

Lecture 4:

① 31 Aug 2016.

Absolute values:

→ Def: for any  $a \in \mathbb{R}$ , we define its absolute value as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

→ Obs.: (i)  $|a| \geq 0$   $\forall a \in \mathbb{R}$ , with equality only if  $a=0$ .

(ii)  $-|a| \leq a \leq |a|, \forall a \in \mathbb{R}$ .

(iii)  $|-a| = |a|, \forall a \in \mathbb{R}$ .

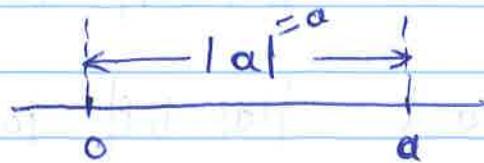
(iv) If  $a, b \in \mathbb{R}$  with  $b \geq 0$ , then

\*  $|a| \leq b \iff -b \leq a \leq b$  (exercise!).

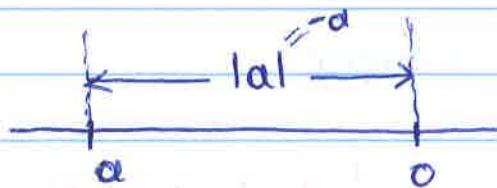
(2)

All the above become obvious once one realises that  $|a|$  is actually the distance of  $a$  from 0:

- $a > 0$  :



- $a < 0$  :



→ Prop. (triangle inequality):

If  $a, b \in \mathbb{R}$ , then  $|a+b| \leq |a| + |b|$

Proof: It suffices (by ④) to show that

$$-(|a| + |b|) \leq a + b \leq |a| + |b| \quad (\text{since } |a| + |b| \geq 0)$$

$$-|a| - |b|$$

Indeed:  $-|a| \leq a \leq |a|$     and     $-|b| \leq b \leq |b|$

by properties  
of ordered field  $\mathbb{R}$

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→ Corollary: If  $a, b \in \mathbb{R}$ , then

$$| |a| - |b| | \leq |a - b|$$

and  $| |a| - |b| | \leq |a + b|$ .

Proof: Exercise.

### Some useful equalities and inequalities:

① Bernoulli's inequality: If  $a \geq 1$ , then  $(1+a)^n \geq 1+na$ , then  $n \in \mathbb{N}$ .

Proof: Exercise (by induction).

② Binomial expansion: If  $a, b \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,

Note:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{then} \quad (a+b)^n = \sum_{k=0}^n \binom{n}{k} \cdot a^k \cdot b^{n-k}$$

Proof: You don't have to know this proof. However, it would be good practice if you tried it. You can do it by induction, where you will need that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad (\text{which can be proved directly})$$

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Another way to see it is to use the fact that

$\binom{n}{k}$  is the number of ways one can choose  $k$  elements out of  $n$ . Since

$$(a+b)^n = \underbrace{(a+b) \cdot \dots \cdot (a+b)}_{n \text{ times}},$$

$(a+b)^n$  will be the sum of all possible terms created by picking  $a$  from  $k$  of the brackets and  $b$  from the rest  $n-k$ , for all  $k=0, 1, \dots, n$ . Each such term

will equal  $a^k b^{n-k}$ ; so, since there are  $\binom{n}{k}$  ways to choose the  $k$  brackets  $a$  will be picked from,  $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

Once you learn combinatorics, this will be immediate.

### ③ Cauchy-Schwarz inequality:

If  $a_1, a_2, \dots, a_n \in \mathbb{R}$  and  $b_1, b_2, \dots, b_n \in \mathbb{R}$ , then

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2) \cdot (b_1^2 + b_2^2 + \dots + b_n^2).$$

I.e.,  $\left| \sum_{k=1}^n a_k b_k \right| \leq \sqrt{\sum_{k=1}^n a_k^2} \cdot \sqrt{\sum_{k=1}^n b_k^2}.$

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Proof: Exercise. You can prove this by induction. Another, more imaginative way is to consider the polynomial

$$P(\lambda) = (a_1 + \lambda b_1)^2 + (a_2 + \lambda b_2)^2 + \dots + (a_n + \lambda b_n)^2, \quad \lambda \in \mathbb{R}$$

What do we know about the sign of  $p(n)$ ?  
What is the discriminant of  $p(n)$ ?

Sidenote: Cauchy-Schwarz is an inequality that generalises in every inner product space. You will learn more about this in your mathematical future.

## ④ Arithmetic - geometric - harmonic mean

Inequality: If  $x_1, x_2, \dots, x_n > 0$ , then

Equality holds only if  $x_1 = x_2 = \dots = x_n$ .

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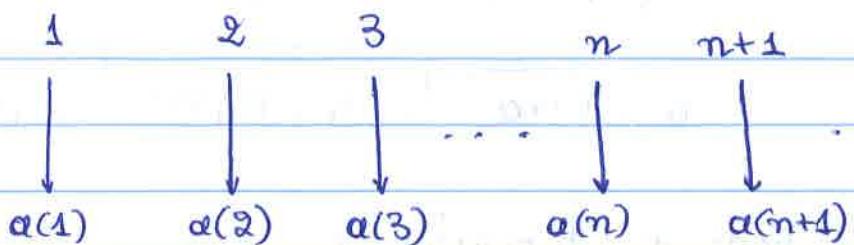
Proof: You don't need to know the geometric - arithmetic mean inequality proof. However, you can try it by induction (it is easy for  $n=2^k, k \in \mathbb{N}$ , but trickier for all  $n$ ). You can find where to read in the Further Reading chapter of Spivak's book (3rd edition).

The harmonic - geometric mean inequality is a straightforward application of the geometric - arithmetic mean inequality, and is left as an exercise.



## Sequences of real numbers.

→ Def: A sequence is a map  $\alpha : \mathbb{N} \rightarrow \mathbb{R}$ .



We denote each  $\alpha(n)$  by  $a_n$ , for simplicity.

We also denote the sequence  $\alpha$  by :

$(a_n)_{n \in \mathbb{N}}$ ,  $(\alpha_n)_{n=1}^{\infty}$ ,  $(a_n)$ ,  $(a_1, a_2, a_3, \dots)$ .

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→ ex:

(i) Let  $c \in \mathbb{R}$ . The sequence  $a_n = c$   $\forall n \in \mathbb{N}$  is
$$(c, c, c, \dots, c, \dots)$$
, a constant sequence.  

$$\begin{array}{cccc} \overset{\text{||}}{c} & \overset{\text{||}}{c} & \overset{\text{||}}{c} & \dots \\ a_1 & a_2 & a_3 & \dots \end{array}$$
(ii)  $a_n = n$ ,  $\forall n \in \mathbb{N}$ :  $(a_n)_{n \in \mathbb{N}} = (1, 2, 3, \dots, n, \dots)$ .(iii)  $a_n = \frac{1}{n}$   $\forall n \in \mathbb{N}$ :  $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$ .(iv)  $a_n = n^2 - n + 1$ ,  $\forall n \in \mathbb{N}$ :  $(a_n) = (1^2 - 1 + 1, 2^2 - 2 + 1,$  $3^2 - 3 + 1, \dots, n^2 - n + 1, \dots)$ .(v)  $a_1 = 1$ ,  $a_{n+1} = \sqrt{1+a_n}$ ,  $\forall n \in \mathbb{N}$  (this sequence is defined inductively).

Then,  $a_2 = \sqrt{1+a_1} = \sqrt{1+1} = \sqrt{2}$ ,

$a_3 = \sqrt{1+a_2} = \sqrt{1+\sqrt{2}}$ ,

$a_4 = \sqrt{1+a_3} = \sqrt{1+\sqrt{1+\sqrt{2}}}$ , etc.

→ Definitions + Observations:① The set of terms of the sequence  $(a_n)_{n \in \mathbb{N}}$ is the set  $\{a_n : n \in \mathbb{N}\}$ .

⑧

⚠ The set of terms of a sequence is not the same as the sequence! Indeed:

- First of all, a sequence is a map, not a set. More particularly, a sequence contains the information of where each  $n \in \mathbb{N}$  is sent to, while that information is not preserved in the set of terms. ex: for  $a_n = (-1)^n \forall n \in \mathbb{N}$ ,  
 $(a_n)_{n \in \mathbb{N}} = (-1, 1, -1, 1, -1, 1, \dots)$ ,

while  $\{a_n : n \in \mathbb{N}\} = \{-1, 1\}$ .

- Two different sequences may have the same set of terms. ex: both  
 $(-1, 1, -1, 1, -1, 1, \dots)$   
and  $(1, -1, 1, -1, 1, -1, \dots)$   
have the same set of terms.

② If  $(a_n)_{n=1}^{\infty}$  is a sequence and  $m \in \mathbb{N}$ , then

the sequence  $(a_m, a_{m+1}, a_{m+2}, \dots)$  is called

a final part of  $(a_n)_{n=1}^{\infty}$ .

$(a_1, a_2, a_3, \dots, a_{m-1}, a_m, a_{m+1}, \dots)$   
↓  
a final part  
of  $(a_n)_{n \in \mathbb{N}}$

Note that  $(a_m, a_{m+1}, \dots) = (a_{m+n-1})_{n \in \mathbb{N}}$ .

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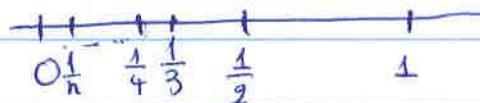
ex: The sequence  $a_n = \frac{1}{n}$   $n \in \mathbb{N}$  has final

parts  $(1, \frac{1}{2}, \frac{1}{3}, \dots)$  (the sequence itself),

$(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ ,  $(\frac{1}{3}, \frac{1}{10}, \frac{1}{11}, \dots)$  (among others).

→ Limit of a sequence:

ex:  $a_n = \frac{1}{n}$ ,  $n \in \mathbb{N}$ .



I want to have a definition for the limit of a sequence that will allow me to say that

$$a_n = \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Observe that what really happens for  $a_n = \frac{1}{n}$

is the following:

"No matter how close to 0 I look at, I can find  $a_n = \frac{1}{n}$  for large  $n$  there."

→ "for large  $n$ " means "for all  $n \in \mathbb{N}$  from some natural number onwards".

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→ "No matter how close to 0 I look at" means

"no matter how small an interval I pick around 0";

or, more specifically,

"no matter how small a neighbourhood of 0 I pick",

where:

Def: For any  $a \in \mathbb{R}$ , we define a neighbourhood of a to be any interval of the form  $(a-\varepsilon, a+\varepsilon)$ , for  $\varepsilon > 0$ .



**⚠** Side note: In topology, a neighbourhood of a point is defined as any open set containing the point. So, for instance, every open interval containing the point is a neighbourhood of the point (whether it is symmetric around the point or not). However, in the case of limits in  $\mathbb{R}$  it suffices to consider only symmetric intervals around each point, so, for simplicity, we will call "neighbourhoods" only such symmetric intervals.

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Thus, I require the following:

"For any neighbourhood of 0, I can find in the neighbourhood all  $a_n = \frac{1}{n}$  from some  $n$  natural number onwards"

↓  
a whole final part of  $(a_n)_{n \in \mathbb{N}}$ !

I.e.:

"For any neighbourhood of 0,  
contained in the neighbourhood".

there exists  
some final part of  $(a_n)_{n \in \mathbb{N}}$

$(a_{n_0}, a_{n_0+1}, a_{n_0+2}, \dots)$ ,  
for some  $n_0 \in \mathbb{N}$  that depends on the neighbourhood.

I.e.:

"For any interval of the form  $(0-\varepsilon, 0+\varepsilon)$ , where  $\varepsilon > 0$ ,

there exists some  $n_0 \in \mathbb{N}$ , depending on  $\varepsilon$ , s.t.  
 $|a_n - 0| < \varepsilon$   $\forall n \geq n_0$ ".

I.e.: For all  $\varepsilon > 0$ ,  $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ , s.t.:  $(|a_n - 0| < \varepsilon, \forall n \geq n_0)$

This is exactly the definition of " $a_n \rightarrow 0$  as  $n \rightarrow +\infty$ ".  
In general:

(19).

