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Series in \mathbb{R}

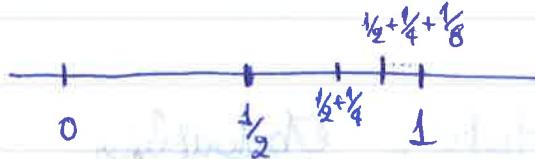
→ Problem: How do we add infinitely many real numbers together? I.e.:

Given a sequence $(a_n)_{n \in \mathbb{N}}$, what do we mean by $a_1 + a_2 + a_3 + \dots$?

For example:

- When $a_n = \frac{1}{2^n}$, $n \in \mathbb{N}$, what is

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ equal to? (What is it even defined as?)



It is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$, so one would think that we start from 0, then add $\frac{1}{2}$, then $\frac{1}{4}$, then $\frac{1}{8}$, etc.

Since at every step we add half the distance of where we are from 1, it may not come as a surprise that the infinite sum will eventually be shown to equal 1:

$1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$ gets closer and closer to 1 for n large.

- When $a_n = (-1)^n$, $n \in \mathbb{N}$, what is

$$\begin{aligned}
 & a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \dots \\
 & = (-1) + 1 + (-1) + 1 + (-1) + 1 + \dots \text{ equal to?}
 \end{aligned}$$

We have: $\alpha_1 = -1$,
 $\alpha_1 + \alpha_2 = 0$,
 $\alpha_1 + \alpha_2 + \alpha_3 = -1$,
 $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, etc.

So, as we add more terms, it doesn't look like $\alpha_1 + \alpha_2 + \dots + \alpha_n$ "stabilises" around some fixed value for n large. Thus, it may not come as a surprise that $(-1) + 1 + (-1) + 1 + (-1) + 1 + \dots$ is not defined.

From these examples, it looks like the infinite sum $\alpha_1 + \alpha_2 + \alpha_3 + \dots$ is determined by the behaviour of $\alpha_1 + \alpha_2 + \dots + \alpha_n$ for n large. Indeed, this is how we define the infinite sum $\alpha_1 + \alpha_2 + \alpha_3 + \dots$:

→ Def: Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

- We call the symbol $\sum_{k=1}^{\infty} \alpha_k$ "the series of α_k ".

- Let $s_1 := \alpha_1$,
 $s_2 := \alpha_1 + \alpha_2$,
 $s_3 := \alpha_1 + \alpha_2 + \alpha_3$,
 \vdots

$$s_n := \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n, \quad n \in \mathbb{N}.$$

We call s_n the n -th partial sum of the series $\sum_{k=1}^{\infty} \alpha_k$.

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- If the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums of

$\sum_{k=1}^{+\infty} a_k$ converges to some $s \in \mathbb{R}$, this was just a symbol up to now!

then we define $\sum_{k=1}^{+\infty} a_k$ to be s .

We say that the series $\sum_{k=1}^{+\infty} a_k$ converges to s ,

and we write $\sum_{k=1}^{+\infty} a_k = s$.

In other words: When $(s_n)_{n \in \mathbb{N}}$ converges,

$$\text{then } \sum_{k=1}^{+\infty} a_k := \lim_{n \rightarrow +\infty} s_n,$$

$$\text{i.e. } s_n \xrightarrow[n \rightarrow +\infty]{} \sum_{k=1}^{+\infty} a_k.$$

- If the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums of

$\sum_{k=1}^{+\infty} a_k$ is not convergent, then we say that

the series $\sum_{k=1}^{+\infty} a_k$ diverges. In particular:

- If $s_n \rightarrow +\infty$, then we define $\sum_{k=1}^{+\infty} a_k$ to be $+\infty$.

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We say that the series $\sum_{k=1}^{+\infty} a_k$ diverges to $+\infty$,

and we write $\sum_{k=1}^{+\infty} a_k = +\infty$.

In other words: When $s_n \rightarrow +\infty$,

then $\sum_{k=1}^{+\infty} a_k := \lim_{n \rightarrow +\infty} s_n = +\infty$,

i.e. $s_n \xrightarrow{n \rightarrow +\infty} \sum_{k=1}^{+\infty} a_k$.

- If $s_n \rightarrow -\infty$, then we define $\sum_{k=1}^{+\infty} a_k$ to be $-\infty$.

We say that the series $\sum_{k=1}^{+\infty} a_k$ diverges to $-\infty$,

and we write $\sum_{k=1}^{+\infty} a_k = -\infty$.

In other words: When $s_n \rightarrow -\infty$,

then $\sum_{k=1}^{+\infty} a_k := \lim_{n \rightarrow +\infty} s_n = -\infty$,

i.e. $s_n \xrightarrow{n \rightarrow +\infty} \sum_{k=1}^{+\infty} a_k$.

- If $\lim_{n \rightarrow +\infty} s_n$ doesn't exist (in $\mathbb{R} \cup \{+\infty, -\infty\}$), then $\sum_{k=1}^{+\infty} a_k$

diverges, and the infinite sum $a_1 + a_2 + \dots$ is not defined.

To sum up:

- If $\lim_{n \rightarrow \infty} s_n$ exists (in $\text{RV}\{+\infty, -\infty\}$), then $\sum_{k=1}^{+\infty} a_k := \lim_{n \rightarrow \infty} s_n$.
- If $\lim_{n \rightarrow \infty} s_n$ doesn't exist, then $\sum_{k=1}^{+\infty} a_k$ is not defined.



Sometimes, a sequence may be given in the form

$$(a_n)_{n=0}^{+\infty}, \text{ or } (a_n)_{n=4}^{+\infty}, \text{ etc.}$$

!!

!!

$$(a_0, a_1, a_2, \dots) \quad (a_4, a_5, a_6, \dots)$$

the number of terms in the sum

No matter what, the n -th partial sum s_n of the series corresponding to the sequence is the sum of the first n terms of the sequence.

(for the sequences above, for instance, s_3 is $a_0+a_1+a_2$ and $a_4+a_5+a_6$, respectively).

→ So, to find $\sum_{k=1}^{+\infty} a_k$, we start adding up the terms one by one in the order that they appear (first we have a_1 , then a_1+a_2 , then $a_1+a_2+a_3$, etc), and see whether, as the sum gets longer and longer, it has a limit.

Let us see some important examples:

Geometric series:

The geometric series with ratio $x \in \mathbb{R}$ is :

$$\sum_{k=0}^{+\infty} x^k$$

(i.e., the series $\sum_{k=0}^{+\infty} a_k$, for $(a_n)_{n=0}^{+\infty} = (1, x, x^2, x^3, \dots)$.)

for this series, $s_n = a_0 + a_1 + \dots + a_{n-1} =$

$$= 1 + x + \dots + x^{n-1} = \begin{cases} \frac{1-x^n}{1-x} \\ n, \text{ if } x=1. \end{cases}$$

So:

- for $x=1$, $s_n = n \rightarrow +\infty$, so $\sum_{k=0}^{+\infty} x^k = +\infty$.

- for $|x| < 1$, $\underbrace{|x|^n}_{\substack{\parallel \\ |x^n|}} \xrightarrow{n \rightarrow +\infty} 0$, i.e., $x^n \xrightarrow{n \rightarrow +\infty} 0$,

$$\text{so } s_n = \frac{1-x^n}{1-x} \xrightarrow{n \rightarrow +\infty} \frac{1-0}{1-x} = \frac{1}{1-x}.$$

Thus, for $|x| < 1$, $\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}$; the series converges.

- for $x \geq 1$, $x^n \xrightarrow{n \rightarrow +\infty} +\infty$, so $s_n = \frac{1-x^n}{(1-x)_0} \xrightarrow{n \rightarrow +\infty} +\infty$.

Thus, for $x \geq 1$, $\sum_{k=0}^{+\infty} x^k = +\infty$; the series diverges.

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- for $x \leq -1$: $(x^n)_{n \in \mathbb{N}}$ doesn't converge, so $(s_n)_{n \in \mathbb{N}}$ doesn't converge either.

Thus, $\sum_{k=0}^{+\infty} x^k$ diverges (and the infinite sum $1+x+x^2+\dots$ is not defined).

- To sum up:

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1,$$

and $\sum_{k=0}^{+\infty} x^k$ diverges for $|x| \geq 1$. In particular,
 $\sum_{k=0}^{+\infty} x^k = +\infty$ for $x \geq 1$

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- for $x \leq -1$: $(x^n)_{n \in \mathbb{N}}$ doesn't converge, so $(s_n)_{n \in \mathbb{N}}$ doesn't converge either.

Thus, $\sum_{k=0}^{+\infty} x^k$ diverges (and the infinite sum $1+x+x^2+\dots$ is not defined).

- To sum up:

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1,$$

and $\sum_{k=0}^{+\infty} x^k$ diverges for $|x| \geq 1$. (In particular, $\sum_{k=0}^{+\infty} x^k = +\infty$ for $x \geq 1$)

Lecture 10:

16 Sep 2016. ①

→ **Telescopic series:** The series $\sum_{k=1}^{+\infty} a_k$ is called

telescopic if there exists a sequence $(b_n)_{n \in \mathbb{N}}$, s.t.:

$$a_k = b_{k+1} - b_k, \quad k \in \mathbb{N}.$$

In that case: $s_n = a_1 + a_2 + \dots + a_n = b_2 - b_1 + b_3 - b_2 + \dots + b_n - b_{n-1} + b_{n+1} - b_n = b_{n+1} - b_1$

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$(s_n)_{n \in \mathbb{N}}$ converges $\iff (b_n)_{n \in \mathbb{N}}$ converges.

