ is homogenous with respect to $\hbar$ after a shift in the input by $\frac{c_{top}(L_i^{-1})}{c(L_i^{-1})}$ for all points in the unit sector. This means contribution of the first two terms in the Riemann-Hurwitz formula cancel out, after applying the shift to the Euler-Satake potentials.

The dilaton shift in the Euler-theoretic potential is $\frac{c_{top}((L_i^{-1})^{inv})}{c((L_i^{-1})^{inv})}$, which will coincide with the Euler-Satake theoretic shift when $\zeta = 1$, and be equal to 1 otherwise, accounting for the correction term introduced earlier.

This means the following formula holds *projectively* (i.e. ignoring scale factors as a result of the homogeneity):

$$\mathcal{D}_X(\boldsymbol{t},\hbar,Q) =$$

$$e^{\sum_r \frac{r\hbar^r}{2}\sum_{\nu_+\nu_-\neq 1}\sum_{a,b} g^{a,b}\partial_{a,\nu_+,r}\partial_{b,\nu_-^{-1},r}}[\bigotimes_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}(t_{M,1}\sum_{\zeta^M=1}\frac{t_{r(\zeta),\zeta}}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta),1,Q^M)]_{\boldsymbol{t_r}=t_{r,\zeta}} \tag{7.5}$$

7.6 Incomplete Potentials

We can rewrite the term inside the exponential of the edge operator as the difference:

$$\frac{r\hbar^r}{2}\sum_{a,b} g^{a,b}((\sum_\zeta \partial_{\zeta,a})(\sum_\eta \partial_{\eta^{-1},b}) - \sum_\zeta \partial_{\zeta,a}\partial_{\zeta^{-1},b})$$

.

The contributions of the second term of the difference do not involve any edges, since they only deal with balanced nodes. Restricted to a vertex of order M the operator is:

$$\Delta_M = exp(\sum_r -\frac{r\hbar^r}{2}\sum_{a,b} g^{a,b}\sum_{ord(\zeta)=M/r}\partial_{r,\zeta,a}\partial_{r,\zeta^{-1},b})$$

If we apply this to the vertex potential *before* correcting the inputs, we get:

$$\Delta_M = exp(\frac{\hbar}{2}\sum_{a,b} g^{a,b}\sum_{\zeta^M=1}\partial_{r(\zeta),\zeta,a}\partial_{r(\zeta),\zeta^{-1},b})$$

Since the input corrections account for the missing factor of r, and correct the exponent of $\hbar$.

Dividing the edge operator this ways leaves us with the formula:

$$\mathcal{D}^E_X(\boldsymbol{t},\hbar,Q) = exp(\sum_r \frac{r\hbar^r}{2}\sum_{a,b}(\sum_\zeta \partial_{r,\zeta,a})(\sum_\eta \partial_{r,\eta,b})\big(\bigotimes_M \Delta_M \bar{\mathcal{D}}^{ES}_{(X\times B\mathbb{Z}_M)}(\dots)\big) \tag{7.6}$$

The operator $exp(\sum_r r\frac{\hbar^r}{2}\sum_{a,b}(\sum_\zeta \partial_{r,\zeta,a})(\sum_\eta \partial_{r,\eta,b}))$ when applied *after* the restriction to the subspace $t_{\zeta,r} = \boldsymbol{t_r}$ takes the form:

$$\nabla := exp(\sum_r \hbar^r/2\sum_{a,b} g^{a,b}\partial_{a,r}\partial_{b,r})$$

.

Renaming ∇^{-1} to Δ and moving it to the other side of the formula gives:

$$\Delta\mathcal{D}_X = \bigotimes_M \Delta_M \mathcal{D}^{ES}_{X\times B\mathbb{Z}_M}(\dots). \tag{7.7}$$

The interpretation of this formula is as follows, Δ is the operator introduced earlier that subtracts the contributions from nodal curves to the Euler-theoretic integrals. Δ_M is the corresponding Euler-Satake-theoretic operator. So as a corollary (with the same input corrections), we obtain:

$$\mathcal{E}_X = \bigotimes_M ES_{X\times B\mathbb{Z}_M}(\dots)$$

8 Dilaton and String Equations

Here we state and prove the dilaton and string equations for the non S_n-equivariant versions of all the invariants defined previously.