→ Def.: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers, and  $a \in \mathbb{R}$ . We say that  $(a_n)_{n \in \mathbb{N}}$  converges to  $a$ , and that  $a$  is the limit of  $(a_n)_{n \in \mathbb{N}}$ , and we write: " $a_n \xrightarrow{\text{optional}} a \text{ as } n \rightarrow \infty$ ", if:

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0, |a_n - a| < \varepsilon$$

⚠ Note that this definition can be rephrased as:

$(a_n)_{n \in \mathbb{N}}$  converges to  $a$  if:

for any neighbourhood  $(a-\varepsilon, a+\varepsilon)$  of  $a$ , there exists a final part of  $(a_n)_{n \in \mathbb{N}}$  contained in  $(a-\varepsilon, a+\varepsilon)$ .

(12).

→ Def.: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers, and  $a \in \mathbb{R}$ . We say that  $(a_n)_{n \in \mathbb{N}}$  converges to  $a$ , and that  $a$  is the limit of  $(a_n)_{n \in \mathbb{N}}$ , and we write: " $a_n \xrightarrow{\text{optional}} a \text{ as } n \rightarrow \infty$ ", if:

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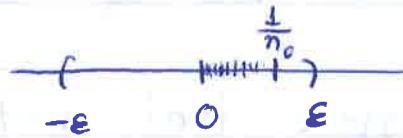
Lecture 5

8 Sep 2016

ex:

$$a_n = \frac{1}{n} \xrightarrow{} 0 \text{ as } n \rightarrow \infty.$$

Proof: Let  $\varepsilon > 0$ .



I am looking for some  $n_0 \in \mathbb{N}$  (depending on the  $\varepsilon$ ), such that:

$$\text{for all } n \geq n_0, -\varepsilon < \frac{1}{n} < \varepsilon.$$

In fact, I know that  $\frac{1}{n} > 0 \forall n \in \mathbb{N}$  (as  $n > 0$ ), so

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I just want  $n_0 \in \mathbb{N}$  s.t.  $\frac{1}{n} < \varepsilon$ ,  $\forall n \geq n_0$ .

By the Archimedean property of the reals, we know that  
 $\exists n_0 \in \mathbb{N} : \frac{1}{n_0} < \varepsilon$ .

Then,  $\forall n \geq n_0$ , we have  $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$ .

So, indeed  $\exists n_0 \in \mathbb{N}$  s.t.  $\forall n \geq n_0$ ,  $\frac{1}{n} < \varepsilon$ .

Since  $\varepsilon$  was arbitrary, the proof is complete. ■

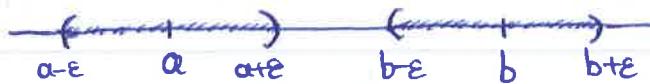
→ Obs.: The smallest  $n_0$  satisfying  $\frac{1}{n_0} < \varepsilon \Leftrightarrow n_0 > \frac{1}{\varepsilon}$

is  $\lfloor \frac{1}{\varepsilon} \rfloor + 1$ .

→ Prop. (Uniqueness of limits) : Let  $(a_n)_{n \in \mathbb{N}}$  a sequence.

If  $a_n \rightarrow a$  and  $a_n \rightarrow b$ , then  $a = b$ .

Proof: Suppose that  $a < b$ .



Ideas: Since  $a, b$  are far apart, I can find 2 neighbourhoods that are disjoint. Since  $a_n \rightarrow a$ , I can find some final part of  $(a_n)$  in the neigh. of  $a$ . Since  $a_n \rightarrow b$ , I can find some final part of  $(a_n)$  in the neigh. of  $b$ . The smallest of the 2 final parts will be simultaneously in both neighbourhoods, contradiction.

Pick  $\varepsilon > 0$  s.t.  $a + \varepsilon < b - \varepsilon$  (this is  $\Leftrightarrow 2\varepsilon < b - a \Leftrightarrow \varepsilon < \frac{b-a}{2}$ ; so,  $\varepsilon = \frac{b-a}{3}$  will do)

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Then, the neighbourhoods  $(a-\varepsilon, a+\varepsilon)$   
and  $(b-\varepsilon, b+\varepsilon)$  are disjoint.

Since  $a_n \rightarrow a$ , there exists  $n_1 \in \mathbb{N}$  s.t.:  
 $\forall n \geq n_1, a_n \in (a-\varepsilon, a+\varepsilon)$ .

Since  $a_n \rightarrow b$ , there exists  $n_2 \in \mathbb{N}$  s.t.:  
 $\forall n \geq n_2, a_n \in (b-\varepsilon, b+\varepsilon)$ .

Take  $n_0 = \max\{n_1, n_2\}$ . Then,  $a_{n_0} \in (a-\varepsilon, a+\varepsilon) \cap (b-\varepsilon, b+\varepsilon)$ ,

a contradiction, as  $(a-\varepsilon, a+\varepsilon) \cap (b-\varepsilon, b+\varepsilon) = \emptyset$ . Similarly,  
we get a contradiction when we assume  $b < a$ . So,  $b = a$ . ■

→ Notation: When  $a_n \rightarrow a \in \mathbb{R}$ , we denote this unique  
limit of  $(a_n)_{n \in \mathbb{N}}$  by

$$\lim_{n \rightarrow \infty} a_n \text{ or } \lim a_n.$$

→ Observation 1: let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, and

$$(a_m, a_{m+1}, a_{m+2}, \dots) = (a_{m+n-1})_{n \in \mathbb{N}}$$

The proof  
is left  
as an  
exercise.

a final part of  $(a_n)_{n \in \mathbb{N}}$ .

Then:  $a_n \rightarrow a \iff a_{m+n-1} \rightarrow a$   
as  $n \rightarrow \infty$ .

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→ Observation 2: If  $a_n \rightarrow a$ ,  $b_n \rightarrow b$  and

$(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  are eventually equal (that is, they differ for at most finitely many indices),

then  $a = b$ .

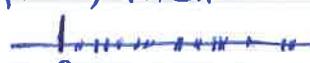
Proof: Since  $(a_n)$ ,  $(b_n)$  are eventually equal, there exists some  $m \in \mathbb{N}$  s.t. :  $a_n = b_n$ ,  $\forall n \geq m$ .

$$\text{So: } \underbrace{(a_m, a_{m+1}, \dots)}_{\text{a final part of } (a_n), \text{ so } \rightarrow a} = \underbrace{(b_m, b_{m+1}, \dots)}_{\text{a final part of } (b_n), \text{ so } \rightarrow b}.$$

By uniqueness of limits,  $a = b$ .

→ Def: The sequence  $(a_n)_{n \in \mathbb{N}}$  is :

- bounded from above if  $\exists b \in \mathbb{R}$  st.  $a_n \leq b$ ,  $\forall n \in \mathbb{N}$ .  

- bounded from below if  $\exists c \in \mathbb{R}$  st.  $a_n \geq c$ ,  $\forall n \in \mathbb{N}$ .  

- bounded if  $\exists b, c \in \mathbb{R}$  st.  $c \leq a_n \leq b$ ,  $\forall n \in \mathbb{N}$ .  


(5)

→ Observation:  $(a_n)_{n \in \mathbb{N}}$  is bounded  $\iff \exists M > 0$  s.t.  $|a_n| \leq M, \forall n \in \mathbb{N}$ .

Proof: Exercise.

→ Prop: Every convergent sequence is bounded.

Proof: Idea: Let  $a_n \rightarrow a$ . Pick some neighbourhood of  $a$ .  
 the  $a_n$ 's for large  $n$  (say,  $n > n_0$ ).  
  
 For large  $n$ , the  $a_n$ 's cluster in that neighbourhood.  
 The  $a_n$ 's outside the neighbourhood are only finitely many. So, the  $a_n$ 's cannot go infinitely far from  $a$ .

Let  $(a_n)_{n \in \mathbb{N}}$  a convergent sequence, and  $a$  its limit.

Let  $\epsilon = 1 (> 0)$ . Since  $a_n \rightarrow a$ , there exists  $n_0 \in \mathbb{N}$  s.t.  
 $|a_n - a| < 1$ ,  $\forall n > n_0$ ;  
 i.e.,  $a - 1 < a_n < a + 1$ ,  $\forall n > n_0$ .

So,  $\forall n \in \mathbb{N}$ :  $\min\{a_1, a_2, \dots, a_{n_0}, a - 1\} \in \mathbb{R}$

$$\begin{aligned} &\leq \\ &a_n \\ &\leq \\ &\max\{a_1, a_2, \dots, a_{n_0}, a + 1\} \in \mathbb{R} \end{aligned}$$

So,  $(a_n)_{n \in \mathbb{N}}$  bounded. ■

→ Algebra of limits:

→ Prop:  $a_n \rightarrow a \Leftrightarrow a_n - a \rightarrow 0 \Leftrightarrow |a_n - a| \rightarrow 0.$

Proof: (A)  $\Leftarrow \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0, |a_n - a| < \varepsilon.$

$$(a_n - a) - 0 \quad |a_n - a| - 0$$

Since (B)  $\Leftarrow \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0, |(a_n - a) - 0| < \varepsilon.$

and (C)  $\Leftarrow \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0, |a_n - a - 0| < \varepsilon,$

we have that (A)  $\Leftarrow$  (B)  $\Leftarrow$  (C) (the same index  
no happens to work  
for the same  $\varepsilon$  in all  
three cases.) ■

→ Corollary:  $a_n \rightarrow 0 \Leftrightarrow |a_n| \rightarrow 0.$

→ Proof: If  $a_n \rightarrow a$ , then  $|a_n| \rightarrow |a|.$

Proof: I want to show:  $|a_n| \rightarrow |a|.$

Let  $\varepsilon > 0$ . I am looking for  $n_0 \in \mathbb{N}$  s.t.:  $\forall n \geq n_0, |a_n - a| < \varepsilon$ .

I know that  $a_n \rightarrow a$ ; so, for this  $\varepsilon > 0$ ,  $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$  s.t.:  $\forall n \geq n_0, |a_n - a| < \varepsilon$ .

And:  $||a_n - l| \leq |a_n - a| \quad \forall n \in \mathbb{N}$  (by properties of absolute value)

So,  $\forall n \geq n_0 : ||a_n - l|| < \varepsilon$ .

Since  $\varepsilon$  was arbitrary, the proof is complete. ■

⚠ It is not true in general that  $|a_n| \rightarrow |a| \Rightarrow a_n \rightarrow a$ .

For example, for  $a_n = (-1)^n \quad \forall n \in \mathbb{N}$ , we have

$|a_n| = 1 \rightarrow 1$ , but  $(a_n)_{n \in \mathbb{N}}$  doesn't converge (exercise!).

→ Prop.: Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$  be sequences.

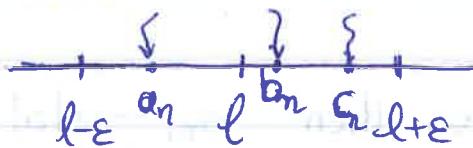
Suppose that: (i)  $\forall n \in \mathbb{N}, a_n \leq b_n \leq c_n$ .

and (ii)  $a_n \rightarrow l$  and  $b_n \rightarrow l$ .

Squeeze Theorem  
or  
Sandwich Lemma.

Then:  $b_n \rightarrow l$ .

Proof: Let  $\varepsilon > 0$ .



Idea: If for large  $n$ ,  $a_n$  is close to  $l$ , and for large  $n$ ,  $c_n$  is close to  $l$ , and  $b_n$  is squeezed between  $a_n$  and  $c_n$ , then  $b_n$  should also be close to  $l$  for large  $n$ .

Since  $a_n \rightarrow l$ , there exists  $n_1 (=n_1(\varepsilon)) \in \mathbb{N}$ , s.t.  
 $\forall n \geq n_1, l - \varepsilon < a_n < l + \varepsilon$ .