And : $\sum_{k=1}^{+\infty} a_k = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (b_{n+1} - b_1) = \lim_{n \rightarrow \infty} b_n - b_1.$

ex: $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} : a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \forall k \in \mathbb{N}.$

So: $a_k = b_{k+1} - b_k \quad \forall k \in \mathbb{N},$

where $b_k = -\frac{1}{k} \quad \forall k \in \mathbb{N}.$

Thus: $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} = \lim_{k \rightarrow \infty} b_{k+1} - b_1 = 0 - (-1) = 1.$



Prop: Let $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}}$ be two sequences,

and let $\lambda, \mu \in \mathbb{R}$. We consider the sequence $(\lambda a_k + \mu b_k)_{k \in \mathbb{N}}.$

If $\sum_{k=1}^{+\infty} a_k, \sum_{k=1}^{+\infty} b_k$ converge,

then $\sum_{k=1}^{+\infty} (\lambda a_k + \mu b_k)$ converges as well,

and $\sum_{k=1}^{+\infty} (\lambda a_k + \mu b_k) = \lambda \cdot \sum_{k=1}^{+\infty} a_k + \mu \cdot \sum_{k=1}^{+\infty} b_k.$

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Proof: Since $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ converge,

we have (by definition of series convergence) that

$$s_n := a_1 + a_2 + \dots + a_n \xrightarrow{n \rightarrow \infty} a$$

$$\text{and } t_n := b_1 + b_2 + \dots + b_n \xrightarrow{n \rightarrow \infty} b$$

for some $a, b \in \mathbb{R}$. Note that

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} s_n = a, \text{ and } \sum_{k=1}^{\infty} b_k = \lim_{n \rightarrow \infty} t_n = b.$$

We now consider the n -th partial sum of $\sum_{k=1}^{\infty} (\lambda a_k + \mu b_k)$:

$$\begin{aligned} \text{it equals } u_n &= (\lambda a_1 + \mu b_1) + (\lambda a_2 + \mu b_2) + \dots + (\lambda a_n + \mu b_n) \\ &= \lambda \cdot (a_1 + a_2 + \dots + a_n) + \mu \cdot (b_1 + b_2 + \dots + b_n) = \\ &= \lambda \cdot s_n + \mu \cdot t_n \xrightarrow{n \rightarrow \infty} \lambda a + \mu b \end{aligned}$$

So, $\sum_{k=1}^{\infty} (\lambda a_k + \mu b_k)$ converges, and

$$\sum_{k=1}^{\infty} (\lambda a_k + \mu b_k) = \lim_{n \rightarrow \infty} u_n = \lambda a + \mu b = \lambda \cdot \sum_{k=1}^{\infty} a_k + \mu \cdot \sum_{k=1}^{\infty} b_k.$$


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→ Prop: For any $m \in \mathbb{N}$,

$$\sum_{k=1}^{+\infty} a_k \text{ converges} \iff \sum_{k=m}^{+\infty} a_k \text{ converges}$$



This tells us that convergence of a series doesn't depend on the first terms of the series.

Proof: We have that, $\forall n > m$:

$$\underbrace{a_1 + a_2 + \dots + a_n}_{\begin{array}{l} \parallel \\ S_n, \\ \text{the } n\text{-th partial} \\ \text{sum of } \sum_{k=1}^{+\infty} a_k \end{array}} = (a_1 + \dots + a_{m-1}) + \underbrace{(a_m + a_{m+1} + \dots + a_n)}_{\begin{array}{l} \parallel \\ \text{the sum of the} \\ \text{first } n-(m-1) \\ \text{terms of } \sum_{k=m}^{+\infty} a_k \\ \parallel \\ t_{n-(m-1)}, \end{array}}$$

where t_k is the k -th partial sum of $\sum_{k=1}^{+\infty} b_k$.

I.e. : $S_n = \underbrace{(a_1 + a_2 + \dots + a_{m-1})}_{\text{a constant}} + t_{n-(m-1)}$, $\forall n \in \mathbb{N}$.

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Thus: $(s_n)_{n \in \mathbb{N}}$ converges $\Leftrightarrow (t_{n-(m-1)})_{n \in \mathbb{N}}$ converges.

$\underbrace{\quad}_{\text{a final part}}$

of $(t_n)_{n \in \mathbb{N}}$)

so converges

$\Leftrightarrow (t_n)_{n \in \mathbb{N}}$ converges

(in fact, they have the same limit).

So:

$(s_n)_{n \in \mathbb{N}}$ converges



$(t_n)_{n \in \mathbb{N}}$ converges,

i.e. $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow \sum_{k=m}^{\infty} a_k$ converges.



Notice, in particular, that

$$\lim_{n \rightarrow \infty} s_n = (a_1 + \dots + a_{m-1}) + \lim_{n \rightarrow \infty} t_{n-(m-1)}, \quad \text{if } \lim_{n \rightarrow \infty} t_n$$

so

$$\sum_{k=1}^{\infty} a_k = (a_1 + \dots + a_{m-1}) + \sum_{k=m}^{\infty} a_k, \quad \text{if } m \in \mathbb{N},$$

when $\sum_{k=1}^{\infty} a_k$ is a convergent series.

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→ Corollary: If the sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ are eventually equal (i.e., if $\exists n_0 \in \mathbb{N}$ s.t. $a_n = b_n \forall n \geq n_0$),

then $\sum_{k=1}^{\infty} a_k$ converges $\iff \sum_{k=1}^{\infty} b_k$ converges.

Proof: $\sum_{k=1}^{\infty} a_k$ converges $\iff \sum_{k=n_0+1}^{\infty} a_k$ converges
 previous Proposition

$\iff \sum_{k=n_0}^{\infty} b_k$ converges

$\iff \sum_{k=1}^{\infty} b_k$ converges.
 previous Proposition



Prop:

Let $(a_k)_{k \in \mathbb{N}}$ be a sequence.

If $\sum_{k=1}^{\infty} a_k$ converges, then: $a_k \xrightarrow{k \rightarrow \infty} 0$.

Proof: Let $s_n := a_1 + a_2 + \dots + a_n$, the n -th partial sum of $\sum_{k=1}^{\infty} a_k$.

" $\sum_{k=1}^{\infty} a_k$ converges" means that $s_n \xrightarrow{n \rightarrow \infty} s$,

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for some $s \in \mathbb{R}$. Now: $a_{n+1} = s_{n+1} - s_n \xrightarrow{n \rightarrow \infty} 0$.

$$\begin{matrix} & & \\ \downarrow n \rightarrow \infty & \downarrow n \rightarrow \infty \\ s & s \end{matrix}$$

So, $a_n \xrightarrow{n \rightarrow \infty} 0$ (as $(a_{n+1})_{n \in \mathbb{N}}$ is just a final part of $(a_n)_{n \in \mathbb{N}}$). ■

The idea for the above is that, to go from one partial sum to the next, we just add ~~the~~ one more term of the sequence. So, since, for large n , the s_n 's all cluster around some point (as $(s_n)_{n \in \mathbb{N}}$ converges), we cannot possibly be adding a lot to go from s_n to s_{n+1} !



The above Proposition is very important! It provides the simplest, most basic way to test if a series converges. It is formulated in the following way:

test

Preliminary test:

If $\sum_{k=1}^{\infty} a_k \rightarrow 0$, then $\sum_{k=1}^{\infty} a_k$ diverges

if $\sum_{k=1}^{\infty} a_k$ converged,
then $a_k \rightarrow 0$, contradiction.

Note that this is equivalent to the Proposition above: it just uses the simple fact that

 $A \Rightarrow B$ 

$(\text{not } B) \Rightarrow (\text{not } A)$ (easy by contradiction, for instance)

→ Examples:

• $\sum_{k=1}^{+\infty} (-1)^k$: $(-1)^k \xrightarrow[k \rightarrow \infty]{} 0$, so $\sum_{k=1}^{+\infty} (-1)^k$ diverges.

• $\sum_{k=1}^{+\infty} x^k$: When $|x| < 1$, then $|x^k| = |x|^k \xrightarrow[k \rightarrow \infty]{} 0$,
 so $x^k \xrightarrow[k \rightarrow \infty]{} 0$.

This tells me nothing! Notice that

So, even though
 I know that the
 geometric series above
converges when $|x| < 1$,
 this doesn't follow
 from the preliminary
 test.

the preliminary test doesn't imply
 convergence if the sequence goes to 0!

It just implies divergence if the sequence
 doesn't go to 0.

But: When $|x| \geq 1$, then $|x^k| \xrightarrow[k \rightarrow \infty]{} +\infty \neq 0$,

so $x^k \xrightarrow[k \rightarrow \infty]{} 0$, so $\sum_{k=1}^{+\infty} x^k$ diverges.

• $\sum_{k=1}^{+\infty} \frac{1}{k}$: $\frac{1}{k} \xrightarrow[k \rightarrow \infty]{} 0$, so the preliminary test

tells me nothing! In fact, we will later
 see that this series diverges.

Thus: When $a_k \xrightarrow[k \rightarrow \infty]{} 0$, ANYTHING CAN HAPPEN.