8.1 The Dilaton Equation

The usual Gromov-Witten potential $\mathcal{D}_X$ satisfies the dilaton equation, which states that $t\partial_t + h\partial_h\mathcal{D}^H_X = \frac{-\chi(X)}{24}\mathcal{D}^H_X$, provided the inputs to $\mathcal{D}^H_X$ have been shifted by $-z$, which literally means $-c_1(L_i)$ in the ith correlator seat. The quantity $-z$ is referred to as the *dilaton shift*.

The dilaton equation is equivalent to the statement

$$\langle\alpha_1,\dots,\alpha_n,c_1(L^{-1})\rangle = (2-2g-n)\langle\alpha_1,\dots,\alpha_n,c_1(L)\rangle_{g,n,d}.$$

The Euler and Euler-Satake potentials introduced previously satisfy various analogues of this equations, for different choices of dilaton shift.

8.1.1 Euler-Satake Invariants

The dilaton shift for the Euler-Satake invariants is translating the ith input by $\frac{c_1(L_i^{-1})}{c(L_i^{-1})} = \frac{-\psi_i}{1-\psi_i}$. We will denote this quanatity by s for this section. This choice of s comes from the theory of twisted Gromov-Witten invariants.

Theorem 8.1. *The dilaton equations for Euler-Satake invariants are:*

$$\overline{\langle\alpha_1,\dots,\alpha_n,s\rangle}^{ES}_{g,n+1,d} = (2-2g-n)\overline{\langle\alpha_1,\dots,\alpha_n\rangle}^{ES}_{g,n,d}$$

and

$$\langle\alpha_1,\dots,\alpha_n,s\rangle^{ES}_{g,n+1,d} = (2-2g-n)\langle\alpha_1,\dots,\alpha_n\rangle^{ES}_{g,n,d}$$

8.1.1.1 Completed Euler-Satake Invariants

For completed invariants, this follows directly from the dilaton equation for twisted Gromov-Witten invariants, the operator relating $\mathcal{D}_X^{ES}$ and $\mathcal{D}_X$ identifies s with $-\psi_{n+1}$, and preserves the homogeneity. However we prove it directly nonetheless.

Inserting the shift into the correlator gives:

$$\int_{X_{g,n+1,d}} c(TX_{g,n+1,d})(\prod_{i=1}^{n} ev_i^*\alpha)\frac{c_1(L_{n+1}^*)}{c(L_{n+1}^*)}$$

.

We can rewrite this as:

$$\int_{X_{g,n+1,d}} c(ft^{-1}TX_{g,n,d})c(T_{ft})\frac{c_1}{c}(L_{n+1}^{-1})(\prod_{i=1}^{n} ev_i^*\alpha)$$

$\frac{c_1}{c}(L_{n+1})$ is a multiple of $c_1(L_{n+1})$, hence the components of T_{ft} coming from the singular locus are ignored, since $c_1(L_{n+1})$ vanishes there, so we can replace T_{ft} with L_{n+1}^{-1}. The resulting integral becomes:

$$\int_{X_{g,n+1,d}} c(ft^*TX_{g,n,d})c_1(L_{n+1}^{-1})(\prod_{i=1}^{n} ev_i^*\alpha)$$

By the projection formula and the fact $ft_*c_1(L_{n+1}) = (2g-2+n)[X_{g,n,d}]^{vir}$, the result is equal to

$$(2-2g-n)\int_{X_{g,n,d}} c(TX_{g,n,d})(\prod_{i=1}^{n} ev_i^*\alpha)$$

It is equivalent to the statement that $t\partial_t + \hbar\partial_\hbar \mathcal{D}_X^{ES} = \frac{-\chi(X)}{24}\mathcal{D}_X^{ES}$ provided the inputs are shifted by $\frac{c_1}{c}(L_i^{-1})$.

