Conformal mapping in 2-dim fluid mechanics:

Def: Let $f : U \rightarrow \mathbb{C}$ be holomorphic. We say that $f$ is conformal at $z_0 \in U$ if $f'(z_0) \neq 0$.

The amazing property of conformal mappings is that they preserve angles. In particular, let $U$ be two curves, that meet at $z_0 \in U$ (i.e., $z_0 = \gamma_1(t) = \gamma_2(t)$ for $t_1, t_2 \in [a, b]$). Let $\theta$ be the angle between the tangents of $\gamma_1$ and $\gamma_2$ at $z_0$ (i.e., the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$).
Then, the images \( f(y_1) \) and \( f(y_2) \) of the two curves through \( f \) (\( f_0 y_1 : [a,b] \rightarrow V \), \( f_0 y_2 : [a,b] \rightarrow V \)),

which obviously pass through \( f(z_0) \) \( (z_0 = f_0 y_1(t_1)) \)

also have tangents at \( f(z_0) \) with angle \( \theta \) between them (just like in the preimage of \( f \)).

In other words, when \( f \) is conformal and \( y_1(t_1) = y_2(t_2) = z_0 \), then

angle \( (y_1'(t_1), y_2'(t_2)) = \) angle \( (f y_1'(t_1), f y_2'(t_2)) \)

(see p. 703 in textbook for a sort of explanation).

If \( f : U \rightarrow V \) is conformal,

then also \( f^{-1} : V \rightarrow U \) is conformal

\[
\begin{align*}
\text{(because} \quad (f^{-1} \circ f)'(z_0) &= (f^{-1})'(f(z_0)) \cdot f'(z_0), \quad \text{for all} \quad z_0) \\
\text{so} \quad (f^{-1})'(f(z_0)) &= \frac{(Id)'(z_0)}{f'(z_0)} \cdot \frac{1}{Id(z_0) \rightarrow z_0} = \frac{1}{f'(z_0) \neq 0} \\
+ f(z_0) &\in V.
\end{align*}
\]
for that reason, conformal mappings can be very useful when solving partial differential equations in 2-dim domains.

As an example, we will consider the following general problem in fluid mechanics:

Suppose that we have a fluid mechanics problem on a 2-dim domain \( \Omega \), such that the fluid at any \( x \in \Omega \) has velocity \( \mathbf{V}(x) \). Suppose the flow is:

incompressible
(i.e., \( \nabla \cdot \mathbf{V} = 0 \),

i.e., \( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \) on \( \partial \Omega \))

irrotational
(i.e., \( \text{curl} \mathbf{V} = 0 \),

i.e., \( \nabla \times \mathbf{V} = 0 \) on \( \partial \Omega \))

and we have the boundary condition that

\( \mathbf{V}(x) \cdot \mathbf{n}(x) = 0, \quad t \in \mathbb{R}. \)

i.e., \( \mathbf{V}(x) \) is tangent to \( \partial \Omega \) at \( x \).

steady
(i.e., there is no time dependence; at any time \( t \), the particle at \( x \in \Omega \) will have velocity \( \mathbf{V}(x) \) independent of \( t \)).
What is \( \vec{v}(x) \) \( \forall x \in U \)?

If \( U \) has no holes, the fact that the flow is irrotational actually implies that there exists some \( \Phi : U \rightarrow \mathbb{R} \) known as the velocity potential, s.t. \( \vec{v}(x) = \nabla \Phi(x) \), \( \forall x \in U \) (i.e., the desired velocity is the gradient of some function).

We go to the conditions in the previous page, and we plug in \( \nabla \Phi \) in the place of \( \vec{v} \) (in all the conditions apart \( \nabla \times \vec{v} = 0 \), as we have used that already).

We need to find \( \Phi : U \rightarrow \mathbb{R} \), s.t.

(i) \( \nabla \cdot \nabla \Phi = 0 \Leftrightarrow \Delta \Phi = 0 \), i.e. \( \Phi \) satisfies Laplace's equation!

(ii) \( \nabla \Phi \cdot \vec{n}(x) = 0 \), \( \forall x \in \partial U \) (i.e., \( \Phi \) doesn't change in the direction of the boundary).