Since  $c_n \rightarrow l$ , there exists  $n_2 (=n_2(\varepsilon)) \in \mathbb{N}$ , s.t.  
 $\forall n \geq n_2, l - \varepsilon < c_n < l + \varepsilon$ .

above this index, both the  $a_n$ 's and the  $b_n$ 's are in  $(l-\varepsilon, l+\varepsilon)$ .  
 Let  $n_0 := \max\{n_1, n_2\}$ . Then,  $\forall n \geq n_0$ :

$$l-\varepsilon < a_n \leq b_n \leq c_n < l+\varepsilon$$

by assumption.

So, we have shown that :

$$\exists n_0 \in \mathbb{N} \text{ s.t. } l-\varepsilon < b_n < l+\varepsilon,$$

i.e.  $|b_n - l| < \varepsilon$ .

Since  $\varepsilon$  was arbitrary,  $b_n \rightarrow l$ . ■

above this index, both the  $a_n$ 's and the  $b_n$ 's are in  $(l-\varepsilon, l+\varepsilon)$ .

Let  $n_0 := \max\{n_1, n_2\}$ . Then,  $\forall n \geq n_0$ :

$$l-\varepsilon < a_n \leq b_n \leq c_n < l+\varepsilon$$

by assumption.

So, we have shown that:

$$\exists n_0 \in \mathbb{N} \text{ s.t. } l-\varepsilon < b_n < l+\varepsilon, \\ \text{i.e. } |b_n - l| < \varepsilon.$$

Since  $\varepsilon$  was arbitrary,  $b_n \rightarrow l$ .

①   $\neq$  Sep 2016

Lecture 6

→ Prop.:

Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  be sequences in  $\mathbb{R}$ .

Suppose that: (i)  $a_n \rightarrow 0$

and (ii)  $(b_n)_{n \in \mathbb{N}}$  bounded.

Then,  $a_n \cdot b_n \rightarrow 0$ .

Proof: Let  $\varepsilon > 0$ . I am looking for  $n_0 \in \mathbb{N}$  s.t.:  $\forall n \geq n_0, |a_n b_n| < \varepsilon$ .

$(b_n)_{n \in \mathbb{N}}$  is bounded  $\Rightarrow \exists M > 0$  s.t.:  $\forall n \in \mathbb{N}, |b_n| < M$ .

Since  $a_n \rightarrow 0$ , there exists  $n_0 (= n_0(\frac{\varepsilon}{M})) \in \mathbb{N}$ , s.t.:

$\forall n \geq n_0, |a_n| < \frac{\varepsilon}{M}$ . (I apply the definition of limit for the positive number  $\frac{\varepsilon}{M}$ , i.e. the neighbourhood  $(0 - \frac{\varepsilon}{M}, 0 + \frac{\varepsilon}{M})$  of 0.)

②

Then, for all  $n \geq n_0$  we have :

$$|a_n b_n| = |a_n| \cdot |b_n| < \frac{\epsilon}{M} \cdot M = \epsilon, \text{ i.e. :}$$

$$\forall n \geq n_0, |a_n b_n| < \epsilon.$$

Since  $\epsilon$  was arbitrary, the proof is complete. ■

→ Prop.: If  $a_n \rightarrow a$   
and  $b_n \rightarrow b$ , then  $a_n + b_n \rightarrow a+b$ .

Proof: Let  $\epsilon > 0$ . I know that:

$$a_n \rightarrow a, \text{ so } \exists n_1 \in \mathbb{N} : \forall n \geq n_1, |a_n - a| < \frac{\epsilon}{2},$$

and

$$b_n \rightarrow b, \text{ so } \exists n_2 \in \mathbb{N} : \forall n \geq n_2, |b_n - b| < \frac{\epsilon}{2}.$$

I define  $n_0 := \max\{n_1, n_2\}$ ; then,

$\forall n \geq n_0$ , we simultaneously have  $|a_n - a| < \frac{\epsilon}{2}$  and  $|b_n - b| < \frac{\epsilon}{2}$ ,

$$\begin{aligned} \text{and thus } |(a_n + b_n) - (a+b)| &= |(a_n - a) + (b_n - b)| \stackrel{\text{triangle inequality}}{\leq} \\ &\leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$$\text{So, } \forall n \geq n_0 : |(a_n + b_n) - (a+b)| < \epsilon.$$

Since  $\epsilon$  was arbitrary, we have  $a_n + b_n \rightarrow a+b$ . ■

(3)

→ Prop.: If  $a_n \rightarrow a$   
and  $b_n \rightarrow b$ , then  $a_n \cdot b_n \rightarrow a \cdot b$ .

Proof:

This is the first time we won't use the  $\varepsilon$ -definition of the limit, but simply the algebra of limits we've so far seen. If you want an  $\varepsilon$ -proof, use this idea together with the first 3 lines of this proof.

I notice that:

$$\begin{aligned} a_n b_n - ab &= a_n b_n - a b_n + a b_n - ab = \\ &= b_n \cdot (a_n - a) + a \cdot (b_n - b). \end{aligned}$$

Idea: if for large  $n$   $a_n$  is close to  $a$ , and for large  $n$   $b_n$  is close to  $b$ , then for large  $n$   $a_n b_n$  is close to  $a \cdot b$ .

Let's look at the sequence  $(b_n \cdot (a_n - a))_{n \in \mathbb{N}}$ . We have:

- $a_n - a \rightarrow 0$ , because  $a_n \rightarrow a$ .
- $(b_n)_{n \in \mathbb{N}}$  is bounded, because it is convergent.

So,  $b_n \cdot (a_n - a) \rightarrow 0$ .  $\textcircled{*}_1$

Similarly for  $a \cdot (b_n - b)$ :

- $b_n - b \rightarrow 0$ , because  $b_n \rightarrow b$ .
- $(a)_{n \in \mathbb{N}}$  is bounded, because it is a constant sequence

So,  $a \cdot (b_n - b) \rightarrow 0$ .  $\textcircled{*}_2$

By  $\textcircled{*}_1$  and  $\textcircled{*}_2$ ,  $b_n(a_n - a) + a(b_n - b) \rightarrow 0$ , i.e.  $a_n b_n \rightarrow a \cdot b$ . 

(4)

→ Corollary:

If  $a_n \rightarrow a$  as  $n \rightarrow \infty$  and  $k \in \mathbb{N}$ ,  
then  $a_n^k \rightarrow a^k$  as  $n \rightarrow \infty$ .

Proof: By the previous proposition :

$$\underbrace{a_n^2}_{\substack{\| \\ a_n \cdot a_n}} \rightarrow \underbrace{a^2}_{\substack{\| \\ a \cdot a}}, \quad \underbrace{a_n^2 \cdot a_n}_{\substack{\| \\ a_n^3}} \rightarrow \underbrace{a^2 \cdot a}_{\substack{\| \\ a^3}}, \text{ etc.}$$

ex:  $\frac{1}{n^2} \rightarrow 0, \frac{1}{n^3} \rightarrow 0, \frac{1}{n^{10}} \rightarrow 0.$

→ Prop:

If  $b_n \neq 0 \forall n \in \mathbb{N}$   
and  $b_n \rightarrow b$ , then  $\frac{1}{b_n} \rightarrow \frac{1}{b}$ .

Proof:

Idea: We notice that  $\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b_n b|} =$

$$= \frac{|b_n - b|}{|b_n| \cdot |b|} \quad \text{If I show that this quantity is small for large } n, \text{ I am done. I notice that if}$$

the denominator is larger than some constant, then the fraction is at most  $|b_n - b|$  (times a constant) which is small. So, I just need to bound the denominator from below.

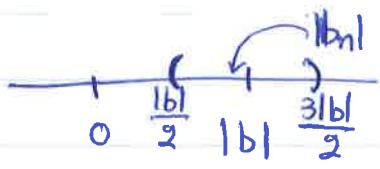
(5)

Let  $\varepsilon > 0$ . I want to show that there exists some  $n_0 \in \mathbb{N}$ ,

$$\text{s.t. : } \forall n \geq n_0, \left| \frac{1}{b_n} - \frac{1}{b} \right| < \varepsilon$$

$\underbrace{\frac{|b_n - b|}{|b_n| \cdot |b|}}$

I know that  $b_n \rightarrow b$ ; so,  $|b_n| \rightarrow |b|$ . By the



definition of limit for the neighbourhood  $(\frac{|b|}{2}, \frac{3|b|}{2})$  of  $|b|$

(i.e., for  $\varepsilon' = \frac{|b|}{2}$ ),

there exists some  $n_1 \in \mathbb{N}$  s.t. :  $\forall n \geq n_1, |b_n| \in (\frac{|b|}{2}, \frac{3|b|}{2})$ .

In particular :  $|b_n| > \frac{|b|}{2}$ ,  $\forall n \geq n_1$ .

So, 
$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{|b_n - b|}{|b_n| \cdot |b|} \right| < \frac{|b_n - b|}{\frac{|b|}{2} \cdot |b|} = \frac{2}{|b|^2} \cdot |b_n - b|$$

for the  $\varepsilon > 0$  I originally picked,  $\exists n_2 \in \mathbb{N}$  s.t. :

$$\forall n \geq n_2, |b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}$$

⑥

I now combine  $\textcircled{*}_1$  and  $\textcircled{*}_2$ :

$$\left. \begin{aligned} \left| \frac{1}{b_n} - \frac{1}{b} \right| &< \frac{2}{|b|^2} \cdot |b_n - b|, \quad \forall n \geq n_1, \\ \text{and } |b_n - b| &< \frac{\varepsilon \cdot |b|^2}{2}, \quad \forall n \geq n_2 \end{aligned} \right\} \Rightarrow$$

$\Rightarrow$  for  $n \geq n_0 := \max \{n_1, n_2\}$ , we simultaneously have that  $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \frac{2}{|b|^2} |b_n - b|$ , and thus and  $|b_n - b| < \frac{\varepsilon |b|^2}{2}$

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| < \frac{2}{|b|^2} \cdot \frac{\varepsilon |b|^2}{2} = \varepsilon, \quad \forall n \geq n_0.$$

Since  $\varepsilon$  was arbitrary,  $\frac{1}{b_n} \rightarrow \frac{1}{b}$ . ■

$\rightarrow$  Corollary: If  $b_n \neq 0 \quad \forall n \in \mathbb{N}$ ,  
 $a_n \rightarrow a$ , then  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$   
and  $b_n \rightarrow b$

Proof:  $\frac{a_n}{b_n} = \boxed{a_n} \cdot \boxed{\frac{1}{b_n}} \rightarrow a \cdot \frac{1}{b} = \frac{a}{b}$ . ■

this is just another notation for  $a^{1/k}$ , the unique positive real  $x$  with  $x^k = a$  (see Lecture 3)

(7)

→ Prop.: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with non-negative terms, and  $k \geq 2$ .

Imagine  $k$  to be fixed here (such as 2, 3, 10...).

If  $a_n \rightarrow a$ , then  $\sqrt[k]{a_n} \rightarrow \sqrt[k]{a}$  as  $n \rightarrow \infty$ .

which will be  $\geq 0$  since  $a_n > 0 \forall n \in \mathbb{N}$

Proof:

Idea: We want to show that  $|\sqrt[k]{a_n} - \sqrt[k]{a}|$  is small for large  $n$ . How can I relate  $|\sqrt[k]{a_n} - \sqrt[k]{a}|$  to the quantity  $|a_n - a|$ , which I know is small? I will use the identity

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \dots + xy^{k-2} + y^{k-1}),$$

which implies that

$$a_n - a = (\sqrt[k]{a_n} - \sqrt[k]{a}) \cdot (\sqrt[k]{a_n^{k-1}} + \sqrt[k]{a_n^{k-2}a} + \dots + \sqrt[k]{a_n a^{k-2}} + \sqrt[k]{a^{k-1}})$$

Case 1:  $a = 0$ . i.e.,  $a_n \rightarrow 0$ , and we want to show that  $\sqrt[k]{a_n} \rightarrow 0$ . Let  $\epsilon > 0$ .