8.1.1.2 Incomplete Euler-Satake Invariants

By the definition of incomplete invariants:

$$\langle \alpha_1, \ldots, \alpha_n, s\rangle_{g,n+1,d}^{ES} := \overline{\langle \alpha_1, \ldots, \alpha_n, s\rangle}_{g,n+1,d}^{ES} + \sum_I (-1)^d C_{\alpha,s}^{ES}(X_I).$$

Given a stratum of $\partial X_{g,n,d}$, denoted by X_I, that is written as a fiber product of k moduli spaces, it has two kinds of strata lying over it in $X_{g,n+1,d}$. The k strata $X_{I,j}$ where the $n+1$st point is in the jth component, and the strata where the $n+1$st point coincides with a marked point or a node. The latter strata all take value 0, since L_i is trivial on $X_{0,3,d}$.

If the jth component has n_j marked points and genus d_j, the contribution from $X_{I,j}$ in the dilaton equaton is equal to $(-1)^d(2-2g_i-n_i)C_\alpha(X_I)$.

So the total contributions from all $X_j = ((\sum_j 2-2g_j-n_j))C_\alpha(X_I)$.

If X_I has k components and s genus reductions. We have $\sum g_j = g-s$, $\sum n_i = n+2(k-1)-2s$, since adding an irreducible component adds two additional marked points corresponding to the node, and performing 1 genus reduction similarly adds 2 marked points.

Thus $\sum_{j=1}^{k}(2-2g_j-n_j) = 2-2g-n$.

8.1.2 Euler Invariants

To address the case of Euler invariants, we use the following notation: Given an orbifold $\mathcal{X}$ and an orbibundle V on X, let V^{inv} denote the bundle on $I\mathcal{X}$ defined on a sector labelled μ by the μ-invariant part of $V|_{I\mathcal{X}}$.

The dilaton shift for Euler-theoretic invariants is $\frac{c_{top}}{c}((L_i^{-1})^{inv})$, which agrees with the previous shift on the unit sector of $IX_{g,n,d}$, and is 1 elsewhere, since a nontrivial automorphism of the curve fixing the marked points induces a nonzero eigenvalue at the tangent lines to those points (the motivation for this choice is to simply the adelic formulas that appear in a later section).

The potentials for the Euler invariants are no longer homogenous after the scaling. Intead, the dilaton shift scales the contribution from each connected component of $IX_{g,n,d}$ by a different quantity. This is equivalent to integrating against a weighted version of $[IX_{g,n,d}]^{vir}$, Correlators integrated against a weighted class W will be denoted $\langle \alpha_1, \dots, \alpha_n; W \rangle$.

A general sector of $IX_{g,n,d}$ is a stratum of maps from $n-$pointed genus g curves together with an automorphism μ fixing the map and the marked points. It is determined by the discrete characteristics of the quotient curve (irreducible components, degree, genus), and the order of μ on each component, denoted M_v, and the orbifold structure of the quotient curve (i.e. all orbifold points, and the eigenvalue with which μ acts on the cotangent line, for nodes, this will be a pair of eigenvalues, one for each branch.)

We require that the eigenvalues to each branch at the node not be inverses, otherwise the node would be smoothable within that component. For a given component S, let $n(S)$ be the number of orbifold points whose isotropy group has order M_v, for v the component containing the point.