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→ I am allowed to write this:
 then $\sum_{k=1}^{+\infty} a_k$ converges, as it just misses some initial terms of $\sum_{k=1}^{+\infty} a_k$.

→ Prop.: If $\sum_{k=1}^{+\infty} a_k$ converges, then:

$$\sum_{k=n+1}^{+\infty} a_k \xrightarrow{n \rightarrow +\infty} 0.$$

Proof: Idea: Since $\sum_{k=1}^{+\infty} a_k$ converges, we have that

$$a_1 + a_2 + \dots + a_n \xrightarrow{n \rightarrow +\infty} \sum_{k=1}^{+\infty} a_k \in \mathbb{R}; \text{ so, for}$$

large n , $\sum_{k=1}^{+\infty} a_k$ is practically $a_1 + a_2 + \dots + a_n$.

So, their difference, which is $\sum_{k=n+1}^{+\infty} a_k$, must be very small:

$$\underbrace{a_1 + a_2 + a_3 + \dots}_{\text{pretty much equal}} = \underbrace{(a_1 + a_2 + \dots + a_n)}_{\text{pretty much equal}} + \underbrace{(a_{n+1} + a_{n+2} + \dots)}_{\text{pretty much } 0}.$$

We have shown that, if $\sum_{k=1}^{+\infty} a_k$ converges, then

$$\sum_{k=1}^{+\infty} a_k \in \mathbb{R} = (a_1 + a_2 + \dots + a_n) + \sum_{k=n+1}^{+\infty} a_k \in \mathbb{R}, \text{ then N.}$$

$$\text{So: } \sum_{k=n+1}^{+\infty} a_k = \underbrace{\sum_{k=1}^{+\infty} a_k}_{\text{constant}} - \underbrace{(a_1 + a_2 + \dots + a_n)}_{\substack{\downarrow n \rightarrow +\infty}}.$$

$$\xrightarrow{n \rightarrow +\infty} \underbrace{\sum_{k=1}^{+\infty} a_k}_{\in \mathbb{R}} - \underbrace{\sum_{k=1}^{+\infty} a_k}_{\in \mathbb{R}} = 0. \quad \blacksquare$$

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The following test for convergence is very important; it is used to prove other basic tests. It says really that $(s_n)_{n \in \mathbb{N}}$ converges $\Leftrightarrow (s_n)_{n \in \mathbb{N}}$ Cauchy

test

Cauchy criterion:

The series $\sum_{k=1}^{+\infty} a_k$ converges

Remember: this is just saying that $(s_n)_{n \in \mathbb{N}}$ is Cauchy!

$$\left\{ \begin{array}{l} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n > m \geq n_0, \\ |a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon \end{array} \right.$$

Proof: $\sum_{k=1}^{+\infty} a_k$ converges \Leftrightarrow the sequence $(s_n)_{n \in \mathbb{N}}$

of partial sums converges.

the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums is Cauchy.

a sequence in \mathbb{R}
converges
 \Leftrightarrow
it is Cauchy

$\xrightarrow{\text{definition of Cauchy sequence}}$

And: $(s_n)_{n \in \mathbb{N}}$ is Cauchy $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t.}$

$$\forall n, m \geq n_0, |s_n - s_m| < \varepsilon.$$

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Notice that this is equivalent to saying that:

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. : $\forall n > m \geq n_0$, $|s_n - s_m| < \varepsilon$

(because $|s_n - s_m| = |s_m - s_n|$).

And : for $n > m \geq n_0$, $s_n - s_m =$

$$= (a_1 + a_2 + \dots + a_m + a_{m+1} + \dots + a_n) - (a_1 + a_2 + \dots + a_m) = \\ = a_{m+1} + a_{m+2} + \dots + a_n.$$

So, $\sum_{k=1}^{+\infty} a_k$ converges \Leftarrow

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. : $\forall n > m \geq n_0$, $|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$.