Since  $a_n \rightarrow 0$ , by the definition of limit for  $\epsilon' = \epsilon^k$  there exists  $n_0 \in \mathbb{N}$  s.t.:  $\forall n \geq n_0, |a_n| < \epsilon^k \Leftrightarrow \sqrt[k]{|a_n|} < \sqrt[k]{\epsilon^k} \Leftrightarrow \sqrt[k]{|a_n|} < \epsilon$

show this! (by another direction)

(8)

$\Leftrightarrow |\sqrt[k]{a_n}| < \epsilon$ . Since  $\epsilon$  was arbitrary,  $\sqrt[k]{a_n} \rightarrow 0$ .

Case 2:  $a > 0$ .

We have:

$$a_n - a = (\sqrt[k]{a_n} - \sqrt[k]{a}) \cdot (\sqrt[k]{a_n}^{k-1} + \sqrt[k]{a_n}^{\frac{k-2}{k}} \sqrt[k]{a} + \dots + \sqrt[k]{a_n}^{\frac{1}{k}} \sqrt[k]{a} + \sqrt[k]{a}^{\frac{k-1}{k}})$$

so  $|\sqrt[k]{a_n} - \sqrt[k]{a}| = \frac{|a_n - a|}{(\sqrt[k]{a_n}^{k-1} + \sqrt[k]{a_n}^{\frac{k-2}{k}} \sqrt[k]{a} + \dots + \sqrt[k]{a_n}^{\frac{1}{k}} \sqrt[k]{a} + \sqrt[k]{a}^{\frac{k-1}{k}})}$

$\geq \sqrt[k]{a}^{k-1}$

thus  $0 \leq |\sqrt[k]{a_n} - \sqrt[k]{a}| \leq \frac{|a_n - a|}{\sqrt[k]{a}^{k-1}}$ ,  $\forall n \in \mathbb{N}$ .

$\downarrow n \rightarrow +\infty$   
 0  
 $\downarrow$   
 0

By the sandwich lemma,  $|\sqrt[k]{a_n} - \sqrt[k]{a}| \rightarrow 0 \Leftrightarrow \sqrt[k]{a_n} \rightarrow \sqrt[k]{a}$ .

We can use the definition of limit instead.

ex:  $\frac{1}{\sqrt{n!}} \rightarrow 0, \left(\frac{1}{n}\right)^{\frac{1}{10}} \rightarrow 0$ .



(9)

→ Prop.: If  $\alpha > 0$ , then  $\sqrt[n]{\alpha} \rightarrow 1$  as  $n \rightarrow \infty$ .

Proof: We want:  $\sqrt[n]{\alpha} \rightarrow 1 \iff \underbrace{\sqrt[n]{\alpha} - 1}_{\vartheta_n} \rightarrow 0$ .

Case 1:  $\alpha \geq 1$ .

In this case,  $\textcircled{1} \quad \vartheta_n \geq 0, \forall n \in \mathbb{N}$  (as  $\alpha \geq 1 \rightarrow \sqrt[n]{\alpha} \geq 1, \forall n \in \mathbb{N}$ ).

So, we can apply Bernoulli's inequality for  $\vartheta_n$ :

$$\underbrace{(1 + \vartheta_n)^n}_{\substack{\parallel \\ \sqrt[n]{\alpha}^n = \alpha}} \geq 1 + n \cdot \vartheta_n, \forall n \in \mathbb{N},$$

Notice that it would have sufficed to just use that  $(1 + \vartheta_n)^n \geq n \cdot \vartheta_n$ , which comes from Bernoulli's inequality (weaker).

$$\text{i.e. } n \vartheta_n \leq \alpha - 1 \iff \vartheta_n \leq \frac{\alpha - 1}{n}, \forall n \in \mathbb{N} \quad \textcircled{2}$$

It is for this that we required  $\alpha \geq 1$ ; Bernoulli's inequality holds for  $\alpha \geq -1$ .

By  $\textcircled{1}, \textcircled{2}$ , we have  $0 \leq \vartheta_n \leq \frac{\alpha - 1}{n}, \forall n \in \mathbb{N}$ .

$$0 \leq \vartheta_n \leq \frac{\alpha - 1}{n} \quad \begin{matrix} \downarrow \\ n \rightarrow \infty \\ 0 \end{matrix} \quad \begin{matrix} \downarrow \\ n \rightarrow \infty \\ 0 \end{matrix}$$

By the sandwich lemma,  $\vartheta_n \rightarrow 0 \iff \sqrt[n]{\alpha} \rightarrow 1$ .

Case 2:  $\alpha < 1$ . In this case,  $\frac{1}{\alpha} > 1 \quad \text{by case 1} \quad \sqrt[n]{\frac{1}{\alpha}} = \frac{1}{\sqrt[n]{\alpha}} \rightarrow 1 \rightarrow \sqrt[n]{\alpha} \rightarrow 1$ . ■

(10)

- ex:  $\left(\frac{1}{2}\right)^{\frac{1}{n}} \rightarrow 1, 3^{\frac{1}{n}} \rightarrow 1, 1000^{\frac{1}{n}} \rightarrow 1.$

$$\rightarrow \boxed{\sqrt[n]{n} \rightarrow 1 \text{ as } n \rightarrow \infty.}$$

Proof: Let's try to work as for  $\sqrt[n]{x}$ :

We want:  $\sqrt[n]{n} \rightarrow 1 \iff \underbrace{\sqrt[n]{n} - 1}_{\approx 0}$

We have  $\boxed{\text{(*)} \quad \Omega_n \geq 0, \forall n \in \mathbb{N}}$  (as  $n \geq 1 \Rightarrow \sqrt[n]{n} \geq 1$ )

however, if we apply Bernoulli's inequality for  $\Omega_n$ , then we get  $\underbrace{(1+\Omega_n)}_{\approx \Omega_n}^n \geq 1+n\cdot\Omega_n$ , i.e.

$$\sqrt[n]{n} = n \quad \Omega_n \leq \frac{n-1}{n} = \frac{x(1-\frac{1}{n})}{x} = \\ = 1 - \frac{1}{n} \xrightarrow{n \rightarrow \infty} 1 \neq 0.$$

Thoughts...

So, Bernoulli's inequality doesn't provide an upper bound for  $(\Omega_n)_{n \in \mathbb{N}}$  good enough for the sandwich lemma to work; we need an even better upper bound for  $(\Omega_n)_{n \in \mathbb{N}}$ , which will actually converge to 0. We get this from the binomial expansion of  $(1+\Omega_n)^n$  (which, notice, implies also Bernoulli's inequality for  $\Omega_n \geq 0$  (rather than  $\geq -1$ ):

(11).

We have:  $\forall n \in \mathbb{N}, n \geq 2$ :

$$\begin{aligned}
 (1+\vartheta_n)^n &= \sum_{k=0}^n \binom{n}{k} \cdot \vartheta_n^k = \frac{n \cdot (n-1)}{2} \\
 &= \underbrace{1 + n\vartheta_n}_{\geq 0, \text{ as } \vartheta_n \geq 0 \text{ then } n} + \underbrace{\binom{n}{2} \cdot \vartheta_n^2 + \binom{n}{3} \cdot \vartheta_n^3 + \dots + \binom{n}{n-1} \vartheta_n^{n-1} + \binom{n}{n} \vartheta_n^n}_{\geq 0, \text{ as } \vartheta_n \geq 0 \text{ then } n}.
 \end{aligned}$$

$$\text{So, } \underbrace{(1+\vartheta_n)^n}_{\sqrt[n]{n^n} = n} \geq \binom{n}{2} \vartheta_n^2 = \frac{n \cdot (n-1)}{2} \cdot \vartheta_n^2$$

$$\rightarrow \frac{n \cdot (n-1)}{2} \cdot \vartheta_n^2 \leq n \iff \vartheta_n^2 \leq \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

$$\boxed{\vartheta_n \leq \sqrt{\frac{2}{n-1}}, \quad \forall n \in \mathbb{N}, \quad n \geq 2} \quad \textcircled{*}_2.$$

$\leftarrow \vartheta_n \geq 0 \text{ then } n$

By  $\textcircled{*}_1$  and  $\textcircled{*}_2$ , we have

$$0 \leq \vartheta_n \leq \sqrt{\frac{2}{n-1}}, \quad \forall n \in \mathbb{N}, \quad n \geq 2.$$

$\downarrow n \rightarrow \infty$        $\downarrow n \rightarrow \infty$   
 0                    0

This is by the algebra of limits  
we have proved so far:

$$\begin{aligned}
 \frac{1}{n} &\rightarrow 0 \rightarrow \frac{1}{n-1} \rightarrow 0 \rightarrow \\
 \rightarrow \frac{2}{n-1} &\rightarrow 2 \cdot 0 = 0 \rightarrow \sqrt{\frac{2}{n-1}} \rightarrow 0
 \end{aligned}$$

By the sandwich lemma,  
 $\vartheta_n \rightarrow 0$ . So,  $\sqrt[n]{n} \rightarrow 1$ .

(12)



Observe that, in the proof of  $\sqrt[n]{n} \rightarrow 1$ ,

the problem that made things harder than for  $\sqrt[n]{a} \rightarrow 1$

is that  $(1+\delta_n)^n = n$ , rather than a constant.

So,  $1+n\delta_n$ , which is also linear in  $n$ ,

cannot help us. We need to use something

like  $\underbrace{(1+\delta_n)^n}_n \geq n^k \cdot \delta_n^k$ , for some fixed  $k \geq 1$   
(or something like that),

to demonstrate how truly small  $\delta_n$  is

(indeed, notice that the above implies that

$$\delta_n^k \leq \frac{n}{n^k} = \frac{1}{n^{k-1}} \xrightarrow{n \rightarrow \infty} 0, \text{ as } k \geq 1$$

so that  
 $n^k$  grows faster than  $n$

That's why we chose  $\binom{n}{2} \delta_n^2$  as the appropriate lower bound for  $(1+\delta_n)^n = n$ ;  $\binom{n}{2} = \frac{n(n-1)}{2}$ ,

which should "behave like"  $n^2$  for  $n$  large; whatever that means.

Notice that, instead of  $\binom{n}{2} \delta_n^2$ , we could have used  $\binom{n}{3} \delta_n^3$ , or  $\binom{n}{4} \delta_n^4$ , ..., or  $\binom{n}{k} \delta_n^k$ , for explicit  $k \in \mathbb{N}$  independent of  $n$ . (of course with  $k < n$ )

## Theorems that guarantee convergence

①

### Theorem:

Every monotone, bounded sequence converges.

uses  
completeness  
of  $\mathbb{R}$ !

More precisely:

(a) If  $(a_n)_{n \in \mathbb{N}}$  is increasing (i.e.  $a_1 \leq a_2 \leq a_3 \leq \dots$ )

and bounded from above,

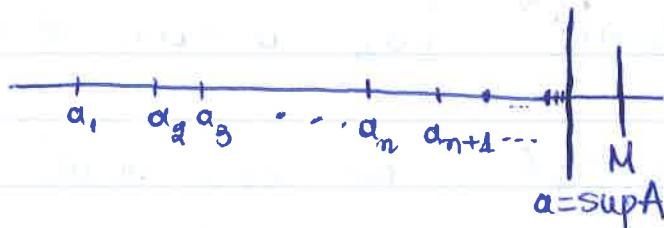
then  $a_n \rightarrow a$ , for some  $a \in \mathbb{R}$ .

(b) If  $(a_n)_{n \in \mathbb{N}}$  is decreasing (i.e.  $a_1 \geq a_2 \geq a_3 \geq \dots$ )

and bounded from below,

then  $a_n \rightarrow a$ , for some  $a \in \mathbb{R}$ .

Proof: (a)



Idea: the limit of  $(a_n)_{n \in \mathbb{N}}$  will be the  $\sup\{a_n : n \in \mathbb{N}\}$ .