Theorem 8.2. *The dilaton equations for Euler-Satake Invariants are:*

$$\overline{\langle \alpha_1, \dots, \alpha_n, s \rangle}^E_{g,n+1,d} = \overline{\langle \alpha_1, \dots, \alpha_n; W - n(1_{Id}) \rangle}^E_{g,n,d}$$

$$\langle \alpha_1, \dots, \alpha_n, s \rangle^E_{g,n+1,d} = \langle \alpha_1, \dots, \alpha_n; W - n \rangle^E_{g,n+1,d}$$

Here W is the dilaton weight function *that has value* $2-2g$ *on the identity component, and n_S on the component S. 1_{Id} is the function with weight 1 on the identity component, and 0 elsewhere.*

The dilaton-shifted completed correlator is equal to:

$$\int_{IX} \prod_{i=1^n} Iev_i^* Ch(\alpha_i) \frac{c_{top}}{c}((L_{n+1}^{-1})^{inv})$$

We can compute the above integral on $IX_{g,n,d}$ by pushing forward by Ift. The preimage of the unit sector is just the unit sector of $IX_{g,n+1,d}$, so the contribution from that sector is identical to the case of Euler-Satake invariants, meaning that W is equal to $2g-2-n$ on the identity component.

A general sector of $IX_{g,n,d}$ is a stratum of maps from $n-$pointed genus g curves together with an automorphism μ fixing the map and the marked points. It is determined by the order of μ, denoted M, the discrete characteristics of the quotient curve (irreducible components, degree, genus), and the orbifold structure of the quotient curve.

The preimage of such a sector under Ift consists of strata of $n+1$ pointed stable maps, labelled by the same automorphism μ. Necessarily, the $n+1$st marked point is fixed by μ, hence the preimage forms a (virtual) finite covering of $IX^{\mu}_{g,n,d}$. If we call the degree of this covering d, the contribution from $IX^{\mu}_{g,n,d}$ obtained by pushing forward the correlator $\overline{\langle a_1, \dots, a_n, \frac{c_{top}}{c}((L_i^{-1})^{inv}) \rangle}^E$, is $d\overline{\langle a_1, \dots, a_n \rangle}^E$.

The fixed points of μ are precisely the preimages of the orbifold points whose isotropy group has maximal order, so $d = n(S)$

The assignemnt $W(S)$ is compatible with the structure of the boundary strata as fiber products, so the dilaton equation for incomplete Euler invariants follows from applying the dilaton equation for completed ones to the contributions from each stratum. The only terms that are not accounted for are when the $n+1$st point coincides with one of the n marked points, and the dilaton shift is placed into a genus-0 correlator with 3 marked points.

There are n such terms lying over each boundary stratum of $IX_{g,n,d}$, and thus subtracting them reduces the weight on each stratum by n. So we replace $W - nId$ with $W - n$.

8.2 The String Equation

Similarly to the dilaton equation, usual Gromov-Witten invariants also satisfy the string equation, which deals with the case where the $n+1$th input is equal to 1. We prove below the analogues of this equation for the four kinds of invariants we are interested in.

8.2.1 Euler-Satake Invariants

Theorem 8.3. *Given a boundary stratum determined by I, let $\mu(I)$ be the number of nodes of a general curve in X_I. The string equations for Euler-Satake Invariants are:*

$$\overline{\langle\alpha_1,\ldots,\alpha_n,1\rangle}^{ES}_{g,n+1,d} = (2-2g)\overline{\langle\alpha_1,\ldots,\alpha_n\rangle}^{ES}_{g,n,d} + \sum_{I\in\partial X_{g,n,d}} \mu(I)C(M_I)$$

$$\langle\alpha_1,\ldots,\alpha_n,1\rangle^{ES}_{g,n+1,d} = (2-2g-n)\langle\alpha_1,\ldots,\alpha_n\rangle^{ES}_{g,n,d}$$

The interpretation of this theorem is that if the correlator with inputs $\alpha_1,\ldots,\alpha_n$ represents the Euler characteristic of some space S, the correlator with inputs $\alpha_1,\ldots,\alpha_n,1$ represents the Euler characteristic of $ft^{-1}(S)$. Generically, ft^{-1} is a fiber bundle with fibers genus-g curves, with Euler characteristic $2-2g$. The exception is at the singular locus, where the Euler characteristic of the curve is increased by the number of nodes.