So, we just need to solve Laplace's equation on \( U \), and then take the gradient of the solution; this will be \( \vec{v} \)!
What if $U$ was the upper half-plane? People know how to solve this problem in the upper half-plane. It is known that, if $U$ = upper half-plane, then the flow moves along all horizontal lines (streamlines).

More precisely: $\Phi$ is constant along any vertical line, and it changes linearly in the horizontal direction. (Thus, $\nabla \Phi$, which is perpendicular to $\{ \Phi = \text{const} \}$, is horizontal, and constant).

Notice indeed how conditions (i) and (ii) are satisfied for this $\nabla \Phi = \nabla \Phi_x = D \Phi$, for $\Phi$ as above. Here, $\nabla \Phi + i \{ \Phi = \text{constant} \}$ is obvious ($\{ \Phi = \text{const} \}$ is a vertical line, so on it $\frac{\partial \Phi}{\partial y} = 0$, thus $\nabla \Phi$ is horizontal). But this would hold even if $\{ \Phi = \text{const} \}$ was a curve (by the implicit function theorem). So, $\nabla \Phi(x) = \nabla \Phi(y)$ is always perpendicular to $\{ \Phi = \text{const} \}$. Another perspective: Since $\Phi$ satisfies Laplace's equation, it has a harmonic conjugate $\Psi$, and, as it happens for harmonic conjugates (by Cauchy-Riemann conditions for holomorphic $\Phi + i \Psi$), we have that the curves $\{ \Phi = \text{constant} \}$ and $\{ \Psi = \text{constant} \}$ are perpendicular when they meet. And $\{ \Psi = \text{constant} \}$ are exactly the streamlines (it can be proved).
Many of the above hold for $\Phi: U \to \mathbb{R}$ that satisfies $\nabla \Phi = 0$ on $U$, no matter how crazy $U$ looks! In particular, always:

If $\Psi$ is the harmonic conjugate of $\Phi$, then:

- At $x_0$, the curves $\{\Phi = \text{const}\}$ and $\{\Psi = \text{const}\}$ are perpendicular when they meet (i.e. $\nabla \Phi(x_0) \cdot \nabla \Psi(x_0) = 0$) as $\nabla \Psi(x_0)$ is always perpendicular to $\{\Phi = \text{const}\}$ and $\nabla \Phi(x_0)$ to $\{\Psi = \text{const}\}$.

- Clearly, by the above we have that $\nabla \Phi(x_0) = \mathbf{J}(x_0)$ is tangent to $\{\Phi = \text{const}\}$ as this is always (when $\Psi = \text{const}$ passes through $x_0$) perpendicular to $\{\Phi = \text{const}\}$, by implicit function theorem.
But, even better, \( \{ \psi = \text{const} \} \) is the path a particle at \( x_0 \) will follow in this flow! That is why the curves \( \{ \psi = \text{const} \} \) are called streamlines; they are the paths the fluid particles follow during the flow.

So, already complex analysis (the existence of harmonic conjugates and the Cauchy–Riemann conditions) has revealed a lot. Conformal mappings will now help us to take any

\[ U \]

conformally change it into the upper half-plane, where we know what happens with incompressible, irrotational flows, and then transfer the solution back to \( U \).

This is based on a simple observation and an important theorem:
Let $\phi : U \rightarrow \mathbb{R}$, with

(i) $\Delta \phi = 0$ on $U$

and (ii) $\nabla \phi (x) \cdot \vec{n}(x) = 0$, $x \in \partial U$. What is $\nabla \phi$?

**Observation:** Suppose that we can find a conformal mapping $f$ that sends $U$ to the upper half-plane, such that all is sent to $\partial (\text{upper half-plane}) = \text{real line}$.

If $\phi : (\text{upper half-plane}) \rightarrow \mathbb{R}$ satisfies $\Delta \phi = 0$ on upper half-plane

and $\nabla \phi(x) \cdot \vec{n}(x) = 0$ $\forall x$ on $\partial (\text{upper half-plane})$

then, $\phi := \tilde{\phi} \circ f$ (i.e. $\phi(y) = \tilde{\phi}(f(y)) \forall y \in U$)

satisfies conditions (i) and (ii).

\[ \text{[Problem 16, p. 746]} \]
So, all we need is to find such $f$; then, $\Gamma$ will be $\hat{\phi} \circ f$. And then, $\nabla^2 \phi = \nabla \phi$.