Let  $A := \{a_n : n \in \mathbb{N}\}$ . We have:

$A \neq \emptyset$  (as  $a_1 \in A$ , for instance) and  $A$  bounded from above

(2)

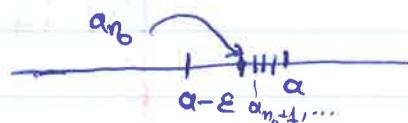
$(a_n)_{n \in \mathbb{N}}$  is bounded, so  $\exists N > 0$   
 s.t.  $a_n \leq N, \forall n \in \mathbb{N}$ . This  $N$  is an  
 upper bound of  $A$ .

Since  $\mathbb{R}$  is complete,  $A$  has a least upper bound in  $\mathbb{R}$ ,  
 $\sup A$ .

Let  $a = \sup A$ ; we will show that  $a_n \rightarrow a$ :

Let  $\varepsilon > 0$ .  $a - \varepsilon < a$ , so  $a - \varepsilon$  is not an upper bound

↓  
the least  
upper bound  
of  $A$



of  $A$ , so  $\exists n_0 \in \mathbb{N}$  s.t.  $a_{n_0} > a - \varepsilon$ .

Now, since  $(a_n)_{n \in \mathbb{N}}$  is increasing, we have

$a_n \geq a_{n_0} \quad \forall n \geq n_0$ , so  $a_n > a - \varepsilon \quad \forall n \geq n_0$ . (1)

And clearly

$$a_n \leq a < a + \varepsilon \quad \forall n \geq n_0 \quad (2)$$

as  $a = \sup A$  is an upper bound of  $A$ .

By (1), (2), we have that

$$a - \varepsilon < a_n < a + \varepsilon, \quad \forall n \geq n_0.$$

Since  $\varepsilon$  was arbitrary, we have that  
 $a_n \rightarrow a$ .

(3)

(b) Exercise.

→ ex: The sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$ ,  $\forall n \in \mathbb{N}$ , converges.

Proof:

- $(a_n)_{n \in \mathbb{N}}$  is increasing (in fact, we can show it is strictly increasing):

We want to check if  $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$ ,  $\forall n \in \mathbb{N}$ .

$$\Leftrightarrow \left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^n \cdot \frac{n+2}{n+1}$$

$$\Leftrightarrow \underbrace{\frac{n+1}{n+2}}_{\substack{\parallel \\ 1 - \frac{1}{n+2}}} < \underbrace{\left(\frac{n \cdot (n+2)}{(n+1)^2}\right)^n}_{\substack{\parallel \\ 1 - \frac{1}{(n+1)^2}}} \Leftrightarrow 1 - \frac{1}{n+2} < \left(1 - \frac{1}{(n+1)^2}\right)^n$$

By Bernoulli's inequality for  $a = -\frac{1}{(n+1)^2} > -1$ , we have:

$$\left(1 - \frac{1}{(n+1)^2}\right)^n \geq 1 - \frac{n}{(n+1)^2}, \text{ so it suffices to}$$

check that  $1 - \frac{n}{(n+1)^2} > 1 - \frac{1}{n+2}$ , which is true (check it).  $\forall n \in \mathbb{N}$ .

(4).

- $(a_n)_{n \in \mathbb{N}}$  is bounded from above:

To show this we introduce the sequence

$$b_n = \boxed{\left(1 + \frac{1}{n}\right)^{n+1}}, \quad \forall n \in \mathbb{N}.$$

$(b_n)_{n \in \mathbb{N}}$  is decreasing (exercise; use Bernoulli's inequality again),

and clearly  $a_n < b_n \quad \forall n \in \mathbb{N}$ ,

$$\hookrightarrow a_n < b_1, \text{ as } (b_n)_{n \in \mathbb{N}} \text{ is decreasing}$$

so  $a_n < b_1 \quad \forall n \in \mathbb{N}$ , which means that

$b_1 = (1+1)^2 = 4$  is an upper bound for  $(a_n)_{n \in \mathbb{N}}$ .

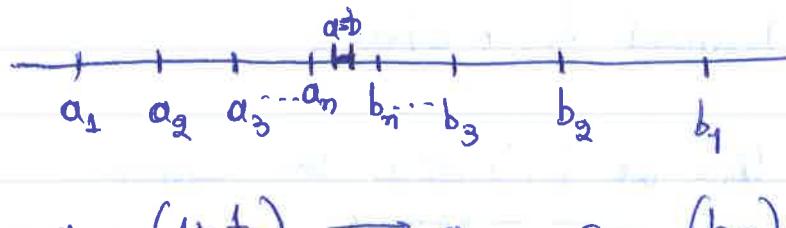
So: since  $(a_n)_{n \in \mathbb{N}}$  is increasing and bounded from above, it converges.

⚠ Similarly,  $(b_n)_{n \in \mathbb{N}}$  is decreasing and bounded below by  $a_1$  ( $a_1 < a_n < b_n, \forall n \in \mathbb{N}$ ), so  $(b_n)_{n \in \mathbb{N}}$  converges too.

In fact: let  $a := \lim_{n \rightarrow \infty} a_n$  and  $b := \lim_{n \rightarrow \infty} b_n$ . Then,

$$\boxed{a=b}:$$

(5)



$$b_n = a_n \cdot \left(1 + \frac{1}{n}\right) \rightarrow a, \text{ so } (b =) \lim_{n \rightarrow \infty} b_n = a.$$

→ Def:  $\boxed{e} := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad \left(= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1}\right).$



Using that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \quad \forall n \in \mathbb{N},$$

you can find very good approximations of  $e$  (as good as you wish). In particular:  $e = 2.718\dots$

→  **$\pm\infty, -\infty$  as limits:** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

I want a definition of  $a_n \rightarrow \pm\infty$  that will allow me to say that: "a<sub>n</sub> is large for large n".

In particular I want the following to hold:

(6)

- No matter how large a number someone gives me, I can find a whole final part of  $(a_n)_{n \in \mathbb{N}}$  above that number".



→ Def: We say that  $a_n \rightarrow +\infty$  as  $n \rightarrow +\infty$

if:  $\forall M > 0, \exists n_0 = n_0(M) \in \mathbb{N}$  s.t.:  $a_n > M, \forall n \geq n_0$ .

This is equivalent to saying:

$a_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  if

$\forall M > 0$ , there exists a whole final part of  $(a_n)_{n \in \mathbb{N}}$  in  $(M, +\infty)$ .

Similarly: We say that  $a_n \rightarrow -\infty$  as  $n \rightarrow +\infty$

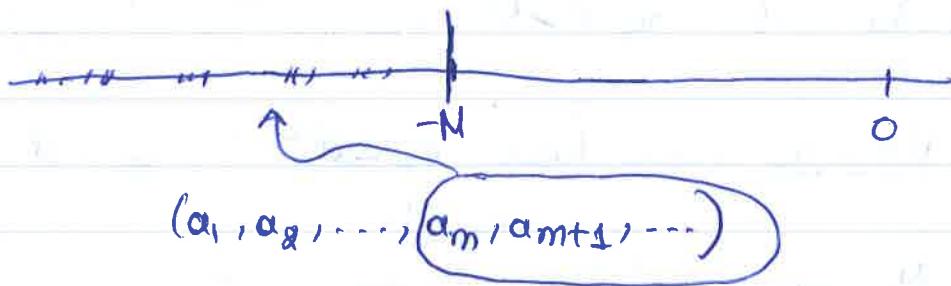
if:  $\forall M < 0, \exists n_0 = n_0(M) \in \mathbb{N}$  s.t.:  $a_n < M, \forall n \geq n_0$ .

7

This is equivalent to saying:

$a_n \rightarrow -\infty$  as  $n \rightarrow \infty$  if

If  $M > 0$ , there exists some final part of  $(a_n)_{n \in \mathbb{N}}$  in  $(-\infty, -M)$ .



→ Prop: Let  $a > 0$ . We define  $a_n := a^n$ , then  $\forall N$ . Then:

- If  $0 < a < 1$ , then  $a_n = a^n \rightarrow 0$ .
- If  $a = 1$ , then  $a_n = 1 \rightarrow 1$ .
- If  $a > 1$ , then  $a_n = a^n \rightarrow +\infty$ .

Proof: - Let  $a > 1$ . Let  $M > 0$ . We will show that there exists some  $n_0 \in \mathbb{N}$  s.t.:  $\forall n \geq n_0$ ,  $a_n > M$ . Indeed:

Since  $a > 1$ , there exists some  $\delta > 0$  s.t.  $a = 1 + \delta$  (actually,  $\delta = a - 1$ ).

$$\text{Then, } a_n = a^n = (1+\delta)^n \geq 1 + n\delta$$

(8)

you can just use  
② viii and vi from the 3rd  
weekly assignment, instead  
of what follows)

↓  
by Bernoulli's inequality,  
which we can apply because  
 $\delta > 0$  (and thus  $>-1$ )

By the Archimedean property of the reals, there exists some  $n_0 \in \mathbb{N}$  s.t.  $n_0 \cdot \delta > N$  (i.e., we can make  $n\delta$  as large as we want).

Then,  $\forall n \geq n_0$ , we have  $a_n > n\delta \geq n_0\delta > N$ ,  
so  $a_n > N$ .

Since  $N$  was arbitrary,  $a_n \rightarrow +\infty$ .

- Let  $0 < a < 1$ . Then,  $\frac{1}{a} > 1 \Rightarrow \left(\frac{1}{a}\right)^n \rightarrow +\infty \Rightarrow a^n \rightarrow 0$

use that  
if  $c_n \rightarrow +\infty$   
then  $\frac{1}{c_n} \rightarrow 0$   
(exercise).

A second proof: Since  $0 < a < 1$ , there exists

some  $\delta > 0$  s.t.  $a = \frac{1}{1+\delta}$ . Then:  $\forall n \in \mathbb{N}$ ,

$$0 < a_n = a^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} < \frac{1}{n\delta} = \frac{1}{\delta} \cdot \frac{1}{n}$$

Bernoulli's inequality

(9)

$$\left. \begin{array}{l} \text{So, } 0 < a_n < \left( \frac{1}{\theta} \cdot \frac{1}{n} \right), \forall n \in \mathbb{N}. \\ \downarrow \quad \quad \quad \downarrow \\ 0 \qquad \qquad \qquad 0 \end{array} \right)$$

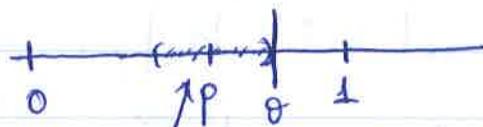
By the sandwich lemma,  $a_n \rightarrow 0$ . ■

→ **The ratio test:** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, with  $a_n \neq 0 \ \forall n \in \mathbb{N}$ .

Consider  $b_n = \frac{|a_{n+1}|}{|a_n|}, \forall n \in \mathbb{N}$ .

- If  $\lim_{n \rightarrow \infty} b_n = p < 1$ , then  $a_n \rightarrow 0$ .
- If  $\lim_{n \rightarrow \infty} b_n = p \stackrel{\text{(maybe }+\infty)}{>} 1$ , then  $|a_n| \rightarrow \infty$ .
- If  $\lim_{n \rightarrow \infty} b_n = 1$ , then the test is inconclusive.

Proof: - Suppose that  $(0 \leq) \lim_{n \rightarrow \infty} b_n = p < 1$ .



or final part of  $(b_n)_{n \in \mathbb{N}}$  in here.

Pick some number  $\delta$  between  $p$  and  $1$ ; say,  $\delta = \frac{1-p}{2}$ ,

(10)

and indeed for this  $\delta$  we have  $\rho < \delta < 1$ .