In the incomplete case, points of S representing singular curves are ignored, as are the points of $ft^{-1}(S)$ lying in the marked point divisors D_i, so the correction term at the nodes is unnecessary, and the fibers of $M_{g,n+1,d}(X)$ over $M_{g,n,d}(X)$ are smooth curves with n marked points removed.

Of course in general, this interpretation is not literally true, so the above does not constitute a proof of the string equations. Hence we prove them directly.

8.2.1.1 Completed Euler-Satake Invariants

For completed invariants, the relevant correlator is:

$$\overline{\langle\alpha_1,\ldots,\alpha_n,1\rangle}^{ES}_{g,n+1,d} = \int_{X_{g,n+1,d}} c(TX_{g,n+1,d})(\prod_{i=1}^{n} ev_i^*\alpha)$$

We can rewrite this as:

$$\int_{X_{g,n+1,d}} ft^*c(TX_{g,n,d})D(\Omega_{ft})(ft^*\prod_{i=1}^{n} ev_i^*\alpha).$$

Here Ω_{ft} denotes the relative cotangent bundle, and the characteristic class D is defined by $D(V) := c(V^*)$, for V a vector bundle, and extended via multiplicativity.

By the projection formula this is equal to

$$\int_{X_{g,n,d}} ft_*D(\Omega_{ft})c(TX_{g,n,d})\prod_i a_i$$

Using the equation for Ω_{ft} calculated by Coates in [3], it splits as:

$$L_{n+1} - \bigoplus_i \mathcal{O}_{D_i} - i_*\mathcal{O}_{\mathcal{Z}}$$

Consequentially $D(\Omega_{ft}) = D(L_{n+1})D(-\bigoplus_i \mathcal{O}_{D_i})D(-i_*\mathcal{O}_{\mathcal{Z}})$

However L_{n+1} is trivial on D_i and $\mathcal{Z}$. which are mutually disjoint, thus all cross-terms in the expansion of the above expression vanish, leaving us with: $1 + (D(L_{n+1}) - 1) + (\sum D(\mathcal{O}_{D_i}) - 1) + (D(-i_*\mathcal{O}_{\mathcal{Z}}) - 1)$. The pushforward of 1 vanishes for dimension reasons. We address each remaining term individually:

$D(L_{n+1}) = c(L^*_{n+1}) = 1 - c_1(L_{n+1})$. So the corresponding part of $D(\omega_{ft})$ is $-c_1(L_{n+1})$, which, as in the dilaton equation, gives a scale factor of $(2 - 2g - n)$.

$D(-i_*\mathcal{O}_{\mathcal{Z}}) = \frac{1}{c((i_*\mathcal{O}_{\mathcal{Z}})^*)}$

Coates proved in [3] the following lemma that applies to any multiplicative characteristic class D.

Lemma 8.4.

$$D(-i_*\mathcal{O}_{\mathcal{Z}}) = 1 + i_*\frac{1}{\psi_+\psi_-}(\frac{D(L_+)D(L_-)}{D(L_1 \otimes L_2)} - 1)$$

This gives us that the nodal term contributes:

$$(\frac{D(L_+)D(L_-)}{D(L_1 \otimes L_2)} - 1) = \frac{(1-\psi_+)(1-\psi_-)}{1-\psi_+-\psi_-} - 1 = \frac{\psi_+\psi_-}{1-\psi_+-\psi_-}$$

So the total contribution is: $i_*\frac{1}{1-\psi_+-\psi_-}$.

Since the conormal bundle to $ft \circ i$ is $L_+ \otimes L_+$, pushing the nodal contribution forward by ft gives: $ft_*i_*c_{total}(T_{ft\circ i})$.