→ Example:

~~~ **Harmonic series:** This is the series

$$\sum_{k=1}^{+\infty} \frac{1}{k}.$$

It diverges:

Suppose that it converges. Then, by the Cauchy criterion for  $\varepsilon = \frac{1}{4}$ , there should exist  $n_0 \in \mathbb{N}$  s.t. :

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$$\text{If } n > m \geq n_0, \quad \left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| < \frac{1}{4}$$

Notice that  $\left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| =$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \underset{n \geq m+1}{\geq} \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n-m \text{ terms}} = \frac{n-m}{n}$$

So, we should have that

$$\frac{n-m}{n} < \frac{1}{4}, \quad \text{if } n > m \geq n_0. \quad \textcircled{*}$$

You can probably already see that this is impossible: if  $m = n_0$  and  $n \rightarrow \infty$ , then  $\frac{n-m}{n} \xrightarrow{n \rightarrow \infty} 1$ , which is

larger than  $\frac{1}{4}$ . One can also prove that

$\textcircled{*}$  cannot hold this way: Since  $\textcircled{*}$  should

hold  $\forall n > m \geq n_0$ , it should hold in particular for  $m = n_0$  and  $n = 2n_0$ ; so,

$$\frac{1}{2} = \frac{2n_0 - n_0}{2n_0} < \frac{1}{4}, \text{ contradiction. So, } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.} \blacksquare$$

Lecture 11.

19 Sep 2016

When it comes to series convergence behaviour, it is much easier to classify all possible scenarios for series with non-negative terms. So, we start

now with such series (and, in fact, we will later see some tests for convergence for general series that

↓  
with positive  
and negative  
terms

actually rely on comparing our general series with other series that only have non-negative terms; for instance, when testing for absolute convergence, or when using comparison tests).

### → Series with non-negative terms:

test

→ Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence with  $a_k \geq 0, \forall k \in \mathbb{N}$ .

Then,  $s_{n+1} - s_n = a_{n+1} \geq 0, \forall n \in \mathbb{N}$  ;

so,  $(s_n)_{n \in \mathbb{N}}$  is increasing. So, there are only two options:

- If  $s_n \uparrow s \in \mathbb{R}$ , then  $\sum_{k=1}^{+\infty} a_k = s$  (the series converges).  
Actually,  $\sum_{k=1}^{+\infty} a_k = \sup\{s_n : n \in \mathbb{N}\}$
- If  $s_n \uparrow +\infty$ , then  $\sum_{k=1}^{+\infty} a_k = +\infty$ .

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Proof:  $(s_n)_{n \in \mathbb{N}}$  is increasing (explained already).

So, there are two cases:

Case 1:  $(s_n)_{n \in \mathbb{N}}$  is bounded from above.  
Then, we know that  $(s_n)_{n \in \mathbb{N}}$  converges to some  $s \in \mathbb{R} \Rightarrow \sum_{k=1}^{\infty} a_k$  converges.

Case 2:  $(s_n)_{n \in \mathbb{N}}$  is not bounded from above.

Any increasing sequence that is not bounded from above goes to  $\infty$  (easy), so  $s_n \xrightarrow[n \rightarrow \infty]{} \infty$ , i.e.  $\sum_{k=1}^{\infty} a_k = \infty$ .

Application:

→ 2nd proof that  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges : this proof also shows that  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ , which is more information than divergence.

$\sum_{k=1}^{\infty} \frac{1}{k}$  has positive terms, so  $(s_n)_{n \in \mathbb{N}}$  is increasing.

$$\text{And: } s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_2 = s_4 = 1 + \underbrace{\frac{1}{2}}_{\geq \frac{1}{4}} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 + 2 \cdot \frac{1}{2}$$

$$s_3 = s_8 = s_4 + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} \geq s_4 + 4 \cdot \frac{1}{8} \geq 1 + 2 \cdot \frac{1}{2} + \frac{1}{2} = 1 + 3 \cdot \frac{1}{2}, \text{ and so on.}$$

(3)

Inductively, we can prove that

$$s_{2^n} \geq 1 + n \cdot \frac{1}{2}, \quad \forall n \in \mathbb{N}.$$

So,  $s_{2^n} \xrightarrow{n \rightarrow \infty} +\infty$ ; in particular,  $(s_n)_{n \in \mathbb{N}}$  is not bounded from above. So, since  $(s_n)_{n \in \mathbb{N}}$  is increasing, we have that  $s_n \xrightarrow{n \rightarrow \infty} +\infty$ . ■

Test



### Cauchy condensation test:

Let  $(\alpha_k)_{k \in \mathbb{N}}$  be a decreasing sequence, with  $\alpha_k \geq 0 \quad \forall k \in \mathbb{N}$ . Then:

$$\sum_{k=0}^{+\infty} \alpha_k \text{ converges} \iff$$

$$\sum_{k=0}^{+\infty} 2^k \alpha_{2^k} \text{ converges.}$$

Proof:

Idea: Since  $(\alpha_k)_{k \in \mathbb{N}} \downarrow$ , we have:

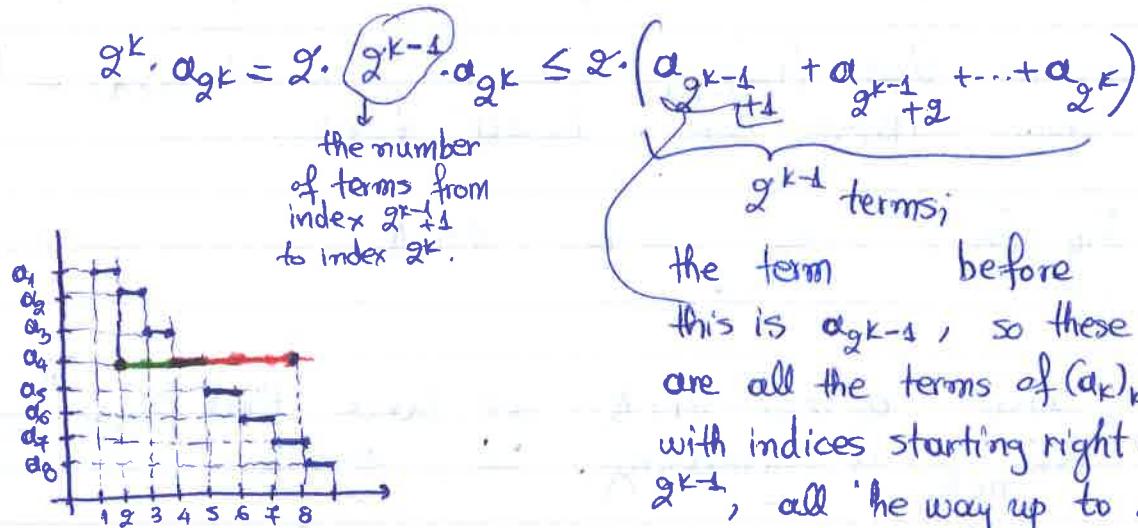
$$2^k \cdot \alpha_{2^k} \geq \underbrace{\alpha_{2^k} + \alpha_{2^k+1} + \dots + \alpha_{2^k+(2^k-1)}}_{2^k \text{ terms}}$$

The number of terms from index  $2^k$  to  $2^{k+1}-1$

the next after this is  $\alpha_{2 \cdot 2^k} = \alpha_{2^{k+1}}$ , so these are all the terms of  $(\alpha_k)_{k \in \mathbb{N}}$  with indices from  $2^k$  up to right before  $2^{k+1}$ .

(4)

On the other hand,



Let  $s_m$  denote the  $n$ -th partial sum of  $\sum_{k=0}^{+\infty} \alpha_k$ ,  
and  $t_n$  denote the  $n$ -th partial sum of  $\sum_{k=0}^{+\infty} 2^k \alpha_{2^k}$ .

Notice that:

$$\begin{aligned} \Rightarrow t_n &= 2^0 \alpha_0 + 2^1 \alpha_2 + 2^2 \alpha_4 + \dots + 2^{n-1} \alpha_{2^{n-1}} = \\ &= \alpha_1 + 2\alpha_2 + 4\alpha_4 + \dots + 2^{n-1} \alpha_{2^{n-1}} = \\ &\geq \alpha_1 + (\alpha_2 + \alpha_3) + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \dots + (\alpha_{2^{n-1}} + \alpha_{2^{n-1}+1} + \dots + \alpha_{2^n-1}) \\ &\quad \text{since } (\alpha_k)_{k \in \mathbb{N}} \downarrow \\ &= S_{2 \cdot 2^{n-1}-1} = S_{2^n-1}, \forall n \in \mathbb{N}. \end{aligned}$$

So: 
$$S_{2^n-1} \leq t_n, \forall n \in \mathbb{N}. \quad \text{④}$$

(5)

Now, suppose that  $\sum_{k=0}^{+\infty} 2^k \alpha_{2^k}$  converges. This means that  $(t_n)_{n \in \mathbb{N}}$  converges, so  $(t_n)_{n \in \mathbb{N}}$  is bounded. Thus,  $\exists M > 0$  s.t.  $t_n \leq M, \forall n \in \mathbb{N}$ .

By ④,  $s_{2^n-1} \leq M, \forall n \in \mathbb{N}$

Since  $\alpha_k \geq 0 \quad \forall k \in \mathbb{N}$ , we have that  $(s_n)_{n \in \mathbb{N}}$  is increasing. So,  $\forall n \in \mathbb{N}$ :

$$\underbrace{s_n}_{\substack{n \leq 2^m-1 \\ \forall m \in \mathbb{N} \\ (\text{check it by induction})}} \leq s_{2^n-1} \leq M.$$

Thus,  $(s_n)_{n \in \mathbb{N}}$  is increasing and bounded from above

$\Downarrow$   
 $(s_n)_{n \in \mathbb{N}}$  converges,  
i.e.  $\sum_{k=1}^{+\infty} 2^k \alpha_k$  converges.

$$\begin{aligned} t_{n+1} &= 2^0 \alpha_{2^0} + 2^1 \alpha_{2^1} + 2^2 \alpha_{2^2} + \dots + 2^n \alpha_{2^n} \leq \\ &\leq 2\alpha_1 + 2\alpha_2 + 4\alpha_4 + \dots + 2^n \alpha_{2^n} = \\ &= 2 \cdot (\alpha_1 + \alpha_2 + 2\alpha_4 + \dots + 2^{n-1} \alpha_{2^n}) \stackrel{\text{as } (\alpha_k)_{k \in \mathbb{N}}}{\leq} \\ &\leq 2 \cdot (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \dots + \alpha_{2^{n-1}+1} + \dots + \alpha_{2^n}) = \\ &= 2 \cdot s_{2^n}. \end{aligned}$$

(6)

So:

$$\boxed{t_{n+1} \leq 2 \cdot s_{2^n}, \forall n \in \mathbb{N}} \quad \text{⑥}$$

Now, suppose that  $\sum_{k=1}^{+\infty} a_k$  converges. This means that

$(s_n)_{n \in \mathbb{N}}$  converges, so  $(s_{2^n})_{n \in \mathbb{N}}$  converges,

so  $(s_{2^n})_{n \in \mathbb{N}}$  is bounded. Thus,  $\exists N > 0$  s.t.

$$s_{2^n} \leq N \quad \forall n \in \mathbb{N} \Rightarrow t_{n+1} \leq 2N, \quad \forall n \in \mathbb{N} \quad \text{⑦}$$

$(t_n)_{n \in \mathbb{N}}$  bounded from above.

Since  $a_k \geq 0 \quad \forall n \in \mathbb{N}$ , we also have that  $(t_n)_{n \in \mathbb{N}}$  is increasing.

So,  $(t_n)_{n \in \mathbb{N}}$  converges, i.e.  $\sum_{k=0}^{+\infty} a_{2^k}$  converges.

Application:



**p-series:**

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \quad , \text{ for } p > 0$$

We have that