Since  $b_n \rightarrow \rho < \theta$ , there exists some  $n_0 \in \mathbb{N}$  st.:

$$n \geq n_0, b_n < \theta$$

(apply the definition of limit for  $\epsilon = \theta - \rho$ ,  
 i.e. the neighbourhood  $(\rho - (\theta - \rho), \rho + (\theta - \rho))$  of  $\rho$ )

i.e.,  $n \geq n_0$ , we have:  $\frac{|a_{n+1}|}{|a_n|} < \theta \Leftrightarrow |a_{n+1}| < \theta \cdot |a_n|$ .

$$\text{So: } \left\{ \begin{array}{l} |a_{n_0+1}| < \theta \cdot |a_{n_0}| \\ |a_{n_0+2}| < \theta \cdot |a_{n_0+1}| \\ \vdots \\ |a_{n_0+k}| < \theta \cdot |a_{n_0+k-1}| \end{array} \right. , \forall k \in \mathbb{N} \quad \Rightarrow$$

$$\Rightarrow |a_{n_0+k}| < \theta^k \cdot |a_{n_0}|, \forall k \in \mathbb{N}.$$

$$\text{So: } 0 < |a_{n_0+k}| < \theta^k \cdot |a_{n_0}|, \text{ so, by the sandwich lemma,}$$

$$\downarrow \quad \downarrow$$

$$0 \quad 0$$

$$a_{n_0+k} \xrightarrow[k \rightarrow \infty]{} 0.$$

(11)

Notice that the sequence  $(a_{n_0+k})_{k \in \mathbb{N}} = (a_{n_0+1}, a_{n_0+2}, \dots)$   
 is a final part of  $(a_n)_{n \in \mathbb{N}}$ , so  $(a_n)_{n \in \mathbb{N}}$  has the same  
 limit as  $(a_{n_0+k})_{k \in \mathbb{N}}$ , i.e.  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

- Suppose that  $p > 1$ . Whether  $p \in \mathbb{R}$  or  $p = \infty$ , since

$b_n \rightarrow p$  there exists some  $\delta > 1$  s.t. :  
 $\forall n \geq n_0, b_n > \delta \rightarrow$

$\leftarrow \forall n \geq n_0, |a_{n+1}| > \delta \cdot |a_n|$ .

Work as before to show that  $|a_n| \rightarrow \infty$ . ■



The root test : Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence.

Consider  $b_n = \sqrt[n]{|a_n|}$ ,  $\forall n \in \mathbb{N}$ .

- If  $b_n \rightarrow p < 1$ , then  $a_n \rightarrow 0$ .
- If  $b_n \rightarrow \underset{\text{maybe } \infty}{p} > 1$ , then  $|a_n| \rightarrow \infty$
- If  $b_n \rightarrow 1$ , then the test is inconclusive.

(18)

Proof: As for the ratio test (exercise).

→ Examples:

1) find the limit of :  $a_n = \frac{n^3 + 5n^2 + 2}{2n^3 + 9}$ ,  $n \in \mathbb{N}$ .

$$a_n = \frac{n^3 + 5n^2 + 2}{2n^3 + 9} = \frac{n^3 \cdot \left(1 + \frac{5}{n} + \frac{2}{n^3}\right)}{n^3 \cdot \left(2 + \frac{9}{n^3}\right)} = \frac{1 + \frac{5}{n} + \frac{2}{n^3}}{2 + \frac{9}{n^3}} \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}$$

extract the largest power in the numerator and the denominator; because a polynomial behaves like its monomial of largest power.

2) find the limit of :  $b_n = \underbrace{\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}}$ ,  $n \in \mathbb{N}$ ,

generally, when the number of terms depends on  $n$ , we think of the sandwich lemma.

$n+1$  terms; so, even though each converges to 0, we cannot deduce their sum converges to 0.

Notice that:  $\frac{1}{(2n)^2} \leq \frac{1}{(n+k)^2} \leq \frac{1}{n^2}$ ,  $\forall k=0, 1, \dots, n$

$$\text{So: } \underbrace{\frac{1}{(2n)^2} + \dots + \frac{1}{(2n)^2}}_{n+1 \text{ terms}} \leq \underbrace{\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}}_{b_n} \leq \underbrace{\frac{1}{n^2} + \dots + \frac{1}{n^2}}_{n+1 \text{ terms}}$$

(13)

i.e.  $\frac{n+1}{(2n)^2} \leq b_n \leq \frac{n+1}{n^2}$ ,  $\forall n \in \mathbb{N}$

$$\frac{n(1+\frac{1}{n})}{n^2(4)} = \frac{n(1+\frac{1}{n})}{n^2} = \left(\frac{1}{n}\right)\left(1+\frac{1}{n}\right) \xrightarrow[n \rightarrow \infty]{\substack{\nearrow 0 \\ \searrow 1}} 0$$

$$\frac{1}{4n} \cdot \left(1+\frac{1}{n}\right) \xrightarrow[n \rightarrow \infty]{\substack{\nearrow 0 \\ \searrow 1}} 0.$$

So,  $b_n \rightarrow 0$  (by the sandwich lemma).

3 | find the limit of:  $c_n = \frac{n}{2^n}$ ,  $n \in \mathbb{N}$ .

this is a ratio, so we hope that, if we try the ratio test, a lot will cancel out.

We have:  $\frac{|c_{n+1}|}{|c_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{(n+1) \cdot 2^n}{n \cdot 2^{n+1}} = \frac{1}{2} \cdot \frac{n+1}{n} =$

$$= \frac{1}{2} \cdot \frac{n(1+\frac{1}{n})}{n} = \frac{1}{2} \cdot \left(1+\frac{1}{n}\right) \xrightarrow[n \rightarrow \infty]{\substack{\nearrow 0 \\ \searrow 1}} \frac{1}{2} < 1.$$

So,  $c_n \rightarrow 0$ .

(14)

4 Show that  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

1st way:  $\left(1 + \frac{1}{n}\right)^{n^2} = \underbrace{\left[\left(1 + \frac{1}{n}\right)^n\right]}_e^n$

$\downarrow$

$e, \text{ so } e > \frac{1+e}{2} (> 1) \text{ for large } n.$

$\begin{array}{c} + \\ \downarrow \\ 1 \\ \frac{1+e}{2} > 1 \\ e \end{array}$

So,  $\left(1 + \frac{1}{n}\right)^{n^2} > \left(\frac{1+e}{2}\right)^n \rightarrow +\infty \text{ as } n \rightarrow +\infty$   
 (since  $\frac{1+e}{2} > 1$ )

$\downarrow$

for large  $n$

So,  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

2nd way:  $\sqrt[n]{\left(1 + \frac{1}{n}\right)^{n^2}} = \left(1 + \frac{1}{n}\right)^n \rightarrow e > 1,$

so, by the root test,  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

(14)

4 Show that  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

1st way:  $\left(1 + \frac{1}{n}\right)^{n^2} = \left[\underbrace{\left(1 + \frac{1}{n}\right)^n}_{e, \text{ so } > \frac{1+e}{2} (>1) \text{ for large } n}\right]^n$

$$\begin{array}{c} + \\ \downarrow \\ \frac{1+e}{2} > 1 \end{array}$$

So,  $\left(1 + \frac{1}{n}\right)^{n^2} > \left(\frac{1+e}{2}\right)^n \rightarrow +\infty \text{ as } n \rightarrow +\infty$  (since  $\frac{1+e}{2} > 1$ )  
 for large  $n$

So,  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

2nd way:  $\sqrt[n]{\left(1 + \frac{1}{n}\right)^{n^2}} = \left(1 + \frac{1}{n}\right)^n \rightarrow e > 1,$

so, by the root test,  $\left(1 + \frac{1}{n}\right)^{n^2} \rightarrow +\infty$ .

Lecture 8:

12 Sep 2015. (1)

→ Def: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers.

A sequence  $(b_n)_{n \in \mathbb{N}}$  is called a subsequence of  $(a_n)_{n \in \mathbb{N}}$  indices of the original sequence

if there exist  $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots$  in  $\mathbb{N}$ ,

s.t.  $b_n = a_{k_n}$ ,  $\forall n \in \mathbb{N}$ .

(2)

In other words, the subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  of the sequence  $(a_n)_{n \in \mathbb{N}}$

(which was a map  $a : 1 \downarrow \alpha(1) 2 \downarrow \alpha(2) \dots \boxed{k_1} \downarrow \alpha(k_1) \dots \boxed{k_2} \downarrow \alpha(k_2) \dots$ )

is another map from  $\mathbb{N}$  to  $\mathbb{R}$ , that only keeps the information of where  $a$  sends  $k_1, k_2, k_3, \dots$ ,

and which we see as the map

$$(a_{k_n})_{n \in \mathbb{N}} : \begin{matrix} 1 \\ \downarrow \\ a(k_1) \end{matrix} \quad \begin{matrix} 2 \\ \downarrow \\ a(k_2) \end{matrix} \quad \begin{matrix} 3 \\ \downarrow \\ a(k_3) \end{matrix} \quad \dots$$

That is,  $(a_n)_{n \in \mathbb{N}}$  is  $(a_1, a_2, \dots, \color{red}{a_{k_1}, \dots, a_{k_2}, \dots, a_{k_3}, \dots})$

and  $(a_{k_n})_{n \in \mathbb{N}}$  is  $(a_{k_1}, a_{k_2}, a_{k_3}, \dots)$ .

**⚠** I am NOT allowed to jump from the index  $k_2$  back to  $k_1$ : the terms will appear in the subsequence in the same order as in the sequence (by definition)

strictly increasing  
the indices we keep for the subsequence?

**Ex.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, and  $k_n = 2n$ . What is  $(a_{k_n})_{n \in \mathbb{N}}$ ?

It is the sequence  $(a_{2n})_{n \in \mathbb{N}} = (a_2, a_4, a_6, \dots)$ .

(3)

- Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, and  $k_n = 2n - 1$ . What is  $(a_{k_n})_{n \in \mathbb{N}}$ ? strictly increasing

It is the sequence  $(a_{2n-1})_{n \in \mathbb{N}} = (a_1, a_3, a_5, \dots)$

- Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, and  $k_n = n^2$ . What is  $(a_{k_n})_{n \in \mathbb{N}}$ ? strictly increasing.

It is the sequence  $(a_{n^2})_{n \in \mathbb{N}} = (a_1, a_4, a_9, a_{16}, \dots)$ .

- Every final part of  $(a_n)_{n \in \mathbb{N}}$  is a subsequence of  $(a_n)_{n \in \mathbb{N}}$ :

Let  $(a_m, a_{m+1}, \dots)$  be a final part of  $(a_n)_{n \in \mathbb{N}}$ .

Then,  $(a_m, a_{m+1}, \dots) = (a_{m+n-1})_{n \in \mathbb{N}}$ ,

and  $m+1-1 < m+2-1 < m+3-1 < \dots$ ,

so  $(a_{m+n-1})_{n \in \mathbb{N}}$  is indeed a subsequence of  $(a_n)_{n \in \mathbb{N}}$ .



We require  $k_1 \leq k_2 \leq k_3 \leq \dots$  in the definition of a subsequence so that the same index isn't repeated.

(4)

→ From the above, we see that, to create a subsequence of  $(a_n)_{n \in \mathbb{N}}$ :

- We pick some  $k_1 \in \mathbb{N}$ ;  $a_{k_1}$  will be the first term of the subsequence.
- We pick some  $k_2 > k_1$  in  $\mathbb{N}$ ;  $a_{k_2}$  will be the second term of the subsequence.
- We pick some  $k_3 > k_2$  in  $\mathbb{N}$ ;  $a_{k_3}$  will be the third term of the subsequence.
- ⋮

→ Observation: If  $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots$  in  $\mathbb{N}$ ,

then  $k_n \geq n$ ,  $n \in \mathbb{N}$ .

i.e., the  $n$ -th term of the subsequence comes after the  $n$ -th term of the original sequence (or they are the same).

Proof:

-  $k_1 \geq 1$ , since  $k_1 \in \mathbb{N}$ .

- Suppose that  $k_m \geq m$ , for some  $m \in \mathbb{N}$ . Then,

(5)

$k_{m+1} > k_m$  *in N*, i.e.  $k_{m+1} \geq k_m + 1 \geq m + 1$ .

→ Observation: Every sequence has infinitely many subsequences.

Proof: A sequence has infinitely many final points, each of which is a subsequence of the sequence. ■

→ Prop:

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence, with  $a_n \rightarrow a$   
( $a \in \mathbb{R}$  or  $a = +\infty$  or  $a = -\infty$ ).

i.e.:

If a sequence has a limit, then all its subsequences have the same limit, that of the original sequence.