So the contribution to the correlator is the integral over $\mathcal{Z}$ of the inputs, weighted by $c_{total}(T^{vir}_{\mathcal{Z}})$.

This is precisely the sum of $C(X_I)$ for X_I the top-dimensional closed boundary strata of $X_{g,n,d}$, not accounting for overlap. Rewriting this in terms of the open strata we get $\sum_{I\in\partial X_{g,n,d}} C(M_I)\mu(I)$, where $\mu(I)$ is the number of nodes of a generic curve in the stratum, and is equal to the number of top-dimensional strata contaning $M(I)$.

The contribution from the terms corresponding to the marked point divisors is:

$D(\mathcal{O}_{D_i}) = \frac{1}{c(\mathcal{O}^*_{D_i})} = c(O(-D_i)) = 1 + [D_i]$. So the correlator is restricted to D_i. Since ft is an isomorphism from D_i and $X_{g,n,d}$, each divisor contributions $\overline{\alpha_1, \dots, \alpha_n}^{ES}_{g,n,d}$, with a total of $n\overline{\alpha_1, \dots, \alpha_n}^{ES}_{g,n,d}$.

8.2.1.2 Incomplete Euler-Satake Invariants

For incomplete invariants, we induct on n and g. The base case of $n = 3, g = 0$ is done as follows:

Since there is no locus of nodes in $X_{0,4,d}$, the correlator $\overline{\langle\alpha_1, \alpha_2, \alpha_3, 1\rangle}^{ES}_{0,4,d} = (-1)\overline{\langle\alpha_1, \alpha_2, \alpha_3\rangle}^{ES}_{0,3,d} + 3\overline{\langle\alpha_1, \alpha_2, \alpha_3\rangle}^{ES}_{0,3,d}$.

However $3\langle\alpha_1, \alpha_2, \alpha_3\rangle^{ES}$ is exactly equal to the boundary contribution from $X_{0,4,d}$, since the only boundary components are the ones from the divisors of marked points, and the incomplete and complete invariants coincide for $n = 3, g = 0$, so we have: $\langle\alpha_1, \alpha_2, \alpha_3, 1\rangle^{ES}_{0,4,d} = (-1)\langle\alpha_1, \alpha_2, \alpha_3\rangle^{ES}$

To complete the induction step, recall

$$\langle\alpha_1, \dots, \alpha_n, 1\rangle^{ES}_{g,n+1,d} = \overline{\alpha_1, \dots, \alpha_n, 1\rangle}^{ES}_{g,n+1,d} - \sum_{I\in\partial X_{g,n+1,d}} C_{\alpha,1}(M_I).$$

By the string equation for completed invariants we can rewrite this as:

$$(2-2g-n)\overline{\langle\alpha_1, \dots, \alpha_n\rangle}^{ES}_{g,n,d} + n\overline{\langle\alpha_1, \dots, \alpha_n\rangle}^{ES}_{g,n,d} + \sum_{I\in\partial X_{g,n,d}} \mu(I)C(M_I) - \sum_{I\in\partial X_{g,n+1,d}} C_{\alpha,1}(M_I).$$

For a given boundary stratum whose closure is a fiber product of moduli spaces X_{g_i,n_i,d_i}, let the jth component contain the $n+1$st marked point (which carries the input 1). By the induction hypothesis, if we denote by I' the boundary stratum of $X_{g,n,d}$ corresponding to the image of M_I, we get $C_{\alpha,1}(M_I) = (2 - 2g_j - n_j)C_\alpha(M_{I'})$. This means for a given stratum in $X_{g,n,d}$, the total contributions from all the correlators lying above it are: $\sum_i 2 - 2g_i - n_i$, the Euler characteristics of the component curves of the stratum, with marked points and nodes removed. This sum is equal to $2 - 2g - n$.

The remaining strata lying above M_I that are unaccounted for are ones where the $n+1$st marked point lies on a new component, i.e. when it coincides with a marked point or a node.