Notice that, since conformal mappings preserve angles, we have that, since $\nabla \hat{\phi}$ is perpendicular to the real line, then also $\nabla \phi$ is perpendicular to $\partial U$ (which is why $(\hat{u})$ is satisfied).

Also, $\{\phi = \text{const}\}$ maps to $\{\hat{\phi} = \text{const}\}$ via $f^4$; similarly for $\hat{\phi}$.

So, the streamlines $\{\psi = \text{const}\}$ in $U$ are just the horizontal lines via $f$! I.e., to find the streamlines for our flow, we just calculate $f^{-1}(\text{each horizontal line})$. The flow is just tangent to these streamlines.

$\nabla \phi = 3 \nabla \phi_0$ to these streamlines.

We don't need to find $\psi$ at all.

Not a surprise! $\nabla \phi$ should be tangent to these curves, as $f^{-1}$ preserves angles, and $\nabla \phi$ was tangent to the horizontal lines.
Finally, notice that the curves $\{\Phi = \text{const}\}$ and $\{\Phi = \text{const}\}$ are perpendicular, as $
abla \Phi$ and $\nabla \Phi$ are harmonic conjugates. However, this is also verified by the fact that, since $f^{-1}$ is conformal, and $\{\Phi = \text{const}\} \perp \{\Psi = \text{const}\}$, then $f^{-1}(\{\Phi = \text{const}\}) \perp f^{-1}(\{\Psi = \text{const}\})$ as well.

Theorem: (Riemann mapping theorem)

If $U \subseteq \mathbb{C}$ is simply connected (i.e., it has no holes), then there exists a conformal mapping $f: U \rightarrow \text{upper half-plane}.$

It is this theorem that tells us that what we described above is not vacuous nonsense; we know that, no matter how crazy our $U$ is as long as it has no holes, we can map it conformally to the upper half-plane!
So, the method we describe above to find a flow on a general \( U \) will always work, as long as we can find the appropriate conformal \( f \) that sends \( U \) to the upper half-plane (if we can't find it, it's just due to our inability; it exists!).

Here is a chart on which holomorphic maps send some standard shapes to the upper half-plane:

- Flow past a corner
- Flow past a cylindrical bump

\[ z^2 \]

\[ z + \frac{a}{2} \]
You can also look up **Nobius transformations.**

They are of the form $f: \mathbb{C} \to \mathbb{C}$, or $\mathbb{C} \setminus \text{point}$.

With

$$f(z) = \frac{az + b}{cz + d}$$

These send (lines and circles) to (lines or circles).

For example, the unit circle $\mathbb{D}$ is sent to a line through $\frac{4}{2-4}$ (it can only be sent to $\mathbb{C}$ unbounded, due to the blowing up at $1$), while it is sent to a circle (itself actually) through $\frac{1}{2}$. 
Some more mappings, that can further lead to the upper half-plane:

\[ \frac{z-a}{z-b} \]

intersection of discs

\[ \frac{1}{z-a} \]

\[ \frac{z-i}{z+i} \]
Polygons

\[ \text{Schwarz - Christoffel formula} \]
Of course, the above ideas can be used in any problem where we are required to find solutions \( \phi \) to Laplace's equation on a domain without holes, with some angular boundary conditions (i.e., \( \nabla \phi \) doesn't have to be the velocity of any fluid).

A situation like this is when we want to find how heat is distributed in a domain (with insulated boundary, with no sources of heat other than particular parts of the boundary where we keep the temperature constant). In this case, it is known that the temperature satisfies Laplace's equation (See p. T11 for such an example).

If \( \phi \) is the temperature at each point, then \( L = \text{const} \) are the curves along which heat is constant (the "isothermals"). The boundary condition is that \( \nabla \phi \) perpendicular to the boundary (i.e., \( \phi \) doesn't change in the direction of the boundary), so this also holds for
$\Delta \Phi$ and the boundary of $f(U)$, for $f$

$\Phi = \phi \circ f$ for any holomorphic conformal map $\phi$

**Conformally**

So, we can just transform $U$ to a parallelogram, find the isothermals there (easy), and get their preimages via $f$ to find the isothermals in the original domain $U$. 