Then,  $a_{k_n} \xrightarrow{n \rightarrow \infty} a$ , for any subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  of  $(a_n)_{n \in \mathbb{N}}$ .

Proof: - Suppose that  $a \in \mathbb{R}$ . Let  $(a_{k_n})_{n \in \mathbb{N}}$  be a subsequence of  $(a_n)_{n \in \mathbb{N}}$ .

Let  $\epsilon > 0$ . We want to show that, for some  $n_0 \in \mathbb{N}$ :  $|a_{k_n} - a| < \epsilon$ ,  $\forall n \geq n_0$ .

Since  $a_n \rightarrow a$ , there exists some  $n_0 \in \mathbb{N}$ :  $|a_n - a| < \epsilon$ ,  $\forall n \geq n_0$ .

Now, by the observation earlier, we have that:

$\forall n \in \mathbb{N}$ ,  $k_n \geq n$ ; in particular,  $k_n \geq n \geq n_0$ ,  $\forall n \geq n_0$ .

(6)

So,  $\forall n \geq n_0 : |a_{k_n} - a| < \epsilon$ . Since  $\epsilon$  was arbitrary,

we have that  $a_{k_n} \rightarrow a$  as  $n \rightarrow \infty$ .

- Work similarly for  $a = +\infty$  and  $a = -\infty$ .



→ Application: The sequence  $a_n = (-1)^n$ ,  $n \in \mathbb{N}$ , doesn't converge, and also  $a_n \not\rightarrow +\infty$ ,  $a_n \not\rightarrow -\infty$ .

Proof: Suppose  $a_n \rightarrow a$ , for some  $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ . Then, for any subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  we also

have  $a_{k_n} \xrightarrow[n \rightarrow \infty]{} a$ . But:

$$a_{2n} = (-1)^{2n} = 1 \text{ } \forall n \in \mathbb{N}, \text{ so } a_{2n} \rightarrow 1,$$

$$\text{while } a_{2n-1} = (-1)^{2n-1} = -1 \text{ } \forall n \in \mathbb{N}, \text{ so } a_{2n-1} \rightarrow -1,$$

and  $1 \neq -1$ , a contradiction.

(7)

→ Bolzano - Weierstrass theorem :

Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

Proof: For the proof we need the Proposition that follows, that states that: every sequence in  $\mathbb{R}$  has a monotone subsequence.

Once we know this, the proof of the Bolzano-Weierstrass theorem follows as such:

- Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded real sequence.  
(that is,  $\exists N > 0$  s.t.  $|a_n| \leq N, \forall n \in \mathbb{N}$ ).
- By the Proposition that follows,  $(a_n)_{n \in \mathbb{N}}$  has a monotone subsequence  $(a_{k_n})_{n \in \mathbb{N}}$   
(note that boundedness of  $(a_n)_{n \in \mathbb{N}}$  is not required for this).
- $(a_{k_n})_{n \in \mathbb{N}}$  is bounded (as  $|a_{k_n}| \leq N \quad \forall n \in \mathbb{N}$ )  
 $\Rightarrow |a_{k_n}| < N \quad \forall n \in \mathbb{N}$

So,  $(a_{k_n})_{n \in \mathbb{N}}$  is monotone and bounded  $\Rightarrow (a_{k_n})_{n \in \mathbb{N}}$  converges

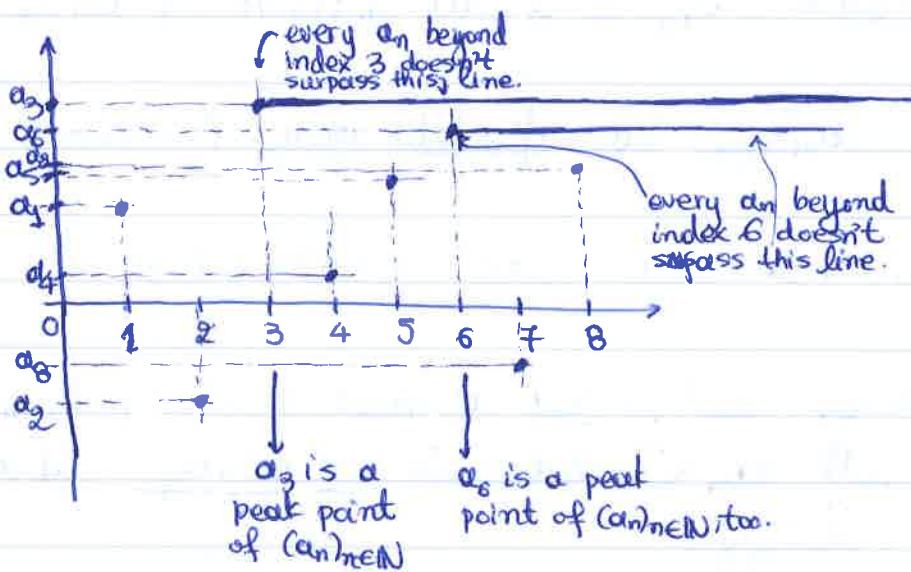
(8)

So, the proof will be complete once we prove the following Proposition.

→ Prop: Every sequence in  $\mathbb{R}$  has a monotone subsequence.

Proof: First, we need the following definition:

→ Def: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . We say that a term  $a_m$  of  $(a_n)_{n \in \mathbb{N}}$  is a peak point of  $(a_n)_{n \in \mathbb{N}}$  if:  $a_m \geq a_n, \forall n \geq m$ :



ex: • The sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = \frac{1}{n}$   $\forall n \in \mathbb{N}$  is decreasing, so all its terms are peak points.

• The sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = -\frac{1}{n}$   $\forall n \in \mathbb{N}$  is increasing, so it has no peak points.

(9)

Let us now go back to our proof:

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

Case 1:  $(a_n)_{n \in \mathbb{N}}$  has infinitely many peak points.

I.e.,  $\exists k_1 < k_2 < \dots < k_n < k_{n+1} < \dots \in \mathbb{N}$  s.t.:

$a_{k_n}$  is a peak point of  $(a_n)_{n \in \mathbb{N}}$ , for all  $n \in \mathbb{N}$ .

Then,  $(a_{k_n})_{n \in \mathbb{N}}$  is decreasing:  $\forall n \in \mathbb{N}, a_{k_n} \geq a_{k_{n+1}}$ ,

because  $a_{k_n}$  is a peak point of  $(a_n)_{n \in \mathbb{N}}$ .

Case 2:  $(a_n)_{n \in \mathbb{N}}$  has finitely many (or none) peak points.

Then, there exists some  $n_0 \in \mathbb{N}$  s.t.:

$\forall n \geq n_0$ ,  $a_n$  is not a peak point of  $(a_n)_{n \in \mathbb{N}}$ .

We will now construct an increasing subsequence of  $(a_n)_{n \in \mathbb{N}}$ :

We set  $k_1 = n_0$ . Since  $a_{k_1}$  is not a peak point of  $(a_n)_{n \in \mathbb{N}}$ , there exists  $k_2 > k_1$  s.t.  $a_{k_2} > a_{k_1}$ .

Since  $a_{k_2}$  is not a peak point of  $(a_n)_{n \in \mathbb{N}}$ ,

there exists  $k_3 > k_2$  s.t.  $a_{k_3} > a_{k_2}$ ,  
and so on.

(10)

We thus inductively find  $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots \in \mathbb{N}$ ,

s.t.  $a_{k_1} < a_{k_2} < \dots < a_{k_n} < a_{k_{n+1}} < \dots$

The subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  of  $(a_n)_{n \in \mathbb{N}}$  is (strictly) increasing.

!

We will later prove that every sequence has a convergent subsequence, in a compact metric space. The above is a corollary of this more general result because a bounded sequence in  $\mathbb{R}$  is always contained in some closed interval, which is a compact metric space.

Let us now see a second proof of the Bolzano-Weierstrass theorem. It will follow from the following theorem on nested intervals, which generalises to any metric space for nested compact sets (Cantor's intersection theorem).

(11)

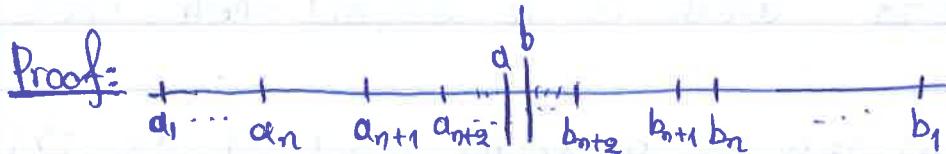
→ **Nested intervals theorem :**

Let  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots \supseteq [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] \supseteq \dots$ ,  
 a sequence of nested intervals. Then:

- $\bigcap_{n=1}^{+\infty} [a_n, b_n] \neq \emptyset$  (In fact,  $\bigcap_{n=1}^{+\infty} [a_n, b_n] = [a, b]$ , where  
 $a = \lim_{n \rightarrow +\infty} a_n = \sup \{a_n : n \in \mathbb{N}\}$   
 and  $b = \lim_{n \rightarrow +\infty} b_n = \inf \{b_n : n \in \mathbb{N}\}$ )
- If, in particular,  $b_n - a_n \rightarrow 0$ , then

$$\bigcap_{n=1}^{+\infty} [a_n, b_n] = \{x\}, \text{ for some } x \in \mathbb{R}.$$

$\left( \begin{array}{l} \lim a_n = \lim b_n \\ \parallel \end{array} \right)$



Since  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots \supseteq [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] \supseteq \dots$ ,

we have that  $(a_n)_{n \in \mathbb{N}}$  is increasing  
 and  $(b_n)_{n \in \mathbb{N}}$  is decreasing,

and  $(a_n)_{n \in \mathbb{N}}$  bounded from above (as  $a_n \leq b_1$   $\forall n \in \mathbb{N}$ ),

and  $(b_n)_{n \in \mathbb{N}}$  bounded from below (as  $b_n \geq a_1$   $\forall n \in \mathbb{N}$ ).

(19)

So:  $a_n \rightarrow a$  for  $a = \sup \{a_n : n \in \mathbb{N}\}$ ,  
 and  $b_n \rightarrow b$  for  $b = \inf \{b_n : n \in \mathbb{N}\}$ .

Since  $a_n \leq b_n \quad \forall n \in \mathbb{N}$ , we have  $a \leq b$  (exercise 10(i) in Weekly Assignment 2).

We will now show that  $\bigcap_{n=1}^{+\infty} [a_n, b_n] = [\alpha, \beta]$ :

- Let  $x \in \bigcap_{n=1}^{+\infty} [a_n, b_n]$ . Then,  $x \in [a_n, b_n], \forall n \in \mathbb{N}$ ; i.e.:

$a_n \leq x \leq b_n, \quad \forall n \in \mathbb{N}$ . Thus,  $a \leq x \leq b$

$$\begin{array}{ccc} \downarrow n \rightarrow +\infty & \downarrow n \rightarrow +\infty & \downarrow n \rightarrow +\infty \\ a & x & b \end{array} \quad (\text{again by 10(i) in Weekly Assignment 2}).$$

So,  $x \in [\alpha, \beta]$ . Therefore:  $\boxed{\bigcap_{n=1}^{+\infty} [a_n, b_n] \subseteq [\alpha, \beta]} \quad (*)_1$

- Let  $x \in [\alpha, \beta]$ . Then,  $a_n \leq \alpha \leq x \leq \beta \leq b_n \quad \forall n \in \mathbb{N}$ ,

so  $a_n \leq x \leq b_n, \quad \forall n \in \mathbb{N}$ ,

i.e.  $x \in [a_n, b_n], \forall n \in \mathbb{N}$ .

So,  $x \in \bigcap_{n=1}^{+\infty} [a_n, b_n]$ . Therefore,  $\boxed{[\alpha, \beta] \subseteq \bigcap_{n=1}^{+\infty} [a_n, b_n]} \quad (*)_2$

By  $(*)_1$  and  $(*)_2$ ,  $\bigcap_{n=1}^{+\infty} [a_n, b_n] = [\alpha, \beta] \Rightarrow \bigcap_{n=1}^{+\infty} [a_n, b_n] \neq \emptyset$ .