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ converges} \Rightarrow p > 1$$

For instance :

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \text{ converges,}$$

$$\sum_{k=1}^{+\infty} \frac{1}{k} \text{ diverges (harmonic series = 1-series),}$$

$$\sum_{k=1}^{+\infty} \frac{1}{\sqrt{k}} \text{ diverges.}$$

(7)

Proof: Since  $\sum_{k=1}^{+\infty} \frac{1}{k^p}$  has non-negative terms,  
 which decrease in  $k$   $\left( \text{since } p \geq 0, k^p < (k+1)^p \forall k \in \mathbb{N} \right)$   
 $\Rightarrow \frac{1}{(k+1)^p} \leq \frac{1}{k^p} \forall k \in \mathbb{N}$

we know, by the Cauchy condensation test,  
 that

$\sum_{k=1}^{+\infty} \frac{1}{k^p}$  converges  $\Leftrightarrow \sum_{k=1}^{+\infty} 2^k \cdot \frac{1}{(2^k)^p}$  converges.

$$\text{And: } \sum_{k=1}^{+\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=1}^{+\infty} \frac{1}{(2^k)^{p-1}} = \sum_{k=1}^{+\infty} \left( \frac{1}{2^{p-1}} \right)^k,$$

which is a geometric series with ratio  $\frac{1}{2^{p-1}}$ ;

$$\text{so, it converges} \Leftrightarrow \left| \frac{1}{\frac{1}{2^{p-1}}} \right| < 1 \Leftrightarrow \frac{1}{2^{p-1}} < 1 \Leftrightarrow 2^{p-1} > 1 \Leftrightarrow p-1 > 0 \Leftrightarrow p > 1$$

### General series :

(not necessarily with non-negative terms).

→ Def: Let  $(\alpha_k)_{k \in \mathbb{N}}$  be a sequence.

(i) We say that  $\sum_{k=1}^{+\infty} \alpha_k$  converges **absolutely** if

$\sum_{k=1}^{+\infty} |\alpha_k|$  converges.

(7)

Proof: Since  $\sum_{k=1}^{+\infty} \frac{1}{k^p}$  has non-negative terms, which decrease in  $k$   $\left( \text{since } p > 0, k^p < (k+1)^p \forall k \in \mathbb{N} \right)$

$$\Rightarrow \frac{1}{(k+1)^p} \leq \frac{1}{k^p} \quad \forall k \in \mathbb{N}$$

we know, by the Cauchy condensation test,

$\sum_{k=1}^{+\infty} \frac{1}{k^p}$  converges  $\Leftrightarrow \sum_{k=1}^{+\infty} 2^k \cdot \frac{1}{(2^k)^p}$  converges.

$$\text{And: } \sum_{k=1}^{+\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=1}^{+\infty} \frac{1}{(2^k)^{p-1}} = \sum_{k=1}^{+\infty} \left( \frac{1}{2^{p-1}} \right)^k,$$

which is a geometric series with ratio  $\frac{1}{2^{p-1}}$ ;

$$\text{So, it converges} \Leftrightarrow \left| \frac{\frac{1}{2^{p-1}}}{\frac{1}{2^{p-1}}} \right| < 1 \Leftrightarrow \frac{1}{2^{p-1}} < 1 \Leftrightarrow 2^{p-1} > 1 \Leftrightarrow p-1 > 0 \Leftrightarrow p > 1$$

### Lecture 12:

① 21 Sep 2016. ■

#### General series :

(not necessarily with non-negative terms).

→ Def: Let  $(\alpha_k)_{k \in \mathbb{N}}$  be a sequence.

(i) We say that  $\sum_{k=1}^{+\infty} \alpha_k$  converges **absolutely** if  $\sum_{k=1}^{+\infty} |\alpha_k|$  converges.

(2)

(ii) If  $\sum_{k=1}^{\infty} a_k$  converges, but does not converge absolutely,

we say that  $\sum_{k=1}^{\infty} a_k$  converges conditionally.

Test

→ Prop: Any absolutely convergent series converges.

⚠ Converse  
not true!!!

i.e.: if  $\sum_{k=1}^{\infty} |a_k|$  converges, then  $\sum_{k=1}^{\infty} a_k$  converges.

Proof: Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence, with  $\sum_{k=1}^{\infty} |a_k|$  convergent.

We want to show that  $\sum_{k=1}^{\infty} a_k$  converges.

We will do so using the Cauchy criterion.

Notice that,  $\forall n > m \in \mathbb{N}$ , we have that

$$|a_{m+1} + a_{m+2} + \dots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n| =$$

triangle  
inequality

$$= |a_{m+1}| + |a_{m+2}| + \dots + |a_n|$$

Let  $\epsilon > 0$ .

Since  $\sum_{k=1}^{\infty} |a_k|$  converges, we have by the Cauchy criterion that:

$$\exists n_0 \in \mathbb{N} \text{ s.t. } \forall n > m \geq n_0, |a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$$

$|a_{m+1} + \dots + a_n|, \text{ by } \textcircled{*}$

(3)

Thus,  $\forall n > m \geq n_0$ ,  $|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$ .

By the Cauchy criterion,  $\sum_{k=1}^{+\infty} a_k$  converges. ■



### Examples:

(i)

$$\left| \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2} \right| = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Since  $\sum_{k=1}^{+\infty} \left| \frac{(-1)^{k-1}}{k^2} \right| = \sum_{k=1}^{+\infty} \frac{1}{k^2}$  converges,

we have also that  $\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2}$  converges.

(ii)

$$\left| \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \right| = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$\sum_{k=1}^{+\infty} \left| \frac{(-1)^{k-1}}{k} \right| = \sum_{k=1}^{+\infty} \frac{1}{k}$  diverges. However,

→ this proves that convergence  $\not\Rightarrow$  absolute convergence!

$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k}$  converges. We will see later a theorem

that implies the convergence of the series in both (i) and (ii). Let's see separately here though a proof

that

$$\left| \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \right| \text{ converges.}$$

(4)

Let  $s_n$  be the  $n$ -th partial sum of  $\sum_{k=1}^{+\infty} \frac{(k-1)^{k-1}}{k}$ .

We will show that  $(s_{2n})_{n \in \mathbb{N}}$  and  $(s_{2n-1})_{n \in \mathbb{N}}$  converge and have the same limit:

$$\bullet \quad s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{2(2n-1)} - \frac{1}{2n} =$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) =$$

$$\textcircled{*} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \dots + \frac{1}{(2n-1) \cdot 2n}, \quad \forall n \in \mathbb{N}$$

$$\text{So, } s_{2(n+1)} - s_{2n} = \frac{1}{(2(n+1)-1) \cdot 2(n+1)} = \frac{1}{(2n+1)(2n+2)} > 0,$$

thus  $(s_{2n})_{n \in \mathbb{N}}$  is an increasing sequence.

$$\begin{aligned} \text{Moreover, by } \textcircled{*}, \quad 0 < s_{2n} &\leq \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} \leq \\ &\leq 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n)^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^2} \end{aligned}$$

Notice that the last inequality holds because the left-hand side equals  $t_{2n}$ , where

$t_n$  is the  $n$ -th partial sum of  $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ . Now,  $\sum_{k=1}^{+\infty} \frac{1}{k^2}$

only has non-negative terms, and thus increasing

(5)

partial sums. Since we know that  $\lim_{n \rightarrow +\infty} \sum_{k=1}^{+\infty} \frac{1}{k^2} \in \mathbb{R}$ , increasing

it must be that  $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \sup \{t_n : n \in \mathbb{N}\}$ . Thus,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n)^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^2}.$$

Thus,  $(s_{2n})_{n \in \mathbb{N}}$  is a bounded, monotone sequence

$\Rightarrow s_{2n} \xrightarrow{n \rightarrow +\infty} s$ , for some  $s \in \mathbb{R}$ .

$$\bullet \quad s_{2n+1} = s_{2n} + \frac{(-1)^{(2n+1)-1}}{2n+1} = s_{2n} + \frac{1}{2n+1} \xrightarrow{n \rightarrow +\infty} s.$$

$\downarrow \quad \downarrow$   
n  $\rightarrow \infty$       n  $\rightarrow \infty$   
s      0

Since  $(s_{2n})_{n \in \mathbb{N}}$  and  $(s_{2n+1})_{n \in \mathbb{N}}$  both converge to the same element of  $\mathbb{R}$ ,  $(s_n)_{n \in \mathbb{N}}$  also converges (to the same limit)  
(Exercise!)

Thus,  $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$  converges. ■

⚠ This last example tells us that there exist conditionally convergent series (i.e. convergent series  $\sum_{k=1}^{+\infty} a_k$  s.t.  $\sum_{k=1}^{+\infty} |a_k|$  doesn't converge). So:

Convergence  $\not\Rightarrow$  Absolute convergence !!!

(5)

## Comparison tests:

These are tests for convergence of a series  $\sum_{k=1}^{\infty} a_k$ , that rely on comparing  $\sum_{k=1}^{\infty} a_k$  with another series whose convergence behaviour we know, and we ~~guess~~ is similar to that of  $\sum_{k=1}^{\infty} b_k$ .

Comparison tests can only test  $\sum_{k=1}^{\infty} a_k$  for

absolute convergence (so, they cannot imply convergence for a series that is only conditionally convergent).