When the $n+1$st point coincides with a marked point, that stratum M_J is contained in the divisor D_i, and contributes $C_{\alpha,1}(M_J)$. These contributions total to

$$\sum_{i=1}^{n} \sum_{M_J \subset D_i} C_{\alpha,1}(M_J) = \sum_{i=1}^{n} C_{\alpha,1}(D_i) = n\overline{\langle \alpha_1, \ldots, \alpha_n \rangle}^{ES}_{g,n,d}$$

.

Similarly, when the $n+1$st point is a node, the contribution $C_{\alpha,1}(M_J) = C_\alpha(M_I)$, and there are $\mu(I)$ such strata lying above M_I, so the total contribution is $\sum_I \mu(I) C_\alpha(M_I)$, which cancels with the nodal correction term in the string equation for completed invariants.

So

$$\sum_{I \in \partial X_{g,n+1,d}} C_{\alpha,1}(M_I) = \sum_{I \in \partial X_{g,n,d}} (2 - 2g - n) C_\alpha(M_I) + \overline{\langle \alpha_1, \ldots, \alpha_n \rangle}^{ES}_{g,n,d} + \sum_{I \in \partial X_{g,n,d}} \mu(I) C(M_I).$$

The extra terms cancel with the correction terms from the string equation for the completed invariants, giving the desired result.

8.2.2 Euler Invariants

The calculations for the additional components of the inertia stack are completely identical to the ones for the dilaton equation, since the additional input is still 1, so the final result is:

Theorem 8.5. *The string equations for Euler invariants are:*

$$\overline{\langle \alpha_1, \ldots, \alpha_n, 1 \rangle}^{E}_{g,n+1,d} = \overline{\langle \alpha_1, \ldots, \alpha_n; W \rangle}^{E}_{g,n,d} + \sum_{I \in \partial X_{g,n,d}} \mu(I) C^{ES}_\alpha(M_I)$$

$$\langle \alpha_1, \ldots, \alpha_n, 1 \rangle^{E}_{g,n+1,d} = \langle \alpha_1, \ldots, \alpha_n; W - n \rangle^{E}_{g,n,d}$$

Here W is the dilaton weight function introduced previously.

Remark. This means the string and dilaton equations for the incomplete invariants coincide exactly.

References

[1] G. Bini and J. Harer. Euler characteristics of moduli spaces of curves. *Journal of the European Mathematical Society*, 13:487–512, 2005.

[2] Jim Bryan and Jason Fulman. Orbifold euler characteristics and the number of commutingm-tuples in the symmetric groups. *Annals of Combinatorics*, 2(1):1–6, 1998.

[3] Tom Coates. *Riemann–Roch theorems in Gromov–Witten theory.* PhD thesis, UC Berkeley, 2003.

[4] Alexander Givental. Permutation-equivariant quantum K-theory IX: Quantum Hirzebruch-Riemann-Roch in all genera. 2017.

[5] J. Harer and D. Zagier. The euler characteristic of the moduli space of curves. *Inventiones mathematicae*, 85(3):457–485, Oct 1986.

[6] Irit Huq-Kuruvilla. Multiplicative quantum cobordism theory, 2021.

[7] Sören Illman. The equivariant triangulation theorem for actions of compact lie groups. *Mathematische Annalen*, 262:487–502, 1983.

[8] Tyler J. Jarvis and Takashi Kimura. The orbifold quantum cohomology of the classifying space of a finite group. In *Orbifolds in Mathematics and Physics, Contemp*, 2002.

[9] C. Murolo, D. J. A. Trotman, and A. A. Du Plessis. Stratified transversality by isotopy. *Transactions of the American Mathematical Society*, 355(12):4881–4900, 2003.

[10] Ichiro SATAKE. The Gauss-Bonnet Theorem for V-manifolds. *Journal of the Mathematical Society of Japan*, 9(4):464 – 492, 1957.