In particular, if  $b_n - a_n \rightarrow 0$ , then  $\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} b_n$ ,

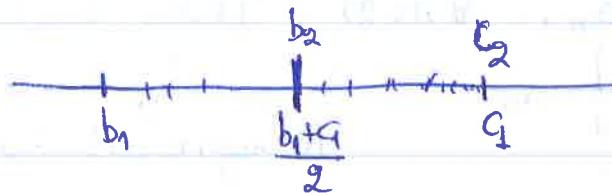
(13)

i.e.  $a=b$ , and  $[a,b]=\{a\}$ . So, if  $b_n-a_n \rightarrow 0$ ,

then  $\bigcap_{n=1}^{\infty} [a_n, b_n]$  is a singleton.

### → Second proof of Bolzano-Weierstrass theorem :

Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence. We will show that it has a convergent subsequence:



Since  $(a_n)_{n \in \mathbb{N}}$  is bounded, there exist  $b_1 < c_1 \in \mathbb{R}$  s.t.:

$$b_1 \leq a_n \leq c_1, \quad \forall n \in \mathbb{N}.$$

We split  $[b_1, c_1]$  in two intervals of equal length,

$[b_1, \frac{b_1+c_1}{2}]$  and  $[\frac{b_1+c_1}{2}, c_1]$ . At least one of these

two intervals contains infinitely many terms of  $(a_n)_{n \in \mathbb{N}}$ . We pick one such interval, and denote it by  $[b_2, c_2]$ .

Inductively, we find  $[b_1, c_1] \supseteq [b_2, c_2] \supseteq \dots \supseteq [b_n, c_n] \supseteq \dots$ , s.t.:

①  $[b_n, c_n]$  contains infinitely terms of  $(a_n)_{n \in \mathbb{N}}$ ,  $\forall n \in \mathbb{N}$

②  $[b_n, c_n]$  has length  $\frac{c_1 - b_1}{2^{n-1}}$ ,  $\forall n \in \mathbb{N}$ .

By the nested intervals theorem,  $\bigcap_{n=1}^{+\infty} [b_n, c_n] \neq \emptyset$ . In particular, since

$$c_n - b_n = \frac{c_1 - b_1}{2^{n-1}} \xrightarrow[n \rightarrow +\infty]{} 0, \text{ we have that}$$

$$\bigcap_{n=1}^{+\infty} [b_n, c_n] = \{x\}, \text{ where } x = \lim_{n \rightarrow +\infty} b_n = \lim_{n \rightarrow +\infty} c_n.$$

We will now define a subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  of  $(a_n)_{n \in \mathbb{N}}$  with  $a_{k_n} \xrightarrow[n \rightarrow +\infty]{} x$ :

- Let  $k_1 = 1$ ; clearly,  $b_1 \leq a_{k_1} \leq c_1$ .

- In  $[b_2, c_2]$ , there are infinitely many terms of  $(a_n)_{n \in \mathbb{N}}$ ; so, in particular  $\exists a_{k_2} \in [b_2, c_2]$ , with  $k_2 > k_1$ .

- In  $[b_3, c_3]$ , there are infinitely many terms of  $(a_n)_{n \in \mathbb{N}}$ ; so, in particular  $\exists a_{k_3} \in [b_3, c_3]$ , with  $k_3 > k_2$ ,

and so on. Eventually:

$b_n \leq a_{k_n} \leq c_n \quad \forall n \in \mathbb{N}$ . By the sandwich lemma,

(15)

$$\lim_{n \rightarrow \infty} a_{k_n} = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = x.$$

So,  $(a_n)_{n \in \mathbb{N}}$  is convergent. □



Notice that both the proofs of the Bolzano-Weierstrass that we provided rely on the total order in  $\mathbb{R}$ . When we generalise the theorem to compact metric spaces, we will not have that advantage any more. So, we'll have to find a better way to exploit the generalisation of the nested intervals theorem that we mentioned earlier.

(15)

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Notice that both the proofs of the Bolzano-Weierstrass that we provided rely on the total order in  $\mathbb{R}$ . When we generalise the theorem to compact metric spaces, we will not have that advantage any more. So, we'll have to find a better way to exploit the generalisation of the nested intervals theorem that we mentioned earlier.

### Lecture 8:

14 Sep 2016. ①



### Cauchy sequences:

This is another notion we will generalise to all metric spaces later.

The main observation that leads to the notion of a Cauchy sequence is that, if a sequence converges, then all its terms are close to the limit for large so, in particular, these terms should be close to each other. But does the converse hold ??

(2)

→ Def: A sequence  $(a_n)_{n \in \mathbb{N}}$  is a **Cauchy sequence**

**if** :  $\forall \epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  s.t.:  $\forall n \geq n_0, |a_n - a_m| < \epsilon$ .



We should think each  $\epsilon > 0$  in the definition above as the "level of closeness" that we want the terms of  $(a_n)_{n \in \mathbb{N}}$  to be achieving eventually (from some index onwards). There is no neighbourhood of any point involved.

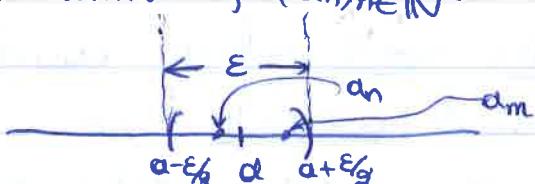
for instance, when  $n \geq n_0$ , we don't just have  $|a_{n+1} - a_n| < \epsilon$ , but also  $|a_{2n} - a_n| < \epsilon$ ,  $|a_{n+1000} - a_n| < \epsilon$ , etc.

→ Prop: Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

If  $(a_n)_{n \in \mathbb{N}}$  converges, then  $(a_n)_{n \in \mathbb{N}}$  is Cauchy.

Proof: Let  $a$  be the limit of  $(a_n)_{n \in \mathbb{N}}$ .

Let  $\epsilon > 0$ .



Since  $a_n \rightarrow a$ , there exists some  $n_0 \in \mathbb{N}$  s.t. :

$$\forall n \geq n_0, |a_n - a| < \frac{\epsilon}{2}.$$

$$\begin{aligned} \text{So, } \forall n, m \geq n_0 : |a_n - a_m| &= |(a_n - a) + (a - a_m)| \leq \\ &\leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

triangle inequality

(3)

Since  $\epsilon > 0$  was arbitrary,  $(a_n)$  is Cauchy.



In fact, we will later see that, in any metric space, convergent sequences are Cauchy.

However, the converse is not true in general; i.e., in a general metric space, Cauchy sequences don't necessarily converge

(i.e., the terms of a sequence being eventually as close as we want to each other doesn't mean that they are also all close to a fixed element of the metric space.)

A metric space where Cauchy sequences converge is called complete. We will now see that

$\mathbb{R}$  is a complete metric space (don't confuse this with the notion of a complete ordered field!)

(4)

→ Thm: In  $\mathbb{R}$ , a sequence converges  $\iff$  it is Cauchy.

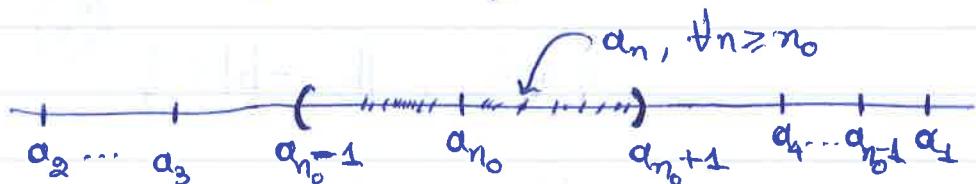
Proof:

Due to the previous Proposition, we just need to show that every Cauchy sequence in  $\mathbb{R}$  converges.

Let  $(a_n)_{n \in \mathbb{N}}$  a Cauchy sequence in  $\mathbb{R}$ . We finish the proof in 2 steps:

Step 1: Since  $(a_n)_{n \in \mathbb{N}}$  is Cauchy, there exists some  $n_0 \in \mathbb{N}$  s.t.  $\forall n, m \geq n_0, |a_n - a_m| < 1$  (we apply the definition of a Cauchy sequence for  $\varepsilon = 1$ ).  
 ↓ Show that  $(a_n)_{n \in \mathbb{N}}$  is bounded.  
 (works in every metric space)

In particular,  $|a_n - a_{n_0}| < 1 \quad \forall n \geq n_0$ ,  
 i.e.  $a_n \in (a_{n_0} - 1, a_{n_0} + 1)$ ,  $\forall n \geq n_0$ .



So:  $a_n \in (\min\{a_1, a_2, \dots, a_{n_0-1}, a_{n_0}-1\}, \max\{a_1, a_2, \dots, a_{n_0-1}, a_{n_0}+1\})$

$\forall n \in \mathbb{N}$ .

So,  $(a_n)_{n \in \mathbb{N}}$  is bounded.

(5)

Step 2: Since  $(a_n)_{n \in \mathbb{N}}$  is bounded, it has a convergent subsequence (by the Bolzano-Weierstrass theorem).

⚠ Not true in every metric space!

It thus suffices to show the following:

Holds in every metric space.

If  $(b_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, and it has a subsequence  $(b_{k_n})_{n \in \mathbb{N}}$  with  $b_{k_n} \xrightarrow{n \rightarrow \infty} b \in \mathbb{R}$ , then  $b_n \xrightarrow{n \rightarrow \infty} b$ .



Proof: Let  $\varepsilon > 0$ .

Since  $b_{k_n} \xrightarrow{n \rightarrow \infty} b$ , there exists some  $n_1 \in \mathbb{N}$ , s.t.:

$$\forall n \geq n_1, |b_{k_n} - b| < \frac{\varepsilon}{2} \quad \text{①}$$

And:

Since  $(b_n)_{n \in \mathbb{N}}$  is Cauchy, there exists  $n_2 \in \mathbb{N}$ , s.t.:

$$\forall n, m \geq n_2, |b_n - b_m| < \frac{\varepsilon}{2} \quad \text{②}$$

So, for all  $n \geq n_0 := \max\{n_1, n_2\}$ ,

(6)

$$|b_n - b| = |(b_n - b_{k_n}) + (b_{k_n} - b)| \leq |b_n - b_{k_n}| + |b_{k_n} - b|$$

$$< \frac{\epsilon}{2},$$

because  $k_n \geq n$  then  $N$ ,

so  $k_n, n \geq n_0 \geq n_0$   
when  $n \geq n_0$ .

$$< \frac{\epsilon}{2},$$

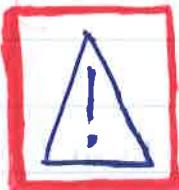
because  $k_n \geq n$  then  $N$ ,

so  $k_n \geq n_0 \geq n_1$

$\forall n \geq n_0$ . ■

By Step 2,  $(a_n)_{n \in \mathbb{N}}$  has a subsequence  $(a_{k_n})_{n \in \mathbb{N}}$ ,

with  $a_{k_n} \rightarrow a$ , for some  $a \in \mathbb{R}$ . By the above,  
 $a_n \rightarrow a$ , so  $(a_n)_{n \in \mathbb{N}}$  converges. ■



A common mistake:

It holds that, if  $(a_n)_{n \in \mathbb{N}}$  is Cauchy,

then  $a_{n+1} - a_n \xrightarrow{n \rightarrow \infty} 0$ . (Exercise!)

However, the converse is (not) true:

find one!

there exists  $(a_n)_{n \in \mathbb{N}}$  with  $a_{n+1} - a_n \xrightarrow{n \rightarrow \infty} 0$ ,  
but with  $(a_n)_{n \in \mathbb{N}}$  not Cauchy!

Thus: for  $(a_n)_{n \in \mathbb{N}}$  to be Cauchy, we need all terms from  
some index onwards to be close to each other; not just consecutive  
terms!!!