The simplest comparison test is commonly known as the "comparison test".

test

## Comparison test:

Let  $\sum_{k=1}^{\infty} a_k$  to test,  $\sum_{k=1}^{\infty} b_k$  known behaviour, with  $b_k \geq 0, \forall k \in \mathbb{N}$

(i) If  $|a_k| \leq b_k \forall k \in \mathbb{N}$  and  $\sum_{k=1}^{\infty} b_k$  converges, then  $\sum_{k=1}^{\infty} |a_k|$  converges (and, in fact,  $\sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} b_k$ )

(ii) If  $a_k \geq b_k \forall k \in \mathbb{N}$  and  $\sum_{k=1}^{\infty} b_k = \infty$ , then

$$\sum_{k=1}^{\infty} a_k = \infty.$$

(7)



Notice that, since

$\sum_{k=1}^{\infty} b_k$  has only non-negative terms,

the only two possibilities are  $\sum_{k=1}^{\infty} b_k \in \mathbb{R}$  or  $\sum_{k=1}^{\infty} b_k = +\infty$

Proof: (i) Since  $\sum_{k=1}^{\infty} |a_k|$  has only non-negative terms, to show it converges, it suffices to show that its sequence  $(|a_1| + |a_2| + \dots + |a_n|)_{n \in \mathbb{N}}$  of partial sums is bounded from above. Indeed:

(8)

$$\text{then } n \in \mathbb{N}, \quad |a_1| + |a_2| + \dots + |a_n| \leq b_1 + b_2 + \dots + b_n \leq \sum_{k=1}^{\infty} b_k \in \mathbb{R},$$

$\downarrow$

$|a_k| \leq b_k \quad \forall k \in \mathbb{N}$

$\downarrow$

$b_k \geq 0 \quad \forall k \in \mathbb{N}$

as  $\sum_{k=1}^{\infty} b_k$  converges.

Thus,  $(|a_1| + |a_2| + \dots + |a_n|)_{n \in \mathbb{N}}$  is an increasing sequence, bounded from above (by  $\sum_{k=1}^{\infty} b_k (\in \mathbb{R})$ , by (8)).

So,

$\sum_{k=1}^{\infty} |a_k|$  converges.

In particular, (8) implies that

$$\lim_{n \rightarrow \infty} (|a_1| + |a_2| + \dots + |a_n|) \leq \sum_{k=1}^{\infty} b_k, \text{ thus}$$

$\sum_{k=1}^{\infty} |a_k|$

$$\sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} b_k \quad \left( \begin{array}{l} \text{! Notice that we first need to prove} \\ \text{that } (|a_1| + \dots + |a_n|)_{n \in \mathbb{N}} \text{ converges, before} \\ \text{saying that its limit is } \sum_{k=1}^{\infty} |a_k| \end{array} \right)$$

(8)

(iii)

$$|a_1| + |a_2| + \dots + |a_n| \geq b_1 + b_2 + \dots + b_n \xrightarrow{n \rightarrow \infty} +\infty,$$

$\downarrow$   
 $|a_k| \geq b_k$   
 for all  $k \in \mathbb{N}$

$\downarrow$   
 as  $\sum_{k=1}^{+\infty} b_k = +\infty$

so

$$|a_1| + |a_2| + \dots + |a_n| \xrightarrow{n \rightarrow \infty} +\infty,$$

i.e.

$$\sum_{k=1}^{+\infty} |a_k| = +\infty.$$

**(Test)**

### → Limiting comparison test:

Let  $\sum_{k=1}^{+\infty} |a_k|$  to test

known behaviour

$\sum_{k=1}^{+\infty} b_k$ , with  $b_k \geq 0$   
 for  $k \in \mathbb{N}$

(i) If  $\sum_{k=1}^{+\infty} b_k$  converges, and  $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$  exists in  $\mathbb{R}$ ,

then  $\sum_{k=1}^{+\infty} |a_k|$  converges.

(really, it can only be in  $[0, +\infty]$ ,  
 so we just exclude  $+\infty$ )

(ii) If  $\sum_{k=1}^{+\infty} b_k = +\infty$ , and  $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$  exists in  $(0, +\infty]$ ,

then  $\sum_{k=1}^{+\infty} |a_k| = +\infty$ .

Proof: (i) Since  $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k} = l \in \mathbb{R}$ , there exists  $\exists N \in \mathbb{N}$  s.t.  $\forall k \geq N$ ,  $\frac{|a_k|}{b_k} > l$  (as the sequence has terms  $> 0$ )

$$\forall k \geq N, \frac{|a_k|}{b_k} > l \Rightarrow |a_k| > l \cdot b_k$$

$\frac{|a_k|}{b_k} > l$

(9)

Since  $\sum_{k=1}^{+\infty} b_k$  converges, we also have that

$$\sum_{k=1}^{+\infty} \underbrace{(l+1) b_k}_{\in \mathbb{R}} \text{ converges} \Rightarrow \sum_{k=k_0}^{+\infty} \underbrace{(l+1) b_k}_{\geq 0} \text{ converges}$$

initial terms  
don't affect  
convergence  
behaviour

by (i) and

$\rightarrow$   
the comparison  
test

$$\sum_{k=1}^{+\infty} |a_k| \text{ converges.}$$

Idea: Since  $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k} \in \mathbb{R}$ ,  
 $|a_k|$  should be "pretty  
much" equal to

a constant times  $b_k$   
(the limit) for  $k$  large,  
it should have the same  
behaviour as  $\sum b_k$ .

(ii) Suppose that  $\sum_{k=1}^{+\infty} |a_k|$  converges.

Since

$$\frac{|a_k|}{b_k} \xrightarrow[k \rightarrow \infty]{} l \in (0, +\infty], \text{ we have}$$

that the terms of  $(a_k)_{k \in \mathbb{N}}$  are eventually  $> 0$ ,

and thus  $\frac{b_k}{|a_k|} \xrightarrow[k \rightarrow \infty]{} \frac{1}{l} \in \mathbb{R}$ , notice that when  $l = +\infty$ , this follows from the exercise on the algebra of limits  $\xrightarrow{k \rightarrow \infty}$  in Weekly Assignment 3, that says that if  $a_k \xrightarrow[k \rightarrow \infty]{} +\infty$  and  $a_k \neq 0 \forall k$ , then  $\frac{1}{a_k} \xrightarrow[k \rightarrow \infty]{} 0$ .

for the sequence  $\left(\frac{b_k}{|a_k|}\right)_{k=k_0}^{+\infty}$

Since  $\sum_{k=k_0}^{+\infty} |a_k|$  converges, by (i) we

have that  $\sum_{k=k_0}^{+\infty} b_k$  converges,

thus  $\sum_{k=1}^{+\infty} b_k$  converges,

contradiction.

(10)

You may prefer a proof as in (i) :

Since  $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k} \in (0, \infty]$ , there exists

some  $N > 0$  s.t.  $\frac{|a_k|}{b_k} > N, \forall k \geq k_0$ ,

for some  $k_0 \in \mathbb{N}$ .

(check this when the limit is in  $(0, \infty)$  and when it is  $\infty$ )

So,  $|a_k| > N \cdot b_k, \forall k \geq k_0$ . (\*)

Since  $\sum_{k=1}^{+\infty} b_k = \infty$ , we also have that  $\sum_{k=1}^{+\infty} N \cdot b_k = \infty$ ,

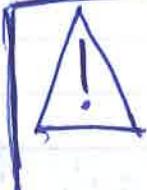
so, by (\*) and the comparison test,  $\sum_{k=1}^{+\infty} |a_k| = \infty$ . ■



Whenever we want to apply a comparison test, we need to guess what series  $\sum_{k=1}^{+\infty} b_k$  to compare our series with. To do that:

- We look at the  $k$ -th term  $|a_k|$  of our series, and intuitively guess a simpler quantity  $b_k$  that we expect  $a_k$  to behave like (ex.: we guess that  $\frac{k^2+k}{k^3}$  should behave like  $\frac{k^2}{k^3} = \frac{1}{k}$ ) .

- We then take  $\frac{|a_k|}{b_k}$ , and see if the limit exists. If yes, perhaps the limiting comparison test will work.



Notice that, when we guess what quantity  $b_k$

$|a_k|$  behaves like, we are truly looking for  $b_k$

s.t.  $\frac{|a_k|}{b_k} \xrightarrow{k \rightarrow \infty} l$  ! See that this is exactly

what happens for  $|a_k| = \frac{k^2+k}{k^3}$  and  $b_k = \frac{1}{k^3}$ .

And if  $\frac{|a_k|}{b_k} \xrightarrow{k \rightarrow \infty} l \in (0, +\infty)$ ,

notice that then the <sup>limiting</sup> comparison test

will definitely tell us if  $\sum_{k=1}^{+\infty} |a_k|$  converges or not.

(as long as we know what happens for  $\sum_{k=1}^{+\infty} b_k$ )  
 So, a good guess for  $(b_k)_{k \in \mathbb{N}}$  will solve our problem for  $\sum_{k=1}^{+\infty} |a_k|$ .



Remember this : amongst other things, the limiting

comparison test tells us that, if  $\frac{|a_k|}{b_k} \xrightarrow{k \rightarrow \infty} l \in (0, +\infty)$ ,

then  $\sum_{k=1}^{+\infty} |a_k|$  and  $\sum_{k=1}^{+\infty} b_k$  have the same

convergence behaviour.

(12)

→ Examples:

$$\textcircled{1} \quad \sum_{k=1}^{+\infty} \left| \frac{\cos(kx)}{k^2} \right| = "a_k"$$

$$|a_k| = \left| \frac{\cos(kx)}{k^2} \right| = \frac{|\cos(kx)|}{k^2} \leq \frac{1}{k^2} \quad \forall k \in \mathbb{N},$$

and  $\sum_{k=1}^{+\infty} \frac{1}{k^2}$  converges, so  $\sum_{k=1}^{+\infty} \left| \frac{\cos(kx)}{k^2} \right|$  converges,

so  $\sum_{k=1}^{+\infty} \frac{\cos(kx)}{k^2}$  converges.

$$\textcircled{2} \quad \sum_{k=1}^{+\infty} \left| \frac{k+1}{k^4+k^2+3} \right| = "a_k"$$

We guess that

[ we expect  $|a_k|$  should "behave like"  $\frac{1}{k^3}$  for large  $k$  ]

$$\frac{1}{k^3} > 0 \quad \text{is appropriate}$$

to compare with

(which means truly that we expect that

$$\frac{|a_k|}{\frac{1}{k^3}} \xrightarrow[k \rightarrow +\infty]{} 1$$

$$\text{We have: } \frac{|a_k|}{b_k} = \frac{\frac{1}{k^3}}{\frac{k+1}{k^4+k^2+3}} = \frac{\frac{1}{k^3}}{\frac{1}{k^4}(1+\frac{1}{k^2}+\frac{3}{k^4})} =$$

$$= \frac{1 + \frac{1}{k}}{1 + \frac{1}{k^2} + \frac{3}{k^4}} \xrightarrow[k \rightarrow +\infty]{} \frac{1+0}{1+0+0} = 1 \in \mathbb{R}.$$

So, since  $\sum_{k=1}^{+\infty} \frac{1}{k^3}$  converges, we have that

$$\sum_{k=1}^{+\infty} \frac{k+1}{k^4+k^2+3} \text{ converges too.}$$

(13)

③

$$\sum_{k=1}^{+\infty} \frac{k+1}{k^2+2}$$

$\parallel a_k$

: We guess this series should behave like

$$\sum_{k=1}^{+\infty} \frac{1}{k}$$

$\parallel b_k$

$$\frac{|a_k|}{b_k} = \frac{\frac{k+1}{k^2+2}}{\frac{1}{k}} = \frac{\frac{k(1+\frac{1}{k})}{k^2(1+\frac{2}{k^2})}}{\frac{1}{k}} = \frac{\frac{1+\frac{1}{k}}{1+\frac{2}{k^2}}}{\frac{1}{k}} \xrightarrow[k \rightarrow +\infty]{} \frac{1+0}{1+0} = 1 \text{ (P)}$$

So, since  $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$ , we have

$$\sum_{k=1}^{+\infty} |a_k| = \sum_{k=1}^{+\infty} \frac{k+1}{k^2+2} = +\infty$$

(13)

(3)

$$\text{too} \sum_{k=1}^{+\infty} \frac{k+1}{k^2+2}$$

$\parallel$   
 $a_k$

: We guess this series should behave like  
 $\sum_{k=1}^{+\infty} \left(\frac{1}{k}\right)$ . Indeed:

$$\sum_{k=1}^{+\infty} \left(\frac{1}{k}\right)$$

$\parallel$   
 $b_k$

$$\frac{|a_k|}{b_k} = \frac{\frac{k+1}{k^2+2}}{\frac{1}{k}} = \frac{\frac{(1+\frac{1}{k})}{(1+\frac{2}{k^2})}}{\frac{1}{k}} = \frac{\frac{1+\frac{1}{k}}{1+\frac{2}{k^2}}}{\frac{1}{k}} \xrightarrow[k \rightarrow +\infty]{} \frac{1+0}{1+0} = 1 \text{ (P)}$$

So, since  $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$ , we have

$$\sum_{k=1}^{+\infty} |a_k| = \sum_{k=1}^{+\infty} \frac{k+1}{k^2+2} = +\infty \text{ too.}$$

23 Sep 2016. (1)

Lecture 13

Test

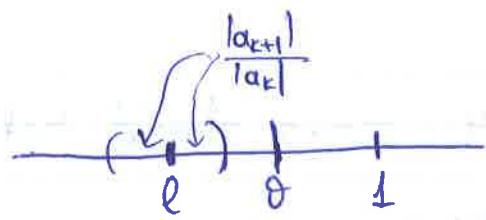
## Ratio test (D'Alembert) :

Let  $\sum_{k=1}^{+\infty} a_k$  be a series with  $a_k \neq 0, \forall k \in \mathbb{N}$ .

- If  $\lim_{k \rightarrow +\infty} \frac{|a_{k+1}|}{|a_k|} = l < 1$ , then  $\sum_{k=1}^{+\infty} |a_k|$  converges  
(and thus  $\sum_{k=1}^{+\infty} a_k$  converges).  
maybe too.
- If  $\lim_{k \rightarrow +\infty} \frac{|a_{k+1}|}{|a_k|} = l > 1$ , then  $\sum_{k=1}^{+\infty} a_k$  diverges.
- If  $\lim_{k \rightarrow +\infty} \frac{|a_{k+1}|}{|a_k|} = 1$ , then the test is inconclusive.

Proof: - Suppose that  $(0 \leq) \lim_{k \rightarrow +\infty} \frac{a_{k+1}}{a_k} = l < 1$ .

(2)



fix some  $\theta \in (1, 1)$  (for instance,  $\theta = \frac{l+1}{2}$ ).

Since  $\frac{|a_{k+1}|}{|a_k|} \xrightarrow[k \rightarrow \infty]{} l < \theta$ , there exists  $k_0 \in \mathbb{N}$  st.:

$$\forall k \geq k_0, \quad \frac{|a_{k+1}|}{|a_k|} < \theta \iff |a_{k+1}| < \theta |a_k|, \quad \forall k \geq k_0.$$

Idea: Since, to go from each term to the next, we multiply with at most  $\theta$ , our series should be smaller than a constant times the geom. series with ratio  $\theta$ .

$$\begin{aligned} \text{Thus: } |a_{k_0+1}| &< \theta |a_{k_0}|, \\ |a_{k_0+2}| &< \theta |a_{k_0+1}|, \\ &\vdots \\ |a_{k_0+N}| &< \theta \cdot |a_{k_0+N-1}| \end{aligned} \quad \Rightarrow |a_{k_0+N}| < \theta \cdot |a_{k_0}|, \quad \forall N \in \mathbb{N}. \text{ So:}$$

$$|a_k| < \underbrace{|a_{k_0}|}_{\text{fixed}} \cdot \theta^{k-k_0}, \quad \forall k \geq k_0.$$

Now, as  $\theta < 1$ ,  $\sum_{k=k_0}^{+\infty} \theta^{k-k_0} = \sum_{k=0}^{+\infty} \theta^k$  converges.

By the comparison test,  $\sum_{k=k_0}^{+\infty} |a_k|$  converges  $\Rightarrow \sum_{k=1}^{+\infty} |a_k|$  converges.

- Suppose that  $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = l > 1$  ↳ maybe too

By the ratio test for sequences,  $|a_k| \xrightarrow[k \rightarrow \infty]{} 0$ , so

(3)

$|a_k| \xrightarrow[k \rightarrow \infty]{\nexists} 0 \implies a_k \xrightarrow[k \rightarrow \infty]{\nexists} 0$ , so, by the preliminary test,  
 $\sum_{k=1}^{+\infty} a_k$  diverges.



Following a similar proof as in the first case, we can bound  $\sum_{k=1}^{+\infty} |a_k|$  from below by a divergent

geometric series. This gives that  $\sum_{k=1}^{+\infty} |a_k|$  diverges. But that doesn't imply that  $\sum_{k=1}^{+\infty} a_k$  diverges too! So, we need to do something different.

- Suppose that  $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = 1$ . Then the test is

inconclusive:

- $\sum_{k=1}^{+\infty} \left( \frac{1}{k} \right) \underset{\text{"a}_k\text{}}{\text{diverges}}$ , and  $\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \xrightarrow[k \rightarrow \infty]{} 1$
- $\sum_{k=1}^{+\infty} \left( \frac{1}{k^2} \right) \underset{\text{"a}_k\text{}}{\text{converges}}$ , and  $\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \frac{k^2}{(k+1)^2} = \frac{k^2}{k^2 + 2k + 1} = \frac{k^2}{k^2(1 + \frac{2}{k} + \frac{1}{k^2})} = \frac{1}{1 + \frac{2}{k} + \frac{1}{k^2}} \xrightarrow[k \rightarrow \infty]{} 1$

(4)

→ Examples:

(i)  $\left[ \sum_{k=1}^{+\infty} \frac{1}{k!} \right] : \frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \xrightarrow{k \rightarrow +\infty} 0 < 1,$

so  $\sum_{k=1}^{+\infty} \frac{1}{k!}$  converges.

(ii)  $\left[ \sum_{k=1}^{+\infty} \frac{2^k}{k!} \right] : \frac{|a_{k+1}|}{|a_k|} = \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \frac{2^{k+1} \cdot k!}{(k+1)! \cdot 2^k} = \frac{2}{k+1} \xrightarrow{k \rightarrow +\infty} 0 < 1,$

so  $\sum_{k=1}^{+\infty} \frac{2^k}{k!}$  converges.

(iii)  $\left[ \sum_{k=1}^{+\infty} k^k \cdot x^k \right], x \neq 0 : \frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)^{k+1} \cdot |x|^{k+1}}{k^k \cdot |x|^k} =$

(this is actually a power series).

$$= \left(\frac{k+1}{k}\right)^k \cdot (k+1) \cdot |x| =$$

$$= \underbrace{\left(1 + \frac{1}{k}\right)^k}_{\xrightarrow{k \rightarrow +\infty} e} \cdot \underbrace{(k+1)}_{\xrightarrow{k \rightarrow +\infty} +\infty} \cdot \underbrace{|x|}_{\text{fixed}} \xrightarrow{k \rightarrow +\infty} +\infty > 1,$$

so  $\sum_{k=1}^{+\infty} k^k \cdot x^k$  diverges  $\forall x \neq 0$

Notice that, for  $x=0$ ,

$$\sum_{k=1}^{+\infty} k^k \cdot 0^k = \sum_{k=1}^{+\infty} 0 = 0, \text{ so the series converges for } x=0$$

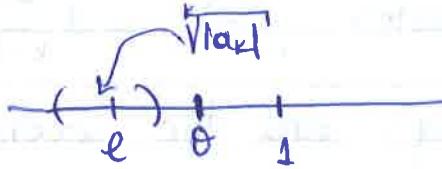
(5)

Test

→ Root test (Cauchy): Let  $\sum_{k=1}^{+\infty} |a_k|$  be a series.

- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = l < 1$ , then  $\sum_{k=1}^{+\infty} |a_k|$  converges  
(and thus  $\sum_{k=1}^{+\infty} a_k$  converges).
- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = l > 1$ , then  $\sum_{k=1}^{+\infty} |a_k|$  diverges.
- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = 1$ , then the test is inconclusive.

Proof: - Suppose that  $(0 \leq) \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = l < 1$ .



Fix some  $\delta \in (l, 1)$   
(for instance,  $\delta = \frac{l+1}{2}$ ).

Since  $\sqrt[k]{|a_k|} \xrightarrow{k \rightarrow \infty} l < \delta$ , there exists some  $k_0 \in \mathbb{N}$  s.t.:

$$\forall k \geq k_0, \underbrace{\sqrt[k]{|a_k|}}_0 < \delta \iff |a_k| < \delta^k, \forall k \geq k_0.$$

Now, since  $|\delta| < 1$ , the series  $\sum_{k=k_0}^{+\infty} \delta^k$  converges, so,  
by the comparison test,  $\sum_{k=k_0}^{+\infty} |a_k|$  converges  $\Rightarrow \sum_{k=1}^{+\infty} |a_k|$  converges.

(6)

- Suppose that  $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = l > 1$ .  
↳ maybe too.

By the root test for sequences,  $|a_k| \xrightarrow{k \rightarrow \infty} +\infty$

$$\Rightarrow |a_k| \xrightarrow{k \rightarrow \infty} 0 \Rightarrow a_k \xrightarrow{k \rightarrow \infty} 0.$$

By the preliminary test,  $\sum_{k=1}^{+\infty} a_k$  diverges.

- Suppose that  $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = 1$ . Then the test is inconclusive. Indeed, the above is satisfied for the terms of the series  $\sum_{k=1}^{+\infty} \frac{1}{k}$  and  $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ , yet the first series diverges and the second converges.

→ Example:

$$\sum_{k=1}^{+\infty} \frac{x^k}{k}, \quad x \neq 0$$

(Diagram: A circle divided into two equal halves by a vertical line. The left half is labeled  $x^k$  and the right half is labeled  $k$ .)

→ obviously converges for  $x=0$ .

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{|x|^{k+1}}{k+1}}{\frac{|x|^k}{k}} = \frac{k}{k+1} \cdot |x| \xrightarrow{k \rightarrow \infty} |x|. \text{ So:}$$

- If  $|x| < 1$ , the series converges.
- If  $|x| > 1$ , the series diverges.

(7)

test

## Dirichlet criterion:

Consider the series  $\sum_{k=1}^{\infty} a_k b_k$ ,

where  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  are sequences of real numbers.

If:

- $(b_k)_{k \in \mathbb{N}}$  is decreasing and  $b_k \xrightarrow{k \rightarrow \infty} 0$   
(we usually denote this by  $b_k \searrow 0$ ) , and

- The sequence  $s_n = a_1 + a_2 + \dots + a_n$ ,  $n \in \mathbb{N}$ , is bounded.

Then,  $\sum_{k=1}^{\infty} a_k b_k$  converges.

→ notice that these imply that  $b_k > 0$ ,  $a_k \in \mathbb{N}$ !

i.e.,  $\exists N > 0$  s.t.:  
 $\forall n \in \mathbb{N}, |s_n| \leq N$ .

Proof: for the proof, we will use the following:

→ **Abel's lemma:** For any sequences  $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}}$ , have that: for any  $n > m$  in  $\mathbb{N}$ ,

$$a_m b_m + a_{m+1} b_{m+1} + \dots + a_n b_n$$

=

$$\sum_{k=m}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n - s_{m-1} b_m,$$

where  $s_k = a_1 + a_2 + \dots + a_k$ ,  $\forall k \in \mathbb{N}$ .

(8)

If  $|x| \neq 1 \Leftrightarrow x = 1$  or  $x = -1$ , the root test tells me nothing, so I need to test further:

→ When  $x=1$ , the series is  $\sum_{k=1}^{\infty} \frac{1}{k}$ , so it diverges.

→ When  $x=-1$ , the series is  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ , so it converges.



Apart from • looking at limits of partial sums,  
• the Cauchy criterion, and  
• identifying our series as telescopic,

the ratio and root tests are the only tests

we have seen so far that can actually imply

divergence for a series whose terms are not all non-negative! All other tests can imply divergence

only for  $\sum_{k=1}^{\infty} |a_k|$ . (weaker).

The next test is very useful: it guarantees

(under conditions) conditional convergence. It implies,

for instance, that the series  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ ,  $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln k}$ ,  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$

are convergent (while we know that they are not absolutely convergent).

(3)

We will see later the proof of this lemma (very simple).

The reason we are interested in a lemma like this is that we want to try the Cauchy criterion to show that  $\sum_{k=1}^{+\infty} a_k b_k$  converges. To do that, we need to understand the behaviour of  $a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + \dots + a_n b_n$ , for  $n > m$ . Since our hypotheses involve information only for  $(b_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$ , we should just express that sum with respect to these sequences.

Accepting Abel's lemma, let us prove that  $\sum_{k=1}^{+\infty} a_k b_k$  satisfies the Cauchy criterion (and thus is convergent). We want:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t.: } \forall n > m \geq n_0, |a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + \dots + a_n b_n| < \varepsilon.$$

$$\begin{aligned} & \text{Let } \varepsilon > 0. |a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + \dots + a_n b_n| \xrightarrow{\text{by Abel's lemma}} \\ & = \left| \sum_{k=m+1}^{n-1} s_k \cdot (b_k - b_{k+1}) + s_n b_n - s_m b_{m+1} \right| \leq \xrightarrow{\text{triangle inequality}} \\ & \leq \sum_{k=m+1}^{n-1} \underbrace{|s_k|}_{\leq M} \cdot \underbrace{|b_k - b_{k+1}|}_{b_k - b_{k+1}, \text{ as } (b_k)_{k \in \mathbb{N}}} + \underbrace{|s_n|}_{\leq M} \cdot \underbrace{|b_n|}_{b_n} + \underbrace{|s_m|}_{\leq M} \cdot \underbrace{|b_{m+1}|}_{b_{m+1}} \leq \\ & \leq M \cdot \left( \sum_{k=m+1}^{n-1} (b_k - b_{k+1}) + b_n + b_{m+1} \right) = \end{aligned}$$

(10)

$$\begin{aligned}
 & M \cdot (b_{m+1} - b_n + b_n + b_{m+1}) = M \cdot b_{m+1} = \underbrace{M \cdot |b_{m+1}|}_{\rightarrow 0} \\
 & \downarrow \\
 & \sum_{k=m+1}^{n-1} (b_k - b_{k+1}) = \\
 & = b_{m+1} - b_{m+2} \\
 & + b_{m+2} - b_{m+3} \\
 & + b_{m+3} - b_{m+4} = b_{m+1} - b_n \\
 & + \vdots \\
 & + b_{n-1} - b_n
 \end{aligned}$$

Now:  $b_{m+1} \xrightarrow[m \rightarrow \infty]{} 0$ . So,

for the  $\varepsilon > 0$  we have fixed, there exists some  $n_0 \in \mathbb{N}$  s.t.: if  $m \geq n_0$ ,  $|b_{m+1}| < \frac{\varepsilon}{M}$ .

Thus, for all  $n > m \geq n_0$ ,

$$|a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \dots + a_n b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary,  $\sum_{k=1}^{+\infty} a_k b_k$  satisfies the Cauchy criterion, and is thus convergent.

→ Proof of Abel's lemma:

Idea: We want to express  $a_m b_m + a_{m+1} b_{m+1} + \dots + a_n b_n$  only in terms of  $(b_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$ , where  $s_n = a_1 + a_2 + \dots + a_n, \forall n \in \mathbb{N}$ . So, we just need to express each  $a_n$  in terms of  $s_k$ 's, which we know how to do:  $a_n = s_n - s_{n-1}, \forall n \in \mathbb{N}$

Let  $n > m$  in  $\mathbb{N}$ . We have that  $a_n = s_n - s_{n-1}, \forall n \in \mathbb{N}$ .

$$\begin{aligned}
 & \text{So, } a_m b_m + a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + \dots + a_n b_n = \\
 & = (\underbrace{s_m - s_{m-1}}_{\sim} b_m + (\underbrace{s_{m+1} - s_m}_{\sim} b_{m+1} + (\underbrace{s_{m+2} - s_{m+1}}_{\sim} b_{m+2} + \dots \\
 & \quad \sim + (\underbrace{s_{n-1} - s_{n-2}}_{\sim} b_{n-1} + (\underbrace{s_n - s_{n-1}}_{\sim} b_n = \\
 & = s_m (b_m - b_{m+1}) + s_{m+1} \cdot (b_{m+1} - b_{m+2}) + \dots + s_{n-1} \cdot (b_{n-1} - b_n) + \\
 & \quad + s_n b_n - s_{m-1} b_m = \sum_{k=m}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n - s_{m-1} b_m.
 \end{aligned}$$

(12)

Application

→ Alternating series with terms decreasing to 0 in

absolute value:  $\sum_{k=1}^{\infty} (-1)^k b_k$ , where  $b_k \downarrow 0$ .

These always converge, as  $b_k \downarrow 0$  and

$$s_n = (-1)^1 + (-1)^2 + (-1)^3 + \dots + (-1)^n = \begin{cases} -1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases},$$

thus  $(s_n)_{n \in \mathbb{N}}$  bounded.

ex: •  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$  converges if  $p > 0$

(as, for  $p > 0$ ,  $\frac{1}{k^p} \downarrow 0$  as  $k \rightarrow \infty$ ).

(while of course it doesn't converge absolutely for  $p \leq 1$ ).



Apart from • looking at limits of partial sums

and • the Cauchy criterion,

the Dirichlet criterion is the only test we have

seen that can imply convergence for

a series that doesn't converge absolutely!



So, let  $\sum_{k=1}^{\infty} a_k$  be a series with infinitely many positive and infinitely many negative terms, that does not converge absolutely.

terms. A safe way to test it for convergence is

- to see if  $\lim_{n \rightarrow \infty} s_n$  exists or not , or
- to try the Cauchy criterion, or
- to see if the series is telescopic (uncommon).  
If we can't see how to get an answer from those, or we want something simpler, we are only allowed to use
- the ratio/root tests ( if we are aiming to show that  $\sum_{k=1}^{\infty} a_k$  diverges)
- the Dirichlet criterion -and its application (if we are aiming to show that  $\sum_{k=1}^{\infty} a_k$  converges)

Any other test would test only for absolute convergence/divergence, which would be pointless since our series doesn't converge absolutely.